

Existentially closed CSA-groups

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Abstract

We study existentially closed CSA-groups. We prove that existentially closed CSA-groups without involutions are simple and divisible, and that their maximal abelian subgroups are conjugate. We also prove that every countable CSA-group without involutions embeds into a finitely generated one having, up to conjugacy, the same maximal abelian subgroups, except maybe the infinite cyclic ones. We deduce from this that there exist $2^{\aleph_0}$ countable existentially closed CSA-groups without involutions and that their first-order theories have $2^{\aleph_0}$ types over $\emptyset$.

1 Introduction

A subgroup H of a group G is *conjugately separated* in G , or *malnormal* in G , if $H \cap H^x = 1$ for every $x \in G \setminus H$. A *CSA-group* (“Conjugately Separated Abelian”) is a group in which every maximal abelian subgroup is malnormal. CSA-groups have been studied in [7] and [16]. The task we set ourselves in this paper is to study existentially closed groups in some classes of CSA-groups.

An alternative definition of CSA-groups is the following. An *ASC-group* (“Abelian Selfnormalizing Centralizers”) is a group in which the centralizer of every nontrivial element is abelian and selfnormalizing. A proof of the equivalence between the definitions of CSA-groups and ASC-groups can easily be extracted from section 5 of [16]:

Fact 1.1 “CSA=ASC”.

CSA-groups are in particular *commutative transitive*, i.e. groups in which the relation $[x, y] = 1$ is an equivalence relation on the set of nontrivial elements, or equivalently in which the centralizer of every nontrivial element is abelian. The commutative transitivity does not suffice to characterize CSA-groups however: dihedral groups are commutative transitive without being ASC-groups.

Free groups are principal examples of nonabelian CSA-groups. More generally, the class of CSA-groups contains torsion-free hyperbolic groups [8], groups acting freely on Λ -trees [1], and $\exists$ -free groups [17], [6].

It is shown in [16] that a CSA-group with an *involution*, i.e. an element of order 2, must be abelian (cf. Fact 2.5 below). Therefore we will restrict our attention to the subclass of CSA-groups without involutions, as the abelian case of our work simply fits into the theory of abelian groups (see Lemma 2.8 below). Following the notation of [16], we denote by *CSA*-group* any CSA-group without involutions. More generally, we will restrict the possibilities with regards to torsion in the following way. We denote by π the set of all prime numbers and we fix a function f from π to $\mathbb{N} \cup \{\infty\}$. We denote by CSA_f the class of CSA-groups in which $(\mathbb{Z}_p)^{f(p)+1}$ does not embed for every prime p such that $f(p) \in \mathbb{N}$. Thus, if f takes only value 0, we are dealing with torsion-free CSA-groups, and if f takes only value ∞ , we are dealing with the class of all CSA-groups. If $f(2) = 0$, we are dealing with a class of CSA-groups without involutions. The function f will be fixed throughout the paper and as we are mainly concerned with CSA-groups without involutions, most of our theorems will begin by the sentence "Assume $f(2) = 0$ ".

Existentially closed groups in specific classes of groups are usually interesting both from an algebraic and a model-theoretic point of view. The reader unfamiliar with existential closedness might like to find some help in subsection 2.1 below and, in [14], a fairly complete account of results on existentially closed groups in specific classes, such as classes of nilpotent or soluble groups. From the algebraic point of view, we prove the following theorem for existentially closed groups in the class of CSA_f -groups:

Theorem 1.2 *Assume $f(2) = 0$ and let G be an existentially closed group in the class of CSA_f -groups. Then G is simple and if C is a maximal abelian subgroup of G , then:*

- (i) $G = \cup_{g \in G} C^g$.
- (ii) C is a direct sum $(\oplus_{p \in \pi} (\oplus_{I_p} \mathbb{Z}_{p^\infty})) \oplus (\oplus_I \mathbb{Q})$ for some indices sets I_p of infinite cardinality if $f(p) = \infty$ and of cardinality $f(p)$ otherwise, and some indices set I .

In particular G is divisible by (i) and (ii).

The present work was motivated by Bruno Poizat who asked whether existentially closed groups in the class of (torsion-free, say) ASC-groups were good candidate for being simple *bad* groups of finite Morley rank. The existence of such groups is a major open question in the theory of groups under the influence of model-theoretic stability, in Shelah's sense, and a bad group of Morley rank 3 would notably be a CSA-group (cf. [2]). It is also highly connected to the possible extension of the Feit-Thompson Theorem in that context of infinite groups (cf. [12]).

Among other properties, the so-called bad groups of finite Morley rank, if they exist, must have a (definable) subgroup C satisfying the condition:

$$(*) \quad G = \cup_{g \in G} C^g \text{ and } C \text{ is malnormal in } G.$$

It is very noticeable that pairs of groups $C < G$ satisfying condition $(*)$ exist for any group C without involutions and with an element of sufficiently large order.

This is stated as Theorem 17 of [11] (see also [10]), but the proofs seem to use some quite elaborate combinatorial group theory, and a priori they say nothing about the first-order theories of such groups. A more elementary attempt to build such pairs has also been taken in [3] by mean of *HNN*-extensions of free groups: this produces CSA-groups G satisfying condition $(*)$ with C isomorphic to the infinite cyclic group.

Theorem 1.2 produces, relatively elementarily and with a divisibility property, pairs of groups $C < G$ satisfying condition $(*)$, with G a CSA-group and C a maximal abelian subgroup of G . Thus, existentially closed CSA-groups of Theorem 1.2 have the same internal algebraic structure of some potential bad groups of finite (3, say) Morley rank. But we will show that they don't have ω -stable first-order theories, and that they are thus certainly not of finite Morley rank. This will be a corollary of the following theorem, which will imply the existence of $2^{\aleph_0}$ countable existentially closed CSA_f -groups, and that the first-order theories of such groups have $2^{\aleph_0}$ types over $\emptyset$.

Theorem 1.3 *Assume $f(2) = 0$. Then every countable CSA_f -group G embeds into a finitely generated one having, up to conjugacy, the same maximal abelian subgroups, except maybe the infinite cyclic ones.*

Our proof of Theorem 1.2 is based on a study of the preservation of the CSA-property under *HNN*-extensions and free products with amalgamation. For that purpose, the main results of [7] would suffice to us (except maybe for our torsion restrictions). We nevertheless improve some of these results, and give different proofs using some characterizations of [13] of subgroups with nontrivial centers in *HNN*-extensions and free products. We also note, as mentionned to us by the referee of the present paper, that A. Ecker showed in his thesis [4, Section 6] the divisibility and the conjugacy of maximal abelian subgroups of existentially closed CSA^* -groups. His proof mainly uses the results of [7]. He also proves some necessary conditions for free products with amalgamation of CSA^* -groups to be CSA^* -groups. These results have the same flavour as ours. His proofs are in the style of [7] and do not provide necessary and sufficient conditions, contrary to ours, but they are shorter.

Theorem 1.3 has an analogue in the class of all groups, namely the fact that every countable group embeds into a finitely generated one having, up to conjugacy, the same elements of finite order [15, Chapter IV, Theorem 3.1]. Our proof of Theorem 1.3 follows the general line of argument for that more classical theorem. For that purpose, we need here a strong form of the Conjugacy Theorem for Free Products, and for that reason we prove a criterion for conjugacy in such products which gives solutions to the equation $a^x = b$. We also have to prove the malnormality of some subgroups in some free products. These two steps involve some quite lengthy computations.

The paper is organized as follows. In the next section we record the basic facts, and some direct corollaries, needed in the sequel. In sections 3 and 4 we study respectively *HNN*-extensions and free products (with amalgamation) in the classes of CSA_f -groups, with variations from [7] in each case. In section 5

we prove (a strong form of) the simplicity of existentially closed CSA_f -groups, as well as Theorem 1.2. In section 6 we prove the criterion for conjugacy of elements in free products (without amalgamation). In section 7 we show the malnormality of some specific subgroups of some free products. These groups are then used in section 8 which is devoted to finitely generated CSA_f -groups: we prove Theorem 1.3 and we deduce from it that there exist $2^{\aleph_0}$ countable existentially closed CSA_f -groups, and that their first-order theories have $2^{\aleph_0}$ types over $\emptyset$.

We denote by $X^\#$ the set of nontrivial elements of a subset X of a group G . If g and h are elements of a group, then g^h denotes $h^{-1}gh$.

2 Prerequisites

2.1 Model theory

Our reference book for model-theoretic notions is [9]. If $\mathcal{C}$ is a class of groups and $G \in \mathcal{C}$, then G is *existentially closed* in $\mathcal{C}$ if the following condition is satisfied: every existential formula with parameters in G and true in an extension of G which is in $\mathcal{C}$ is also true in G .

Each class of groups considered in the sequel will be the class of all models of some universal theory of groups, i.e. the class of all groups satisfying some universal sentences in the language of groups. Thus the following fact will always apply:

Fact 2.1 [9, Corollary 8.2.2] *Let $\mathcal{C}$ be an inductive class of groups and G a group in $\mathcal{C}$. Then there is an existentially closed group H in $\mathcal{C}$ such that $G \leq H$. If $\mathcal{C}$ is furthermore the class of all models of a universal theory of groups, then one can insure that $|H| \leq \max(|G|, \aleph_0)$.*

Our proof of the existence of a lot of (consistent) types in first-order theories of existentially closed CSA-groups will use the following elementary result from model theory.

Fact 2.2 [9, Corollary 6.5.3] *If T is a first-order theory, then the models of the universal theory $T_\forall$ are precisely the substructures of models of T .*

2.2 CSA-groups

As CSA-groups are commutative transitive, the following elementary fact will be used without any reference.

