Superstable groups acting on trees

Abderezak Ould Houcine

Camille Jordan Institute, University Lyon 1, France

Modnet Bercelona Conference, November 4, 2008

Groups acting on trees

Definition

Groups acting on trees

Definition

- A *simplicial tree* is a connected graph without circuits.

Groups acting on trees

Definition

- A *simplicial tree* is a connected graph without circuits.
- A group G *acts on a simplicial tree* T if it acts by automorphisms on T.

Groups acting on trees

Definition

- A *simplicial tree* is a connected graph without circuits.
- A group G *acts on a simplicial tree* T if it acts by automorphisms on T.
- A *real tree* X is a geodesic metric space such that any two points are joined by an unique arc.

Groups acting on trees

Definition

- A *simplicial tree* is a connected graph without circuits.
- A group G *acts on a simplicial tree* T if it acts by automorphisms on T.
- A *real tree* X is a geodesic metric space such that any two points are joined by an unique arc. This equivalent to saying that X is a 0-hyperbolic geodesic space.

Groups acting on trees

Definition

- A *simplicial tree* is a connected graph without circuits.
- A group G *acts on a simplicial tree* T if it acts by automorphisms on T.
- A *real tree* X is a geodesic metric space such that any two points are joined by an unique arc. This equivalent to saying that X is a 0-hyperbolic geodesic space.
- A group G *acts on a real tree* if it acts by isometries.

Definition (Classification of elements)

Groups acting on trees

Definition (Classification of elements)

Let G be a group acting on a simplicial or a real tree T.

Definition (Classification of elements)

Let G be a group acting on a simplicial or a real tree T.

- An element g is said an *inversion* if $ge = \bar{e}$ for some edge e (when T is simplicial).

Definition (Classification of elements)

Let G be a group acting on a simplicial or a real tree T.

- An element g is said an *inversion* if $ge = \bar{e}$ for some edge e (when T is simplicial).
- An element g is said *elliptic* if $gx = x$ for some x of T.

Definition (Classification of elements)

Let G be a group acting on a simplicial or a real tree T.

- An element g is said an *inversion* if $ge = \bar{e}$ for some edge e (when T is simplicial).
- An element g is said *elliptic* if $gx = x$ for some x of T.
- An element g is said *hyperbolic* if it is neither an inversion nor elliptic.

Groups acting on trees

Definition (Classification of elements)

Let G be a group acting on a simplicial or a real tree T.

- An element g is said an *inversion* if $ge = \bar{e}$ for some edge e (when T is simplicial).
- An element g is said *elliptic* if $gx = x$ for some x of T.
- An element g is said *hyperbolic* if it is neither an inversion nor elliptic.
- A group G acts *freely* if every nontrivial element of G is hyperbolic.

Motivation

Motivation

Theorem 1 (Sela)

Motivation

Theorem 1 (Sela)

A free group is stable. □

Motivation

Theorem 1 (Sela)

A free group is stable. □

Reformulation:

Motivation

Theorem 1 (Sela)

A free group is stable. □

Reformulation: A group acting freely on a simplicial tree is stable.

Motivation

Theorem 1 (Sela)

A free group is stable. □

Reformulation: A group acting freely on a simplicial tree is stable.

Remark. A superstable group acting freely on a real (or simplicial) tree is abelian.

Motivation

Definition

Motivation

Definition

The action of G is *trivial* if there is no hyperbolic elements. □

Motivation

Definition

The action of G is *trivial* if there is no hyperbolic elements. □

Question

Motivation

Definition

The action of G is *trivial* if there is no hyperbolic elements. □

Question

What can be said about the model theory of groups acting nontrivially on simplicial trees?

Motivation

Definition

The action of G is *trivial* if there is no hyperbolic elements. □

Question

What can be said about the model theory of groups acting nontrivially on simplicial trees? Is it possible for such groups to be ω-stable or superstable?

Bass-Serre theorem

Bass-Serre theorem

Theorem 2 (Bass-Serre)

Bass-Serre theorem

Theorem 2 (Bass-Serre)

A group acts without inversions and nontrivially on a simplicial tree if and only if either G splits as a free product with amalgamation or G has an infinite cyclic quotient. □

Free products

Free products

What about the superstability of free products?

Free products

What about the superstability of free products?

