SATISFACTION OF EXISTENTIAL THEORIES IN FINITELY PRESENTED GROUPS AND SOME EMBEDDINGS THEOREMS

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Abstract. The main result is that for every recursively enumerable existential consistent theory Γ (in the usual language of group theory), there exists a finitely presented SQ-universal group H such that Γ is satisfied in every nontrivial quotient of H. Furthermore if Γ is satisfied in some group with soluble word problem, then one can take H with soluble word problem. We characterize the finitely generated groups with soluble word problem as the finitely generated groups G for which there exists a finitely presented group H whose all nontrivial quotients embeds G. We prove also that for every countable group G, there exists a 2-finitely generated SQ-universal group H such that every nontrivial quotient of H embeds G.

1. Introduction

In this paper, we denote by $\mathcal{L}_{gp}$ the usual language of group theory with multiplication, inverse, and neutral element. We denote by T_{gp}, the theory of groups in the language $\mathcal{L}_{gp}$. By a theory in $\mathcal{L}_{gp}$, we mean a theory consistent with T_{gp}. In [OH03], we studied groups G for which there exists an existential theory Γ such that G is embeddable in every group satisfying Γ. Part of our interest on this groups comes from the fact that the finitely generated ones are exactly those having a soluble word problem [OH03]. Since every finitely generated group with soluble word problem is embeddable in a finitely presented group, it becames natural to study satisfaction of existential theories in finitely presented groups. It is well-known that an existential sentence, in group theory language, is satisfied in some finitely presented group. What can be said on the satisfaction of existential theories in finitely presented groups and their quotients in general ? This question was discussed in the Ph.D. thesis of the author [OH03], and most of the results of the present paper are extracted from that thesis.

We recall some notions needed in the sequel. Let $X = \{x_i | i \in \mathbb{N}\}$. A countable group G is said *recursively presented*, if G admits a presentation $G = \langle X | P(X) \rangle$ such that $P(X)$ is recursively enumerable; and it is said to have a *soluble word problem*, if it admits a presentation $G = \langle X | P(X) \rangle$ for which the set of words $w(\bar{x})$ on $X^{\pm 1}$ such that $w(\bar{x}) = 1$ in G is recursive. A subgroup $H \leq G$ is said to have a *soluble generalized word problem in* G (relatively to the presentation $G = \langle X | P(X) \rangle$) if the set of words $w(\bar{x})$ on $X^{\pm 1}$ such that $w \in H$ is recursive.

Recall also that a group G is called *SQ-universal* if every countable group is embeddable in a quotient of G. Higman, Neumann, and Neumann proved that the free group of rank 2 is SQ-universal [HBN49]. More generally, the free product of two nontrivial groups, with the single exception of the free product of two copies of the cyclic group of order two, is SQ-universal [LS77]. Roughly speaking, the SQ-universality can be seen as measure of the bigness of the group. In this sense, the

following theorem shows that any existential and recursively enumerable theory is true in every nontrivial quotient of a finitely presented group having many quotients.

Theorem 1.1. [OH03] *For every existential and recursively enumerable theory Γ, there exists a finitely presented 2-finitely generated SQ-universal group H such that Γ is satisfied in every nontrivial quotient of H. Furthermore if Γ is satisfied in some group with soluble word problem, then one can take H with soluble word problem.*

In Section 8 (1) we give an example of an existential theory Γ for which there is no finitely presented group whose nontrivial quotients satisfy Γ. However, given an existential theory Γ, we have the following property, which follows from the above theorem: there exists an existential and recursively enumerable theory Γ' such that $\Gamma \subseteq \Gamma'$ if and only if there exists a finitely presented group H whose all nontrivial quotients satisfy Γ (see Section 8 (1) for more details).

There are several characterizations of finitely generated groups with soluble word problem in the literature. One of them, due to Boone and Higman [BH74], states that the finitely generated groups with soluble word problem are those which can be embedded in a simple subgroup of a finitely presented group. In this spirit, the next theorem gives a new characterization of finitely generated groups with soluble word problem.

Theorem 1.2. [OH03] *A finitely generated group G has a soluble word problem, if and only if, there exists a finitely presented group H such that G is embeddable in every nontrivial quotient of H.*

It is a classical theorem of Higman, Neumann and Neumann that every countable group is embeddable in a finitely generated one [LS77]. The following is a strong version of that theorem.

Theorem 1.3. *For every countable group G, there exists a 2-finitely generated SQ-universal group H such that G is embeddable in every nontrivial quotient of H.*

Also we notice that we cannot ensure that H is recursively presented when G is recursively presented (see Section 8 (3)).

The paper is organized as follows. In the next section we record the material that we require from Model Theory and Combinatorial Group Theory. In Section 3 we prove preparatory lemmas about existential theories. The goal of that section is to reduce Γ of Theorem 1.1 to a one existential sentence. In Section 4 we prove two lemmas about length functions in groups, which are interesting independently of the subject of the paper. Lemma 4.2 is needed to prove that some subgroups of free groups or free products have a soluble generalized word problem and this will be done in Section 5. This is then used, to prove the second half part of Theorem 1.1; that is, one can take H with soluble word problem whenever Γ is satisfied in some group with soluble word problem. Section 6 is devoted to some embeddings in groups. In Section 7 we prove the main results, and in Section 8 we finish with some remarks.

2. Prerequisites

Our reference book for model theoretics notions is [Hod93], but the reader can also consult [CK73, Mar02]. As noticed in the introduction, we work in the language $\mathcal{L}_{gp}$ of group theory with multiplication, inverse, and neutral element and we denote

by T_{gp} the theory of groups. By an *existential theory* we mean a set of existential sentences consistent with the group theory T_{gp}, in the language $\mathcal{L}_{gp}$. Recall that an existential sentence is said *primitive*, if it is of the form

$$\exists \bar{x}(\bigwedge_{w \in P} w(\bar{x}) = 1 \wedge \bigwedge_{v \in N} v(\bar{x}) \neq 1)),$$

where P, N are finite sets of words on the variables $\bar{x} = \{x_1, \ldots, x_n\}$ and their inverses. In the sequel, to simplify notations, if $\bar{x} = \{x_1, \ldots, x_n\}$ is a set of variables we denote by $\bar{x}^{\pm 1}$ the union of $\bar{x}$ and its inverses.

A theory T is said to have the *Joint Embedding Property* (abbreviated JEP) if any two models of T are embeddables in a third. If a theory T has the JEP, and when two existential sentences ψ_1, ψ_2 are consistent with T, then $\psi_1 \wedge \psi_2$ is also consistent with T. This last property will be used freely without explicit mention.

We recall some well-known theorems in combinatorial group theory, which can all be found in [LS77]. We assume that the reader is familiar with free groups, free product and free product with amalgamation. If G_1, G_2 are groups, it is well-known that the natural morphisms $\phi_1 : G_1 \to G_1 * G_2$, $\phi_2 : G_2 \to G_1 * G_2$ are embeddings. Consequently Group Theory has the JEP. We will need the following famous theorem of Higman.

Fact 2.1. [Hig61] (Higman's Embedding Theorem) *Every recursively presented group is embeddable in a finitely presented one.* □

We will also use the following theorem of Clapham.

Fact 2.2. [Cla67] *Every countable group with soluble word problem is embeddable in a finitely presented group with soluble word problem.* □

Fact 2.3. [BH74] (Boone-Higman) *Every countable group with soluble word problem is embeddable in a simple group with soluble word problem, which is embeddable in a finitely presented group.* □

By combining Boone-Higman's theorem and Clapham's theorem we get:

Fact 2.4. *Every countable group with soluble word problem is embeddable in a simple group with soluble word problem, which is embeddable in a finitely presented group with soluble word problem.* □

There is characterization of finitely generated group with soluble word problem in term of embeddings in *existentially closed groups*. In this paper, one will not need the definition of existentially closed groups but one will use some of their well-known properties. The raider interested by existentially closed groups might consults the book of Higman and Scott [HS88]. B.H. Newmann proved that every finitely generated group with soluble word problem is embeddable in every existentially closed group [LS77]. Macintyre proved the converse of Neumann's theorem.