Fact 2.3 [16, Proposition 8] *Let G be a commutative transitive group. Then the centralizers of nontrivial elements of G are exactly the maximal abelian subgroups of G .*

Fact 2.4 [16, Proposition 10] *The class of CSA-groups is a universal class, axiomatized by the following axioms:*

- (i) $\forall x \forall y \forall z (x \neq 1 \wedge [x, y] = 1 \wedge [x, z] = 1) \Rightarrow [y, z] = 1.$
- (ii) $\forall x \forall y (x \neq 1 \wedge [x, x^y] = 1) \Rightarrow [x, y] = 1.$

If f is a function from the set π of prime numbers to $\mathbb{N} \cup \{\infty\}$, then the class of CSA_f -groups is universally axiomatizable. It suffices to add to the axioms of Fact 2.4 the universal axioms expressing, for each prime p such that $f(p) \in \mathbb{N}$, that at most $p^{f(p)} - 1$ elements of order p can commute pairwise. In particular Fact 2.1 applies to the class of CSA_f -groups.

Fact 2.5 [16, Remark 7] *Let G be a CSA-group with an involution. Then G is abelian.*

2.3 Abelian groups

Fact 2.6 [5, Theorem 23.1] *Every divisible abelian group D is isomorphic to a direct sum $(\bigoplus_{p \in \pi} (\bigoplus_{I_p} \mathbb{Z}_{p^\infty})) \oplus (\bigoplus_I \mathbb{Q})$ for some indices sets I_p and I .*

Fact 2.7 [5, Theorem 21.2; Baer] *Let A be an abelian group and D a divisible subgroup of A . Then D is a direct summand in A , i.e. $A = D \oplus C$ for some subgroup C of A .*

Lemma 2.8 *Let A be an abelian CSA_f -group. Then A embeds into an existentially closed abelian CSA_f -group D . Furthermore, any existentially closed abelian CSA_f -group C is divisible and isomorphic to $(\bigoplus_{p \in \pi} (\bigoplus_{I_p} \mathbb{Z}_{p^\infty})) \oplus (\bigoplus_I \mathbb{Q})$, where the sets I_p have cardinality $f(p)$ if $f(p) \in \mathbb{N}$ and are infinite otherwise.*

Proof

As the class of abelian CSA_f -groups is inductive, A embeds into a group D which is existentially closed in this class by Fact 2.1. We will show that any existentially closed abelian CSA_f -group C has the required properties. This can most probably be extracted from elsewhere, but for the reader's convenience we provide a direct proof.

For p a prime, let C_p denote the p -primary component of C , i.e. the subgroup of elements whose order is a power of p . As C is existentially closed, it is clear that $\mathbb{Z}_p^{f(p)}$ embeds into C_p .

We claim now that C_p is p -divisible. Suppose towards a contradiction that an element $x \in C_p^\#$ has no p^{th} -root and consider the group

$$C_0 = \langle C, y \mid y^p = x ; [y, C] = 1 \rangle.$$

It is clear that C_0 is abelian and that C has index p in C_0 . Thus every element of $C_0 \setminus C$ has the form $y^s c$ for some $s \in \{1, \dots, p-1\}$ and some $c \in C$. If such an element $y^s c$ has order p for some $c \in C$, then $1 = (y^s c)^p = y^{ps} c^p = (y^p)^s c^p = x^s c^p$ and $x^s = c^{-p}$. As $x \in C_p$, $x^s \in C_p$. Thus c^{-p} has order a power of p , as well as c . Thus x and c are in C_p . As s and p are relatively prime,

there exist integers α and β such that $1 = \alpha s + \beta p$ by Bezout's Theorem. Thus $x = x^{\alpha s + \beta p} = (x^s)^\alpha (x^p)^\beta = (c^{-p})^\alpha (x^\beta)^p = (c^{-\alpha} x^\beta)^p$ has a p^{th} -root in C_p , a contradiction. This shows that an element $y^s c \in C_0 \setminus C$ cannot have order p . Thus C_0 has no more elements of order p than those already in C . As y is a p -element of C_0 , C_0 has no more elements of prime order than those already in C . Thus C_0 is an abelian CSA_f -group and as C is existentially closed in the class of abelian CSA_f -groups, x must be p -divisible in C_p , a contradiction. Thus C_p is p -divisible as claimed and as finite p -groups are q -divisible for every prime q different from p , it follows that C_p is divisible.

Hence the torsion subgroup $\text{Tor}(C)$ of C is divisible and it has the required form $\bigoplus_{p \in \pi} (\bigoplus_{I_p} \mathbb{Z}_{p^\infty})$ by Fact 2.6. Now Fact 2.7 shows that $C = \text{Tor}(C) \oplus C_1$ for some torsion-free subgroup C_1 of C . Adding roots to elements of C_1 as before does not produce new torsion elements, thus C_1 must be divisible again as C is existentially closed in the class of abelian CSA_f -groups. Thus C is divisible and has the required form. $\square$

2.4 HNN-extensions

We denote by $G^* = \langle G, t \mid t^{-1} A t = B \rangle$ the *HNN-extension* of a group G relatively to an isomorphism between two subgroups A and B of G . A sequence $g_0, t^{\epsilon_1}, g_1, \dots, t^{\epsilon_n}, g_n$ (with $n \geq 0$, $g_i \in G$, $\epsilon_i = \pm 1$) is *reduced* if it has no subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_j, t^{-1} with $g_j \in B$. An element $x = g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n} g_n$ of G^* is in *normal form* if the associated sequence is reduced.

Fact 2.9 [15, Britton's Lemma] *If $g_0, t^{\epsilon_1}, g_1, \dots, t^{\epsilon_n}, g_n$ ($n \geq 1$) is a reduced sequence, then the product $g_0 t^{\epsilon_1} g_1 \dots t^{\epsilon_n} g_n$ is nontrivial in the HNN-extension $G^* = \langle G, t \mid A^t = B \rangle$.*

Fact 2.10 [15, Chapter IV, Theorem 2.4] *Every torsion element of $G^* = \langle G, t \mid A^t = B \rangle$ is a conjugate of a (torsion) element of G .*

Fact 2.11 [13, Theorem 1 (ii)] *Suppose that $C \leq \langle G, t \mid A^t = B \rangle$, that $c \in Z(C)$ and that c has no conjugate in B . Then C is conjugate to an HNN-group $\langle B', s \mid B'^s = B' \rangle$ where $B' \leq B$ (i.e. C is an infinite cyclic extension of a subgroup of a conjugate of B), unless C is in a conjugate of G .*

It is also well known that there is a unicity of normal forms in *HNN-extension*, by fixing representatives of cosets of the subgroups involved. Nevertheless, our proofs will not use such a unicity.

2.5 Free Products

We denote by $G *_A H$ the free product of two groups G and H amalgamated over A , i.e. relatively to an isomorphism between two subgroups of G and H isomorphic to A . A sequence $c_1, \dots, c_n$, $n \geq 1$, of elements of $G *_A H$ is *reduced* if:

- (1) Each c_i is in a factor G or H .
- (2) Two consecutive elements c_i, c_{i+1} are in two different factors.
- (3) If $n > 1$, then no c_i is in A .
- (4) If $n = 1$, then $c_1 \neq 1$.

A nontrivial element $x = c_1 \cdots c_n$ of P is in *normal form* if the associated sequence is reduced.

Fact 2.12 [15, Chapter IV, Theorem 2.6] *If $c_1, \dots, c_n$ is a reduced sequence, $n \geq 1$, then the product $c_1 \cdots c_n$ is nontrivial in $G *_A H$.*

Fact 2.13 [15, Chapter IV, Theorem 2.7] *Every torsion element of $G *_A H$ is a conjugate of a (torsion) element of G or H .*

Fact 2.14 [13, Theorem 1 (i)] *Suppose that $C \leq G *_A H$, that $c \in Z(C)$ and that c has no conjugate in A . Then C is conjugate to an HNN-group $\langle A', s \mid A'^s = A' \rangle$ where $A' \leq A$ (i.e. C is an infinite cyclic extension of a subgroup of a conjugate of A), unless C is in a conjugate of a factor.*

We will also need the following lemma elsewhere.

Lemma 2.15 *Let $P = G *_A H$ be a free product with amalgamated subgroup A . Then, for every $g \in G$, $C_P(g) \leq G$ or $C_P(g)$ is conjugate to $C_P(a)$ for some $a \in A$.*

Proof

We may assume $g \in G \setminus A$. Suppose $C_P(g) \not\leq G$ and let $x \in C_P(g) \setminus G$. As x cannot be in H , we can write $x = c_1 \cdots c_n$ in normal form with $n \geq 2$. Then $c_n^{-1} \cdots c_1^{-1} g c_1 \cdots c_n g^{-1} = 1$. Hence $c_1 \in G$ and $c_1^{-1} g c_1 \in A$. Thus $C_P(g)$ is conjugate to $C_P(g^{c_1})$ with $g^{c_1} \in A$. $\square$

In case of a free product F (with trivial amalgamated subgroup), a nontrivial element x has a unique normal form. The *length* of x , denoted by $|x|$, is the length of its normal form.

If A and B are elements of a free product F , then we say that AB is in *reduced form* if $A = 1$ or $B = 1$, or the normal form of AB can be obtained by adjusting the normal form of A and the normal form of B . Similarly, we say that AB is in *semi-reduced form* if $A = 1$ or $B = 1$, or the following condition is satisfied: if $a_1, \dots, a_n$ is the normal form of A and $b_1, \dots, b_m$ is the normal form of B , then $a_1, \dots, (a_n b_1), \dots, b_m$ is the normal form of AB .

A nontrivial element A of F is *cyclically reduced* (abbreviated c.r.) if $|A| = 1$ or AA is in reduced form, and *weakly cyclically reduced* (abbreviated w.c.r.) if $|A| > 1$ and AA is in semi-reduced form. (Note that our definition of w.c.r. is slightly different from the one of [15].) It is well known that every nontrivial element of F is conjugate to a cyclically reduced one.

