Embeddings in finitely presented groups which preserve the centre

Abderezak OULD HOUCINE
Université Claude Bernard Lyon 1
43 boulevard du 11 novembre 1918
69622 Villeurbanne cedex
France
ould@math.univ-lyon1.fr

Abstract

V.N. Remeslennikov proposed in 1976 the following problem: *is any countable abelian group a subgroup of the centre of some finitely presented group* ? We prove that every finitely generated recursively presented group G is embeddable in a finitely presented group K such that the centre of G coincide with that of K. We prove also that there exists a finitely presented group H with soluble word problem such that every countable abelian group is embeddable in the centre of H. This gives a strong positive answer to the question raised by V.N. Remeslennikov.

1 Introduction

V.N. Remeslennikov proposed in 1976 the following problem: *is any countable abelian group a subgroup of the centre of some finitely presented group* ? The problem, which is natural in the context of Higman's famous embedding theorem, is listed recently as open [3, 8]. However in 1980, B.M. Hurley [5] announced, without proof, the following proposition which yields a positive answer to the above problem: *a necessary and sufficient condition for an abelian group to be the centre of some finitely presented group is that it should be recursively presentable.* In this paper we prove various results on embeddings in finitely presented groups which preserve the centre, including the proposition stated by B.M. Hurley and the fact that there exists *a finitely presented group H with soluble word problem such that every countable abelian group is embeddable in the centre of H*. This gives of course a positive answer to the question raised by V.N. Remeslennikov.

We shall now state the main results of the paper which will be proved in Section 4,5,6, while Section 2 is devoted to presenting the terminology and the tools used, and Section 3 contains preparatory propositions. The main results of this paper are as follows.

Theorem I. *Let G be a countable group. Then G is embeddable in a finitely generated group K such that $Z(G) = Z(K)$ and:*

(1) *If G is recursively presented and $Z(G)$ is recursively enumerable in G, then we can take K to be recursively presented.*

(2) *If G has a soluble word problem and $Z(G)$ has a generalized soluble word problem in G, then we can take K with soluble word problem.*

Theorem II. *Let G be a finitely generated recursively presented group. Then G is embeddable in a finitely presented group K such that $Z(G) = Z(K)$ and if G has a soluble word problem, then we can take K with soluble word problem.*

Corollary 1. *An abelian group is the centre of a finitely presented group if and only if it is recursively presentable.*

Corollary 2. *An abelian group is the centre of a finitely presented group with soluble word problem if and only if it has a presentation admitting a soluble word problem.*

Corollary 3. *There exists a finitely presented group H with soluble word problem such that every countable abelian group is embeddable in the centre of H.*

One can also deduce the following corollary which is a generalization of Theorem II.

Corollary 4. *Let G be a finitely generated recursively presented group. Let A be a countable recursively presented abelian group. Then G is embeddable in a finitely presented group K such that $Z(K) = A$ and if G has a soluble word problem and A has a presentation admitting a soluble word problem, then we can take K with soluble word problem.*

2 Preliminaries

The goal of this section is to fix the definitions that are going to be used and to present the small cancellation theory over amalgamated free products. We work in the following context. Let G_1, G_2 be groups and A a common subgroup of G_1 and G_2. One considers the free product of G_1 and G_2 amalgamating the subgroup A and one notes it $F = G_1 *_A G_2$. We call G_1 and G_2 the factors of F. Then for every element $w \in F$ such that $w \notin A$, there exists a sequence $(g_1, \cdots, g_n)$ of elements of $G_1 \cup G_2$ such that $w = g_1 \cdots g_n$ and:

(i) g_i, g_{i+1} come from different factors,

(ii) $g_i \notin A$.

A sequence which satisfies the conditions (i)-(ii) is called reduced. It is well known that if $(g_1, \cdots, g_n)$, $(h_1, \cdots, h_m)$ are reduced sequences such that $g_1 \cdots g_n = h_1 \cdots h_m$, then $m = n$. Then for every element $w \in F$ we define the length of w denoted $|w|$ by: $|w| = 0$ if $w \in A$ and $|w| = n$ (if $w \notin A$) where n is the length of some reduced sequence $(g_1, \cdots, g_n)$ such that $w = g_1 \cdots g_n$.

Let $w \in F$. A ***normal form*** of w is a sequence $(g_1, \cdots, g_n)$ such that $w = g_1 \cdots g_n$ and if $w \in A$, then $n = 1$, otherwise $(g_1, \cdots, g_n)$ is reduced. Notice that an element w of F can have several normal forms.

Let $g = g_1 \cdots g_n$. We say that $g = g_1 \cdots g_n$ is in ***normal form*** if $(g_1, \cdots, g_n)$ is a normal form of g.

A normal form $(g_1, \cdots, g_n)$ of an element w is ***cyclically reduced*** if $n = 1$ or if g_n and g_1 are in different factors. Then one normal form of w is cyclically reduced if and only if all normal forms of w are cyclically reduced, which allows us to define cyclically reduced elements.

A normal form $(g_1, \cdots, g_n)$ of an element w is ***weakly cyclically reduced*** if $n = 1$ or if $g_n g_1 \notin A$. Then one normal form of w is weakly cyclically reduced if and only if all normal forms of w are weakly cyclically reduced. As before, this allows us to define weakly cyclically reduced elements.

A subset W of F is ***symmetrized*** if:

(i) *every element of W is weakly cyclically reduced,*

(ii) *if $w \in W$ then $w^{-1} \in W$,*

(iii) *every weakly cyclically reduced conjugate of every element of W is in W.*

Given a group G and a subset $X \subseteq G$, we denote by $X^{\pm 1}$ the set $X \cup X^{-1}$. Let R be a subset of F such that every element of R is weakly cyclically reduced. The ***symmetrized closure*** of R, denoted by $W(R)$, is the smallest symmetrized subset of F which contains R. We denote by $W_0(R)$ the set of cyclically reduced conjugates of elements of $R^{\pm 1}$.

One has the following lemma that summarizes some properties of normal forms and symmetrized sets.

Lemma 2.1 *Let $F = G_1 *_A G_2$ be a free product with amalgamation.*

(1) *Let R be a subset of F such that every element of R is weakly cyclically reduced. Then the symmetrized closure of R is the set of all weakly cyclically reduced conjugates of elements of $R^{\pm 1}$.*

(2) *If $(g_1, \cdots, g_n)$, $(h_1, \cdots, h_n)$ are normal forms such that $g_1 \cdots g_n = h_1 \cdots h_n$, then there exists a sequence $(a_1, \cdots, a_n, a_{n+1})$ of elements of A such that $a_1 = a_{n+1} = 1$ and for every $i = 1, \cdots, n$, $g_i = a_i h_i a_{i+1}^{-1}$.* □

Let $u, v \in F$ with normal form $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_m)$ respectively. Let $g = uv$. We say that g is in ***semi-reduced form*** (u, v) if $u_n v_1 \notin A$. We say that g is in ***reduced form*** (u, v) if u_n, v_1 are in different factors.

One of the objects of small cancellation theory is to see, when we have a normal subgroup N of F, what conditions insure that N does not have *short* elements and in particular guarantee $N \cap G_1 = N \cap G_2 = 1$, so that in the quotient F/N *short* elements are not *hurt*.

Let W be a subset of F. An element $b \in F$ is said to be a ***piece*** (relative to W) if there exists distinct elements $w_1, w_2 \in W$ such that $w_1 = bc_1$ and

$w_2 = bc_2$ in semi-reduced form. This means that b is cancelled in the product $w_2^{-1}w_1$.

For a positive real number λ we define the following condition:

$C'(\lambda)$: *if* $w \in W$ *is in semi-reduced form* (b, c) *where* b *is a piece then* $|b| < \lambda\,|w|$. *Further, for every* $w \in W$, $|w| > (1/\lambda)$.

In practice, to verify that a set W with $W = W^{-1}$satisfies $C'(\lambda)$, one takes two elements w_1, w_2 of W such that $w_1w_2 \neq 1$ and one proves that the length of the element which is cancelled in the product w_1w_2 is smaller than $\lambda\,|w_1|$ and $\lambda\,|w_2|$. Using normal forms this is equivalent to the following:

if $w_1 = a_m a_{m-1} \cdots a_1$ *and* $w_2 = b_1 \cdots b_n$ *are in normal forms and*

$$a_i \cdots a_1.b_1 \cdots b_i \in G_1 \cup G_2,$$

then $i < \lambda m$, $i < \lambda n$. (*Of course also the condition for every* $w \in W$, $|w| > (1/\lambda)$.)

We will use frequently the following principal theorem.

Theorem 2.2 [6, Theorem 11.2, Chapter V] *Let* $F = G_1 *_A G_2$ *be a free product with amalgamation,* W *be a symmetrized subset of* F *and let* N *be the normal closure of* W *in* F. *Suppose that* W *satisfies* $C'(\lambda)$ *with* $\lambda \leq \frac{1}{6}$. *If* $w \in N$, *with* $w \neq 1$, *then* $w = usv$ *in reduced form where there is a cyclically reduced* $r \in W$, *with* $r = st$ *in reduced form and* $|s| > (1 - 3\lambda)|r|$.

In particular, the natural map $\pi\colon F \to F/N$ *embeds each factor of* F. □

We need also the following theorem.

Theorem 2.3 [6, Theorem 2.8, Chapter IV] *Let* $F = G_1 *_A G_2$ *be a free product with amalgamation. Let* $u = u_1 \cdots u_n$ *be a cyclically reduced element of* F *where* $(u_1, \cdots, u_n)$ *is a normal form and* $n \geq 2$. *Then every cyclically reduced conjugate of* u *can be written as* ava^{-1} *where* $a \in A$ *and* v *is the product of some cyclic permutation of* $(u_1, \cdots, u_n)$.

Lemma 2.4 *Let* $F = G_1 *_A G_2$ *be a free product with amalgamation. Let* λ,α *be positive real numbers such that* $\lambda \leq \alpha$. *Let* R *be a subset of* F *which satisfies:*

(1) *Every element of* R *is cyclically reduced.*

(2) *For every* $r \in R$, $\lambda|r| + 1 \leq \alpha\,|r|$, *and* $|r| > \frac{1}{\alpha}$.

If $W_0(R)$ *satisfies* $C'(\lambda)$, *then* $W(R)$ *satisfies* $C'(\alpha)$.

Proof

Observe that since every element of R is cyclically reduced and for every $r \in R$, $|r| > \frac{1}{\alpha}$, then every element w of $W(R)$ satisfies $|w| > \frac{1}{\alpha}$.

By Lemma 2.1 (1), we know that the elements of $W(R)$ are the weakly cyclically reduced conjugates of elements of $R^{\pm 1}$. Let $w_1, w_2 \in W(R)$ such that $w_1w_2 \neq 1$. We are going to prove that if some element is cancelled in the product w_1w_2, then its length is smaller than $\alpha\,|w_1|$ and $\alpha\,|w_2|$.

We have to consider two cases. The first case where w_1,w_2 are not cyclically reduced, and the second case where w_1 cyclically reduced and w_2 is not cyclically reduced. The other cases can be reduced to the previous ones, or they are obvious as the case where w_1, w_2 are cyclically reduced.