Fact 2.5. [Mac72] (Macintyre's theorem) *A finitely generated group which is embeddable in every exitentially closed group has a soluble word problem.* □

We shall make use of the following well-known property in model theory.

Fact 2.6. *Every existentially closed group satisfies every existential (consistent) sentence.* □

The following result can be extracted from [LS77, Chapter IV, Corollary 2.2].

Fact 2.7. [LS77] *Let $G_1 = \langle X|P_1(X)\rangle$, $G_2 = \langle X|P_2(X)\rangle$ be groups and $\mathcal{A} \leq G_1$, $\mathcal{B} \leq G_2$ be subgroups. Let $\phi : \mathcal{A} \cong \mathcal{B}$ be an isomorphism. If the following conditions are satisfied,*

(1) *G_1 and G_2 have a soluble word problem,*
(2) *$\mathcal{A}$ (resp. $\mathcal{B}$) has a soluble generalized word problem in G_1(resp. G_2),*
(3) *ϕ and ϕ^{-1} are computable; that is for any word $w \in \mathcal{A}$ one can effectively calculates $\phi(w)$ (similarly for ϕ^{-1}),*

then the group $G_1 *_{\mathcal{A}=\mathcal{B}} G_2$ has a soluble word problem. □

To get the solvability of the generalized word problem in some groups, we prove Lemma 5.1 and Lemma 5.2 in Section 4. For that raison, we shall make use of some results of Lyndon about length functions in groups [Lyn63]. Lyndon has introduced five axioms to study length functions on groups in abstract setting. In fact we need just three of this axioms, as that was considered in [Chi01].

A *valuated group* G is a group with a length function $\ell : G \to \mathbb{N}$ satisfying the following axioms (considered by Lyndon [Lyn63]).

$\mathcal{A}_1$. $\ell(1) = 0$,
$\mathcal{A}_2$. for all $x \in G$, $\ell(x^{-1}) = \ell(x)$,
$\mathcal{A}_3$. for all $x, y, z \in G$, $c(x, y) \geq \min\{c(x, z), c(z, y)\}$,
where $c(x, y) = \frac{1}{2}(\ell(x) + \ell(y) - \ell(xy^{-1}))$, for all $x, y \in G$.

Any nontrivial group G can be equipped by a length function which return it as a valuated group. By defining

$$\ell(g) = \begin{cases} 1, & \text{if } g \neq 1; \\ 0, & \text{if } g = 1. \end{cases}$$

we get that (G, ℓ) is a valuated group. This length function will be called the *elementary length function* of G.

More natural examples of valuated groups can be found in the class of free groups and free products and this was the origin of the notion of the length function introduced by Lyndon. If G is a free group and by denoting $|g|$ the length of the reduced word representing g relatively to a given basis of G, then $(G, |.|)$ satisfies the five axioms introduced by Lyndon ([Lyn63, Proposition (2.7)]), and thus $(G, |.|)$ is a valuated group. Similarly, if G is a free product and by denoting $|g|$ the length of the normal form of g, relatively to a given free decomposition of G, then $(G, |.|)$ satisfies the five axioms introduced by Lyndon ([Lyn63, Proposition (2.8)]), and $(G, |.|)$ is also a valuated group. For more details we refer the reader to [Lyn63] or to [LS77, Pages 66-67].

We will need the following fact which can be extracted from [Chi01].

Fact 2.8. [Chi01, Chapter 5, Proposition 1.1] *Let $(G_1, \ell_1), (G_2, \ell_2)$ be a valuated groups. Let $G = G_1 * G_2$ be their free product and let ℓ be the following length function on G*

$$\ell(g) = \sum_{i=1}^{n} \ell_i(g_i), \quad \textit{where } g_1, \ldots, g_n \textit{ is the normal form of } g,$$

$$\textit{and } \ell_i(g_i) = \ell_1(g_i) \textit{ if } g_i \in G_1, \ \ell_i(g_i) = \ell_2(g_i) \textit{ if } g_i \in G_2.$$

Then (G, ℓ) is a valuated group. □

Thus the natural length function on a free product $G_1 * G_2$, is just the length defined above relatively to the valuated groups $(G_1, \ell_1), (G_2, \ell_2)$ where ℓ_1 (resp. ℓ_2) is the elementary length function of G_1 (resp. G_2).

3. Existential theories

The following lemma can easily be derived from [HS88].

Lemma 3.1. [HS88, Lemma 1.5] *For every group G and for every sequence $(g_i | 1 \leq i \leq n)$ of elements of G and for every g in G, the following properties are equivalents:*
(1) *$G \models (g_i = 1 \Rightarrow g = 1)$, for every $1 \leq i \leq n$.*
(2) *The equations $x_i^{-1} g_i x_i y_i^{-1} g_i y_i = g$ $(1 \leq i \leq n)$ have a solution in some extension of G.* □

Lemma 3.2. *Let Γ be an existential recursively enumerable theory. Then there exists a* recursively enumerable existential primitive *theory Γ^* such that $T_{gp} \cup \Gamma^* \vdash \Gamma$.*

Proof. If ψ is a given existential sentence, then one can determines *effectively* a normal disjunctive form of ψ. Since Γ is recursively enumerable, without loss of generality, we may assume that Γ is enumerated in such a way that every ψ in Γ is in normal disjunctive form.

The idea of the proof will be clear throughout a simple example. Suppose that ψ is of the form

$$\exists \bar{x}_1((w_1(\bar{x}_1) = 1 \wedge v_1(\bar{x}_1) \neq 1) \bigvee \exists \bar{x}_2(w_2(\bar{x}_2) = 1 \wedge v_2(\bar{x}_2) \neq 1)),$$

where w_1, w_2 (resp. v_1, v_2) are words over $\bar{x}_1^{\pm 1}$ (resp. $\bar{x}_2^{\pm 1}$). Renaming the variables if necessary, we may assume without loss of generality that $\bar{x}_1 \cap \bar{x}_2 = \emptyset$. We introduce a new variables s_1, t_1, z_1 and t_2, s_2, z_2 and we consider the sentence ψ^* defined by

$$\exists \bar{x}_1 \exists \bar{x}_2 \exists z_1 \exists z_2 \exists s_1 \exists s_2 \exists t_1 \exists t_2 ((\bigwedge_{1 \leq i \leq 2} w_i(\bar{x}_i) = 1 \wedge s_i^{-1} v_i(\bar{x}_i) s_i t_i^{-1} v_i(\bar{x}_i) t_i = z_i)$$

$$\wedge z_1 . z_2 \neq 1).$$

Then ψ^* is an existential primitive sentence. Clearly $T_{gp} \vdash \psi^* \Rightarrow \psi$. Let us show that ψ^* is consistent. Since ψ is consistent, there exists a nontrivial group G in which we can find $\bar{a}$ and $\bar{b}$ such that

$$G \models (w_1(\bar{a}) = 1 \wedge v_1(\bar{a}) \neq 1), \text{ or } G \models (w_2(\bar{b}) = 1 \wedge v_2(\bar{b}) \neq 1).$$

Let g be an nontrivial element of G. Then

$$G \models (v_1(\bar{a}) = 1 \Rightarrow g = 1), \text{ or } G \models (v_2(\bar{b}) = 1 \Rightarrow g = 1).$$

By Lemma 3.1, there exists an extension H of G containing s, t such that

$$H \models (s^{-1} v_1(\bar{a}) s t^{-1} v_1(\bar{a}) t = g) \tag{1}$$

or

$$H \models (s^{-1} v_2(\bar{b}) s t^{-1} v_2(\bar{b}) t = g). \tag{2}$$

If (1) is satisfied then by taking

$$s_1 = s, t_1 = t, z_1 = g, \bar{x}_1 = \bar{a} \text{ and } s_2 = t_2 = z_2 = 1, \bar{x}_2 = \bar{1},$$

we get

$$H \models ((\bigwedge_{1\leq i\leq 2} w_i(\bar{x}_i) = 1 \wedge s_i^{-1}v_i(\bar{x}_i)s_it_i^{-1}v_i(\bar{x}_i)t_i = z_i) \wedge z_1.z_2 \neq 1).$$

The case (2) can be treated similarly. Therefore ψ^* is consistent.