Fact 2.16 [15, Chapter IV, Theorem 1.10; Kurosh Subgroup Theorem] *Let $G = *A_i$ be a free product and let H be a subgroup of G . Then H is a free product $F * (*H_j)$ where F is a free group and each H_j is the intersection of H with a conjugate of some factor A_i of G .*

Corollary 2.17 *Let F be a free product and g an element of F such that $|g| \geq 2$. If g is c.r. or w.c.r., then $C_F(g)$ is infinite cyclic.*

Proof

We have $g \in Z(C_F(g))$. By Fact 2.16, $C_F(g)$ is a free product $*A_i$ where A_i is infinite cyclic or of the form $A_i = C_F(g) \cap G_i^f$ for some factor G_i and some $f \in F$. As $g \in Z(C_F(g))$ again, $C_F(g) = A_i$ for some i . But $C_F(g)$ cannot be of the form $A_i = C_F(g) \cap G_i^f$ because G_i^f contains no c.r. or w.c.r. elements of length ≥ 2 . Therefore $C_F(g)$ is infinite cyclic. $\square$

3 HNN -extensions of CSA-groups

We prove now a necessary and sufficient condition for an HNN -extension to be a CSA^* -group.

Theorem 3.1 *Let G be a CSA^* -group and $G^* = \langle G, t \mid A^t = B \rangle$ be an HNN -extension of G . Then the following are equivalent:*

- (i) G^* is a CSA^* -group.
- (ii) For every $a \in A^\#$, $C_{G^*}(a)$ is abelian and malnormal in G^* .

In that case, for every nontrivial element $g^ \in G^*$, $C_{G^*}(g^*)$ is infinite cyclic or is conjugate to $C_{G^*}(g)$ for some $g \in G^\#$.*

Proof

It is clear that (i) implies (ii). Assume now (ii). We are going to prove that $C_{G^*}(g)$ is abelian and selfnormalizing in G^* for every nontrivial element g of G^* . By Fact 2.11 there are only three cases to consider after conjugacy:

- (I) $g \in A$.
- (II) $C_{G^*}(g) \leq G$.
- (III) $C_{G^*}(g) = \langle A', s \mid A'^s = A' \rangle$ for some $A' \leq A$.

In case (I), $C_{G^*}(g)$ is abelian and selfnormalizing by assumption.

In case (II), $C_{G^*}(g) = C_G(g)$ is abelian and it is sufficient to prove that $C_{G^*}(g)$ is malnormal in G^* . By case (I) we may assume that g is not in some conjugate of A . Let $x \in G^*$ and $g', g'' \in C_{G^*}(g)^\#$ such that $g'^x = g''$. If $x \in G$, then $x \in C_{G^*}(g)$ because G is a CSA -group. Suppose now towards a contradiction that $x \notin G$, i.e. that $x = g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n$ is in normal form with $n \geq 1$. Then

$$g_n^{-1} t^{-\epsilon_n} \dots t^{-\epsilon_1} g_0^{-1} g' g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n g''^{-1} = 1.$$

Hence $g_0^{-1} g' g_0 \in A$ and $\epsilon_1 = 1$ or $g_0^{-1} g' g_0 \in B$ and $\epsilon_1 = -1$. In both cases g' is in some conjugate of A , a contradiction.

In case (III) with $A' \neq 1$, $g \in C_{G^*}(a)$ for some nontrivial $a \in A$. As $C_{G^*}(a)$ is malnormal in G^* , $C_{G^*}(g) \leq C_{G^*}(a)$, and since $C_{G^*}(a)$ is abelian we find $C_{G^*}(g) = C_{G^*}(a)$. Therefore $C_{G^*}(g)$ is abelian and malnormal.

It remains only to deal with the case (III) with $A' = 1$. Then $C_{G^*}(g) \simeq \mathbb{Z}$ is abelian and it remains only to show that it is selfnormalizing. So assume

now towards a contradiction $C_{G^*}(g) < N_{G^*}(C_{G^*}(g))$. As $N_{G^*}(C_{G^*}(g))$ modulo $C_{G^*}(C_{G^*}(g))$ embeds into $\text{Aut}(\mathbb{Z}) (\simeq \mathbb{Z}_2)$ and $C_{G^*}(C_{G^*}(g)) = C_{G^*}(g)$, we have $N_{G^*}(C_{G^*}(g)) = \langle y \rangle \cdot C_{G^*}(g)$ for some y which inverts $C_{G^*}(g)$ by conjugation and such that $y^2 \in C_{G^*}(g)$. But G^* has no involutions by Fact 2.10 and assumption. Thus y^2 is a nontrivial element in the center of $N_{G^*}(C_{G^*}(x))$. In particular $C_{G^*}(y^2)$ is nonabelian. But the above analysis shows that the centralizer of any nontrivial element of G^* is abelian, a contradiction. Thus $C_{G^*}(g) \simeq \mathbb{Z}$ is selfnormalizing as claimed.

This completes the proof that G^* is an ASC-group and it has no involutions as remarked above. Furthermore the above analysis shows that, for every $g^* \in G^*$, $C_{G^*}(g^*)$ is infinite cyclic or conjugate to $C_{G^*}(g)$ for some $g \in G$. $\square$

The following theorem generalizes Corollary 1 of [7]. Recall that an *HNN*-extension $G^* = \langle G, t \mid A^t = B \rangle$ is said to be *separated* if $A \cap B^x = 1$ for every $x \in G$.

Theorem 3.2 *Assume $f(2) = 0$. Let $G^* = \langle G, t \mid A^t = B \rangle$ be a separated HNN-extension of a CSA_f -group G where A and B are malnormal subgroups of G . Then:*

- (i) *For every $g \in G^\#$, $X_g \subseteq G$ where $X_g = \{x \in G^* \mid g^x \in C_G(g)\}$. In particular $C_{G^*}(g) \leq G$ and $C_{G^*}(g) = C_G(g)$.*
- (ii) *G^* is a CSA -group.*
- (iii) *G^* and G have, up to conjugacy in G^* , the same maximal abelian subgroups, except maybe the infinite cyclic ones. In particular G^* is a CSA_f -group.*

Proof

(i). Assume first $g \in A$. Assume towards a contradiction that $x \in X_g$ and that $|x| \geq 1$. Let $x = g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n$ be in normal form with $n \geq 1$. Then:

$$g_n^{-1} t^{-\epsilon_n} \dots t^{-\epsilon_1} g_0^{-1} g g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n g'^{-1} = 1$$

for some $g' \in C_G(g)$. Hence $g_0^{-1} g g_0 \in A$ and $\epsilon_1 = 1$ or $g_0^{-1} g g_0 \in B$ and $\epsilon_1 = -1$. As G^* is separated, the second case cannot occur. Thus $g_0^{-1} g g_0 \in A$ and $\epsilon_1 = 1$. Since A is malnormal, $g_0 \in A$. Now if $n = 1$, then $g_1^{-1} (g_0^{-1} g g_0)^t g_1 = g'$. Since $g \in A$ and $g' \in C_G(g)$, the malnormality of A implies that $g' \in A$. Therefore $(g_0^{-1} g g_0)^{t g_1} = g'$ belongs to $B^{g_1} \cap A$, a contradiction as G^* is separated. Now if $n \geq 2$, then $g_1^{-1} (g_0^{-1} g g_0)^t g_1 \in A$ or $g_1^{-1} (g_0^{-1} g g_0)^t g_1 \in B$. The first possibility is impossible as G^* is separated. Now $g_1^{-1} (g_0^{-1} g g_0)^t g_1 \in B$ implies that $g_1 \in B$ as B is malnormal. Hence $\epsilon_1 = 1$, $\epsilon_2 = -1$, and the sequence (t, g_1, t^{-1}) is not reduced, a contradiction. Therefore $X_g \subseteq G$ in the case $g \in A$.

The case $g \in B$ can be treated similarly.

Now, if g is conjugate in G to some element of $A \cup B$, then $g^y \in A$ or $g^y \in B$ for some $y \in G$ and $X_g = \{x \in G^* \mid g^x \in C_G(g)\} = (X_{g^y})^{y^{-1}}$. Therefore $X_g \subseteq G$ in that case.

Hence it remains only to treat the case in which g is not conjugate in G to some element in $A \cup B$. We are going to prove that $X_g \subseteq G$ in that case too.

Let $x \in G^*$ such that $g^x = g' \in C_G(g)$ and assume towards a contradiction that $x = g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n$ is in normal form with $n \geq 1$. Then

$$g_n^{-1} t^{-\epsilon_n} \dots t^{-\epsilon_1} g_0^{-1} g g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n g'^{-1} = 1.$$

Hence $g_0^{-1} g g_0 \in A$ or $g_0^{-1} g g_0 \in B$, a contradiction.

(ii). Let $a \in A^\#$. Then $C_{G^*}(a) \leq G$ by (i), and $C_{G^*}(a) = C_G(a)$ is abelian. By Theorem 3.1 it suffices to prove that $C_G(a)$ is malnormal in G^* . Let $x \in G^*$ and $g \in C_G(a)^\#$ such that $g^x \in C_G(a)$. Since G is a CSA-group we have $C_G(g) = C_G(a)$. By (i) we have $x \in X_g = C_G(g)$.

(iii). Let M be a maximal abelian subgroup of G which is not infinite cyclic. Then $M = C_G(g)$ for some g in $G^\#$. By (i), $C_G(g) = C_{G^*}(g)$. As G^* is a CSA-group, M is a maximal abelian subgroup of G^* .

Conversely, let now M be a maximal abelian subgroup of G^* which is not infinite cyclic. By Theorem 3.1, M is conjugate to $C_{G^*}(g)$ for some $g \in G^\#$. But by (i), $C_{G^*}(g) = C_G(g)$ is a maximal abelian subgroup of G . $\square$

4 Free products of CSA-groups

We prove now the analogue of Theorem 3.1 for free products (with amalgamation) of CSA*-groups.

Theorem 4.1 *Let $P = G *_A H$ be a free product, with amalgamated subgroup A , of two CSA*-groups G and H . Then the following are equivalent:*

- (i) P is a CSA*-group.
- (ii) For every $a \in A^\#$, $C_P(a)$ is abelian and malnormal in P .

In that case, for every non trivial element $x \in P$, $C_P(x)$ is infinite cyclic or is conjugate to $C_P(y)$ for some $y \in G \cup H$.