Let $w_1 = a_1 \cdots a_n$ and $w_2 = b_1 \cdots b_m$ in normal form. Since w_1 and w_2 are weakly cyclically reduced we have $a_n a_1 \notin A$ and $b_m b_1 \notin A$.

Case (1). w_1 and w_2 are not cyclically reduced.

We can write

$$w_1 = a_n^{-1}(a_n a_1)a_2 \cdots a_{n-1}a_n \text{ and } w_2 = b_m^{-1}(b_m b_1)b_2 \cdots b_{m-1}b_m.$$

Since $a_n a_1 \notin A$ and $b_m b_1 \notin A$, and a_n, a_1 are in the same factor, b_m, b_1 are in the same factor, the elements $(a_n a_1)a_2 \cdots a_{n-1}$, $(b_m b_1)b_2 \cdots b_{m-1}$ are in reduced form and are cyclically reduced. We see that they are conjugates of elements of $R^{\pm 1}$. Now consider how there can be cancellation in the product $w_1 w_2$. If there is no cancellation we have the result. If $|a_n b_1| = 1$, then the length of any piece is smaller than 1, and since $1 < \alpha\,|w_1|$, $1 < \alpha\,|w_2|$, we get the desired conclusion.

Now suppose that $a_n b_1 \in A$ and let $\gamma = a_n b_1$. We see that $b_2 \cdots b_{m-1}(b_m b_1)$ and $\gamma b_2 \cdots b_{m-1}(b_m b_1)\gamma^{-1}$ are cyclically reduced conjugates of some element of $R^{\pm 1}$. Then

$$w_1 w_2 = a_n^{-1}((a_n a_1)a_2 \cdots a_{n-1})\gamma b_2 \cdots b_{m-1}(b_m b_1)b_1^{-1}$$
$$= a_n^{-1}((a_n a_1)a_2 \cdots a_{n-1})\gamma b_2 \cdots b_{m-1}(b_m b_1)\gamma^{-1}a_n.$$

Let

$$r_1 = (a_n a_1)a_2 \cdots a_{n-1}, \quad r_2 = \gamma b_2 \cdots b_{m-1}(b_m b_1)\gamma^{-1}.$$

Then r_1 and r_2 are in $W_0(R)$. It is enough to look at pieces in the product $r_1 r_2$.

Since $w_1 w_2 \neq 1$, $r_1 r_2 \neq 1$. By hypothesis $W_0(R)$ satisfies $C'(\lambda)$. Therefore, if d is a piece in the product of r_1 and r_2, then $|d| < \lambda |r_1|$ and $|d| < \lambda |r_2|$. But it is not difficult to see that the corresponding piece in the product of w_1 and w_2 is of length $|d| + 1$. Then

$$|d| + 1 < \lambda|r_1| + 1, \quad |d| + 1 < \lambda|r_2| + 1,$$

also,

$$|d| + 1 < \alpha(|r_1| + 1), \quad |d| + 1 < \alpha(|r_2| + 1).$$

But $|w_1| = |r_1| + 1$ and $|w_2| = |r_2| + 1$. Thus we get the desired conclusion.

Case (2). w_1 is not cyclically reduced and w_2 is cyclically reduced.

The proof is similar to the previous one. In this case we see that $w_1 = a_n^{-1}(a_n a_1)a_2 \cdots a_{n-1}a_n$ and $w_2 = b_1 b_2 \cdots b_{m-1}b_m$. As before since $a_n a_1 \notin A$ and a_n, a_1 are in the same factor, then the element $(a_n a_1)a_2 \cdots a_{n-1}$ is in reduced form and it is a cyclically reduced conjugate of an element of $R^{\pm 1}$. If there exists cancellation in the product of w_1 and w_2 then $a_n b_1 \in A$. As before put $\gamma = a_n b_1$. We have

$$w_1 w_2 = a_n^{-1}((a_n a_1)a_2 \cdots a_{n-1})\gamma b_2 \cdots b_{m-1} b_m . b_1 b_1^{-1}$$
$$= a_n^{-1}((a_n a_1)a_2 \cdots a_{n-1})\gamma b_2 \cdots b_{m-1} b_m b_1 \gamma^{-1} a_n.$$

We see also that $\gamma b_2 \cdots b_{m-1} b_m b_1 \gamma^{-1}$ is a cyclically reduced conjugate of some element of $R^{\pm 1}$.

As in the previous case, if d is a piece in the product of $r_1 = (a_n a_1)a_2 \cdots a_{n-1}$ and $r_2 = \gamma b_2 \cdots b_{m-1} b_m b_1 . \gamma^{-1}$, then $|d| < \lambda |r_1|$ and $|d| < \lambda |r_2|$. But, as before, the corresponding piece in the product of w_1 and w_2 is of length $|d| + 1$. Then

$$|d| + 1 < \lambda |r_1| + 1, \quad |d| + 1 < \lambda |r_2| + 1,$$

also,

$$|d| + 1 < \alpha(|r_1| + 1), \quad |d| + 1 < \alpha |r_2|.$$

But $|w_1| = |r_1| + 1$ and $|w_2| = |r_2|$, and thus we get the result.

Therefore, $W(R)$ satisfies $C'(\alpha)$. □

The proof of the following lemma can be extracted from elsewhere, but for completeness we provide a proof.

Lemma 2.5 *Let $F = G_1 *_A G_2$ be a free product with amalgamation such that $G_1 \neq A$ and $G_2 \neq A$. Then $Z(F) \leq Z(A)$.*

Proof

Let $g \in Z(F)$ and let $(g_1, \cdots, g_n)$ be a normal form of g. Suppose that $|g| \geq 2$. Since $g \in Z(F)$ we have $g g_n^{-1} = g_n^{-1} g$, hence $g_1 \cdots g_{n-1} g_n^{-1} \cdots g_2^{-1} g_1^{-1} g_n = 1$. Since g_{n-1} and g_n are in different factors and $n \geq 2$, we see that

$$|g_1 \cdots g_{n-1} g_n^{-1} \cdots g_2^{-1} g_1^{-1} g_n| \geq 1,$$

which is a contradiction.

Suppose now $|g| = 1$. Suppose $g \in G_1$. Since $G_2 \neq A$, there exists $g' \in G_2 \setminus A$. Then $g g' g^{-1} g'^{-1} = 1$, which is a contradiction because the sequence (g, g', g^{-1}, g'^{-1}) is reduced. As $G_1 \neq A$, the argument apply when $g \in G_2$. Hence $|g| = 0$. Therefore $Z(F) \leq A$ and consequently $Z(F) \leq Z(A)$. □

We finish this section with some properties (and definitions) of recursively presented groups and groups with soluble word problem. Let $X = \{x_i | i \in \mathbb{N}\}$. A countable group G is said *recursively presented*, if G has a presentation $G = \langle X | P(X) \rangle$ such that $P(X)$ is recursively enumerable; and it is said to have a *soluble word problem*, if it has a presentation $G = \langle X | P(X) \rangle$ for which the set of words $w(\bar{x})$ on $X^{\pm 1}$ such that $w(\bar{x}) = 1$ in G is recursive. A subgroup $H \leq G$ is said to have a *soluble generalized word problem in G*, if the set of words $w(\bar{x})$ such that $w \in H$ is recursive, and is said *recursively enumerable in G* if the set of words $w(\bar{x})$ such that $w(\bar{x}) \in H$ is recursively enumerable.

Lemma 2.6 *Let G be a finitely generated group.*

(1) *If G is recursively presented, then $Z(G)$ is recursively enumerable in G and it is recursively presented.*

(2) *If G has a soluble word problem, then $Z(G)$ has a soluble word problem and also a soluble generalized word problem in G.*

Proof

Suppose that G is generated by $\bar{a} = \{a_1, \cdots, a_n\}$. Let $W(\bar{y})$ be the set of words over the set $\{y_1, \cdots, y_n\}^{\pm 1}$. Let $V = \{w(\bar{y}) \in W(\bar{y}) \mid w(\bar{a}) \in Z(G)\}$. Then clearly we have $V = \{w(\bar{y}) \in W(\bar{y}) \mid [w(\bar{a}), a_i] = 1, 1 \leq i \leq n\}$.

(1) Since G is recursively presented the set $\{w(\bar{y}) \in W(\bar{y}) \mid w(\bar{a}) = 1\}$ is recursively enumerable and thus V is recursively enumerable. Then $Z(G)$ is recursively enumerable in G. Now let us show that $Z(G)$ is recursively presented. Let $(v_i(\bar{y}) \mid i \in \mathbb{N})$ be an enumeration of V. Let L be the set of words on $\{v_i \mid i \in \mathbb{N}\}^{\pm 1}$, regarding it as a set of variables. Then the set

$$K = \{w(v_{i_1}, \cdots, v_{i_m}) \mid w(v_{i_1}(\bar{a}), \cdots, v_{i_m}(\bar{a})) = 1\}$$

is recursively enumerable.

Let $X = \{x_i \mid i \in \mathbb{N}\}$. If $w(v_{i_1}, \cdots, v_{i_m})$ is a word in L, let $w(x_{i_1}, \cdots, x_{i_m})$ denote the word obtained by replacing each v_{i_j} by x_{i_j}. Let

$$P = \{w(x_{i_1}, \cdots, x_{i_m}) \in L \mid w(v_{i_1}, \cdots, v_{i_m}) \in K\},$$

and let $H = \langle X | P \rangle$. Then H is recursively presented and it is clear that it is isomorphic to $Z(G)$. Indeed,

$$H \models w(x_{i_1}, \cdots, x_{i_n}) = 1 \Leftrightarrow w(x_{i_1}, \cdots, x_{i_n}) \in P \Leftrightarrow G \models w(v_{i_1}, \cdots, v_{i_m}) = 1.$$

(2) The proof is similar to the previous one. Since G has a soluble word problem, V is recursive and then $Z(G)$ has a generalized soluble word problem in G. Similarly the set P is recursive and then the group H has a soluble word problem. □

We are going to use a particular case of some results of C.R.J. Clapham [1, 2]. For this we will need the following definition. Let G be a finitely generated group and H a subgroup of G. We call H ***strongly benign*** ($\mathcal{A}$-strongly benign in the vocabulary of C.R.J. Clapham) if the HNN-extension $G^* = \langle G, t | t^{-1}ht = h \mid h \in H \rangle$ can be embedded in a finitely presented K with soluble word problem such that G and $\langle G, t \rangle$ have a generalized soluble word problem in K.

Lemma 2.7 [1, Corollary 3.8.1] *Let G be a finitely generated group with soluble word problem and φ a recursive isomorphism of a subgroup A into G such that A and $\varphi(A)$ have generalized soluble word problem in G. Then the subgroup $\langle G, t^{-1}Gt \rangle$ has a generalized soluble word problem in $G^* = \langle G, t \mid t^{-1}at = \varphi(a),\ a \in A \rangle$.*

Lemma 2.8 [2, Lemma 11.2] *A subgroup of a finitely generated free group is strongly benign if and only if it is recursive.*

3 Preparatory propositions

Notation. Let $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_m)$ be a sequences. The notation $(u_1, \cdots, u_n) \leq (v_1, \cdots, v_m)$ means that there exists p such that $(u_1, \cdots, u_n) = (v_p, \cdots, v_{p+n-1})$.