Let us return to the general case. If $\psi \in \Gamma$ we can write ψ as follows

$$\psi = \bigvee_{1\leq i\leq n} \exists \bar{x}_i(\bigwedge_{w\in W_i} w(\bar{x}_i) = 1 \wedge \bigwedge_{v\in V_i} v(\bar{x}_i) \neq 1),$$

Where W_i and V_i are finite sets of words over $\bar{x}_i^{\pm 1}$. As before, we may assume without loss of generality that $\bar{x}_i \cap \bar{x}_j = \emptyset$, for $i \neq j$ and $1 \leq i, j \leq n$. If there exists i such that the fragment $\bigwedge_{v\in V_i} v(\bar{x}_i) \neq 1$ does not appears, then we let $\psi^* = \exists x(x = x)$. If it is not the case, we introduce the following new sequences of variables:

$$(t_{i,v}|v \in V_i, 1 \leq i \leq n),$$

$$(s_{i,v}|v \in V_i, 1 \leq i \leq n),$$

$$(z_i|1 \leq i \leq n).$$

For every $1 \leq i \leq n$, let $\varphi_i(\bar{x}_i; z_i)$ be the following formula

$$\exists_{1\leq i\leq n}^{v\in V_i} s_{i,v} \exists_{1\leq i\leq n}^{v\in V_i} t_{i,v}(\bigwedge_{w\in W_i} w(\bar{x}_i) = 1 \wedge \bigwedge_{v\in V_i} s_{i,v}^{-1}v(\bar{x}_i)s_{i,v}t_{i,v}^{-1}v(\bar{x}_i)t_{i,v} = z_i).$$

Let

$$\psi^* = \exists \bar{x}_1 \cdots \exists \bar{x}_n \exists z_1 \cdots \exists z_n(\bigwedge_{1\leq i\leq n} \varphi_i(\bar{x}_i; z_i) \wedge z_1 \cdots z_n \neq 1).$$

Then ψ^* is existential primitive. Since Γ is recursively enumerable and since the procedure which produces ψ^* form ψ is effective, the set $\Gamma^* = \{\psi^* | \psi \in \Gamma\}$ is recursively enumerable.

Let us prove that Γ^* is consistent and $T_{gp} \vdash \psi^* \Rightarrow \psi$ (thus $T_{gp} \cup \Gamma^* \vdash \Gamma$). As it was remarked before, T_{gp} has the JEP and consequently, to prove that Γ^* is consistent it is sufficient to prove that for every $\psi \in \Gamma$, ψ^* is consistent.

Let us show first that for every $\psi \in \Gamma$, ψ^* is consistent. Since ψ is consistent, there exists a nontrivial group G which satisfies ψ. Therefore there exist $1 \leq i_0 \leq n$ and $\bar{a}$ in G such that $G \models (\bigwedge_{w\in W_{i_0}} w(\bar{a}) = 1 \wedge \bigwedge_{v\in V_{i_0}} v(\bar{a}) \neq 1)$.

Let g be a nontrivial element of G. Since for every $v \in V_{i_0}$,

$$G \models (v(\bar{a}) = 1 \Rightarrow g = 1),$$

by Lemma 3.1, there exists an extension H of G, containing two sequences

$$(b_v|v \in V_{i_0}),\ (d_v|v \in V_{i_0})$$

such that

$$H \models \bigwedge_{v\in V_{i_0}} b_v^{-1}v(\bar{a})b_vd_v^{-1}v(\bar{a})d_v = g.$$

Therefore $H \models \varphi_{i_0}(\bar{a}; g)$. Now if we let

$$\bar{x}_{i_0} = \bar{a}, s_{i_0,v} = b_v, t_{i_0,v} = d_v, z_{i_0} = g, \text{ for } v \in V_{i_0},$$

$$\bar{x}_i = \bar{1}, s_{i,v} = t_{i,v} = z_i = 1 \text{ for every } i \neq i_0, \text{ and } v \in V_i,$$

then we get $H \models \psi^*$. Thus ψ^* is consistent.

We show now that $T_{gp} \vdash \psi^* \Rightarrow \psi$. Let $G \models \psi^*$. Then there exists $1 \leq i_0 \leq n$ and $z_{i_0} \in G$ such that $z_{i_0} \neq 1$ and $G \models \exists \bar{x}_{i_0} \varphi_{i_0}(\bar{x}_{i_0}; z_{i_0})$. Thus $G \models \psi$. This ends the proof of the lemma. □

Lemma 3.3. *Let Γ be an existential primitive recursively enumerable theory. Then there exists an existential sentence ψ such that $T_{gp} \cup \{\psi\} \vdash \Gamma$.*

Proof. Put $\Gamma = \{\psi_i \mid i \in \omega\}$. We can write ψ_i as follows.

$$\psi_i = \exists \bar{x}_i (\bigwedge_{w \in W_i} w(\bar{x}_i) = 1 \wedge \bigwedge_{v \in V_i} v(\bar{x}_i) \neq 1).$$

We can suppose without loss of generality that $\bar{x}_i \cap \bar{x}_j = \emptyset$, for $j \neq i$.

Let

$$y,$$
$$(s_{i,v} | v \in V_i, i \in \omega),$$
$$(t_{i,v} | v \in V_i, i \in \omega),$$

be a new sequences of variables.

Let

$$P_i = \{w(\bar{x}_i) \mid w \in W_i\} \cup \{s_{i,v}^{-1} v(\bar{x}_i) s_{i,v} t_{i,v}^{-1} v(\bar{x}_i) t_{i,v} y^{-1} \mid v \in V_i\}.$$

Let $P = \bigcup_{i \in \omega} P_i$.

Since Γ is recursively enumerable, P is recursively enumerable. Therefore the following group

$$G = \langle \{y, \bar{x}_i, s_{i,v}, t_{i,v} | i \in \omega, v \in V_i\} \mid P \rangle.$$

is recursively presented.

We claim that $G \models y \neq 1$. It is sufficient to find a group H and a an morphism $\phi : G \to H$ such that $\phi(y) \neq 1$. Since Γ is consistent there exists a group H_{-1} satisfying Γ. Therefore H_{-1} contains a sequence $(\bar{a}_i | i \in \omega)$ such that

$$H_{-1} \models \bigwedge_{w \in W_i} w(\bar{a}_i) = 1 \wedge \bigwedge_{v \in V_i} v(\bar{a}_i) \neq 1, \text{ for every } i \in \omega.$$

Let b be a nontrivial element of H_{-1}. Then

$$H_{-1} \models \bigwedge_{v \in V_0} (v_0(\bar{a}_0) = 1 \Rightarrow b = 1)$$

and thus, by Lemma 3.1, there exists an extension H_0 of H_{-1} containing two sequences $(c_{0,v} | v \in V_0)$, $(d_{0,v} | v \in V_0)$ such that

$$H_0 \models c_{0,v}^{-1} v(\bar{a}_0) c_{0,v} c_{0,v}^{-1} v(\bar{a}_0) c_{0,v} b^{-1} = 1.$$

Since $H_{-1} \models (\bigwedge_{v \in V_1} (v_1(\bar{a}_1) = 1 \Rightarrow b = 1)$ then

$$H_0 \models (\bigwedge_{v \in V_1} (v_1(\bar{a}_1) = 1 \Rightarrow b = 1)$$

and thus, by the same argument as above, we find an extension H_1 of H_0 containing two sequences $(c_{1,v} | v \in V_1)$, $(d_{1,v} | v \in V_1)$ such that

$$H_1 \models c_{1,v}^{-1} v(\bar{a}_1) c_{1,v} d_{1,v}^{-1} v(\bar{a}_1) d_{1,v} b^{-1} = 1.$$

Containing in this manner, we get a sequence of groups $(H_i | i \in \omega)$ such that H_i is embeddable in H_{i+1} and H_i contains two sequences $(c_{i,v} | v \in V_i)$, $(d_{i,v} | v \in V_i)$ such that

$$H_i \models c_{i,v}^{-1} v(\bar{a}_i) c_{i,v} d_{i,v}^{-1} v(\bar{a}_i) d_{i,v} b^{-1} = 1.$$

Put $H = \bigcup_{i\in\omega} H_i$. Let us show that H has the required property. By the well-known properties of presentation of groups, there exists a morphism $\phi : G \to H$ satisfying

$$\phi(y) = b,\ \phi(\bar{x}_i) = \bar{a}_i,\ \phi(s_{i,v}) = c_{i,v},\ \phi(t_{i,v}) = d_{i,v},\quad \text{for every } i \in \omega, v \in V_i,$$

and since $b \neq 1$ we have the desired conclusion. This ends the proof of our claim.