Proof

It is analogue to the proof of Theorem 3.1, using the corresponding facts of subsection 2.5 instead of those of subsection 2.4. The only care which has to be taken here is in the corresponding case (II), in which case $C_P(g) \leq G$ or H for some $g \in P^\#$. In that case we also want to show that $C_P(g)$ is malnormal and we may also assume that g is not in a conjugate of A . We are going to treat the case $C_P(g) \leq G$, the other one being similar.

We claim that we may also assume that $C_P(g) \cap A^y = 1$ for every $y \in P$. For if $C_P(g) \cap A^y \neq 1$ for some $y \in P$, then there exists some $z \in C_P(g)^\#$ and in a conjugate of A . By assumption, $C_P(z)$ is abelian and malnormal in P . Thus $C_P(g) = C_G(g) \leq C_P(z)$ and we find that $C_P(g) = C_P(z)$ is malnormal, which proves our claim.

Let $x \in P$ and $g', g'' \in C_P(g)^\#$ such that $g'^x = g''$. If $x \in G$, then $x \in C_P(g)$ because G is a CSA-group. Suppose now towards a contradiction that $x \notin G$, i.e. that $x = g_1 \dots g_n$ is in normal form with $g_1 \notin A$. Then

$$g_n^{-1} \dots g_1^{-1} g' g_1 \dots g_n g''^{-1} = 1.$$

If $n = 1$, then $g_1^{-1}g'g_1 = g''$ and $g_1 \in H \setminus A$. This implies that $g' \in A$, a contradiction to our assumption. If $n = 2$, then $g_2^{-1}(g_1^{-1}g'g_1)g_2 = g''$. In this case $g_1^{-1}g'g_1 \in A$ whenever $g_1 \in G \setminus A$ and $g' \in A$ whenever $g_1 \in H \setminus A$, a contradiction to our assumption in both cases. Now, if $n > 2$, then $g_1^{-1}g'g_1 \in A$, which is a contradiction again. This completes the proof of Theorem 4.1. $\square$

The following theorem is an improvement of Theorem 2 of [7].

Theorem 4.2 *Assume $f(2) = 0$, and let $P = G *_A H$ be a free product, with amalgamated subgroup A , of two CSA_f -groups G and H . Assume also that A is malnormal in G . Then $C_P(h) = C_H(h)$ for every $h \in H^\#$, and P is a CSA -group. Furthermore, any maximal abelian subgroup of P , except maybe the infinite cyclic ones, is conjugate in P to some maximal abelian subgroup of G or of H ; in particular P is a CSA_f -group.*

Proof

It is analogue to the proof of Theorem 3.2. For every $h \in H^\#$, let

$$X_h = \{x \in P \mid h^x \in C_H(h)\}.$$

We first show that, for every $h \in H^\#$, $X_h \subseteq H$. Let $x \in P$ and $h' \in C_H(h)^\#$ such that $h^x = h'$. Suppose now towards a contradiction that $x \notin H$, i.e. that $x = c_1 \cdots c_n$ is in normal form with $c_1 \notin A$. Then

$$c_n^{-1} \cdots c_1^{-1} h c_1 \cdots c_n h'^{-1} = 1.$$

If $n = 1$, then $c_1^{-1} h c_1 = h'$ and $c_1 \notin H$. This implies that $h, h' \in A$, and, as A is malnormal in G , we get $c_1 \in A$, a contradiction to our assumption. Assume now $n \geq 2$. Then $c_n^{-1} \cdots c_2^{-1} (c_1^{-1} h c_1) c_2 \cdots c_n h'^{-1} = 1$. If $c_1 \in G \setminus A$, then $h \in A$, $h^{c_1} \in A$, and $c_1 \in A$ by malnormality of A in G , a contradiction. If $c_1 \in H \setminus A$, then $h^{c_1} \in A$ and, as A is malnormal in G , $h^{c_1 c_2} \in G \setminus A$. Then $n > 2$, as otherwise $h^{c_1 c_2} = h' \in G \cap H = A$, and with $n > 2$ we get a contradiction. This completes the proof of the fact that $X_h \subseteq H$. In particular $C_P(h) \leq H$.

We show now that P is a CSA -group. By Theorem 4.1, it is sufficient to show, for every $a \in A^\#$, that $C_P(a)$ is abelian and malnormal in P . As $C_P(a) \subseteq X_a \subseteq H$, $C_P(a) = C_H(a)$ is abelian. It remains only to show that it is malnormal. Let $x \in P$ and $h \in C_H(a)^\#$ such that $h^x \in C_H(a)$. Since H is a CSA -group, we have $C_H(a) = C_H(h)$. Hence $x \in X_h = C_H(h)$, and thus $C_P(a) = C_H(a) = C_H(h)$ is malnormal. Therefore P is a CSA -group.

Let now M be a maximal abelian subgroup of P which is not infinite cyclic. As P is a CSA -group, $M = C_P(x)$ for some x in $P^\#$. Then, by Theorem 4.1, $C_P(x)$ is conjugate to $C_P(y)$ for some nontrivial element $y \in G \cup H$. If $y \in H$, then $C_P(y) \subseteq X_y \subseteq H$, and $C_P(y) = C_H(y)$ is a maximal abelian subgroup of H as H is a CSA -group. Assume now $y \in G$. Then, by Lemma 2.15, $C_P(y) \leq G$, in which case $C_P(y) = C_G(y)$ is a maximal abelian subgroup of G , or $C_P(y)$ is conjugate to $C_P(a)$ for some $a \in A$ ($\leq H$), which is a case treated above. $\square$

5 Algebraic properties

We prove now a strong form of the simplicity of existentially closed CSA_f -groups, provided $f(2) = 0$ as always.

Theorem 5.1 *Assume $f(2) = 0$, and let G be an existentially closed CSA_f -group. Then G satisfies the sentence*

$$\forall a \forall b \exists x \exists t ((a \neq 1 \wedge b \neq 1 \wedge a \neq b) \Rightarrow t^{-1} b x^{-1} a x t = a x^{-1} a x).$$

In particular any group elementary equivalent to G is simple.

Proof

Let a and b be distinct nontrivial elements of G , and let $F = G * \langle x \mid \rangle$. By Theorem 4.2, F is a CSA_f -group. Let A and B be the subgroups of F generated by $g_1 = b x^{-1} a x$ and $g_2 = a x^{-1} a x$ respectively. As g_1 and g_2 are cyclically reduced, $A \simeq B \simeq \mathbb{Z}$. By Corollary 2.17, $C_F(g_1)$ and $C_F(g_2)$ are infinite cyclic and since g_1 and g_2 are not proper powers, $C_F(g_1) = A$ and $C_F(g_2) = B$. Since F is a CSA_f -group, A and B are malnormal in F . Let $F^* = \langle F, t \mid t^{-1} g_1 t = g_2 \rangle$. We claim that F^* is separated. Let $x \in F$ such that $(g_1^n)^x = g_2^m$ for some n and $m \neq 0$. Since g_1^n and g_2^m are cyclically reduced, and by the Conjugacy Theorem for Free Products (Fact 6.2 below), we have $|g_1^n| = |g_2^m| = 4|n| = 4|m|$, and the normal forms of g_1^n and g_2^m are cyclic permutations of each other. Therefore $|m| = |n|$ and $n = \pm m$. Without loss of generality, we can suppose $m > 0$. Then $(a x^{-1} a x) \cdots (a x^{-1} a x)$ (m times) can be obtained by a cyclic permutation of $(b x^{-1} a x) \cdots (b x^{-1} a x)$ or of $(x^{-1} a^{-1} x b^{-1}) \cdots (x^{-1} a^{-1} x b^{-1})$ (m times). But we see that the first possibility is true if and only if $a = b$ and that the second possibility is true if and only if $b^{-1} = a$ and $a = a^{-1}$. But $a \neq a^{-1}$ since G has no involutions. Therefore $a = b$, which contradicts our assumption. Hence F^* is separated as claimed, and it is a CSA_f -group by Theorem 3.2. As G is existentially closed, there exist x and t in G such that $t^{-1} b x^{-1} a x t = a x^{-1} a x$.

In particular $b = (a x)^{t^{-1}} a^{-x}$ is in the normal closure of $\langle a \rangle$. This shows that G , as well as any group elementary equivalent to G , is simple. $\square$

Proof of Theorem 1.2

We consider an existentially closed group G in the class of CSA_f -groups, with $f(2) = 0$, and C a maximal abelian subgroup of G . The simplicity of G was seen in Theorem 5.1.

We first show (ii). By Lemma 2.8, C embeds into a divisible abelian group C_1 whose torsion subgroup is isomorphic to $(\oplus_{p \in \pi} (\oplus_{I_p} \mathbb{Z}_{p^\infty}))$, where the sets I_p have cardinalities $f(p)$ respectively (i.e. an infinite cardinality if $f(p) = \infty$). As C is malnormal in G , $G *_C C_1$ is a CSA_f -group by Theorem 4.2. As G is existentially closed in the class of CSA_f -groups, C must also be divisible and with the form required in statement (ii) of Theorem 1.2.

To prove statement (i) of Theorem 1.2, it remains only to show that every maximal abelian subgroup B of G is conjugate to C . So assume now towards a contradiction that such a B is not conjugate to C . As G is existentially closed in

the class of CSA_f -groups, our assumption means that there is no element, in any CSA_f -extension of G , which conjugates two nontrivial elements commuting with C and B respectively. By the above argument, with B instead of C , we see that B is also divisible. Now by (ii), C is isomorphic to $(\bigoplus_{p \in \pi} (\bigoplus_{I_p} \mathbb{Z}_{p^\infty})) \oplus (\bigoplus_I \mathbb{Q})$, where the sets I_p have cardinality $f(p)$ if $f(p) \in \mathbb{N}$ and are infinite otherwise, and the same property holds also for B . Therefore there exists an abelian CSA_f -group B_1 in which C and B are embeddable.