Proposition 3.1 *Let $F = G_1 *_A G_2$ be a free product with amalgamation and R a subset of F which satisfies:*

(i) Every element of R is cyclically reduced and every $r \in R$, satisfies $|r| > 12$.

(ii) For every $r \in R$ and for every normal form $(g_1, \cdots, g_n)$ of r there are no i, j, $i \neq j$ and $\alpha, \beta \in A$ such that $g_i^{-1} = \alpha g_j \beta$.

(iii) The symmetrized set $W(R)$ satisfies $C'(\lambda)$ with $\lambda \leq \frac{1}{9}$.

Let N be the normal closure of R in F and $\pi : F \to (F/N)$ the natural map. Then $\pi(Z(F)) = Z(F/N)$.

Remark. *We see that if $r \in R$ and if there exists a normal form $(g_1, \cdots, g_n)$ of r which satisfies the condition of (ii), then every other normal form $(h_1, \cdots, h_n)$ of r satisfies also the condition of (ii). It is not difficult to see also that the same property is true for any cyclic permutation of $(g_1, \cdots, g_n)$ and for any normal form for the inverse of r.*

In order to prove Proposition 3.1 we will need the following lemma whose proof is omitted as it follows easily by induction on n.

Lemma 3.2 *Let $F = G_1 *_A G_2$ be a free product with amalgamation. Let $g = g_1 \cdots g_n$ in normal form, $n \geq 2$ and $t \in F$ such that $|t| \leq 1$. Suppose that $|gtg^{-1}| \geq 3$. Then there exists $i \in \{1, \cdots, n\}$ and $\alpha \in F$ such that $|\alpha| = 1$ and $gtg^{-1} = g_1 \cdots g_i \alpha g_i^{-1} \cdots g_1^{-1}$ is in normal form.* □

Proof of Proposition 3.1 We suppose $\pi(Z(F)) \neq Z(F/N)$, and we prove that there exists $r \in R$ that does not satisfy the condition (ii).

Let $\pi(g) \in Z(F/N)$ such that $\pi(g) \notin \pi(Z(F))$. Let $g_0 \in F$ be of minimal length such that $\pi(g_0) = \pi(g)$.

Since $\pi(g_0) \in Z(F/N)$, for every $t \in F$ such that $|t| \leq 1$, $g_0 t g_0^{-1} t^{-1} \in N$. Since $g_0 \notin Z(G)$ there exists $t_0 \in F$, such that $|t_0| \leq 1$ with $g_0 t_0 g_0^{-1} t_0^{-1} \neq 1$.

By Theorem 2.2, there exists $w \in W(R)$, cyclically reduced such that

$$|g_0 t_0 g_0^{-1} t_0^{-1}| > (1 - 3\lambda)|w|.$$

We have $|g_0 t_0 g_0^{-1}| \geq |g_0 t_0 g_0^{-1} t_0^{-1}| - 1 > (1 - 3\lambda)|w| - 1$. Since $\lambda \leq \frac{1}{9}$ and $|w| > 9$ a simple count gives us $(1 - 3\lambda)|w| > \frac{2}{3}.9 = 6$, hence $|g_0 t_0 g_0^{-1}| \geq 3$. We have also $|g_0 t_0 g_0^{-1} t_0^{-1}| \leq 2|g_0| + 2$, and $|g_0 t_0 g_0^{-1} t_0^{-1}| > (1 - 3\lambda)|w|$, hence $|g_0 t_0 g_0^{-1} t_0^{-1}| > 6$ and hence $2|g_0| + 2 \geq 6$ therefore $|g_0| \geq 2$. Therefore the conditions of Lemma 3.2 are satisfied.

Let $(a_1, \cdots, a_n)$ be a normal form of g_0. By Lemma 3.2, there exists $i \in \{1, \cdots, n\}$ and $\alpha \in F$, such that $|\alpha| = 1$ and $g_0 t_0 g_0^{-1} = a_1 \cdots a_i \alpha a_i^{-1} \cdots a_1^{-1}$ is in normal form.

We have three cases to consider: $|a_1^{-1} t_0^{-1}| = 0$, $|a_1^{-1} t_0^{-1}| = 1$, $|a_1^{-1} t_0^{-1}| = 2$. We are going to treat just the case $|a_1^{-1} t_0^{-1}| = 0$, the other cases can be treated similarly.

Let $\gamma = a_1^{-1} t_0^{-1}$. We remark that $i \geq 2$ because $|g_0 t_0 g_0^{-1} t_0^{-1}| > (1 - 3\lambda)|w| > 6$. Then the sequence $(a_1, \cdots, a_i, \alpha, a_i^{-1}, \cdots, a_2^{-1}\gamma)$ is a normal form of $h = g_0 t_0 g_0^{-1} t_0^{-1}$. To simplify notations we denote the previous normal form of h by $(v_1, v_2, \cdots, v_m)$.

By Theorem 2.2, there exists a normal form $(u_1, \cdots, u_m)$ of h, there exists a normal form $(w_1, \cdots, w_q)$ of w, there exists $l > (1 - 3\lambda)|w|$ and there exists $p \in \{1, \cdots, m\}$ such that $(u_p, \cdots, u_{p+l-1}) = (w_1, \cdots, w_l)$.

By Theorem 2.3, there exists $a \in A$ and r where r is the product of a cyclic permutation of some $r' \in R^{\pm 1}$, such that $w = a^{-1} r a$.

Let $(r_1, \cdots, r_q)$ be a normal form of r. Then $(a^{-1} r_1, \cdots, r_q a)$ is a normal form of w and by Lemma 2.1, there exists a sequence $(\alpha_1, \cdots \alpha_{q+1})$ of elements of A such that

$$w_j = \alpha_j r_j \alpha_{j+1}^{-1} \text{ for } j \neq 1, j \neq q, \text{ and } w_1 = \alpha_1 a^{-1} r_1 \alpha_2^{-1}, w_q = \alpha_q r_q a \alpha_{q+1}^{-1}.$$

Similarly there exists a sequence $(\beta_1, \cdots, \beta_{m+1})$ of elements of A such that $u_j = \beta_j v_j \beta_{j+1}^{-1}$. Then

$$(*) \qquad (\beta_p v_p \beta_{p+1}^{-1}, \cdots, \beta_{p+l-1} v_{p+l-1} \beta_{p+l}^{-1}) = (\alpha_1 a^{-1} r_1 \alpha_2^{-1}, \cdots, \alpha_l r_l \alpha_{l+1}^{-1}).$$

Let us show that $(v_p, \cdots, v_{p+l-1}) \nleq (a_1, \cdots, a_i, \alpha)$ and $(v_p, \cdots, v_{p+l-1}) \nleq (\alpha, a_i^{-1}, \cdots, a_2^{-1}\gamma)$.

Suppose that $(v_p, \cdots, v_{p+l-1}) \leq (a_1, \cdots, a_i, \alpha)$ or that $(v_p, \cdots, v_{p+l-1}) \leq (\alpha, a_i^{-1}, \cdots, a_2^{-1}\gamma)$.

Then for some k, x we have $(v_{p+1}, \cdots, v_{p+l-2}) = (a_k, \cdots, a_x)$. By $(*)$ we get

$$(a_k, \cdots, a_x) = (v_{p+1}, \cdots., v_{p+l-2})$$
$$= (\beta_{p+1}^{-1} \alpha_2 r_2 \alpha_3^{-1} \beta_{p+2}, \cdots, \beta_{p+l-2}^{-1} \alpha_{l-1} r_{l-1} \alpha_l^{-1} \beta_{p+l-1}).$$

Then

$$g_0 = a_1 \cdots a_{k-1}.(\beta_{p+1}^{-1}(\alpha_2 r_2 \alpha_3^{-1})\beta_{p+2}\beta_{p+2}^{-1} \cdots \beta_{p+l-2}\beta_{p+l-2}^{-1}(\alpha_{l-1} r_{l-1} \alpha_l^{-1})\beta_{p+l-1}^{-1})$$
$$.a_{x+1} \cdots a_n = a_1 \cdots a_{k-1}.(\beta_{p+1}^{-1}(\alpha_2 r_2 \cdots r_{l-1} \alpha_l^{-1})\beta_{p+l-1}^{-1}).a_{x+1} \cdots a_n.$$

Since $\pi(r_2 \cdots r_{l-1}) = \pi(r_1^{-1} r_q^{-1} \cdots r_l^{-1})$ we find

$$\pi(g_0) = \pi(a_1 \cdots a_{k-1}.(\beta_{p+1}^{-1}(\alpha_2 r_1^{-1} r_q^{-1} \cdots r_l^{-1} \alpha_l^{-1})\beta_{p+l-1}^{-1}).a_{x+1} \cdots a_n).$$

Let

$$d = a_1 \cdots a_{k-1}.(\beta_{p+1}^{-1}(\alpha_2 r_1^{-1} r_q^{-1} \cdots r_l^{-1} \alpha_l^{-1})\beta_{p+l-1}^{-1}).a_{x+1} \cdots a_n.$$

Since $l > (1-3\lambda)|r|$, and $\lambda \leq \frac{1}{9}$ then $l-2 > (1-3\lambda)|r|-2 > \frac{2}{3}|r|-2$ and since $|r| > 12$ a simple count shows us that, $\frac{2}{3}|r|-2 > \frac{1}{2}|r|$, hence $l-2 > \frac{1}{2}|r|$.

Therefore we have $|r_1^{-1}r_q^{-1}\cdots r_l^{-1}| \leq q-l+2 < \frac{1}{2}|r|$. Then we have

$$|d| = |a_1\cdots a_{k-1}.(\beta_{p+1}^{-1}(\alpha_2 r_1^{-1}r_q^{-1}\cdots r_l^{-1}\alpha_l^{-1})\beta_{p+l-1}^{-1}).a_{x+1}\cdots a_n| \leq (k-1)+(q-l+2)+(n-x)$$

$$< (k-1)+\tfrac{1}{2}|r|+(n-x).$$

Since $|g_0| = (k-1)+(l-2)+(n-x)$ and $l-2 > \frac{1}{2}|r|$ we have $|d| < |g_0|$.

We have $\pi(d) = \pi(g) = \pi(g_0)$ and $|d| < |g_0|$. This contradicts the fact that the length of g_0 is minimal.

Hence there exist k, j such that $(v_p, \cdots, v_{p+l-1}) = (a_k, \cdots, a_i, \alpha, a_i^{-1}, \cdots, a_j^{-1})$. Therefore we see that there exist i_1, i_2 and $\delta, \mu \in A$ such that $\delta r_{i_1}\mu = r_{i_2}^{-1}$, which contradicts Condition (ii). $\square$

Definitions. Let $F = G_1 *_A G_2$ be a free product with amalgamation and R a subset of F.