Since G is recursively presented, by Higman's Embedding Theorem, G is embeddable in a finitely presented group $K = \langle \bar{a} \mid u_1(\bar{a}), \cdots, u_n(\bar{a})\rangle$. Let $b(\bar{a})$ be a word such that $K \models b = b(\bar{a})$. Let ψ be the sentence

$$\exists \bar{x}(u_1(\bar{x}) = 1 \wedge \cdots \wedge u_n(\bar{x}) = 1 \wedge b(\bar{x}) \neq 1).$$

We claim that $T_{gp} \cup \{\psi\} \vdash \Gamma$. Let L be a group satisfying ψ and let $\bar{z}$ in L such that

$$L \models u_1(\bar{z}) = 1 \wedge \cdots \wedge u_n(\bar{z}) = 1 \wedge b(\bar{z}) \neq 1.$$

Then there exists a morphism $\phi : K \to L$ such that $\phi(\bar{a}) = \bar{z}, \phi(b) = b(\bar{z}) \neq 1$. Thus there exists a morphism $\xi : G \to L$ such that $\xi(y) = b(\bar{z}) \neq 1$. Therefore L satisfies

$$w(\xi(\bar{x}_i)) = 1,\ \text{for every } i \in \omega,\ w \in W_i,$$
$$\xi(s_{i,v})^{-1} v(\xi(\bar{x}_i))\xi(s_{i,v})\xi(t_{i,v})^{-1} v(\xi(\bar{x}_i))\xi(t_{i,v})\xi(y)^{-1} = 1,\ \text{for every } i \in \omega, v \in V_i,$$
$$\xi(y) \neq 1.$$

Since $\xi(y) \neq 1$ we get $v(\xi(\bar{x}_i)) \neq 1$ for every $i \in \omega$ and $v \in V_i$. Thus $L \models \psi_i$ for every $i \in \omega$. □

4. Length functions

The following lemma was proved by Lyndon [Lyn63, Lemma (6.1)] under strong assumptions. We give here another proof which works for valuated groups.

Lemma 4.1. *Let (G, ℓ) be a valuated group. Let $n \geq 2$ and $(g_1, \ldots, g_n)$ a sequence in G satisfying*

$$c(g_{i-1}, g_i^{-1}) + c(g_i, g_{i+1}^{-1}) < \ell(g_i),\ \textit{for all } i \textit{ with } 1 < i < n. \tag{1}$$

Then

$$\ell(g_1 \cdots g_n) = \sum_{i=1}^{n} \ell(g_i) - 2\sum_{i=1}^{n-1} c(g_i, g_{i+1}^{-1}). \tag{2}$$

Proof. The proof is by induction on $n \geq 2$. The formula (2) is obvious for $n = 2$. Let (g_1, g_2, g_3) in G satisfying (1). By (1) we have $c(g_2, g_3^{-1}) < -c(g_1, g_2^{-1}) + \ell(g_2)$, and since

$$c(g_1g_2, g_2) + c(g_1, g_2^{-1}) = \frac{1}{2}(\ell(g_1g_2) + \ell(g_2) - \ell(g_1) + \ell(g_1) + \ell(g_2) - \ell(g_1g_2)) = \ell(g_2),$$

we get

$$c(g_2, g_3^{-1}) < c(g_1g_2, g_2). \tag{3}$$

By axiom $\mathcal{A}_3$ and (3) we have

$$c(g_1g_2, g_3^{-1}) \geq \min\{c(g_1g_2, g_2), c(g_2, g_3^{-1})\} \geq c(g_2, g_3^{-1}).$$

Now if we suppose that $c(g_2, g_3^{-1}) < c(g_1g_2, g_3^{-1})$ then by (3) we get

$$c(g_2, g_3^{-1}) < \min\{c(g_1g_2, g_3^{-1}), c(g_1g_2, g_2)\},$$

contradicting axiom $\mathcal{A}_3$. Therefore

$$c(g_1g_2, g_3^{-1}) = c(g_2, g_3^{-1}). \tag{4}$$

By simplifying the expression of (4), we find

$$\ell(g_1.g_2.g_3) = \ell(g_1.g_2) + \ell(g_3) - 2c(g_2, g_3^{-1}).$$

Since $\ell(g_1.g_2) = \ell(g_1) + \ell(g_2) - 2c(g_1, g_2^{-1})$, we get

$$\ell(g_1.g_2.g_3) = \ell(g_1) + \ell(g_2) - 2c(g_1, g_2^{-1}) + \ell(g_3) - 2c(g_2, g_3^{-1}),$$

and thus (2) is satisfied as required.

Now suppose that (2) is true for all sequences of length k, $2 \leq k \leq n$, which satisfies (1) and let $(g_1, \ldots, g_{n+1})$ be a sequence satisfying (1) such that $n+1 \geq 4$.

Put $z = g_3 \cdots g_{n+1}$. We claim $c(g_2, z^{-1}) = c(g_2, g_3^{-1})$. By induction we have

$$\ell(g_2 z) = \sum_{i=2}^{n+1} \ell(g_i) - 2\sum_{i=2}^{n} c(g_i, g_{i+1}^{-1}),$$

and

$$\ell(z) = \sum_{i=3}^{n+1} \ell(g_i) - 2\sum_{i=3}^{n} c(g_i, g_{i+1}^{-1}).$$

Therefore we have

$$\ell(z) - \ell(g_2 z) = -\ell(g_2) + (\ell(g_2) + \ell(g_3) - \ell(g_2 g_3)) = \ell(g_3) - \ell(g_2 g_3),$$

and thus

$$c(g_2, z^{-1}) = \frac{1}{2}(\ell(g_2) + \ell(g_3) - \ell(g_2 g_3) = c(g_2, g_3^{-1}),$$

as claimed.

Therefore $c(g_1, g_2^{-1}) + c(g_2, z^{-1}) < \ell(g_2)$. By the case $n = 3$, we get

$$\ell(g_1 g_2 z) = \ell(g_1) + \ell(g_2) + \ell(z) - 2c(g_1, g_2^{-1}) - 2c(g_2, z^{-1}). \tag{5}$$

By replacing in (5), $\ell(z)$ by

$$\sum_{i=3}^{n+1} \ell(g_i) - 2\sum_{i=3}^{n} c(g_i, g_{i+1}^{-1})$$

and $c(g_2, z^{-1})$ by $c(g_2, g_3^{-1})$ we obtain the required formula. □

We are ready to prove the following lemma, needed in the proof of Lemma 5.1 and Lemma 5.2 in the next section.

Lemma 4.2. *Let (G, ℓ) be a valuated group. Let $n \geq 2$ and $(g_1, \ldots, g_n)$ a sequence of G satisfying*

$$\ell(g_i g_{i+1}) > \ell(g_i), \ell(g_{i+1}) \text{ for all } i \text{ with } 1 \leq i \leq n-1. \tag{1}$$

Then

$$\ell(g_1 \cdots g_n) \geq \ell(g_i), \text{ for all } i \text{ with } 1 \leq i \leq n, \text{ and } \ell(g_1 \cdots g_n) \geq n. \tag{2}$$

Proof. We claim that

$$c(g_{i-1}, g_i^{-1}) + c(g_i, g_{i+1}^{-1}) < \ell(g_i), \qquad \text{for all } i \text{ with } 1 < i < n. \tag{3}$$

We have

$$c(g_{i-1}, g_i^{-1}) + c(g_i, g_{i+1}^{-1}) =$$

$$\frac{1}{2}(\ell(g_{i-1}) + \ell(g_i) - \ell(g_{i-1}g_i) + \ell(g_i) + \ell(g_{i+1}) - \ell(g_ig_{i+1})).$$

Since $\ell(g_{i-1}) - \ell(g_{i-1}g_i) < 0$ and $\ell(g_{i+1}) - \ell(g_ig_{i+1}) < 0$, we get

$$\frac{1}{2}(\ell(g_{i-1}) + \ell(g_i) - \ell(g_{i-1}g_i) + \ell(g_i) + \ell(g_{i+1}) - \ell(g_ig_{i+1})) < \frac{1}{2}(\ell(g_i) + \ell(g_i)),$$

and we find (3) as claimed.