By Theorem 4.2, $P_1 = G *_B B_1$ is a CSA_f -group in which B_1 is a maximal abelian subgroup. By our assumption and by Lemma 2.15, C is a maximal abelian subgroup of P_1 . Now C embeds into a group C_1 isomorphic to B_1 . By Theorem 4.2 again, $P_2 = P_1 *_C C_1$ is a CSA_f -group such that $C_{P_2}(c) = C_{C_1}(c) = C_1$ for every $c \in C_1^\#$. In particular C_1 is a malnormal subgroup of P_2 . By our assumption and Lemma 2.15 again, B_1 is a maximal abelian subgroup of P_2 . In particular B_1 is malnormal in P_2 . By our assumption, the HNN -extension $P_2^* = \langle P_2, t \mid t^{-1}B_1t = C_1 \rangle$ is separated. By Theorem 3.2, P_2^* is a CSA_f -extension of G . But B_1 and C_1 are conjugate in P_2^* , a final contradiction to our assumption. This completes the proof of Theorem 1.2. $\square$

We end this section with the following

Question 5.2 *Can the set I in the statement of Theorem 1.2 be finite? empty?*

Of course, the question " $I = \emptyset$?" makes no sense if the class considered is the class of torsion-free CSA -groups, in which case maximal abelian subgroups are all isomorphic to $\bigoplus_I \mathbb{Q}$ for a nonempty fixed set I . Otherwise, one is tempted to answer yes to Question 5.2. In the same vein, one may wonder about the existence of some constraints on the cardinality of a set I_p , in the statement of Theorem 1.2, when $f(p) = \infty$. It seems that there are no such constraints, except the trivial ones of course.

6 Conjugacy in free products

We prove now a strong form of the Conjugacy Theorem for Free Products.

Theorem 6.1 *Let F be a free product (without amalgamation). Let x, y, z in F such that $|y| \geq 2$ and $y^x = z$. Suppose that y is c.r. or w.c.r., as well as z . Then there exist A, B in F , and α and β in $\mathbb{Z}$, with $\alpha \geq 1$, such that $y = (AB)^\alpha$, $z = (BA)^\alpha$, and $x = A(BA)^\beta$. Furthermore one can impose that:*

- (i) *AB and BA are in reduced forms whenever y and z are c.r.*
- (ii) *AB and BA are in semi-reduced forms whenever y and z are w.c.r.*
- (iii) *AB is in reduced form and BA is in semi-reduced form whenever y is w.c.r. and z is c.r.*

Before the proof of Theorem 6.1, we show how it implies the Conjugacy Theorem for Free Products:

Fact 6.2 [15, Chapter IV, Theorem 1.4] *If y and z are cyclically reduced elements which are conjugate in a free product and $|y| \geq 2$, then $|y| = |z|$ and the normal forms of y and z are cyclic permutations of each other.*

Proof

By Theorem 6.1, there exist A, B in F , and $\alpha \geq 1$, such that $y = (AB)^\alpha$, $z = (BA)^\alpha$ and AB and BA are in reduced forms. The result follows then easily. $\square$

The rest of this section is devoted to the proof of Theorem 6.1, so we adopt all the assumptions and the notation of the statement of that theorem for the rest of this section. We may assume $x \neq 1$, the case $x = 1$ being obvious. We shall treat the three cases (i)–(iii) separately.

Case (i): y and z c.r.

Let $x = x_1 \cdots x_p$, $y = y_1 \cdots y_n$, and $z = z_1 \cdots z_m$ in normal forms. We are going to prove the theorem by induction on $p = |x|$.

We first treat the case $p = 1$. Then $x^{-1}y_1 \cdots y_n x = z$. Since y and z are c.r. and $|y| \geq 2$, we have $x^{-1}y_1 = 1$ or $y_n x = 1$. If $x^{-1}y_1 = 1$, then, since y is c.r., y_n and x are in different factors and $z = (y_2 \cdots y_n) \cdot x$ is in reduced form. By putting $B = y_2 \cdots y_n$ and $A = x$, we have $y = AB$, $z = BA$ and $x = A$. Now, if $y_n x = 1$, then, as before, $z = x^{-1} \cdot (y_1 \cdots y_{n-1})$ is in reduced form. By putting $A = y_1 \cdots y_{n-1}$ and $B = y_n$, we have $y = AB$, $z = BA$, and $x^{-1} = B$.

We pass from p to $p + 1$ as follows. We have

$$x_{p+1}^{-1} \cdots x_1^{-1} y_1 \cdots y_n x_1 \cdots x_{p+1} = z.$$

Since y and z are c.r. and $|y| \geq 2$, we have $x_1^{-1}y_1 = 1$ or $y_n x_1 = 1$. We first treat the case $x_1^{-1}y_1 = 1$.

Case (1): $x_1^{-1}y_1 = 1$.

Then we have

$$x_{p+1}^{-1} \cdots x_2^{-1} y_2 \cdots y_n y_1 x_2 \cdots x_{p+1} = z.$$

Put $x' = x_2 \cdots x_{p+1}$ and $y' = y_2 \cdots y_n y_1$. Then y' is c.r. and $|y'| \geq 2$. By induction there exist A_1, B_1 , $\alpha \geq 1$ and β such that $y' = (A_1 B_1)^\alpha$, $z = (B_1 A_1)^\alpha$, and $x' = A_1 (B_1 A_1)^\beta$.

Subcase (1-a): $A_1 = 1$ or $B_1 = 1$.

Then $y' = z = C^\alpha$ and $x' = C^s$ for some $s \in \mathbb{Z}$, where $C = A_1$ whenever $B_1 = 1$ and $C = B_1$ whenever $A_1 = 1$.

Since y' is c.r., C is c.r. Thus we can write $C = C' y_1$ in reduced form for some C' , and $y_1 C'$ is also in reduced form. Put $A = y_1$ and $B = C'$. Then $y = y_1 \cdots y_n = y_1 C^{\alpha-1} C' = y_1 (C' y_1)^{\alpha-1} C' = (y_1 C')^\alpha = (AB)^\alpha$, $z = y' = (C' y_1)^\alpha = (BA)^\alpha$, and $x = x_1 x' = y_1 (C' y_1)^s = A(BA)^s$.

Subcase (1-b): $A_1 \neq 1$ and $B_1 \neq 1$.

Since $B_1 \neq 1$ we can write $B_1 = B'y_1$ in reduced form for some B' . As $y' = (A_1B_1)^\alpha = (A_1B'y_1)^\alpha$, we have $y_2 \cdots y_n = (A_1B'y_1)^{\alpha-1}A_1B'$. Put $A = y_1A_1$ and $B = B'$. Then

$$\begin{aligned} y &= y_1y'y_1^{-1} = y_1(A_1B'y_1)^\alpha y_1^{-1} = y_1(A_1B'y_1)^{\alpha-1}A_1B' = \\ &= (y_1A_1B')^\alpha = (AB)^\alpha, \\ z &= (B_1A_1)^\alpha = (B'y_1A_1)^\alpha = (BA)^\alpha, \end{aligned}$$

where AB and BA are in reduced forms, and

$$x = x_1x' = y_1A_1(B_1A_1)^\beta = y_1A_1(B'y_1A_1)^\beta = A(BA)^\beta.$$

Case (2): $y_nx_1 = 1$.

By taking inverses we get

$$x_{p+1}^{-1} \cdots x_1^{-1} y_n^{-1} \cdots y_1^{-1} x_1 x_2 \cdots x_{p+1} = z^{-1}.$$

Therefore, by case (1), there exist A_1, B_1, α and β such that $y^{-1} = (A_1B_1)^\alpha$, $z^{-1} = (B_1A_1)^\alpha$ and $x = A_1(B_1A_1)^\beta$. Now, by taking $A = B_1^{-1}$ and $B = A_1^{-1}$, we have $y = (AB)^\alpha$, $z = (BA)^\alpha$ and

$$x = A_1(B_1A_1)^\beta = B^{-1}(A^{-1}B^{-1})^\beta = AA^{-1}B^{-1}(A^{-1}B^{-1})^\beta = A(BA)^{-\beta-1}.$$

Case (ii): y and z w.c.r.

Since y and z are w.c.r., we have $|y|, |z| \geq 3$. Let $x = x_1 \cdots x_p$, $y = y_1 \cdots y_n$, and $z = z_1 \cdots z_m$ in normal forms. Let $y' = y_1^{-1}yy_1 = y_2 \cdots (y_ny_1)$ and $z' = z_1^{-1}zz_1 = z_2 \cdots (z_mz_1)$. Then y' and z' are c.r. and $z_1^{-1}x^{-1}y_1(y')y_1^{-1}xz_1 = z'$. Put $x' = y_1^{-1}xz_1$. Then $x'^{-1}y'x' = z'$, $|y'| \geq 2$, and by the previous case there exist A_1, B_1, α , and β such that $y' = (A_1B_1)^\alpha$, $z' = (B_1A_1)^\alpha$ and $x' = A_1(B_1A_1)^\beta$.

Case (1): $A_1 = 1$ or $B_1 = 1$.

Then $y' = z' = C^\alpha$ and $x' = C^s$ for some $s \in \mathbb{Z}$, where $C = A_1$ whenever $B_1 = 1$ and $C = B_1$ whenever $A_1 = 1$.

Since y' is c.r., C is c.r. Thus we can write $C = C'(y_ny_1)$ in reduced form, for some C' . (Remark that $y_ny_1 = z_mz_1$ and that $y_2 \cdots y_{n-1} = z_2 \cdots z_{m-1}$ by the normal forms of y' and z').

Put $A = y_1C'z_m$ and $B = z_1y_1^{-1}$. Then

$$y = y_1C^{\alpha-1}C'y_n = y_1(C'y_ny_1)^{\alpha-1}C'y_n = (y_1C'y_n)^\alpha.$$

But $y_1C'y_n = y_1C'z_mz_1y_1^{-1}$. Therefore $y = (y_1C'z_mz_1y_1^{-1})^\alpha = (AB)^\alpha$.