(1) Let C be a set of normal forms. We say that C ***defines explicitly*** R or that R is ***explicitly defined*** by C if :

(i) If $(c_1, \cdots, c_n) \in C$, then $c_1\cdots c_n \in R^{\pm 1}$.

(ii) For every $r \in R^{\pm 1}$, there exists a normal form $(c_1, \cdots, c_n) \in C$ such that $r = c_1 \cdots c_n$.

(2) For every set C of normal forms we denote by $\overline{C}$ the set of all cyclic permutation of elements of C.

(3) Let C be a set of normal forms and λ a positive real number such that $\lambda \leq \frac{1}{6}$. We define $L(C, \lambda)$ to be

$$L(C,\lambda) = \{(g,c,l) \in F \times \overline{C} \times \mathbb{N} \mid c = (c_1, \cdots, c_n), \quad (1-3\lambda)n < l \leq n,$$

$$\exists \alpha, \beta \in A, \text{ such that } \alpha g \beta = c_1 \cdots c_l\}.$$

Proposition 3.3 *Let $F = G_1 *_A G_2$ be a free product with amalgamation and R a subset of F, explicitly defined by C, such that:*

(i) *G_1 and G_2 have a soluble word problem.*

(ii) *A has a generalized soluble word problem in both G_1 and G_2.*

(1) *Every element of R is cyclically reduced.*

(2) *The symmetrized set $W(R)$ satisfies $C'(\lambda)$ with $\lambda \leq \frac{1}{6}$.*

(3) *For every $n \in \mathbb{N}$, the set $\{c \in C \mid |c| \leq n\}$ is finite.*

(4) *The map defined by $\varphi(n) = \{c \in C \mid |c| \leq n\}$ is recursive.*

(5) *The set $L(C, \lambda)$ is recursive.*

(6) *There exists an algorithm which for every $(g, c, l) \in L(C, \lambda)$ produces $(\alpha, \beta) \in A^2$ such that $\alpha g \beta = c_1 \cdots c_l$.*

Let N be the normal closure of R in F. Then F/N has a soluble word problem.

Lemma 3.4 *Let $F = G_1 *_A G_2$ be a free product with amalgamation and R a subset of F, explicitly defined by C, and such that every element of R is cyclically reduced. Suppose that $W(R)$, the symmetrized closure of R, satisfies $C'(\lambda)$ with $\lambda \leq \frac{1}{6}$. Let N be the normal closure of R in F. Let $w \in N$, with $w \neq 1$, and let $(g_1, \cdots, g_n)$ be a normal form of w. Then there exist i, $1 \leq i \leq n$, $c \in \overline{C}$ and $l \in \mathbb{N}$, such that $(g_i \cdots g_{i+l-1}, c, l) \in L(C, \lambda)$. We call c a* **witness** *of w.*

Proof

By Theorem 2.2, there exists $r \in W(R)$ cyclically reduced such that $r = s.t$ in reduced form and $w = usv$ in reduced form and $|s| > (1 - 3\lambda)|r|$. To simplify notations we write $w = u_1 \cdots u_n$, $r = v_1 \cdots v_m$ where $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_m)$ are normal forms and $(u_i, \cdots, u_{i+l-1}) = (v_1, \cdots, v_l)$, $l > (1-3\lambda)|r|$. By Lemma 2.1, there exists a sequence $(\alpha_1, \cdots, \alpha_{n+1})$ of elements of A such that $g_i = \alpha_i u_i \alpha_{i+1}^{-1}$. Since $r \in W(R)$ is cyclically reduced, by the conjugacy theorem (Theorem 2.3), there exists r_0 a cyclic permutation of an element of $R^{\pm 1}$ and $\gamma \in A$ such that $r = \gamma r_0 \gamma^{-1}$. Since C defines explicitly R, there exists $\bar{c} = (c_1, \cdots, c_m) \in \overline{C}$ such that $(c_1, \cdots, c_m)$ is a normal form of r_0. Therefore we see that the sequence $(\gamma c_1, \cdots, c_m \gamma^{-1})$ is a normal form of r, hence by Lemma 2.1, there exists a sequence $(\beta_1, \cdots, \beta_{m+1})$ of elements of A such that $v_i = \beta_i c_i \beta_{i+1}^{-1}$, for $i \neq 1$ and $i \neq m$, $v_1 = \beta_1 \gamma c_1 \beta_2^{-1}$ and $v_m = \beta_m c_m \gamma^{-1} \beta_{m+1}^{-1}$. A simple count shows us that $g_i \cdots g_{i+l-1} = \alpha_i u_i \cdots u_{i+l-1} \alpha_{i+l}^{-1} = \alpha_i \beta_1 \gamma c_1 \cdots c_l \beta_{l+1}^{-1} \alpha_{i+l}^{-1}$. Therefore $(g_i \cdots g_{i+l-1}, \bar{c}, l) \in L(C, \lambda)$. □

Proof of Proposition 2.3 In this proof the natural map $\pi : F \to F/N$ is written $\pi(w) = \overline{w}$. Let $\overline{w} \in F/N$ written as a word in the generators of F/N. Since F has a soluble word problem, one can determine if $w = 1$ or no. If it is the case, then $\overline{w} = 1$. Otherwise, since A has a generalized soluble word problem, one can calculate a normal form $(g_1, \cdots, g_n)$ of w. If $w \in N$, then by Lemma 3.4, there exists i, $1 \leq i \leq n$, $c \in \overline{C}$ and $l \in \mathbb{N}$, such that $(g_i \cdots g_{i+l-1}, c, l) \in L(C, \lambda)$. We see that we must have $|c| < \frac{|w|}{(1-3\lambda)}$. Since the map $\varphi(n) = \{c \in C \mid |c| \leq n\}$ is recursive we compute the set $K = \{c \in C \mid |c| \leq \frac{|w|}{(1-3\lambda)}\}$ which is finite. Then we compute all cyclic permutations of elements of $\overline{K}$. For every $a \in \overline{K}$, for every l such that $1 \leq l \leq |a|$, $l > (1 - 3\lambda)|a|$, and for every i such that $i + l - 1 \leq n$, let us check whether $(g_i \cdots g_{i+l-1}, a, l) \in L(C, \lambda)$. Since $L(C, \lambda)$ is recursive the above procedure is recursive. If at every stage the answer to the question $(g_i \cdots g_{i+l-1}, a, l) \in L(C, \lambda)$ is no, then $w \notin N$ and hence $\overline{w} \neq 1$. If at some stage the answer to the question $(g_i \cdots g_{i+l-1}, a, l) \in L(C, \lambda)$ is yes, by (6), there exists an algorithm which produces $\alpha, \beta \in A$ such that $g_i \cdots g_{i+l-1} = \alpha a_1 \cdots a_l \beta$. Put $w_1 = g_1 \cdots g_i . \alpha^{-1} a_m^{-1} \cdots a_{l+1}^{-1} \beta^{-1} . g_{l+1} \cdots g_n$. Then we see that $\overline{w} = \overline{w_1}$ and $|w_1| < |w|$. Then we will redo the same thing for w_1.

At the end of the process we have $(w, w_1, \cdots, w_t)$, such that $|w_t| < |w_{t-1}| <$... $|w_1| < |w|$ and w_t does not have any witness in $\overline{C}$. If $w_t = 1$, then $w \in N$, otherwise $w \notin N$. □

4 Proof of Theorem I

Let G be a countable group generated by $\{a_i \mid i \in \mathbb{N} \setminus \{0\}\}$. Let $G_1 = G \times \langle x|\rangle$ and $G_2 = Z(G) \times \langle y|\rangle$ where $\langle x|\rangle$ and $\langle y|\rangle$ are two copies of the free group on one generator. Let $F = G_1 *_{Z(G)} G_2$. By Lemma 2.5, $Z(F) \leq Z(G)$, and since $Z(G) \leq Z(F)$, we find $Z(F) = Z(G)$.

Let $x_1, x_2 \in \langle x|\rangle$, such that $x_1 \neq x_2$, $x_1x_2 \neq 1$, $x_1 \neq 1$ and $x_2 \neq 1$. Let for every $i \in \mathbb{N} \setminus \{0\}$,

$$(*) \qquad w_i = a_i^{-1}(x_1y)^{80(i-1)+1}(x_2y)(x_1y)^{80(i-1)+2}\cdots(x_1y)^{80i}(x_2y).$$

It is clear that w_i is cyclically reduced.

Let $W(R)$ be the symmetrized closure of $R = \{w_i \mid i \in \mathbb{N} \setminus \{0\}\}$. Let $W_0(R)$ be the set of cyclically reduced conjugates of elements of $R^{\pm 1}$. Now we show that $W_0(R)$ satisfies $C'(\frac{1}{10})$.

Let $\alpha_1, \alpha_2 \in W_0(R)$ such that $\alpha_1.\alpha_2 \neq 1$. By Theorem 2.3, there exist $r_1, r_2 \in R^{\pm 1}$ and $a, b \in Z(G)$ such that $\alpha_1 = a.r_1'.a^{-1}$ and $\alpha_2 = b.r_2'.b^{-1}$ where r_1', (resp. r_2') is a cyclic permutation of r_1 (resp. r_2). Since $a, b \in Z(G)$, we have $\alpha_1 = r_1'$, and $\alpha_2 = r_2'$. Since $\alpha_1.\alpha_2 \neq 1$, we find $r_1'.r_2' \neq 1$. A classical argument like the one used in the book of R.C. Lyndon and P.E. Schupp ([6], p. 283 or p. 290) shows that $W_0(R)$ satisfies $C'(\frac{1}{10})$. By Lemma 2.4, $W(R)$ satisfies $C'(1/9)$.

Hence by Theorem 2.2, G is embedded into F/N .

We see that F/N is finitely generated. We see also that if G is recursively presented and $Z(G)$ is recursively enumerable in G then F/N is recursively presented. It is not difficult to see that R satisfies the assumption of Proposition 3.1, hence $Z(F/N) = \pi(Z(F)) = \pi(Z(G))$.

Now suppose that G has a soluble word problem and $Z(G)$ has a generalized soluble word problem in G.

Then we see that $Z(G)$ has soluble generalized soluble word problem in $G_2 = Z(G) \times \langle y|\rangle$. Hence F has a soluble word problem.

Let us show that $W(R)$ satisfies the conditions of the Proposition 3.3. Let C_0 be the set of normal forms given in $(*)$. Let C be the set obtained by adding to C_0 the set of normal forms of the inverses of the elements of C_0. Then it is clear that C defines explicitly R. It is clear that R satisfies the conditions (1)-(4) of Proposition 3.3.

It is sufficient now to show that $L(C, \lambda)$ is recursive and that there is an algorithm which for every $(g, c, l) \in L(C, \lambda)$ produces $(\alpha, \beta) \in Z(G)^2$ such that $\alpha g \beta = c_1 \cdots c_l$.

Let $(g, c, l) \in F \times \overline{C} \times \mathbb{N}$. Then it easy to see that we can calculate a sequence $(g_0, \cdots, g_n)$ such that $g = g_0 \cdots g_n$ and:

(i) $g_0 \in Z(G)$, $(g_1, \cdots, g_n)$ is a normal form,
(ii) if $g_i \in G_1$ then $g_i = \alpha_i.x^{n_i}$, $n_i \in \mathbb{Z}$, $\alpha_i \in G \setminus Z(G)$ or $\alpha_i = 1$,
(iii) if $g_i \in G_2$ then $g_i = y^{p_i}$, $p_i \in \mathbb{Z}$.

