On elementary extensions of free groups

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Formulas and Sentences

Definition 1

- A quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x} = \{x_1, \ldots, x_n\}$, is an expression

$$\varphi(\bar{x}) := \bigvee_{1 \leq i \leq n} \Big(\bigwedge_{1 \leq j \leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1 \leq j \leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

where w_{ij} and v_{ij} are words on the alphabet $\{x_1, \ldots, x_n\}^{\pm 1}$.

- A formula $\phi(\bar{z})$, with free variables $\bar{z}$, is an expression

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \varphi(x_1, y_1, \ldots, x_n, y_n, \bar{z}),$$

where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

Formulas and Sentences

- $\bar{g} \in G \implies \phi(\bar{x}; \bar{g})$ is a formula with parameters from G.
- Sentences := formulas without free variables.
- $G \models \phi \Leftrightarrow G$ satisfies ϕ.
- Theory:= a set of sentences.

Formulas and Sentences

Examples:

- $\phi(x) = \forall y([y, x] = 1)$,

$$G \models \phi(g) \Leftrightarrow g \in Z(G)$$

- $\phi = \forall x \forall y([x, y] = 1)$,

$$G \models \phi \Leftrightarrow G \text{ is abelian}$$

- $\phi_n = \forall x(x^n = 1 \Rightarrow x = 1)$, $n \geq 1$,

$$G \models \phi_n \text{ for any } n \geq 1 \Leftrightarrow G \text{ is torsion-free}$$

Elementary equivalence

First order theory:= set of sentences.

Example: G is a group,

$$Th(G) = \{\phi \text{ a sentence } | G \models \phi\},$$

$Th(G)$ is the *elementary theory* of G.

Elementary equivalence

- H and G are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.
- H is an *elementary subgroup* of G (or G is an *elementary extension* of H), if for any $\bar{h} \in H$, for any formula $\phi(\bar{x})$, if $G \models \phi(\bar{h})$ then $H \models \phi(\bar{h})$.

Notation:

- $H \equiv G$ for G and H are elementary equivalent.
- $H \preceq G$, H is an elementary subgroup of G.

Examples

- Two finite groups are elementarily equivalent if and only if they are isomorphic.
- $\mathbb{Z}^n \equiv \mathbb{Z}^m \Leftrightarrow n = m$.
- (Oger's Theorem) Two finitely generated nilpotent groups G and H are elementarily equivalent if and only if $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$.

Free groups

F_X the free group on X, $|X|$ the cardinal of X.

Theorem 1 (Kharlampovich-Myasnikov, independently Sela)

$F_n \preceq F_m$ for $m \geq n \geq 2$. □

Theorem 2 (Kharlampovich-Myasnikov)

$Th(F_n)$ is decidable for $n \geq 2$. □

Consequence:

- $F_X \preceq F_Y$ for $X \subseteq Y, |X| \geq 2$.
- $Th(F_X)$ is decidable.

Ultrapowers

An **ultrafilter** μ on a set I is a finitely additive measure on I with value in $\{0,1\}$. An ultrafilter is **non-principal** if the measure of any finite set is 0.
Given a group G, consider the set G^I and identify two sequences $(g_i)_{i\in I}$ and $(h_i)_{i\in I}$ if the set of indices i such that $g_i = h_i$ is of measure 1:

$$(g_i)_{i\in I} \sim (h_i)_{i\in I} \Leftrightarrow \mu(\{i \in I \mid g_i = h_i\}) = 1,$$

One defines naturally a product on $G^I/\sim$:

$$\hat{g} \cdot \hat{h} = ((g_i)_{i\in I}/\sim) \cdot ((h_i)_{i\in I}/\sim) = ((g_i h_i)_{i\in I}/\sim).$$

Then $(G^I/\sim, \cdot)$ is a group called the **ultrapower** of G (relative to μ and I) and it is denoted G^μ.