We also have

$$\begin{aligned} z &= z_1(z_2 \cdots z_{m-1})z_m = z_1(C^{\alpha-1}C')z_m = z_1(C'y_ny_1)^{\alpha-1}C'z_m = \\ &= z_1(C'z_mz_1)^{\alpha-1}C'z_m = (z_1C'z_m)^{\alpha-1}z_1C'z_m = (z_1C'z_m)^\alpha. \end{aligned}$$

But $z_1 C' z_m = z_1 y_1^{-1} y_1 C' z_m = BA$. Hence $z = (BA)^\alpha$.

We see that $y = AB$ and $z = BA$ are in semi-reduced forms. We have

$$x = y_1 x' z_1^{-1} = y_1 C^s z_1^{-1}.$$

If $s \geq 0$, then

$$\begin{aligned} x &= y_1 x' z_1^{-1} = y_1 (C' y_n y_1)^s z_1^{-1} = (y_1 C' y_n)^s y_1 z_1^{-1} = \\ &= (y_1 C' z_m z_1 y_1^{-1})^s y_1 z_1^{-1} = (AB)^s B^{-1} = A(BA)^{s-1} \end{aligned}$$

If $s < 0$, then

$$\begin{aligned} x &= y_1 x' z_1^{-1} = y_1 (y_1^{-1} y_n^{-1} C'^{-1})^{-s} z_1^{-1} = (y_n^{-1} C'^{-1} y_1^{-1})^{-s} y_1 z_1^{-1} = \\ &= (y_1 C' z_m z_1 y_1^{-1})^s y_1 z_1^{-1} = (AB)^s B^{-1} = A(BA)^{s-1} \end{aligned}$$

Case (2): $A_1 \neq 1$ and $B_1 \neq 1$.

Since $A_1 \neq 1$ and $B_1 \neq 1$, we can write $B_1 = B'(y_n y_1)$ and $A_1 = A'(z_m z_1)$ in reduced forms for some B' and A' . Put $A = y_1 A' z_m$ and $B = z_1 B' y_n$. Then

$$\begin{aligned} y &= y_1 y' y_1^{-1} = y_1 (A_1 B_1)^\alpha y_1^{-1} = y_1 (A_1 B' y_n y_1)^\alpha y_1^{-1} = \\ &= y_1 (A_1 B' y_n y_1)^{\alpha-1} A_1 B' y_n = (y_1 A_1 B' y_n)^\alpha = \\ &= (y_1 A' z_m z_1 B' y_n)^\alpha = (AB)^\alpha \end{aligned}$$

and

$$\begin{aligned} z &= z_1 z' z_1^{-1} = z_1 (B_1 A_1)^\alpha z_1^{-1} = z_1 (B_1 A' z_m z_1)^\alpha z_1^{-1} = \\ &= (z_1 B_1 A' z_m)^\alpha = (z_1 B' y_n y_1 A' z_m)^\alpha = (BA)^\alpha, \end{aligned}$$

and we see that AB and BA are in semi-reduced forms.

If $x' = A_1 (B_1 A_1)^\beta$ and $\beta \geq 0$, then

$$\begin{aligned} x &= y_1 x' z_1^{-1} = y_1 A_1 (B_1 A_1)^\beta z_1^{-1} = y_1 A' z_m z_1 (B' y_n y_1 A' z_m z_1)^\beta z_1^{-1} = \\ &= y_1 A' z_m (z_1 B' y_n y_1 A' z_m)^\beta = A(BA)^\beta. \end{aligned}$$

The case $x' = A_1 (B_1 A_1)^\beta$ and $\beta < 0$ can be treated similarly.

Case (iii): y w.c.r. and z c.r.

Let $x = x_1 \cdots x_p$, $y = y_1 \cdots y_n$, and $z = z_1 \cdots z_m$ in normal forms. Let $y' = y_1^{-1} y y_1 = y_2 \cdots (y_n y_1)$. Then y' is c.r. and $x^{-1} y_1 (y') y_1^{-1} x = z$. Put $x' = y_1^{-1} x$. Then $x'^{-1} y' x' = z$ and by case (i) there exist A_1 , B_1 , α , and β such that $y' = (A_1 B_1)^\alpha$, $z' = (B_1 A_1)^\alpha$ and $x' = A_1 (B_1 A_1)^\beta$.

Then we consider the case $A_1 = 1$ or $B_1 = 1$, and the case $A_1 \neq 1$ and $B_1 \neq 1$. These two cases can be treated as the corresponding subcases (1-a) and (1-b) of case (i), taking care here of the fact that the corresponding elements A and B satisfy the following condition: AB is in reduced form and BA is in semi-reduced form.

This completes the proof of Theorem 6.1 in all cases. $\square$

7 Some malnormal subgroups of free products

We now show the malnormality of some specific subgroups of some free products.

Theorem 7.1 *Let G be a group and $\{g_i \mid i \in 2\mathbb{N} + 1\}$ be an infinite set of elements of G such that $g_i^{-1} \neq g_j$ whenever $i, j \in 2\mathbb{N} + 1$. Let $F = G * \langle a, b \mid \rangle$ and $K = \langle g_i b^{-i} a^i b^i \mid i \in 2\mathbb{N} + 1 \rangle$. Then K is a free group, with basis the set of indicated elements, and K is malnormal in F .*

The rest of this section is devoted to the proof of Theorem 7.1, so we adopt for the rest of this section all the assumptions and the notation associated to the statement of that theorem. Notice that the condition $g_i^{-1} \neq g_i$ implies that the g_i 's are not involutions.

Throughout the following proof, we view F as the free product $G * \langle a \mid \rangle * \langle b \mid \rangle$ and we consider normal forms relatively to this free product decomposition. Notice that every element of $K^\#$ has length at least 4. It is clear that K is freely generated by the indicated generators. We shall show that K is malnormal in F . We first prove:

Lemma 7.2 *Let y and z in $K^\#$ such that $y = AB$ and $z = BA$ for some elements A and $B \in F$. Then:*

- (i) *If y and z are c.r. (and, thus, AB and BA are in reduced forms), then A and $B \in K$.*
- (ii) *If y and z are w.c.r. (and, thus, AB and BA are in semi-reduced forms), then A and $B \in K$.*
- (iii) *If y is w.c.r. and z is c.r. (and, thus, AB is in reduced form and BA is in semi-reduced form), then A and $B \in K$.*

Proof

We may assume A and B nontrivial. Let $y = y_1 \cdots y_n$ and $z = z_1 \cdots z_m$ in normal forms relatively to the free product decomposition of F , and write

$$y = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \cdots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p}$$

and

$$z = (g_{j_1} b^{-j_1} a^{j_1} b^{j_1})^{\beta_1} \cdots (g_{j_q} b^{-j_q} a^{j_q} b^{j_q})^{\beta_q}$$

with some generators of K and with the condition $i_k \neq i_{k+1}$ and $j_k \neq j_{k+1}$. Then the four following conditions are simultaneously satisfied: $y_1 = g_{i_1}$ or $y_1 = b^{-i_1}$, $y_n = g_{i_p}^{-1}$ or $y_n = b^{i_p}$, $z_1 = g_{j_1}$ or $z_1 = b^{-j_1}$, and $z_m = g_{j_q}^{-1}$ or $z_m = b^{j_q}$.

We shall treat the three cases (i)–(iii) separately. The idea is to determine the last element of the normal form of A and to prove that

$$A = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \cdots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^\beta$$

for some $k \in \{1, \dots, p\}$ and some β such that $|\beta| \leq |\alpha_k|$.

Case (i)

Since $y = AB$ is in reduced form, there exists $s < n$ such that $A = y_1 \cdots y_s$ is in normal form. We shall consider the case division $z_1 = g_{j_1}$ or $z_1 = b^{-j_1}$.

Assume first $z_1 = g_{j_1}$. Then $z_m = b^{j_q}$ as z is c.r. Since $z = BA$ is in reduced form, the last element of the normal form of A is b^{j_q} , and thus $y_s = b^{j_q}$. Now the only elements of the normal form of y which are in $\langle b \rangle$ are of the form b^{i_k} , b^{-i_k} , or $b^{i_k - i_{k+1}}$. Since b has an infinite order and $i_k - i_{k+1} \neq j_q$ (because $i_k - i_{k+1}$ is even and j_q is odd), we have $y_s = b^{j_q} = b^{i_k}$ for some k and $\alpha_k > 0$. Therefore

$$A = y_1 \cdots y_s = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \cdots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^\beta$$

for some $\beta \leq \alpha_k$. Hence A is in K , as well as B .

Assume now $z_1 = b^{-j_1}$. Then $z_m = g_{j_q}^{-1}$ as z is c.r. Since $z = BA$ is in reduced form, the last element of the normal form of A is $g_{j_q}^{-1}$, and thus $y_s = g_{j_q}^{-1}$. Now the only elements of the normal form of y which are in G are of the form g_{i_k} , $g_{i_k}^{-1}$, or $g_{i_k}^{-1} g_{i_{k+1}}$ for some k . By assumption on the set $\{g_i \mid i \in 2\mathbb{N} + 1\}$, we have $y_s = g_{j_q}^{-1} = g_{i_k}^{-1}$ or $y_s = g_{j_q}^{-1} = g_{i_k}^{-1} g_{i_{k+1}}$ for some k .

We claim that the case $y_s = g_{j_q}^{-1} = g_{i_k}^{-1} g_{i_{k+1}}$ cannot occur. Indeed, if $y_s = g_{j_q}^{-1} = g_{i_k}^{-1} g_{i_{k+1}}$, then, since $y_{s+1} = b^{-i_{k+1}}$ and $y = AB$ is in reduced form, the first element of the normal form of B is $b^{-i_{k+1}}$ and $|B| \geq 3$. Since $y_{s+2} = a^{i_{k+1}}$, the second element of the normal form of B is $a^{i_{k+1}}$. Since $z_1 = b^{-j_1}$, the second element of the normal form of z is a^{-j_1} . Now, since $z = BA$ is in reduced form, we have $a^{-j_1} = a^{i_{k+1}}$, which contradicts the fact that a has an infinite order. Therefore the second case cannot occur as claimed.