Therefore by Lemma 4.1, for every sequence $(g_1, \dots, g_n)$ satisfying (1) we have

$$\ell(g_1 \cdots g_n) = \sum_{i=1}^{n} \ell(g_i) - 2\sum_{i=1}^{n-1} c(g_i, g_{i+1}^{-1}). \tag{4}$$

The proof of (2) is by induction on n. The result is clear for $n = 2$.

For $n + 1$. By (4) we have

$$\ell(g_1 \cdots g_{n+1}) = \sum_{i=1}^{n+1} \ell(g_i) - 2\sum_{i=1}^{n} c(g_i, g_{i+1}^{-1}). \tag{5}$$

Developing the expression (5) we find

$$\ell(g_1 \cdots g_{n+1}) = \sum_{i=1}^{n} \ell(g_i) - 2\sum_{i=1}^{n-1} c(g_i, g_{i+1}^{-1}) + \ell(g_{n+1}) - 2c(g_n, g_{n+1}^{-1})$$

$$= \ell(g_1 \cdots g_n) + \ell(g_ng_{n+1}) - \ell(g_n).$$

Since $\ell(g_ng_{n+1}) - \ell(g_n) > 0$ we get

$$\ell(g_1 \cdots g_{n+1}) \geq \ell(g_1 \cdots g_n) + 1. \tag{6}$$

By induction we have

$$\ell(g_1 \cdots g_n) \geq \ell(g_i) \text{ for all } i \text{ with } 1 \leq i \leq n$$

and thus, by (6) we get

$$\ell(g_1 \cdots g_{n+1}) \geq \ell(g_i) \text{ for all } i \text{ with } 1 \leq i \leq n.$$

Now since

$$\ell(g_1 \cdots g_{n+1}) = \ell(g_{n+1}^{-1} \cdots g_1^{-1})$$

we get by (6)

$$\ell(g_1 \cdots g_{n+1}) \geq \ell(g_{n+1}^{-1} \cdots g_2^{-1}),$$

and as before by induction, we have

$$\ell(g_{n+1}^{-1} \cdots g_2^{-1}) \geq \ell(g_{n+1})$$

and thus

$$\ell(g_1 \cdots g_{n+1}) \geq \ell(g_{n+1}).$$

Therefore we get the first part of (2).

The second part of (2)

$$\ell(g_1 \cdots g_n) \geq n$$

follows by induction and by (6). □

5. Generalized word problems in some groups

Lemma 5.1. *Let F_2 be the free group of rank 2 and with basis $\{b, c\}$. Let $\mathcal{A}$ be the subgroup of F_2 generated by the set*

$$X = \{b,\ c^{-1}b^{-1}cbc,\ c^{-2}b^{-1}cbc^2,\ c^{-(3+i)}bc^{(3+i)},\ i \in \lambda\},\ \text{where } \lambda \leq \omega.$$

Then $\mathcal{A}$ is free with basis X and it has a soluble generalized word problem in F_2.

Proof. For every $x \in F_2$, we let $|x|$ denotes the length of x relatively to the basis $\{b, c\}$. Using normal forms, we see that X satisfies

$$\text{if } (x_1, x_2) \text{ is a reduced sequence of } X^{\pm 1},\ \text{then } |x_1x_2| > |x_1|, |x_2|.$$

Since $(F_2, |.|)$ is a valuated group, by Lemma 4.2, if $(x_1, \ldots, x_n)$ is a reduced sequence of $X^{\pm 1}$ then

$$|x_1 \cdots x_n| \geq |x_i| \text{ for every } 1 \leq i \leq n,\ \text{and } |x_1 \cdots x_n| \geq n. \tag{1}$$

Therefore, if $(x_1, \ldots, x_n)$ is a reduced sequence of $X^{\pm 1}$, then $x_1 \cdots x_n \neq 1$ and thus $\mathcal{A}$ is free with basis X.

Let us show that $\mathcal{A}$ has a soluble generalized word problem in F_2. Let w be a word on the set $\{b, c\}^{\pm 1}$. Then $w \in \mathcal{A}$ if and only if there exists a reduced sequence $(x_1, \ldots, x_n)$ of $X^{\pm 1}$ such that $w = x_1 \cdots x_n$. By (1), we get that if $(x_1, \ldots, x_n)$ is a reduced sequence of $X^{\pm 1}$ such that $w = x_1 \cdots x_n$ then $|x_i| \leq |w|$, $1 \leq i \leq n$, and $n \leq |w|$.

Therefore

$$w \in \mathcal{A} \text{ if and only if}$$
$$\text{there exists a sequence } (x_1, \cdots, x_n) \text{ of } X^{\pm 1} \text{ satisfying}$$
$$|x_i| \leq |w| \text{ for } 1 \leq i \leq n,\ \text{and } n \leq |w|.$$

Clearly for every $m \in \mathbb{N}$, the set $\{x \in X^{\pm 1} \mid |x| \leq m\}$ is finite. It is also clear that the function $f(m) = \{x \in X^{\pm 1} \mid |x| \leq |w|\}$ is calculable. Since F_2 has a soluble word problem the function $w \mapsto |w|$ is also calculable. Therefore to decide if $w \in \mathcal{A}$ or not, one calculates all the sequences $(x_1, \cdots, x_n)$ of $X^{\pm 1}$ satisfying $|x_i| \leq |w|$, $1 \leq i \leq n$, and $n \leq |w|$, and one see if $w = x_1 \cdots x_n$ or not; this is possible as F_2 has a soluble word problem and $f(m) = \{x \in X^{\pm 1} \mid |x| \leq m\}$ is calculable. □

Lemma 5.2. *Let $\mathcal{K}$ be a group generated by $\{x_i \mid i \in \lambda\}$, where $\lambda \leq \omega$, and having a soluble word problem. Let F_2 be the free group of rank 2 and with basis $\{a, b\}$. Let $\mathcal{M} = \mathcal{K} * F_2$ and let $w \in \mathcal{K}$ be a nontrivial element. Let $\mathcal{B}$ be the subgroup of $\mathcal{M}$ generated by*

$$Y = \{b,\ a^{-1}ba,\ a^{-2}b^{-1}aba^2,\ a^{-3}[w, b]a^3,\ a^{-(4+i)}x_iba^{(4+i)},\ i \in \lambda\}.$$

Then $\mathcal{B}$ is free with basis Y and it has a soluble generalized word problem in $\mathcal{M}$.

Proof. The proof is very similar to that of the preceding lemma. Let us denote by ℓ_1 the elementary length function of $\mathcal{K}$. Then $(\mathcal{K}, \ell_1)$ is a valuated group. Denote by ℓ_2 the length function defined on F_2 as the length of normal forms relative to the basis $\{a, b\}$. Then the length function ℓ on $\mathcal{M} = \mathcal{K} * F_2$, as defined in Fact 2.8, makes $\mathcal{M}$ as a valuated group.

We see that

$$\text{if } (y_1, y_2) \text{ is a reduced sequence of } Y^{\pm 1},\ \text{then } \ell(y_1y_2) > \ell(y_1), \ell(y_2).$$

Therefore, as in the precedent lemma, by Lemma 4.2, if $(y_1, \dots, y_n)$ is a reduced sequence of $Y^{\pm 1}$ then

$$\ell(y_1 \cdots y_n) \geq \ell(y_i) \text{ for every } 1 \leq i \leq n, \text{ and } \ell(y_1 \cdots y_n) \geq n. \tag{1}$$

Therefore, if $(y_1, \dots, y_n)$ is a reduced sequence of $Y^{\pm 1}$, then $y_1 \cdots y_n \neq 1$ and thus $\mathcal{B}$ is free with basis Y.