Let us prove the following claim:

Claim. Let $(g_0, \cdots, g_n)$ be a sequence which satisfies the conditions (i)-(iii) and let $(c, l) \in \overline{C} \times \mathbb{N}$ with $c = (c_1, \cdots, c_m)$. Then the following properties are equivalents:

(1) *There exist $\alpha, \beta \in Z(G)$, such that $\alpha g \beta = c_1 \cdots c_l$.*

(2) *$n = l$ and one of the following conditions is satisfied:*

(a) *If there exists q such that $c_q = a_i^{-1} x_1$ then:*

- *For every k such that $c_k \in \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$, $\alpha_k = e$ and $g_k = c_k$.*
- *For every k such that $c_k \in \{y, y^{-1}\}$, $g_k = c_k$.*
- *$\alpha_q a_i \in Z(G)$, and $x^{n_q} = x_1$.*

(b) *If there exists q such that $c_q = x_1^{-1} a_i$ then:*

- *For every k such that $c_k \in \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$, $\alpha_k = e$ and $g_k = c_k$.*
- *For every k such that $c_k \in \{y, y^{-1}\}$, $g_k = c_k$.*
- *$\alpha_q a_i^{-1} \in Z(G)$, and $x^{n_q} = x_1$.*

(c) *If there is no q such that $c_q = a_i^{-1} x_1$ or $c_q = x_1^{-1} a_i$ then :*

- *For every k such that $c_k \in \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$, $\alpha_k = e$ and $g_k = c_k$.*
- *For every k such that $c_k \in \{y, y^{-1}\}$, $g_k = c_k$.*

If (a) is satisfied then if we take $\alpha = 1$ and $\beta = a_i^{-1} \alpha_k^{-1} g_0^{-1}$ then $\alpha g \beta = c_1 \cdots c_l$.
If (b) is satisfied then if we take $\alpha = 1$ and $\beta = a_i \alpha_k^{-1} g_0^{-1}$ then $\alpha g \beta = c_1 \cdots c_l$.
If (c) is satisfied then if we take $\alpha = 1$ and $\beta = g_0^{-1}$ then $\alpha g \beta = c_1 \cdots c_l$.

Observe that there is at most one k such that $c_k = a_i^{-1} x_1$ or $c_k = x_1^{-1} a_i$.

Proof.

(1)⇒(2).

Let $\alpha, \beta \in Z(G)$ such that $\alpha g \beta = c_1 \cdots c_l$. Since $\alpha, \beta \in Z(G)$, we have $\alpha g \beta = \alpha \beta g$. Then

$$(\alpha \beta g_0 g_1).g_2 \cdots g_n.c_l^{-1} \cdots c_1^{-1} = 1.$$

Since the sequence $(\alpha \beta g_0 g_1, g_2, \cdots, g_n)$ is a normal form we must have $n = l$.

It is not difficult to see, by induction, that $g_k c_k^{-1} \in Z(G)$ for $k = 1, \cdots, n$.

We only treat the case (a), the other cases can be treated similarly.

(a) If there exists q such that $c_q = a_i^{-1} x_1$ then:

- Let k be such that $c_k \in \{x_1, x_2, x_1^{-1}, x_2^{-1}\}$. Since $g_k c_k^{-1} \in Z(G)$, we have $g_k = \alpha_k.x^{n_k} = a.c_k$ where $a \in Z(G)$. Hence we must have $\alpha_k = 1$ and $g_k = c_k$.
- Let k be such that $c_k \in \{y, y^{-1}\}$. Since $g_k c_k^{-1} \in Z(G)$, $g_k = y^{p_k} = a.c_k$ where $a \in Z(G)$. Then this implies that $a = 1$ and $g_k = y^{p_k} = c_k$.
- Since $g_q c_q^{-1} \in Z(G)$ then $g_q = \alpha_q.x^{n_q} = a.a_i^{-1} x_1$ where $a \in Z(G)$. Hence we have $x^{n_q} = x_1$ and $\alpha_q.a_i = a \in Z(G)$.

(2)⇒(1).

It is sufficient to calculate. We treat only the case (a), the other cases can be treated similarly. Suppose that (a) is satisfied. Then $g_1 \cdots g_n c_n^{-1} \cdots c_1^{-1} = \alpha_q a_i$. Let $\alpha = 1$ and $\beta = a_i^{-1} \alpha_k^{-1} g_0^{-1}$. Then $\alpha g_0 g_1 \cdots g_n c_n^{-1} \cdots c_1^{-1} \beta = g_0 \alpha_q a_i \beta = 1$. Hence $\alpha g \beta = c_1 \cdots c_l$. □

Since F has a soluble word problem and $Z(G)$ has a generalized soluble word problem in F we see that the procedures $(a)(b)(c)$ are recursive. Therefore we see that $L(C, \lambda)$ is recursive and that there is an algorithm which for every $(g, c, l) \in L(C, \lambda)$ produces $(\alpha, \beta) \in Z(G)^2$ such that $\alpha g \beta = c_1 \cdots c_l$. □

5 Proof of Theorem II

The proof is in two stages. In the first stage, we prove the first part of the theorem, that is if G is a finitely generated recursively presented group then G is embeddable in a finitely presented group K such that $Z(G) = Z(K)$. In the second stage, we prove the second part of the theorem, that is if G has a soluble word problem then we can take K with soluble word problem.

Stage 1. [1]

Let G be a finitely generated and recursively presented group. Clearly we may assume that G is non-abelian. Let $\{a_1, \cdots, a_n\}$ be a generating set of G. Let

$$G_0 = \langle G, z |\ z^{-1}gz = g,\ g \in Z(G)\rangle.$$

Since G is recursively presented, we can apply Lemma 2.6 and then $Z(G)$ is recursively enumerable in G. Hence G_0 is recursively presented. Since G is non-abelian, by Lemma 2.5, it is easy to see that $Z(G_0) = Z(G)$. The group G_0 is generated by the set $\{a_1, \cdots, a_n, z\}$. Hence there is an isomorphism $\nu : \mathbb{F}_X/R \cong G_0$, where $\mathbb{F}_X$ is the free non-abelian group of rank $n+1$ with basis $X = \{x_1, \cdots, x_n, x_{n+1}\}$, and R is the normal closure of the presentation (which is recursively enumerable) of G_0 and ν satisfies $\nu(\bar{x}_i) = a_i$, for $i = 1, \cdots, n$ and $\nu(\bar{x}_{n+1}) = z$ where $\bar{x}_i$ is the class of x_i relative to the subgroup R. Let

$$F_R = \langle \mathbb{F}_X, d |\ d^{-1}rd = r,\ r \in R\rangle.$$

By Higman's embedding theorem F_R is embeddable in a finitely presented group say H. Without loss of generality we can assume that $x_1, \cdots, x_n, x_{n+1}$ are included among the generating symbols of the given presentation of H. If w is a word in the generators of $\mathbb{F}_X$, let $\overline{w}$ denote the word of G_0 obtained by replacing each x_i by a_i for $i = 1, \cdots, n$ and x_{n+1} by z.

In F_R the subgroup L generated by $\mathbb{F}_X$ and $d^{-1}\mathbb{F}_X d$ is the free product of $\mathbb{F}_X$ and $d^{-1}\mathbb{F}_X d$ with R amalgamated.

Define a homomorphism $\phi : L \to G_0$ by $\phi(w) = \overline{w}$ and $\phi(d^{-1}wd) = 1$. Since the two definitions agree on the amalgamated part, ϕ is well-defined.

Consider the group $H \times G_0$. We shall use the ordered pair notation to denote elements of this group. Viewing L as a subgroup of H, we consider the subgroup $L \times Z(G_0)$. Define a map $\psi : L \times Z(G_0) \to H \times G_0$ by $\psi((l, g)) = (l, \phi(l).g)$. Let us show that ψ is an injective homomorphism. We have

$$\psi((l_1, g_1).(l_2, g_2)) = \psi((l_1.l_2, g_1.g_2)) = (l_1 l_2, \phi(l_1 l_2).g_1.g_2)$$

$$\psi((l_1, g_1)).\psi((l_2, g_2)) = (l_1, \phi(l_1).g_1).(l_2, \phi(l_2).g_2) = (l_1 l_2, \phi(l_1).g_1\phi(l_2).g_2).$$

Since $g_1, g_2 \in Z(G_0)$ we have $\phi(l_1).g_1\phi(l_2).g_2 = \phi(l_1).\phi(l_2).g_1.g_2$, and since ϕ is an homomorphism we have $\phi(l_1).\phi(l_2) = \phi(l_1.l_2)$. Hence $\psi((l_1, g_1).(l_2, g_2)) =$

[1] The beginning of the proof in this stage is inspired by the proof of Higman's embedding theorem, more precisely the Higman's Rop Trick.

$\psi((l_1, g_1)).\psi((l_2, g_2))$. Hence ψ is an homomorphism and it is clear that ψ is injective.

Therefore we can form the HNN-extension

$$K = \langle H \times G_0, s |\ s^{-1}(l, g)s = (l, \phi(l).g),\ l \in L,\ g \in Z(G_0)\rangle.$$

Viewing G as a subgroup of G_0 and hence as a subgroup of $H \times G_0$ we can form the following free product with amalgamation

$$\Gamma = K *_G G \times \langle t|\rangle.$$

We notice that we view Γ as a free product with amalgamation and not as an HNN-extension. Let

$$r = (s^{-1}z)tzt^2zt^3z \cdots zt^{80}.$$

Let N be the normal closure of $\{r\}$ in Γ. Let us show that Γ/N can be finitely presented. A set of defining relations for Γ/N can be obtained by taking the union of the following relations:

(1) *The defining relations for H.*

(2) *The relation $s = ztzt^2zt^3z \cdots zt^{80}$.*

(3) *The relations saying that the generators of G_0 commute with the generators of H.*

(4) *The relations saying that the generators of G commute with t.*

(5) *The defining relations for G_0.*

(6) *The relations $s^{-1}(l, g)s = (l, \phi(l).g)$, for a set of generators of L, and for every $g \in Z(G_0)$.*

It is clear that Γ/N is finitely generated. We now introduce a set of relations denoted by (7), which is a subset of (6):

(7) $s^{-1}(l, 1)s = (l, \phi(l))$, *where l belongs to a finite generating set of L.*

We are going to prove that the relations (5)-(6) follow from the relations (1)-(4) and (7), and this will show that Γ/N is finitely presented since (1)-(4) are finite as well as (7).