Properties

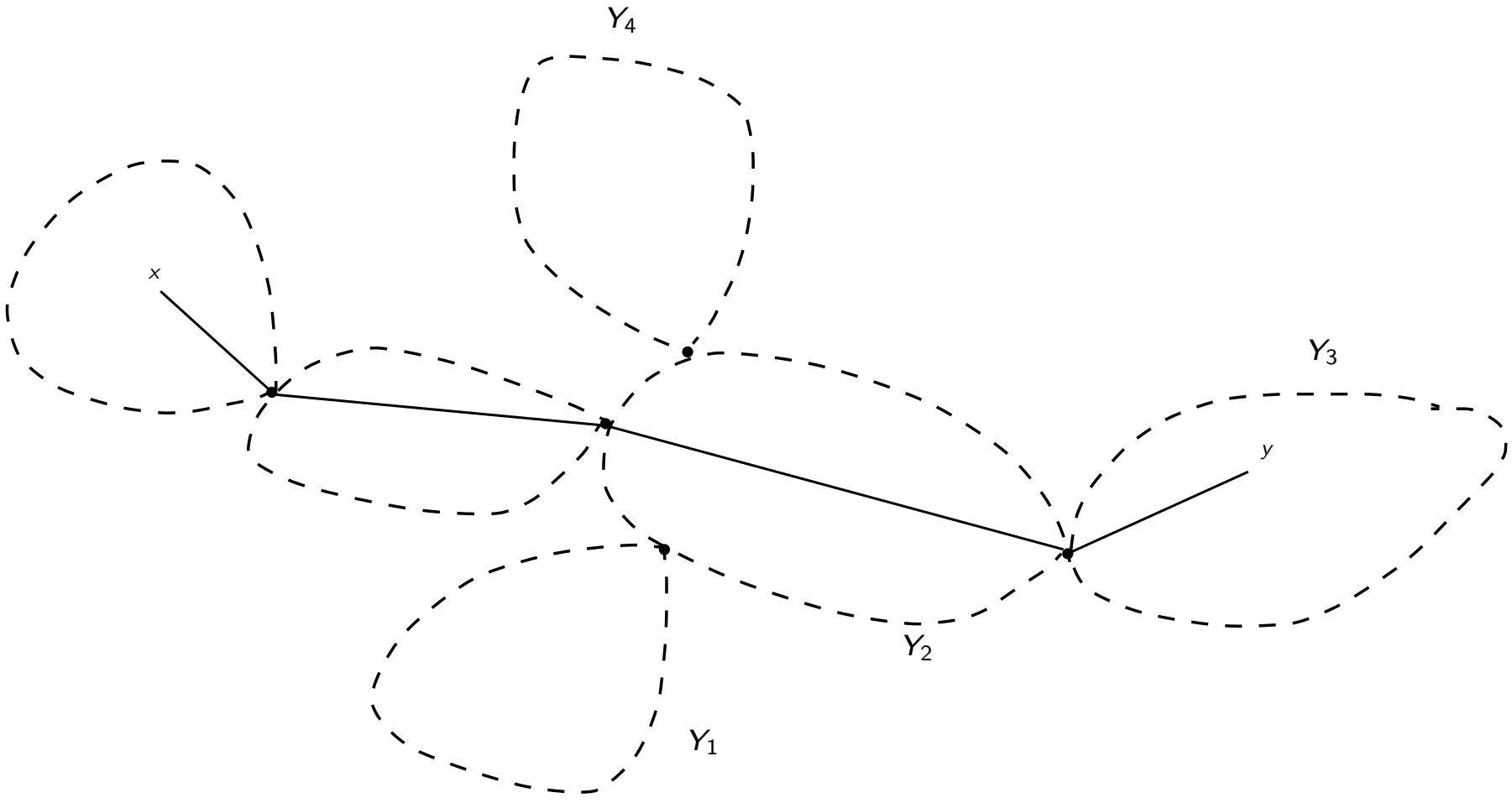
- (Los's Theorem) The natural embedding $g = (g_i = g)_{i \in I} \to g/\sim$ from G to G^μ is an elementary embedding; that is G^μ is an elementary extension of G.
- If $G \equiv H$ then H is elementary embeddable in some ultrapower G^μ of G.
- (Keislar-Shelah's Theorem) H and G are elementarily equivalent if and only if there exists a set I and a non-principal ultrafilter μ on I such that $H^\mu \simeq G^\mu$.