Also we get, as in the previous lemma, that if w is a word on the set $\{x_i \mid i \in \lambda\}^{\pm 1} \cup \{b, c\}^{\pm 1}$, then

$$\begin{gathered} w \in \mathcal{B} \text{ if and only if} \\ \text{there exists a sequence } (y_1, \cdots, y_n) \text{ of } Y^{\pm 1} \text{ satisfying} \\ \ell(y_i) \leq \ell(w) \text{ for } 1 \leq i \leq n, \text{ and } n \leq \ell(w). \end{gathered}$$

We have also, for every $m \in \mathbb{N}$ the set $\{y \in Y^{\pm 1} \mid \ell(y) \leq m\}$ is finite and the function $f(m) = \{y \in Y^{\pm 1} \mid \ell(y) \leq m\}$ is calculable. Since $\mathcal{M}$ has a soluble word problem, the conclusion follows exactly as in the previous lemma. □

6. Some embeddings

We find the essential of the following lemma in [MI91]. We reprove it here as the properties (3) and (6) are not proved in [MI91].

Lemma 6.1. *Let $\mathcal{K}$ be a countable group and w an element of $\mathcal{K}$. Then there exists a group $\mathcal{K}_w$ and a morphism $\varphi : \mathcal{K} \to \mathcal{K}_w$ satisfying the following conditions:*

(1) *If $\mathcal{K} \models w \neq 1$ then φ is an embedding.*
(2) *The normal closure of $\varphi(w)$ in $\mathcal{K}_w$ is $\mathcal{K}_w$; in particular if $\mathcal{K}_w \models \varphi(w) = 1$, then $\mathcal{K}_w \cong 1$.*
(3) *If $\mathcal{K} \models w \neq 1$ then: let B be a normal subgroup of $\mathcal{K}$ such that $w \notin B$, and let $N(B)$ the normal closure of $\varphi(B)$ in $\mathcal{K}_w$, then $(\mathcal{K}/B)_w \cong \mathcal{K}_w/N(B)$.*
(4) *$\mathcal{K}_w$ is generated by two elements.*
(5) *If $\mathcal{K}$ is finitely presented then so is $\mathcal{K}_w$.*
(6) *If $\mathcal{K}$ has a soluble word problem then $\mathcal{K}_w$ has a soluble word problem.*

Proof. Put $\mathcal{K} = \langle \{x_i \mid i \in \lambda\} \mid P \rangle$, where $\lambda \leq \omega$, and such that if $\mathcal{K}$ is finitely generated then $\lambda < \omega$.

Let $\mathcal{K}_w$ be the group generated by $\{x_i \mid i \in \lambda\} \cup \{a, b, c\}$, and having as presentation the presentation of $\mathcal{K}$ together with the following relations

$$\begin{gathered} a^{-1}ba = c^{-1}b^{-1}cbc, \\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2, \\ a^{-3}[w, b]a^3 = c^{-3}bc^3, \\ a^{-(4+i)}x_iba^{(4+i)} = c^{-(4+i)}bc^{(4+i)}, \ i \in \lambda. \end{gathered}$$

Let $\varphi : \mathcal{K} \to \mathcal{K}_w$ be the natural morphism. We shall prove that $\mathcal{K}_w$ satisfies the enounced properties.

(1). Suppose that $w \neq 1$ in $\mathcal{K}$. Let F_2 be the free group with basis $\{b, c\}$. Let $\mathcal{A}$ be the group generated by the set

$$X = \{b, \ c^{-1}b^{-1}cbc, \ c^{-2}b^{-1}cbc^2, \ c^{-(3+i)}bc^{(3+i)}, \ i \in \lambda\}.$$

Then by Lemma 5.1, $\mathcal{A}$ is a free group with basis X.

Similarly, let in the free product $\mathcal{M} = \mathcal{K} * \langle a, b | \rangle$, $\mathcal{B}$ the subgroup generated by

$$Y = \{b, \ a^{-1}ba, \ a^{-2}b^{-1}aba^2, \ a^{-3}[w, b]a^3, \ a^{-(4+i)}x_iba^{(4+i)}, \ i \in \lambda\}.$$

Since $w \neq 1$, by Lemma 5.1 $\mathcal{B}$ is free with basis Y.

Therefore the presentation of $\mathcal{K}_w$, is just the presentation of the free product with amalgamation $\mathcal{M} *_{\mathcal{A}=\mathcal{B}} \langle b, c|\rangle$; thus $\mathcal{K}_w \cong \mathcal{M} *_{\mathcal{A}=\mathcal{B}} \langle b, c|\rangle$.

Consequently, if $\mathcal{K} \models w \neq 1$ then φ is an embedding.

(2). Let N_w be the normal closure of $\varphi(w)$ in $\mathcal{K}_w$. Since $[w, b] \in N_w$, the relation

$$a^{-3}[w, b]a^3 = c^{-3}bc^3$$

implies $b \in N_w$. The relations

$$a^{-1}ba = c^{-1}b^{-1}cbc,\ a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2,$$

show that $c, a \in N_w$. Finally the relations

$$a^{-(4+i)}x_iba^{(4+i)} = c^{-(4+i)}bc^{(4+i)}, i \in \lambda,$$

show that $x_i \in N_w$ for every $i \in \lambda$. Therefore $N_w = \mathcal{K}_w$.

(3). Let B be a normal subgroup of $\mathcal{K}$ such that $w \notin B$. Then $\mathcal{K}_w/N(B)$ has as presentation the presentation of $\mathcal{K}_w$ union the relations $\{b = 1 \mid b \in B\}$. The last presentation is just the presentation of the group $(\mathcal{K}/B)_w$ (as in (1)). Therefore $(\mathcal{K}/B)_w \cong \mathcal{K}_w/N(B)$.

(4) Let C be the subgroup of $\mathcal{K}_w$ generated by $\{b, ca^{-1}\}$. The relation $a^{-1}ba = c^{-1}b^{-1}cbc$ can be written $b(ca^{-1})b(ca^{-1})^{-1}b^{-1} = c$. Thus $c \in C$. Since $ca^{-1} \in C$, we get $a \in C$. Since the x_i's are written on a, b, c we have $C = \mathcal{K}_w$ as desired.

(5) Clearly if $\mathcal{K}$ is finitely presented, then $\mathcal{K}_w$ is finitely presented.

(6) If $\mathcal{K} \models w = 1$, then $\mathcal{K}_w = 1$ and in this case the result is clear. So suppose that $\mathcal{K} \models w \neq 1$. Then as in (1) $\mathcal{K}_w \cong \mathcal{M} *_{\mathcal{A}=\mathcal{B}} \langle b, c|\rangle$. By Lemma 5.1, $\mathcal{A}$ has a generalized word problem in $\{b, c\}$, and by Lemma 5.2 $\mathcal{B}$ has a generalized word problem in $\mathcal{K} * \langle a, b|\rangle$. Therefore, by Fact 2.7, $\mathcal{K}_w$ has a soluble word problem. □

Some techniques used in the following lemma are inspired by the results of [MI81].

Lemma 6.2. *Let $\mathcal{K}$ be a countable group and w a nontrivial element of $\mathcal{K}$. Then $\mathcal{K}$ is embeddable in a 2-finitely generated SQ-universal group $\mathcal{K}(w)$ satisfying the following conditions:*
(1) *If N is a proper normal subgroup of $\mathcal{K}(w)$, then $w \notin N$.*
(2) *If $\mathcal{K}$ is finitely presented, then so is $\mathcal{K}(w)$.*
(3) *If $\mathcal{K}$ has a soluble word problem, then $\mathcal{K}(w)$ has a soluble word problem.*

Proof. Let $L = \mathcal{K} \times F_2$, where F_2 is the free group of rank 2. We may write $\mathcal{K}$ and w again for their images in L. Since $w \neq 1$ in $\mathcal{K}$ we get $w \neq 1$ in L. Therefore, by Lemma 6.1 (1), L is embeddable in L_w and thus $\mathcal{K}$ is embeddable in L_w. We claim that L_w satisfies the conclusions of the lemma; that is one can take $\mathcal{K}(w) = L_w$. The property (1) is just a reformulation of Lemma 6.1 (2). It is clear, by Lemma 6.1 (4)-(5)-(6) that the properties (2)-(3) are satisfied, and that L_w is 2-finitely generated. It remains to show that L_w is SQ-universal.