Thus $y_s = g_{j_q}^{-1} = g_{i_k}^{-1}$. Then $\alpha_k < 0$ and

$$A = y_1 \cdots y_{s-1} g_{j_q}^{-1} = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \cdots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^\beta$$

for some $\beta \geq \alpha_k$. Hence A is in K , as well as B .

Case (ii)

We claim that $|A| \geq 2$ and $|B| \geq 2$. Assume towards a contradiction $|A| = 1$. Then $g_{i_1} = A \cdot g$ for some $g \in G^\#$ whenever $y_1 = g_{i_1}$, and $b^{-i_1} = A \cdot b^\gamma$ for some $\gamma \in \mathbb{Z}^\#$ whenever $y_1 = b^{-i_1}$, because AB is in semi-reduced form. As $z = BA$, this implies

$$z = BA = (g b^{-i_1} a^{i_1} b^{i_1}) \cdot (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1 - 1} \cdots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p} \cdot A = \\ (g_{j_1} b^{-j_1} a^{j_1} b^{j_1})^{\beta_1} \cdots (g_{j_q} b^{-j_q} a^{j_q} b^{j_q})^{\beta_q}$$

whenever $y_1 = g_{i_1}$, and

$$z = BA = (b^\gamma a^{-i_1} b^{i_1} g_{i_1}^{-1}) \cdot (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1 + 1} \cdots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p} \cdot A = \\ (g_{j_1} b^{-j_1} a^{j_1} b^{j_1})^{\beta_1} \cdots (g_{j_q} b^{-j_q} a^{j_q} b^{j_q})^{\beta_q}$$

whenever $y_1 = b^{-i_1}$. A simple counting of normal forms shows that $g = g_{j_1} = g_{i_1} = Ag$ whenever $y_1 = g_{i_1}$ and that $b^\gamma = b^{-j_1} = b^{-i_1} = Ab^\gamma$ whenever $y_1 =$

b^{-i_1} . We get in both cases $A = 1$, which is a contradiction to our assumption. Thus $|A| \geq 2$.

Replacing y, z, A, B by $y^{-1}, z^{-1}, B^{-1}, A^{-1}$ respectively in the above argument, we find $|B| \geq 2$. This completes the proof of our claim that $|A| \geq 2$ and $|B| \geq 2$.

We shall consider the case division $z_1 = g_{j_1}$ or $z_1 = b^{-j_1}$.

Assume first $z_1 = g_{j_1}$. Then $z_m = g_{j_q}^{-1}$ as z is w.c.r. Since $z = BA$ and $|A| \geq 2$, the last element of the normal form of A is $g_{j_q}^{-1}$. As $z = BA$ is in semi-reduced form and $|B| \geq 2$, the first element in the normal form of B is g_{j_1} . Since $y = AB$ is in semi-reduced form, there exists $s < n$ such that $y_s = g_{j_q}^{-1} g_{j_1}$ and $A = y_1 \cdots y_{s-1} g_{j_q}^{-1}$. As before the only elements of the normal form of y which are in G are of the form $g_{i_k}, g_{i_k}^{-1}$, or $g_{i_k}^{-1} g_{i_{k+1}}$ for some k . Therefore, we have $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}$ or $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}^{-1}$ or $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}^{-1} g_{i_{k+1}}$ for some k .

We claim that the cases $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}$ and $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}^{-1}$ cannot occur. Indeed:

If $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}$, then, since AB is in semi-reduced form and $A = y_1 \cdots y_{s-1} g_{j_q}^{-1}$, we have $y_{s-1} = b^{i_{k-1}}$ and $s \geq 5$. Thus $|A| \geq 5$ and $y_{s-2} = a^{i_{k-1}}$. Since $z = BA$ is in semi-reduced form and $|A| \geq 5$, we have $z_{m-1} = y_{s-1} = b^{i_{k-1}}$, thus $z_{m-2} = a^{i_{k-1}}$. Since $z_m = g_{j_q}^{-1}$ we have $z_{m-1} = b^{j_q}$ and $z_{m-2} = a^{-j_q}$. This implies that $a^{-j_q} = a^{i_{k-1}}$, a contradiction as a has an infinite order.

If $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}^{-1}$, then, since AB is in semi-reduced form and $A = y_1 \cdots y_{s-1} g_{j_q}^{-1}$, we have $y_{s-1} = b^{i_k}$ and $s \geq 4$. Thus $|A| \geq 4$. Since $z = BA$ is in semi-reduced form and $|A| \geq 4$, we have $z_{m-1} = y_{s-1} = b^{i_k}$. Now, since $z_m = g_{j_q}^{-1}$ we have $z_{m-1} = b^{j_q}$. Hence $b^{j_q} = b^{i_k}$ and $j_q = i_k$ as b has an infinite order. But $g_{j_q}^{-1} g_{j_1} = g_{i_k}^{-1}$, thus $g_{j_1} = 1$, a contradiction.

Thus $y_s = g_{j_q}^{-1} g_{j_1} = g_{i_k}^{-1} g_{i_{k+1}}$ as claimed. Then $\alpha_k < 0$ and $|A| \geq 4$. Since $z = BA$ and $A = y_1 \cdots y_{s-1} g_{j_q}^{-1}$, $b^{i_k} = y_{s-1} = z_{m-1} = b^{j_q}$. Thus $i_k = j_q$ and

$$A = y_1 \cdots y_{s-1} g_{j_q}^{-1} = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \cdots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^{\beta}$$

for some $\beta \geq \alpha_k$. Hence A is in K , as well as B . This completes the proof in the case $z_1 = g_{j_1}$.

Consider now the case $z_1 = b^{-j_1}$. Then $z_m = b^{j_q}$ as y is w.c.r. As in the previous case, the last element of the normal of A is b^{j_q} , the first element of the normal form of B is b^{-j_1} , and there exists $s < n$ such that $y_s = b^{j_q} b^{-j_1}$ and $A = y_1 \cdots y_{s-1} b^{j_q}$. Now the only elements of the normal form of y which are in $\langle b \rangle$ are of the form b^{i_k}, b^{-i_k} , or $b^{i_k - i_{k+1}}$ for some k . Since b has an infinite order and $j_q - j_1 \neq \pm i_k$ (because $j_q - j_1$ is even and $\pm i_k$ is odd), we have $y_s = b^{j_q} b^{-j_1} = b^{i_k - i_{k+1}}$ for some k and $\alpha_k > 0$. We have $A = y_1 \cdots y_{s-1} b^{j_q}$ and $y_{s-1} = a^{i_k} = a^{j_q}$ as above. Therefore $i_k = j_q$ and

$$A = y_1 \cdots y_{s-1} b^{j_q} = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \cdots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^{\beta}$$

for some $\beta \leq \alpha_k$. Hence A is in K , as well as B .

Case (iii)

We claim that $|A| \geq 2$. Assume towards a contradiction $|A| = 1$. Then $A = g_{i_1}$ whenever $y_1 = g_{i_1}$ and $A = b^{-i_1}$ whenever $y_1 = b^{-i_1}$, since AB is in reduced form. Since $z = BA$, this implies

$$z = BA = (b^{-i_1} a^{i_1} b^{i_1}) \cdot (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1-1} \dots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p} g_{i_1} = \\ (g_{j_1} b^{-j_1} a^{j_1} b^{j_1})^{\beta_1} \dots (g_{j_q} b^{-j_q} a^{j_q} b^{j_q})^{\beta_q}$$

whenever $y_1 = g_{i_1}$ and

$$z = BA = (a^{-i_1} b^{i_1} g_{i_1}^{-1}) \cdot (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1+1} \dots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p} \cdot b^{-i_1} = \\ (g_{j_1} b^{-j_1} a^{j_1} b^{j_1})^{\beta_1} \dots (g_{j_q} b^{-j_q} a^{j_q} b^{j_q})^{\beta_q}$$

whenever $y_1 = b^{-i_1}$. Computing with normal forms, we get: if $y_1 = g_{i_1}$, then $b^{-i_1} = g_{j_1}$ or $b^{-i_1} = b^{-j_1}$ and $a^{i_1} = a^{-j_1}$, and if $y_1 = b^{-i_1}$, then $a^{-i_1} = b^{-j_1}$ or $a^{-i_1} = g_{j_1}$. We get in both cases a contradiction, and thus $|A| \geq 2$ as claimed.

Since $y = AB$ is in reduced form, there exists $s < n$ such that $A = y_1 \dots y_s$ is in normal form. As before we consider the case division $z_1 = g_{j_1}$ or $z_1 = b^{-j_1}$.

If $z_1 = g_{j_1}$, then $z_m = b^{j_q}$ as z is c.r. Since $z = BA$ is in semi-reduced form and $|A| \geq 2$, the last element of the normal form of A is b^{j_q} , and thus $y_s = b^{j_q}$. Therefore we see as in the corresponding subcase of case (i) that $y_s = b^{j_q} = b^{i_k}$ for some k and that $\alpha_k > 0$. Therefore

$$A = y_1 \dots y_s = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \dots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^{\beta}$$

for some $\beta \leq \alpha_k$. Hence A is in K , as well as B .

If $z_1 = b^{-j_1}$, then $z_m = g_{j_q}^{-1}$ as z is c.r. Therefore, as in the corresponding subcase of case (i), the last element of the normal of A is $g_{j_q}^{-1}$, $y_s = g_{j_q}^{-1}$, and $y_s = g_{j_q}^{-1} = g_{i_k}^{-1} g_{i_{k+1}}$ or $y_s = g_{j_q}^{-1} = g_{i_k}^{-1}$ for some k .

We claim that the case $y_s = g_{j_q}^{-1} = g_{i_k}^{-1} g_{i_{k+1}}$ cannot occur. Indeed, if $y_s = g_{j_q}^{-1} = g_{i_k}^{-1} g_{i_{k+1}}$, then the first element of the normal form of B is $b^{-i_{k+1}}$ and $|B| \geq 3$, and the second element of the normal form of B is $a^{i_{k+1}}$, which is a contradiction as the second element of the normal form of z must be a^{-j_1} .