Theorem 3 (Poizat, 1984)

Free products

What about the superstability of free products?

Theorem 3 (Poizat, 1984)

A nontrivial free product $G_1 * G_2$ is superstable if and only if $G_1 = G_2 = \mathbb{Z}_2$. □

ω-stable groups

ω-stable groups

Theorem 4

ω-stable groups

Theorem 4

An action of an ω-stable group on a simplical tree is trivial. □

ω-stable groups

Theorem 4

An action of an ω-stable group on a simplical tree is trivial. □

Using Bass-Serre theorem:

ω-stable groups

Theorem 4

An action of an ω-stable group on a simplical tree is trivial. □

Using Bass-Serre theorem:

Corollary 1

A free product with amalgamtion or an HNN-extension is not ω-stable. □

Superstable groups

Superstable groups

Examples:

Superstable groups

Examples:

- $\mathbb{Z}$ is superstable and acts freely on a simplicial tree.

Superstable groups

Examples:

- $\mathbb{Z}$ is superstable and acts freely on a simplicial tree.
- If G is superstable then $G \oplus \mathbb{Z}$ is superstable and acts nontrivially on a simplicial tree.

Superstable groups

Examples:

- $\mathbb{Z}$ is superstable and acts freely on a simplicial tree.
- If G is superstable then $G \oplus \mathbb{Z}$ is superstable and acts nontrivially on a simplicial tree.
- If G is superstable then $G \oplus (\mathbb{Z}_2 * \mathbb{Z}_2)$ is superstable and acts nontrivially on a simplicial tree.

Superstable groups

Examples:

- $\mathbb{Z}$ is superstable and acts freely on a simplicial tree.
- If G is superstable then $G \oplus \mathbb{Z}$ is superstable and acts nontrivially on a simplicial tree.
- If G is superstable then $G \oplus (\mathbb{Z}_2 * \mathbb{Z}_2)$ is superstable and acts nontrivially on a simplicial tree. Moreover $G \oplus (\mathbb{Z}_2 * \mathbb{Z}_2) = (G \oplus \mathbb{Z}_2) *_G (G \oplus \mathbb{Z}_2)$.

Classifications of actions

Classifications of actions

Let $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Let G be a group acting on a Λ-tree T.

Classifications of actions

Let $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Let G be a group acting on a Λ-tree T. We suppose that if T is simplicial then the action is without inversions.

Classifications of actions

Let $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Let G be a group acting on a Λ-tree T. We suppose that if T is simplicial then the action is without inversions.

Definition

Classifications of actions

Let $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Let G be a group acting on a Λ-tree T. We suppose that if T is simplicial then the action is without inversions.

Definition

- The *hyperbolic lenght* function is defined by:

$$\ell(g) = \inf\{d(x, gx) | x \in T\}.$$

Classifications of actions

Let $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Let G be a group acting on a Λ-tree T. We suppose that if T is simplicial then the action is without inversions.

Definition

- The *hyperbolic lenght* function is defined by:

$$\ell(g) = \inf\{d(x, gx) | x \in T\}.$$

- (Fact) g is hyperbolic if and only if $\ell(g) > 0$.

Classifications of actions

Classifications of actions

One of the following cases holds:

Classifications of actions

One of the following cases holds:

(1)(Abelian actions) the hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \Lambda$ is a homomorphism.

Classifications of actions

One of the following cases holds:

(1)(Abelian actions) the hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \Lambda$ is a homomorphism.

(2)(Dihedral actions) the hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \mathrm{Isom}(\Lambda)$ is a homomorphism whose image contains a reflection and a nontrivial translation, and the absolute value signs denote hyperbolic length for the action of $\mathrm{Isom}(\Lambda)$.

Classifications of actions

One of the following cases holds:

(1)(Abelian actions) the hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \Lambda$ is a homomorphism.

(2)(Dihedral actions) the hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \mathrm{Isom}(\Lambda)$ is a homomorphism whose image contains a reflection and a nontrivial translation, and the absolute value signs denote hyperbolic length for the action of $\mathrm{Isom}(\Lambda)$.

(3)(Irreducible actions) G contains a free subgroup of rank 2 which acts freely, without inversions and properly discontinuously on T.

Superstable groups

Superstable groups

Theorem 5

Superstable groups

Theorem 5

Let G be a superstable group acting nontrivially on a Λ-tree, where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$.