First we prove that the relations (5) follow from the relations (7) and (1)-(4). Let $\overline{w}$ be a word on the generators of G_0 such that $\overline{w} = 1$. Then the corresponding word w on the generators of $\mathbb{F}_X$ is in R. Now from (7) we have

$$s^{-1}(w, 1)s = (w, \phi(w)),$$

and by definition of ϕ we find $s^{-1}(w, 1)s = (w, \overline{w})$. Since $d^{-1}wd = w$ (which is a consequence of (1)) we have

$$(w, 1) = (d^{-1}wd, 1).$$

But by the definition of ϕ and from (7),

$$s^{-1}(d^{-1}wd, 1)s = s^{-1}(w, 1)s = (w, \phi(d^{-1}wd)) = (w, 1) = (w, \overline{w}),$$

and hence $\overline{w} = 1$ follows.

Now let us show that the relations (6) follow from the relations (7) and (1)-(5). By (5) we get that every $g \in Z(G_0)$ satisfies $gz = zg$. By (4) we find that every $g \in Z(G_0)$ satisfies $gt = tg$. Hence by (2) we get that every $g \in Z(G_0)$ satisfies $gs = sg$, which can be written as $(1, g)s = s(1, g)$ in the ordered pair notation. Now

$$s^{-1}(l, g)s = s^{-1}(l, 1)(1, g)s.$$

Hence

$$s^{-1}(l, g)s = s^{-1}(l, 1)s(1, g),$$

and by the relations (7),

$$s^{-1}(l, g)s = (l, \phi(l)).(1, g).$$

Hence $s^{-1}(l, g)s = (l, \phi(l).g)$, for a set of generators of L, and for every $g \in Z(G_0)$. This completes the proof of the fact that Γ/N is finitely presented.

Now we show that the natural map $\pi : \Gamma \to \Gamma/N$ is injective on G_0 and that we have $Z(\Gamma/N) = \pi(Z(G_0))$. Since $Z(G) = Z(G_0)$ and $G \leq G_0$ this completes the proof.

By Lemma 2.5, $Z(\Gamma) \leq Z(G)$. If $g \in Z(G)$, then we see that g commutes with t. From the presentation of K, it is also clear that g commutes with s and all the generators of H. Hence $Z(G) \leq Z(\Gamma)$ and thus $Z(G) = Z(\Gamma)$. We prove now the following claim.

Claim 1. Let $a, b \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$ and $\alpha, \beta \in G$. Then

$$a\alpha b = \beta \text{ in } \Gamma \text{ if and only if } a = b^{-1} \text{ and } \alpha = \beta \in Z(G).$$

Proof.
If $a = b^{-1}$ and $\alpha = \beta \in Z(G)$, then clearly $a\alpha b = \beta$.

We see that if $a\alpha b = \beta$, then $\alpha = a^{-1}\beta b^{-1}$ and so it is sufficient to prove the claim for $a \in \{z, s^{-1}z\}$.

If $a = z$ and $b = s^{-1}z$ (resp. $b = z^{-1}s$) this implies that the sequence $(z\alpha, s^{-1}, z\beta^{-1})$, (resp. $(z\alpha z^{-1}, s, \beta^{-1})$), is not reduced in the HNN-extension K, which is clearly a contradiction. So if $a = z$ then $b \in \{z, z^{-1}\}$. Now if $b = z$, the sequence $(z, \alpha, z, \beta^{-1})$ is not reduced in the HNN-extension G_0, which is a contradiction. So if $a = z$ then $b = z^{-1}$. Hence the sequence $(z, \alpha, z^{-1}, \beta^{-1})$ is not reduced in the HNN-extension G_0, so $\alpha \in Z(G)$, and hence $\alpha = \beta$.

Now if $a = s^{-1}z$ and $b = z$ (resp. $b = z^{-1}$) the sequence $(s^{-1}, z\alpha z\beta^{-1})$, (resp. $(s^{-1}, z\alpha z^{-1}\beta^{-1})$), is not reduced in the HNN-extension K, which is clearly a contradiction. So if $a = s^{-1}z$ then $b \in \{s^{-1}z, z^{-1}s\}$. Now if $b = s^{-1}z$, the sequence $(s^{-1}, z\alpha, s^{-1}, \beta^{-1})$ is not reduced in the HNN-extension K, which is a contradiction. So if $a = s^{-1}z$ then $b = z^{-1}s$. Hence the sequence $(s^{-1}, z\alpha z^{-1}, s, \beta^{-1})$ is not reduced in the HNN-extension K, so $z\alpha z^{-1} \in L \times Z(G)$. So there exists $(l, g) \in L \times Z(G)$ such that $z\alpha z^{-1} = l.g$. So we must have $l = 1$ and $\alpha = g \in Z(G)$. Since $s^{-1}z\alpha z^{-1}s = \beta$ we have $\alpha = \beta$. This completes the proof. □

Let $R_0 = \{r\}$. Then we see easily, using the above claim, that R_0 satisfies the conditions (i)-(ii) of Proposition 3.1. Therefore by Proposition 3.1 and Theorem 2.2, it is sufficient to show that the symmetrized closure of R_0 satisfies $C'(\frac{1}{10})$.

Since we view Γ as a free product with amalgamation and not as an HNN-extension, we see that $|r| = 160$. By Lemma 2.4, it is sufficient to show that the set $W_0(R_0)$ of cyclically reduced conjugates of the elements of $\{r\} \cup \{r^{-1}\}$ satisfies $C'(\frac{1}{70})$.

Let $w_1, w_2 \in W_0(R_0)$ such that $w_1 w_2 \neq 1$. By the conjugacy theorem (Theorem 2.3) there exists $\alpha, \beta \in G$ and r_1, r_2 a cyclic permutations of elements of $\{r\} \cup \{r^{-1}\}$ such that $w_1 = \alpha r_1 \alpha^{-1}$ and $w_2 = \beta r_2 \beta^{-1}$. We can write $r_1 = a_1 \cdots a_n$ and $r_2 = b_1 \cdots b_n$ where $a_i, b_i \in \{z, z^{-1}, t^j, t^{-i}, s^{-1}z, z^{-1}s\}$. Now consider how there can be cancellation in the product $w_1 w_2$.

If there is a cancellation in the product $w_1 w_2$ we must have : a_n and b_1 are in the same factor and $|a_n \alpha^{-1} \beta b_1| = 1$ or $|a_n \alpha^{-1} \beta b_1| = 0$. Let us prove that the length of any piece is at most 2.

If $|a_n \alpha^{-1} \beta b_1| = 1$ then it is clear that the length of the piece which was cancelled is 1. So we consider the case $|a_n \alpha^{-1} \beta b_1| = 0$, so the case $a_n \alpha^{-1} \beta b_1 \in G$. Now if $a_n \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$ and $b_1 \in \{t^{-i}, t^j\}$, we see that $|a_n \alpha^{-1} \beta b_1| = 1$. The same thing holds if $a_n \in \{t^{-i}, t^j\}$ and $b_1 \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$. Therefore we have the following two cases to consider.

Case (1). $a_n, b_1 \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$.

Since $a_n \alpha^{-1} \beta b_1 \in G$, by Claim 1 we have $a_n \alpha^{-1} \beta b_1 = \alpha^{-1} \beta$, and $\alpha^{-1} \beta \in Z(G)$, $b_1 = a_n^{-1}$. So

$$a_{n-1} a_n \alpha^{-1} \beta b_1 b_2 = a_{n-1} \alpha^{-1} \beta b_2,$$

and since $a_n, b_1 \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$ we must have $a_{n-1}, b_2 \in \{t^i, t^{-j}\}$. So

$$a_{n-1} a_n \alpha^{-1} \beta b_1 b_2 = \alpha^{-1} \beta a_{n-1} b_2.$$

Now if $a_{n-1}.b_2 = 1$, then $r_1 = a_1 \cdots a_{n-1}.a_n$ and $r_2 = a_n^{-1}.a_{n-1}^{-1} \cdots b_n$. But a cyclic permutation of

$$(s^{-1}z, t, z, t^2, z, t^3, z, \cdots, z, t^{80})$$

or of

$$(t^{-80}, z^{-1}, \cdots, z^{-1}, t^{-3}, z^{-1}, t^{-2}, z^{-1}, t^{-1}, z^{-1}s)$$

is uniquely determined by the first two of its elements.

(To see what happens we illustrate the situation. If $a_n = z$ and $a_{n-1} = t^i$ then $b_1 = z^{-1}$ and $b_2 = t^{-i}$. Therefore

$$r_1 = t^{i+1} \cdots z t^{80} s^{-1} z t \cdots t^{i-1} z t^i z,$$

$$r_2 = z^{-1} t^{-i} z^{-1} t^{-(i-1)} \cdots t^{-1} z^{-1} s t^{-80} z^{-1} \cdots z^{-1} t^{-(i+1)},$$

and then $r_1r_2 = 1$.) So we have $r_1r_2 = 1$. Since $\alpha^{-1}\beta \in Z(G)$ we have

$$w_1w_2 = \alpha\alpha^{-1}\beta r_1r_2\beta^{-1} = 1,$$

so $w_1w_2 = 1$ which is a contradiction. Therefore $a_{n-1}.b_2 \neq 1$ and hence $|a_{n-1}\alpha^{-1}\beta b_2| = 1$. So the length of the piece which was cancelled is 2.

Case (2). $a_n, b_1 \in \{t^i, t^{-j}\}$.

Since $a_n\alpha^{-1}\beta b_1 \in G$, we have $a_n.b_1 = 1$ and

$$a_{n-1}a_n\alpha^{-1}\beta b_1b_2 = a_{n-1}\alpha^{-1}\beta b_2.$$

Now if $a_{n-1}\alpha^{-1}\beta b_2 \in G$, and since $a_{n-1}, b_2 \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$, then by Claim 1, we have $a_{n-1}\alpha^{-1}\beta b_2 = \alpha^{-1}\beta$, and $\alpha^{-1}\beta \in Z(G)$, $b_2 = a_{n-1}^{-1}$. So as in the previous case we find $r_1r_2 = 1$. Since $\alpha^{-1}\beta \in Z(G)$ we get

$$w_1w_2 = \alpha\alpha^{-1}\beta r_1r_2\beta^{-1} = 1,$$

and thus $w_1w_2 = 1$, which is a contradiction. Therefore $a_{n-1}\alpha^{-1}\beta b_2 \notin G$ and hance $|a_{n-1}\alpha^{-1}\beta b_2| = 1$. So the length of the piece which was cancelled is 2.

Now since $|w_1| = |w_2| = 160$ and the maximal length of the piece which was cancelled is 2, a simple count show that : $2 < \frac{160}{70} = \frac{1}{70}|w|$. Hence $W_0(R_0)$ satisfies $C'(\frac{1}{70})$. This completes the proof of this stage. □

Stage 2. Suppose that G has a soluble word problem. By Lemma 2.6, $Z(G)$ has a generalized soluble word problem in G. So G_0 has a soluble word problem. Hence R is a recursive subgroup of $\mathbb{F}_X$. By Lemma 2.8, R is a strongly benign subgroup. Hence F_R is embeddable in finitely presented group H_1 such that $\mathbb{F}_X$ and $\langle \mathbb{F}_X, d\rangle$ have a soluble generalized soluble word problem in H_1.

It is clear that the proof of the stage 1 is independent of the choice of the finitely presented group H. Therefore we apply the same construction and we assume that $H = H_1$.