Thus $y_s = g_{j_q}^{-1} = g_{i_k}^{-1}$ as claimed. Then $\alpha_k < 0$ and

$$A = y_1 \dots y_{s-1} g_{j_q}^{-1} = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \dots (g_{i_k} b^{-i_k} a^{i_k} b^{i_k})^{\beta}$$

for some $\beta \geq \alpha_k$. Hence A is in K , as well as B . □

Lemma 7.3 *Let y in K , A in F , and $\alpha \geq 1$ such that $y = A^\alpha$. Then:*

- (i) *If y is c.r. (and thus AA is in reduced form), then $A \in K$.*
- (ii) *If y is w.c.r. (and thus AA is in semi-reduced form), then $A \in K$.*

Proof

Put $A' = A$ and $B' = A^{\alpha-1}$. Then $y = A'B' \in K$ and $y = B'A' \in K$.

If y is c.r. and AA is in reduced form, then $A'B'$ and $B'A'$ are in reduced forms and $A \in K$ by Lemma 7.2.

If y is w.c.r. and AA is in semi-reduced form, then $A'B'$ and $B'A'$ are in semi-reduced forms and $A \in K$ by Lemma 7.2. $\square$

We are now ready to prove Theorem 7.1, i.e. that K is malnormal in F . Let $y, z \in K^\#$, and $x \in F$ such that $y^x = z$. We want to show that $x \in K$.

We claim that we can assume y and z cyclically reduced in K (relatively to the free basis of K indicated above). Indeed, it is well-known that every element of a free group is conjugate to a cyclically reduced one. Thus $y = y'^\alpha$ and $z = z'^\beta$ for some elements y' and z' c.r. in K and some $\alpha, \beta \in K$. Thus $\beta x^{-1} \alpha^{-1} y' \alpha x \beta^{-1} = z'$ and x is in K provided $\alpha x \beta^{-1}$ is.

We assume thus y and z cyclically reduced in K (relatively to the basis of K). We claim that y is then c.r. or w.c.r. in F , as well as z . Indeed if

$$y = (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \dots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p}$$

for some generators of K , then yy is equal to

$$(g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \dots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p} \cdot (g_{i_1} b^{-i_1} a^{i_1} b^{i_1})^{\alpha_1} \dots (g_{i_p} b^{-i_p} a^{i_p} b^{i_p})^{\alpha_p}.$$

Now if α_1 and α_p have the same sign, then y is c.r. If $\alpha_1 > 0$ and $\alpha_p < 0$, then the first and the last element of the normal form of y are g_{i_1} and $g_{i_p}^{-1}$ respectively. If $g_{i_p}^{-1} g_{i_1} = 1$, then $i_1 = i_p$, which contradicts the fact that y is c.r. in K . Therefore $g_{i_p}^{-1} g_{i_1} \neq 1$, yy is in semi-reduced form, and y is w.c.r. The case $\alpha_1 < 0$ and $\alpha_p > 0$ can be treated similarly. Thus y is c.r. or w.c.r. in F , as well as z , which completes the proof of our claim.

Since $|y| > 1$, there exist by Theorem 6.1 A and B in F , $\alpha \geq 1$, and β such that $y = (AB)^\alpha$, $z = (BA)^\alpha$, and $x = A(BA)^\beta$ with:

- (i) AB and BA in reduced forms (and, thus, AB and BA c.r.) provided y and z are c.r.
- (ii) AB and BA in semi-reduced forms (and, thus, AB and BA w.c.r.) provided y and z are w.c.r.
- (iii) AB in reduced form and BA in semi-reduced form (and, thus, AB w.c.r. and BA c.r.) provided y is w.c.r. and z is c.r.

Now AB and BA are in K by Lemma 7.3, and A and B are in K by Lemma 7.2. Hence $x = A(BA)^\beta \in K$. This ends the proof of the malnormality of K and, thus, of Theorem 7.1. $\square$

8 Embedding in finitely generated CSA-groups

We are now ready, with the result of the previous section, to prove Theorem 1.3. Recall that we want to embed any countable CSA $_f$ -group G , with $f(2) = 0$,

into a finitely generated one having, up to conjugacy, the same maximal abelian subgroups, except maybe the infinite cyclic ones.

Proof of Theorem 1.3

We may of course assume G not finitely generated. Let $X = 2\mathbb{N} + 1$. As G has no involutions, it has by Zorn's Lemma a generating set $\{g_i \mid i \in X\}$ such that $g_i^{-1} \neq g_j$ for every $i, j \in X$. Consider the free product

$$F = G * \langle a \mid \rangle * \langle b \mid \rangle * \langle c \mid \rangle.$$

Throughout the following proof, we consider normal forms relatively to the indicated free product decomposition of F . Consider the subgroups

$$D = \langle c^i b^{-i} a^i b^i \mid i \in X \rangle \text{ and } E = \langle g_i a^{-i} b^i a^i \mid i \in X \rangle$$

of F . It is clear that D and E are free over their indicated generators and that the function $\varphi : c^i b^{-i} a^i b^i \mapsto g_i a^{-i} b^i a^i$ ($i \in X$) extends to an isomorphism from D to E . Consider the HNN -extension

$$F^* = \langle F, t \mid t^{-1} d t = \varphi(d), d \in D \rangle.$$

We are going to prove that the supergroup F^* of G has the required properties. First notice that F^* is finitely generated by $\{a, b, c, t\}$. By the Kurosh Subgroup Theorem (Fact 2.16), F and G have, up to conjugacy, the same maximal abelian subgroups, except maybe the infinite cyclic ones. By Theorem 3.2, it is thus sufficient to prove that the HNN -extension F^* of F is separated and that D and E are malnormal in F .

By Theorem 7.1, E is malnormal in $G * \langle a, b \rangle$. By well-known properties of free products, $G * \langle a, b \rangle$ is malnormal in F . Hence E is malnormal in F . Now it is clear that the set $\{c^i \mid i \in X\}$ satisfies the assumptions of Theorem 7.1, and therefore D is malnormal in $\langle a, b, c \rangle$. As before, we find D malnormal in F .

It remains only to show that F^* is separated. Assume towards a contradiction that there exist $x \in F$, $y \in E^\#$, and $z \in D^\#$ such that $y^x = z$.

Then there exist y' and z' cyclically reduced in $E^\#$ (relatively to the basis of E) and $D^\#$ (relatively to the basis of D) respectively such that $y = y'^\alpha$ and $z = z'^\beta$ for some elements $\alpha \in E$ and $\beta \in D$. Then $\beta x^{-1} \alpha^{-1} y' \alpha x \beta^{-1} = z'$ and, replacing x by $\alpha x \beta^{-1}$ if necessary, we can assume y and z c.r. in E and D respectively.

As at the end of the proof of Theorem 7.1, we see that y is then c.r. or w.c.r. in F , as well as z . By Theorem 6.1, there exist elements A and B and $\alpha \geq 1$ such that $y = (AB)^\alpha$ and $z = (BA)^\alpha$, with AB and BA in reduced form or in semi-reduced form, depending on the possible cases. As E and D are malnormal in F , we have $AB \in E$ and $BA \in D$. Since $AB \in E$, there exists an element of $G^\#$ which appears in the normal form of AB , and thus the same property holds for A or B as AB is in reduced or in semi-reduced form. Since BA is in reduced or in semi-reduced form, there are no cancellations in the product BA , and thus there exists an element of $G^\#$ which appears in the normal form of BA . But $BA \in D$, which is clearly a contradiction.

Therefore F^* is separated and we have the desired conclusion. $\square$

We derive the following corollaries from Theorem 1.3:

Corollary 8.1 *Assume $f(2) = 0$. Then there exist $2^{\aleph_0}$ finitely generated CSA_f -groups. In particular there exist $2^{\aleph_0}$ countable existentially closed CSA_f -groups.*

Proof

For $X \subseteq \mathbb{N} \setminus \{0, 1\}$, let $G(X) = \ast_{n \in X} \mathbb{Z}^n$. By the Kurosh Subgroup Theorem (Fact 2.16), the maximal abelian subgroups of $G(X)$, except maybe the infinite cyclic ones, are isomorphic to $\mathbb{Z}^n$, with n in X . As any nontrivial element of $G(X)$ lies in the subgroup generated by finitely many factors, one can check easily with Theorem 4.2 that $G(X)$ is a CSA -group. Furthermore $G(X)$ is torsion-free by Fact 2.13. Then, by Theorem 1.3, $G(X)$ embeds into a finitely generated CSA_f -group $F(X)$ having, up to isomorphism, the same maximal abelian subgroups, except maybe the infinite cyclic ones. Thus $F(X) \not\cong F(Y)$ whenever $Y \subseteq \mathbb{N} \setminus \{0, 1\}$ and $Y \neq X$. It follows that there exist $2^{\aleph_0}$ pairwise nonisomorphic finitely generated CSA_f -groups of the form $F(X)$.

The second statement is now a mere corollary of Fact 2.1 and of the remark following Fact 2.4. $\square$

Corollary 8.2 *Assume $f(2) = 0$, and let G be an existentially closed CSA_f -group. Then the first-order theory of G has $2^{\aleph_0}$ types over $\emptyset$.*

Proof

We first prove that the universal theory $\text{Th}_{\forall}(G)$ is true in every CSA_f -group. Suppose towards a contradiction that there exists $\psi \in \text{Th}_{\forall}(G)$ such that $\neg\psi$ is true in some CSA_f -group H . Then the existential sentence $\neg\psi$ is true in $G * H$, which is a CSA_f -group by Theorem 4.2. Since G is existentially closed, $\neg\psi$ is true in G , a contradiction.

Now, by Fact 2.2, every CSA_f -group embeds into a model of the theory $\text{Th}(G)$. But there are $2^{\aleph_0}$ finitely generated such groups by Theorem 1.3. In particular, $\text{Th}(G)$ has $2^{\aleph_0}$ types over $\emptyset$. $\square$

Acknowledgements We thank Bruno Poizat who motivated this work, and Frank Wagner and the referee for some useful information.

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