Let G be a countable group. Since F_2 is SQ-universal, there exists a normal subgroup B of F_2 such that G is embeddable in F_2/B. We write again B for its image in L. Now B is a normal subgroup of $L = \mathcal{K} \times F_2$ and $L/B \cong \mathcal{K} \times (F_2/B)$.

Clearly $w \notin B$. Therefore, by Lemma 6.1 (3), we have $(L/B)_w \cong L_w/N(B)$, where $N(B)$ is the normal closure of B in L_w.

Since G is embeddable in F_2/B and $L/B = \mathcal{K} \times (F_2/B)$, G is embeddable in L/B and therefore, by Lemma 6.1 (1), G is embeddable in $(L/B)_w$. Since

$(L/B)_w \cong L_w/N(B)$, G is embeddable in $L_w/N(B)$. Thus L_w is SQ-universal as desired. □

7. Proof of the Main Results

Proof of Theorem 1.1.

Let Γ be an existential recursively enumerable theory. We treat first the case, when Γ is not necessarily true in some group with soluble word problem. We claim that we may suppose Γ reduced to a one sentence. By Lemma 3.2, there exists a primitive existential theory Γ^* such that $T_{gp} \cup \{\Gamma^*\} \vdash \Gamma$. By Lemma 3.3, there exists an existential sentence ψ such that $T_{gp} \cup \{\psi\} \vdash \Gamma^*$ and thus $T_{gp} \cup \{\psi\} \vdash \Gamma$. Thus we suppose that Γ is reduced to a one sentence ψ.

We may suppose without loss of generality, that ψ is primitive. Put

$$\psi \equiv \exists \bar{x}(\bigwedge_{1 \leq i \leq n} w_i(\bar{x}) = 1 \wedge \bigwedge_{1 \leq i \leq m} v_i(\bar{x}) \neq 1).$$

Let $\mathcal{M} = \langle \bar{x} | w_i(\bar{x}), 1 \leq i \leq n \rangle$. Since ψ is consistent $\mathcal{M} \models \bigwedge_{1 \leq i \leq m} v_i(\bar{x}) \neq 1$ and thus $\mathcal{M} \models \psi$.

Let $w \in \mathcal{M}$ be a nontrivial element. Let $\mathcal{K}$ be the group

$$\mathcal{K} = \langle \bar{x}, t_j, s_j, 1 \leq j \leq m \mid w_i(\bar{x}), 1 \leq i \leq n, t_j^{-1} v_j t_j s_j^{-1} v_j s_j w^{-1}, 1 \leq j \leq m \rangle.$$

By Lemma 3.1, the equations

$$t_j^{-1} v_j t_j s_j^{-1} v_j s_j w^{-1} = 1, 1 \leq j \leq m \tag{1}$$

have a solution in some extension of $\mathcal{K}$, and thus $\mathcal{M}$ is embeddable in $\mathcal{K}$. We may write again $\mathcal{M}$ for its image in $\mathcal{K}$. Therefore $\mathcal{K} \models \psi$ and $w \neq 1$ in $\mathcal{K}$.

Let $\mathcal{K}(w)$ be the 2-finitely generated SQ-universal group obtained by Lemma 6.2. By the same lemma, $\mathcal{K}$ is embeddable in $\mathcal{K}(w)$ and since $\mathcal{K}$ is finitely presented $\mathcal{K}(w)$ is also finitely presented. It remains to show that ψ is true in all nontrivial quotients of $\mathcal{K}(w)$.

Let N be a proper normal subgroup of $\mathcal{K}(w)$ and let $\pi : \mathcal{K}(w) \to \mathcal{K}(w)/N$ be the natural morphism. By Lemma 6.2 (1), $w \notin N$. Therefore $(\mathcal{K}(w)/N) \models \pi(w) \neq 1$. By the relations (1) we have

$$(\mathcal{K}(w)/N) \models \bigwedge_{1 \leq i \leq m} \pi(v_i(\bar{x})) \neq 1.$$

Therefore

$$(\mathcal{K}(w)/N) \models \bigwedge_{1 \leq i \leq n} \pi(w_i(\bar{x})) = 1 \wedge \bigwedge_{1 \leq i \leq m} \pi(v_i(\bar{x})) \neq 1$$

and thus $(\mathcal{K}(w)/N) \models \psi$ as required. Thus ψ is true in all nontrivial quotients of $\mathcal{K}(w)$ as desired. This completes the proof in our first case.

Suppose now that Γ is true in some group with soluble word problem, and let G be a such group. By Fact 2.4, G is embeddable in a simple group S which is embeddable in a finitely presented group H with soluble word problem. Set $H = \langle \bar{x} | P(\bar{x}) \rangle$, where P is finite. Let $w(\bar{x})$ be a word written on the generators of H, such that $w \neq 1$ in S.

Put $\psi = \exists \bar{x}(\bigwedge_{v \in P} v(\bar{x}) = 1 \wedge w(\bar{x}) \neq 1)$. We claim that every group satisfying ψ satisfies Γ. Let L be a group satisfying ψ and let $\bar{a} \in L$ such that $L \models \bigwedge_{v \in P} v(\bar{a}) = 1 \wedge w(\bar{a}) \neq 1)$. Then there exists a morphism ϕ from H to L which sends $\bar{x}$ to

$\bar{a}$ and such that $\phi(w) \neq 1$. Since S is simple, ϕ is injective on S. Therefore S is embeddable in L and thus G is also embeddable in L.

Since G satisfies Γ and Γ is existential, L satisfies also Γ. Thus every group satisfying ψ satisfies Γ.

Let $H(w)$ be the group obtained by Lemma 6.2. As in the previous case, $H(w)$ is 2-finitely generated finitely presented and SQ-universal. Since H has a soluble word problem, by Lemma 6.2 (3), $H(w)$ has also a soluble word problem. It remains to show that ψ is true in all nontrivial quotients of $H(w)$.

Let N be a proper normal subgroup of $H(w)$ and let $\pi : H(w) \to H(w)/N$ be the natural morphism. By Lemma 6.2 (1), $w \notin N$. Therefore $(H(w)/N) \models \pi(w) \neq 1$. Therefore

$$(H(w)/N) \models \bigwedge_{v \in P} \pi(v(\bar{x})) = 1 \wedge \pi(w(\bar{x})) \neq 1$$

and thus $(H(w)/N) \models \psi$ as required.

This ends the proof of the theorem. □

Proof of Theorem 1.2.

Let G be a finitely generated group. Suppose that G has a soluble word problem and let us prove that there exists a finitely presented SQ-universal group H such that G is embeddable in every nontrivial quotient of H. By Fact 2.4, G is embeddable in a simple group S which is embeddable in a finitely presented group H with soluble word problem. Set $H = \langle \bar{x} | P(\bar{x}) \rangle$, where P is finite. Let $w(\bar{x})$ be a nontrivial element of S written on the generators of H.

Put $\psi = \exists \bar{x} (\bigwedge_{v \in P} v(\bar{x}) = 1 \wedge w(\bar{x}) \neq 1)$. As in the proof of Theorem 1.1, every group satisfying ψ embeds G. By Theorem 1.1, there exists a finitely presented group H all whose nontrivial quotients satisfy ψ. Therefore G is embeddable in every nontrivial quotient of H.

Now suppose that there exists a finitely presented group H such that G is embeddable in every nontrivial quotient of H. Set $H = \langle \bar{x} | P(\bar{x}) \rangle$ where $\bar{x} = \{x_1, \cdots, x_n\}$.

Let $\psi = \exists \bar{x} (\bigwedge_{v \in P} v(\bar{x}) = 1 \wedge (x_1 \neq 1 \vee \cdots \vee x_n \neq 1))$. We claim that every group satisfying ψ embeds G. Let L be a group satisfying ψ and let $\bar{a} \in L$ such that $L \models \bigwedge_{v \in P} v(\bar{a}) = 1 \wedge (a_1 \neq 1 \vee \cdots \vee a_n \neq 1))$. Then there exists a morphism ϕ from H to L which sends $\bar{x}$ to $\bar{a}$ and such that $\langle \bar{a} \rangle$ is nontrivial. Therefore $\langle \bar{a} \rangle$ is nontrivial quotient of H and thus contains a copy of G. Thus G is embeddable in L as claimed.