Let us show that Γ and R_0 satisfy the conditions of Proposition 3.3. By Lemma 2.7, the subgroup $L = \langle \mathbb{F}_X, d^{-1}\mathbb{F}_X d\rangle$ has a generalized soluble word problem in $F_R = \langle \mathbb{F}_X, d \mid d^{-1}rd = r,\ r \in R\rangle$. It is easy to see that L has a generalized soluble word problem in H. We see also that $L \times Z(G)$ has a generalized soluble word problem in $H \times G_0$.

Let us show that $\psi(L \times Z(G))$ has a soluble generalized soluble word problem in $H \times G_0$. Let (h, g) in $H \times G_0$. Since L has a generalized soluble word problem in H, one can determine whether $h \in L$. If $h \notin L$ then $(h, g) \notin \psi(L \times Z(G))$. If $h \in L$ then we compute $\phi(h)$ (ϕ is clearly computable). Now if there exists $g_0 \in Z(G)$ such that $\phi(h).g_0 = g$ we must have $\phi(h)^{-1}g \in Z(G)$. Since $Z(G)$ has a generalized soluble word problem in $H \times G_0$ we can determine whether $\phi(h)^{-1}g \in Z(G)$. If $\phi(h)^{-1}g \notin Z(G)$ then $(h, g) \notin \psi(L \times Z(G))$. If $\phi(h)^{-1}g \in Z(G)$ then

$$\psi(h, \phi(h)^{-1}g) = (h, \phi(h)\phi(h)^{-1}g) = (h, g),$$

and so $(h, g) \in \psi(L \times Z(G))$.

The maps ψ, ψ^{-1} are computable and $L \times Z(G)$, $\psi(L \times Z(G))$ have a generalized soluble word problem in $H \times G_0$. Therefore K has a soluble word problem and we can calculate the normal form relative to the structure of the HNN-extension of K.

Hence the group G has a generalized soluble word problem in K. It also has a generalized soluble word problem in $G \times \langle t | \rangle$.

So Γ has a soluble word problem and we can calculate the normal form relative to its structure of free product with amalgamation.

Therefore, it is sufficient to show that the $R_0 = \{r\}$ satisfies the conditions of Proposition 3.3. Let $C = \{((s^{-1}z), t, z, t^2, \cdots, z, t^{80}), (t^{-80}, z^{-1}, \cdots, t^{-1}, (z^{-1}s))\}$. Then we see that C defines explicitly $\{r\}$. We see also that $\{r\}$ satisfies the conditions (1)-(4) of Proposition 3.3. Then it is sufficient to show that the set $L(C, \lambda)$ is recursive and there exists an algorithm which satisfies condition (6) of Proposition 3.3. The conclusion will follows from a sequence of claims. We need first the following claim.

Principal Claim. *Let $w \in \Gamma$ with a normal form $(\alpha_1 a_1 \beta_1, \cdots, \alpha_n a_n \beta_n)$ where $\alpha_i, \beta_i \in G$ and $a_i \in \{z, z^{-1}, s^{-1}z, z^{-1}s, t^{-i}, t^j\}$. Then the following conditions are equivalents:*

(1) There exist $a, b \in G$ such that $awb = a_1 \cdots a_n$.

(2) Let $I = \{i \mid a_i \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}\}$. Then for every $i, j \in I$ such that $i < j$, one has $(\beta_i \alpha_{i+1} \beta_{i+1} \alpha_{i+2} \cdots \beta_{j-1} \alpha_j) \in Z(G)$.

If (2) is satisfied, then

• if $a_n \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$, then we can take $a = \alpha_n^{-1}.\beta_{n-1}^{-1} \cdots \beta_1^{-1} \alpha_1^{-1}$ and $b = \beta_n^{-1}$,

• if $a_n \in \{t^{-i}, t^j\}$ and $n = 1$, then we can take $a = \alpha_1^{-1}$ and $b = \beta_1^{-1}$,

• if $a_n \in \{t^{-i}, t^j\}$ and $n \geq 2$, then we can take $a = \alpha_{n-1}^{-1}.\beta_{n-2}^{-1} \cdots \beta_1^{-1} \alpha_1^{-1}$ and $b = (\beta_{n-1} \alpha_n \beta_n)^{-1}$.

Proof.

(1)⇒(2). By induction on n. We consider two cases: $a_n \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$ and $a_n \in \{t^{-i}, t^j\}$.

***Case* (I)**. $a_n \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$. For $n = 1$, we have

$$awba_1^{-1} = a\alpha_1 a_1 \beta_1 b a_1^{-1} = 1,$$

and in this case, we find $I = \{1\}$ and the property is true. It is not difficult to see that we can take $a = \alpha_1^{-1}$ and $b = \beta_1^{-1}$.

We go from n to $n + 1$. We have

$$awba_{n+1}^{-1} a_n^{-1} \cdots a_1^{-1} = a\alpha_1 a_1 \beta_1 \cdots \alpha_n a_n \beta_n \alpha_{n+1} a_{n+1} (\beta_{n+1} b) a_{n+1}^{-1} a_n^{-1} \cdots a_1^{-1} = 1,$$

and hence we must have $a_{n+1}(\beta_{n+1} b) a_{n+1}^{-1} \in G$. Therefore by Claim 1 we have $\beta_{n+1} b \in Z(G)$. Since $a_{n+1} \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$ then $a_n \in \{t^{-i}, t^j\}$ and we find

$$a\alpha_1 a_1 \beta_1 \cdots a_{n-1} (\beta_{n-1} \alpha_n \beta_n \alpha_{n+1} \beta_{n+1} b) a_{n-1}^{-1} \cdots a_1^{-1} = 1,$$

and thus we get $(\beta_{n-1}\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}b) \in Z(G)$.

Since $a_n \in \{t^{-i}, t^j\}$, we have $a_{n-1} \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$. We see that the sequence $(\alpha_1 a_1 \beta_1, \cdots, \alpha_{n-1}a_{n-1}(\beta_{n-1}\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}))$ satisfies the same conditions of the Claim, and hence by induction hypothesis we get that for every $i, j \in I$ such that $i < j \leq n-1$, $\beta_i\alpha_{i+1}\cdots\beta_{j-1}\alpha_j \in Z(G)$. Hence $\beta_i\alpha_{i+1}\cdots\beta_{n-1-1}\alpha_{n-1} \in Z(G)$.

Since $\beta_{n+1}b \in Z(G)$ and $(\beta_{n-1}\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}b) \in Z(G)$ we find that $\beta_{n-1}\alpha_n\beta_n\alpha_{n+1} \in Z(G)$. Hence for every $i \in I$ such that $i < n+1$, we get

$$\beta_i\alpha_{i+1}\cdots\beta_{n-1-1}\alpha_{n-1}\beta_{n-1}\alpha_n\beta_n\alpha_{n+1} \in Z(G),$$

and it is not hard to see that we can take $a = \alpha_n^{-1}\beta_{n-1}^{-1}\cdots\beta_1^{-1}\alpha_1^{-1}$ and $b = \beta_n^{-1}$.

***Case* (II)**. $a_n \in \{t^{-i}, t^j\}$. For $n = 1$, we have $awba_1^{-1} = a\alpha_1 a_1\beta_1 ba_1^{-1} = a\alpha_1\beta_1 b = 1$ and in this case $I = \emptyset$ and the property is true. It is not hard to see that we can take $a = \alpha_1^{-1}$ and $b = (\beta_1\alpha_2\beta_2)^{-1}$.

We go from n to $n+1$. We have

$$awba_{n+1}^{-1}a_n^{-1}\cdots a_1^{-1} = a\cdots\alpha_n a_n\beta_n\alpha_{n+1}a_{n+1}(\beta_{n+1}b)a_{n+1}^{-1}a_n^{-1}\cdots a_1^{-1}$$

$$= a\cdots\alpha_n a_n(\beta_n\alpha_{n+1}\beta_{n+1}b)a_n^{-1}\cdots a_1^{-1} = 1,$$

and hence we must have $\beta_n\alpha_{n+1}\beta_{n+1}b \in Z(G)$. We see that the sequence $(\alpha_1 a_1\beta_1, \cdots, \alpha_n a_n(\beta_n\alpha_{n+1}\beta_{n+1}))$ satisfies the conditions of the case (I) and hence for every $i, j \in I$ such that $i < j \leq n$, $\beta_i.\alpha_{i+1}\cdots\beta_{j-1}.\alpha_j \in Z(G)$, and the result follows. We easily see that we can take $a = \alpha_{n-1}^{-1}\beta_{n-2}^{-1}\cdots\beta_1^{-1}\alpha_1^{-1}$ and $b = (\beta_{n-1}\alpha_n\beta_n)^{-1}$.

(2)⇒(1). The proof is a straightforward calculation. □

Claim 2. *Let $A(z) = \{g \in K \mid \exists\alpha, \beta \in G$ such that $g = \alpha z\beta\}$. Then $A(z)$ is recursive and there exists an algorithm which for every $g \in A(z)$ produces $\alpha, \beta \in G$ such that $g = \alpha z\beta$.*

Proof.

Let $g \in K$. Then one can effectively calculate a normal form (in the HNN-extension K) of g say $b_1 s^{\varepsilon_1} b_2 \cdots b_n s^{\varepsilon_n} b_{n+1}$ where $\varepsilon_i = \pm 1$ and $b_i \in H \times G_0$. If $n \geq 1$, clearly $g \notin A(z)$. Thus we suppose $g \in H \times G_0$. Therefore $g = hg_0$, where $h \in H$ and $g_0 \in G_0$. If $h \neq 1$, then $g \notin A(z)$. Hence $g \in G_0$. Then one can effectively calculate a normal form (in the HNN-extension G_0) of g say $b_1 z^{\varepsilon_1} b_2 \cdots b_n z^{\varepsilon_n} b_{n+1}$ where $\varepsilon_i = \pm 1$ and $b_i \in G$. If $n \geq 2$ then $g \notin A(z)$, and if $\varepsilon_1 = -1$ then $g \notin A(z)$. Thus we suppose $g = b_1 z b_2$. Hence $g \in A(z)$. And we see that the above procedure is effective and produces $\alpha, \beta \in G$ such that $g = \alpha z\beta$. □

Claim 3. *Let*

$$Q = \{(h_1, h_2, g_1, g_2) \mid h_i \in H,\ g_i \in G_0,\ \exists\alpha, \beta \in G \text{ such that}$$

$$h_1 g_1 s^{-1} h_2 g_2 = \alpha s^{-1} z \beta\}.$$

Then Q is recursive and there exists an algorithm which for every $(h_1, h_2, g_1, g_2) \in Q$ produces $\alpha, \beta \in G$ such that $h_1 g_1 s^{-1} h_2 g_2 = \alpha s^{-1} z \beta$.

Proof. Let us show that the following properties are equivalent:

(1) $(h_1, h_2, g_1, g_2) \in Q$.

(2) $h_1.h_2 = 1$, $h_1, h_2 \in L$, $g_1\phi(h_2) \in G$, $g_2 \in A(z)$, $g_2 = \gamma_1 z \gamma_2$, $\gamma_1 \in Z(G)$ and one can take $\alpha = g_1\phi(h_2)\gamma_1$, $\beta = \gamma_2^{-1}$.