Superstable groups

Theorem 5

Let G be a superstable group acting nontrivially on a Λ-tree, where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. If G is α-connected and $\Lambda = \mathbb{Z}$, or if the action is irreducible, then G interprets a simple group having a nontrivial action on a Λ-tree. □

Superstable groups

Theorem 5

Let G be a superstable group acting nontrivially on a Λ-tree, where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. If G is α-connected and $\Lambda = \mathbb{Z}$, or if the action is irreducible, then G interprets a simple group having a nontrivial action on a Λ-tree. □

Corollary 2

If G is superstable and splits as $G = G_1 *_A G_2$, with the index of A in G_1 different from 2, then G interprets a simple superstable non ω-stable group acting nontrivially on a simplicial tree. □

Minimal Superstable groups

Minimal Superstable groups

Definition

Minimal Superstable groups

Definition

Let G be a group and $\mathcal{B}$ be a family of definable subgroups of G.

Minimal Superstable groups

Definition

Let G be a group and $\mathcal{B}$ be a family of definable subgroups of G. We say that $\mathcal{B}$ is a *Borel* family, if for any $B \in \mathcal{B}$, $N_G(B)/B$ is finite and B is generous, for any $g \in G$, $B^g \in \mathcal{B}$, and any two elements of $\mathcal{B}$ are conjugate to each other.

Minimal Superstable groups

Minimal Superstable groups

Theorem 6

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$.

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if H is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a Λ-tree, then H is nilpotent-by-finite.

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if H is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a Λ-tree, then H is nilpotent-by-finite. Then there are definable subgroups $H_1 \lhd H_2 \lhd G$ such that H_2 is of finite index in G, and one of the following cases holds:

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if H is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a Λ-tree, then H is nilpotent-by-finite. Then there are definable subgroups $H_1 \lhd H_2 \lhd G$ such that H_2 is of finite index in G, and one of the following cases holds:

(1) *H_1 is connected, any action of H_1 on a Λ-tree is trivial, H_2/H_1 is soluble and has a nontrivial action on a Λ-tree.*

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if H is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a Λ-tree, then H is nilpotent-by-finite. Then there are definable subgroups $H_1 \lhd H_2 \lhd G$ such that H_2 is of finite index in G, and one of the following cases holds:

(1) *H_1 is connected, any action of H_1 on a Λ-tree is trivial, H_2/H_1 is soluble and has a nontrivial action on a Λ-tree.*
(2) *H_2/H_1 is simple and acts nontrivially on a Λ-tree,*

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if H is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a Λ-tree, then H is nilpotent-by-finite. Then there are definable subgroups $H_1 \lhd H_2 \lhd G$ such that H_2 is of finite index in G, and one of the following cases holds:

(1) *H_1 is connected, any action of H_1 on a Λ-tree is trivial, H_2/H_1 is soluble and has a nontrivial action on a Λ-tree.*
(2) *H_2/H_1 is simple and acts nontrivially on a Λ-tree, H_2/H_1 has a Borel family of equationally-definable nilpotent subgoups such that there exists $m \in \mathbb{N}$ such that for every hyperbolic element g in H_2/H_1, there is $1 \leq n \leq m$, such that g^n is in some $B \in \mathcal{B}$.*

Minimal Superstable groups

Theorem 6

Let G be a superstable group of finite Lascar rank acting nontrivially on a Λ-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if H is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a Λ-tree, then H is nilpotent-by-finite. Then there are definable subgroups $H_1 \lhd H_2 \lhd G$ such that H_2 is of finite index in G, and one of the following cases holds:

(1) *H_1 is connected, any action of H_1 on a Λ-tree is trivial, H_2/H_1 is soluble and has a nontrivial action on a Λ-tree.*
(2) *H_2/H_1 is simple and acts nontrivially on a Λ-tree, H_2/H_1 has a Borel family of equationally-definable nilpotent subgoups such that there exists $m \in \mathbb{N}$ such that for every hyperbolic element g in H_2/H_1, there is $1 \leq n \leq m$, such that g^n is in some $B \in \mathcal{B}$. If $\Lambda = \mathbb{Z}$ then $H_2/H_1 = G_1 *_A G_2$ with the biindex of A is 2 in both G_1 and G_2.*