Since ψ is consistent, by Fact 2.6, it is true in every existentially closed group and therefore G is embeddable in every existentially closed group. By Macintyre's theorem G has a soluble word problem. □

Proof of Theorem 1.3.

Let G be a countable group and write $G = \langle X = \{x_i | i \in \omega\} | P(X) \rangle$. Let $(v_i | i \in \omega)$ be the list of all nontrivial elements of G.

Let $w \in G$ be a nontrivial element and let

$$\mathcal{K} = \langle X, t_j, s_j, j \in \omega \mid P(X), t_j^{-1} v_j t_j s_j^{-1} v_j s_j w^{-1}, j \in \omega \rangle.$$

By Lemma 3.1, the equations

$$t_j^{-1} v_j t_j s_j^{-1} v_j s_j w^{-1} = 1, j \in \omega, \tag{1}$$

have a solution in some extension of $\mathcal{K}$ and thus G is embeddable in $\mathcal{K}$.

Since $w \neq 1$, by Lemma 6.2, there exists a group $\mathcal{K}(w)$ which embeds $\mathcal{K}$ and $\mathcal{K}(w)$ is 2-finitely generated and SQ-universal. We may write $\mathcal{K}$ and G again for

their images in $\mathcal{K}(w)$; that is $\mathcal{K} \leq \mathcal{K}(w)$, and thus $G \leq \mathcal{K}(w)$. We claim that G is embeddable in every nontrivial quotient of $\mathcal{K}(w)$. Let N be a proper normal subgroup in $\mathcal{K}(w)$ and let $\pi : \mathcal{K}(w) \to \mathcal{K}(w)/N$ denotes the natural morphism. It remains to show that π is injective on G. By Lemma 6.2 (1), $\pi(w)$ is nontrivial in $\mathcal{K}(w)/N$. By the relations of $\mathcal{K}$

$$t_j^{-1} v_j t_j s_j^{-1} v_j s_j w^{-1} = 1, j \in \omega,$$

we get $\pi(v_j) \neq 1$ for every $j \in \omega$. Thus π is injective on G as desired. □

8. Remarks

(1) The condition that Γ is recursively enumerable can not be removed from Theorem 1.1. We give an example. Let Γ be the set of existential sentences consistent with group theory. Suppose towards a contradiction that there exists a finitely presented group H such that Γ is true in all nontrivial quotients of H. Set $H = \langle \bar{x} | P(\bar{x}) \rangle$ where $\bar{x} = \{x_1, \cdots, x_n\}$.

Let $\psi = \exists \bar{x} (\bigwedge_{v \in P} v(\bar{x}) = 1 \wedge (x_1 \neq 1 \vee \cdots \vee x_n \neq 1))$. Then clearly, as in the proof of Theorem 1.2, every group satisfying ψ satisfies Γ. Thus $T_{gp} \cup \{\psi\} \vdash \Gamma$. Since $T_{gp} \cup \{\psi\}$ is a finite theory the set

$$cons(\psi) = \{\phi \text{ existential } \mid T_{gp} \cup \{\psi\} \vdash \phi\}$$

is recursively enumerable. Clearly $cons(\psi) = \Gamma$ and thus Γ is recursively enumerable.

It is sufficient to show that Γ is not recursively enumerable. By a well-known result of Novikov-Boone, there exists a finitely presented group H with insoluble word problem. Let $H = \langle \bar{x} | w_1(\bar{x}), \cdots, w_n(\bar{x}) \rangle$ be a such group. Now we have

$$H \models v(\bar{x}) \neq 1 \Leftrightarrow \exists \bar{x} ((w_1(\bar{x}) = 1 \wedge \cdots \wedge w_n(\bar{x}) = 1 \wedge v(\bar{x}) \neq 1) \in \Gamma.$$

Therefore if Γ is recursively enumerable then the set $\{v(\bar{x}) \mid H \models v(\bar{x}) \neq 1\}$ is recursively enumerable and thus H has a soluble word problem; a contradiction. Thus Γ is not recursively enumerable and we get our final contradiction.

As noticed in the introduction, we have the following proposition.

Proposition 8.1. *Let Γ be an existential theory. Then the following properties are equivalents:*
(1) *There exists an existential recursively enumerable theory Γ' such that $\Gamma \subseteq \Gamma'$.*
(2) *There exists a finitely presented group whose all nontrivial quotients satisfy Γ.*

Proof. (1) ⇒ (2) is a consequence of Theorem 1.1.

(2) ⇒ (1). Let H be a such group and set $H = \langle \bar{x} | P(\bar{x}) \rangle$ where $\bar{x} = \{x_1, \cdots, x_n\}$.

Let $\psi = \exists \bar{x} (\bigwedge_{v \in P} v(\bar{x}) = 1 \wedge (x_1 \neq 1 \vee \cdots \vee x_n \neq 1))$. Then clearly, as before, every group satisfying ψ satisfies Γ. Thus $T_{gp} \cup \{\psi\} \vdash \Gamma$. Since $T_{gp} \cup \{\psi\}$ is a finite theory the set

$$cons(\psi) = \{\phi \text{ existential } \mid T_{gp} \cup \{\psi\} \vdash \phi\}$$

is recursively enumerable. Thus $\Gamma \subseteq cons(\psi)$. □

(2) We give a short proof of the following property (implicit in the proof of Theorem 1.1): *for every existential sentence ψ there exists a finitely presented group H such that ψ is true in all nontrivial quotients of H.*

Let G be a countable group satisfying ψ. By Hall's theorem, G is embeddable in a finitely generated simple group S. Set $S = \langle \bar{x} | P(\bar{x}) \rangle$ where $\bar{x} = \{x_1, \cdots, x_n\}$. We claim that

$$T_{gp} \cup \{w(\bar{c}) = 1 \mid w \in P\} \cup \{c_1 \neq 1 \vee \cdots \vee c_n \neq 1\} \vdash \psi.$$

Let L be a group having a tuple $\bar{a}$ and satisfying $w(\bar{a}) = 1$ for every $w \in P$ and $a_1 \neq 1 \vee \cdots \vee a_n \neq 1$. Then the group A generated by $\{a_1, \cdots, a_n\}$ is nontrivial and it is quotient of S. Since S is simple $A \cong S$ and thus A satisfies ψ and also L as claimed.

By compactness theorem there exists $m \in \omega$ such that

$$(*) \qquad T_{gp} \cup \{w_1(\bar{c}) = 1, \cdots, w_m(\bar{c}) = 1\} \cup \{c_1 \neq 1 \vee \cdots \vee c_n \neq 1\} \vdash \psi.$$

Let $H = \langle \bar{x} \mid w_1(\bar{x}), \cdots, w_m(\bar{x}) \rangle$. Then, by (*), H satisfies the required property.

Notice that this proof is short but it does not guarantee that H is SQ-universal.

(3) The assertion of Theorem 1.3 is the best possible in the following sense: one can not require that H is recursively presented whenever G is recursively presented.

We show the following proposition.

Proposition 8.2. *Let G be a countable group for which there exists a recursively presented group K such that G is embeddable in every nontrivial quotient of K. Then every finitely generated subgroup of G has a soluble word problem.*

Proof. By Higman's Embedding Theorem, K is embeddable in a finitely presented group H. Set $H = \langle \bar{x} | w_1(\bar{x}), \cdots, w_n(\bar{x}) \rangle$, and let w be a nontrivial element of K. By Lemma 6.2, H is embeddable in $H(w)$ and w is nontrivial in every nontrivial quotient of $H(w)$.

Clearly, by the property satisfied by G, G is embeddable in every nontrivial quotient of H. Thus G is embeddable in every group satisfying the sentence $\exists \bar{x}(w_1(\bar{x}) = 1 \wedge \cdots \wedge w_n(\bar{x}) = 1 \wedge w(\bar{x}) \neq 1)$. Therefore, by Fact 2.6, G is embeddable in every existentially closed group. Thus, by Macintyre's theorem, every finitely generated subgroup of G has a soluble word problem. □

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