(1)$\Rightarrow$(2). Let $\alpha, \beta \in G$ such that $h_1 g_1 s^{-1} h_2 g_2 = \alpha s^{-1} z \beta$.

Then the sequence $(h_1 g_1, s^{-1}, h_2 g_2 \beta^{-1} z^{-1}, s, \alpha^{-1})$ is not reduced in the HNN-extension K and thus $h_2 g_2 \beta^{-1} z^{-1} \in L \times Z(G)$. So $h_2 \in L$ and $g_2 \beta^{-1} z^{-1} \in Z(G)$. Hence $g_2 = \delta z \beta \in A(z)$, and $\delta \in Z(G)$. Therefore

$$h_1 g_1 s^{-1} h_2 g_2 \beta^{-1} z^{-1} s \alpha^{-1} = h_1 g_1 h_2 \phi(h_2) \delta \alpha^{-1} = 1,$$

and so $h_1 h_2 = 1$ and $g_1 \phi(h_2) = \alpha \delta^{-1} \in G$.

(2)$\Rightarrow$(1) Let $\beta = \gamma_2^{-1}$. Then

$$h_1 g_1 s^{-1} h_2 g_2 \beta^{-1} z^{-1} s = h_1 g_1 s^{-1} h_2 \gamma_1 z \gamma_2 \gamma_2^{-1} z^{-1} s =$$

$$= h_1 g_1 s^{-1} h_2 \gamma_1 s$$

$$= h_1 g_1 h_2 \phi(h_2) \gamma_1 = g_2 \phi(h_2) \gamma_1 = \alpha \in G,$$

and this ends the proof of the equivalence (1)$\Leftrightarrow$(2).

Since L has a generalized soluble word problem in H, and $A(z)$ is recursive and there exists an algorithm which for every $g \in A(z)$ produces $\alpha, \beta \in G$ such that $g = \alpha z \beta$, the conclusion follows from the above equivalence. $\square$

Claim 4. *For every $a \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$, the set $A(a) = \{g \in K \mid \exists \alpha, \beta \in G$ such that $g = \alpha a \beta\}$ is recursive and there exists an algorithm which for every $g \in A(a)$ produces $\alpha, \beta \in G$ such that $g = \alpha a \beta$. Also for every $a \in \{t^{-i}, t^j\}$, the set $A(a) = \{g \in G \times \langle t \mid \rangle \mid \exists \alpha, \beta \in G$ such that $g = \alpha a \beta\}$ is recursive and there exists an algorithm which for every $g \in A(a)$ produces $\alpha, \beta \in G$ such that $g = \alpha a \beta$.*

Proof. We see that $g \in A(a)$ if and only if $g^{-1} \in A(a^{-1})$ and $g = \alpha a \beta$ if and only if $g^{-1} = \beta^{-1} a^{-1} \alpha^{-1}$. Therefore it is sufficient to show that the above properties are true for $a \in \{z, s^{-1}z\}$. For the case $a = z$ this was proved in Claim 2.

Let $g \in K$. Then one can effectively calculate a normal form (in the HNN-extension K) $b_1 s^{\varepsilon_1} b_2 \cdots b_n s^{\varepsilon_n} b_{n+1}$ where $\varepsilon_i = \pm 1$ and $b_i \in H \times G_0$. If $n \geq 2$ then $g \notin A(s^{-1}z)$. Hence we must have $n = 1$, $\varepsilon_1 = -1$ and thus $g = b_1 s^{-1} b_2$. Then one can effectively calculate $h_1, h_2 \in H$, $g_1, g_2 \in G_0$ such that $b_1 = h_1 g_1$ and $b_2 = h_2 g_2$. We see that $g \in A(s^{-1}z)$ if and only if $(h_1, h_2, g_1, g_2) \in Q$. Since Q is recursive we see that $A(s^{-1}z)$ is recursive. By Claim 3, there exists an algorithm which for $(h_1, h_2, g_1, g_2) \in Q$ produces $\alpha, \beta \in G$ such that

$h_1g_1s^{-1}h_2g_2 = \alpha s^{-1}z\beta$. Hence there exists an algorithm which for every $g \in A(s^{-1}z)$ produces $\alpha, \beta \in G$ such that $g = \alpha s^{-1}z\beta$.

For $a \in \{t^{-i}, t^j\}$, the conclusion is obvious. $\square$

Now we are ready to prove that the set

$$L(C,\lambda) = \{(g,c,l) \in \Gamma \times \overline{C} \times \mathbb{N} \mid c = (c_1, \cdots, c_n),\ (1-3\lambda)n < l \leq n,$$

$$\exists \alpha, \beta \in G,\ \text{ such that } \ \alpha g \beta = c_1 \cdots c_l\},$$

is recursive, where $\lambda = 1/10$, and that there exists an algorithm which for every $(g,c,l) \in L(C,\lambda)$ produces $(\alpha,\beta) \in G^2$ such that $\alpha g \beta = c_1 \cdots c_l$.

Let $(g,c,l) \in \Gamma \times \overline{C} \times \mathbb{N}$. Then one can effectively calculate a normal form $(g_1, \cdots, g_m)$ of g in Γ.

(1). If $l \leq (1-3\lambda)|c|$ then $(g,c,l) \notin L(C,\lambda)$.

(2). Otherwise,

• If $(g,c,l) \in L(C,\lambda)$, then there exists $\alpha, \beta \in G$, $1 \leq l \leq m$, $l > (1-3\lambda)|c|$ such that $\alpha g \beta = c_1 \cdots c_l$ and $m = l$. Then we must have a sequence $(\gamma_1, \cdots, \gamma_m, \gamma_{m+1})$ of G such that $g_1 = \alpha^{-1}\gamma_1 c_1 \gamma_2^{-1}$, $g_i = \gamma_i c_i \gamma_{i+1}^{-1}$, $g_m = \gamma_m c_m \gamma_{m+1}^{-1}\beta^{-1}$. Hence we have $g_i \in A(c_i)$. Then it is sufficient to verify whether $g_i \in A(c_i)$.

• If there is some i such that $g_i \notin A(c_i)$ then $(g,c,l) \notin L(C,\lambda)$.

• If for every i, $g_i \in A(c_i)$ then by Claim 4, one can effectively calculate two sequences $(\alpha_1, \cdots, \alpha_m)$, $(\beta_1, \cdots, \beta_m)$ of G such that $g_i = \alpha_i c_i \beta_i$.

By the Principal Claim, for every $i, j \in I$ such that $i < j$,

$$(*) \qquad \beta_i \alpha_{i+1} \cdots \beta_{j-1}\alpha_j \in Z(G).$$

• If for some $i, j \in I$ such that $i < j$, $(*)$ does not hold then $(g,c,l) \notin L(C,\lambda)$.

• If for every $i, j \in I$ such that $i < j$, $(*)$ holds then, by the Principal Claim, $(g,c,l) \in L(C,\lambda)$ and we can take $\alpha = \alpha_m^{-1}\beta_{m-1}^{-1} \cdots \beta_1^{-1}\alpha_1^{-1}$, $\beta = \beta_m^{-1}$ if $c_m \in \{z, z^{-1}, s^{-1}z, z^{-1}s\}$, and we can take $\alpha = \alpha_{m-1}^{-1}\beta_{m-2}^{-1} \cdots \beta_1^{-1}\alpha_1^{-1}$ and $\beta = (\beta_{m-1}\alpha_m\beta_m)^{-1}$ if $c_m \in \{t^{-i}, t^j\}$.

Hence $L(C,\lambda)$ is recursive and we see, by the above method, that there exists an algorithm which for every $(g,c,l) \in L(C,\lambda)$, produces $(\alpha,\beta) \in G^2$ such that $\alpha g \beta = c_1 \cdots c_l$. $\square$

6 Proofs of corollaries

Proof of Corollary 1. By Lemma 2.6, if H is finitely presented then $Z(H)$ is recursively presented. Conversely, let G be a countable recursively presented abelian group. By Theorem I, G is embeddable in a finitely generated and recursively presented group K such that $G = Z(G) = Z(K)$. By Theorem II, K is embeddable in finitely presented group H such that $Z(K) = Z(H)$, hence $Z(H) = G$ and the result follows. $\square$

Proof of Corollary 2. By Lemma 2.6, if H is finitely presented with soluble word problem then $Z(H)$ is recursively presented and with soluble word problem. Conversely, let G be a countable abelian group with soluble word problem. By Theorem I, G is embeddable in a finitely generated group with soluble word problem K such that $G = Z(G) = Z(K)$. By Theorem II, K is embeddable in a finitely presented group H with soluble word problem such that $Z(K) = Z(H)$, hence $Z(H) = G$ and the result follows. □

Proof of Corollary 3. It follows from Corollary 2 and from the following lemma.

Lemma 6.1 *There exists a countable abelian group K with soluble word problem such that every countable abelian group can be embedded in K.*

Proof. Let $(\pi_n)_{n\in\omega}$ be the sequence of prime numbers. Let $K = (\mathbb{Q})^{(\aleph_0)} \oplus (\oplus_{i\in\omega}(\mathbb{Z}_{\pi_i^\infty})^{(\aleph_0)})$. Then it is not difficult to see that K has a soluble word problem.

Let G be a countable abelian group. By a classical result G is embeddable in a divisible and countable abelian group say G_1. It is also well-known that every divisible abelian group is isomorphic to a direct sum of groups each of which is isomorphic to $\mathbb{Q}$ or a group of the form $\mathbb{Z}_{\pi_n^\infty}$. Hence the groups G_1 and G are embeddable in K. Since K has a soluble word problem the result follows. □

Proof of Corollary 4. Let G be a finitely generated recursively presented group. Let A be a countable recursively presented abelian group. Let $M = (G * \langle x|\rangle) \times A$. Then we see that $Z(M) = A$ and M is recursively presented. By Theorem I, M is embeddable in a finitely generated recursively presented group L such that $Z(L) = Z(M) = A$. The result follows by Theorem II. □

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References

[1] C.R.J. Clapham. *Finitely presented groups with word problems of arbitrary degrees of insolubility.* Proc. London. Math. Soc, (3) **14** (1964) 633-76.

[2] C.R.J. Clapham. *An embedding theorem for finitely presented groups.* Proc. Lond. Math. Soc. (3) **17** (1967), 419-430.

[3] P. de la Harpe. *Topics in geometric group theory.* The University of Chicago Press, 2000.

[4] G. Higman. *Subgroups of finitely presented groups.* Proc. Royal. Soc. Ser.A, **262** (1961), 455-475.

[5] B.M. Hurley. *Small cancellation theory over groups equipped with an integer-valued length function.* In Word Problems II, (North Holland Publishing Co., Amsterdam, 1980), 157-214.

[6] R.G. Lyndon, P.E. Schupp. *Combinatorial group theory.* Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[7] A. Ould Houcine. *Sur quelques problèmes de plongement dans les groupes.* Ph.D. Thesis, Université Paris 7, 2003.

[8] Kourovka Notebook, 1999 edition, problem 5.47.