Actions on trees

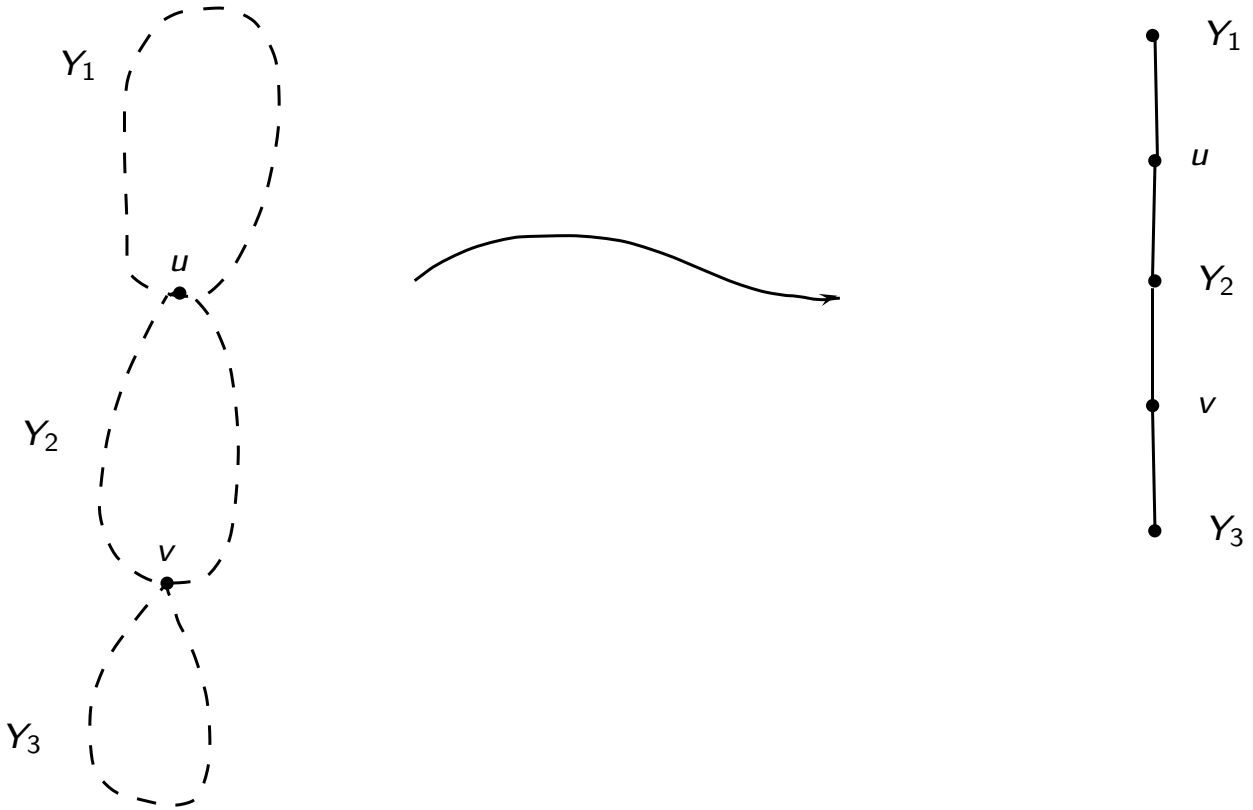
Let G be a group acting on a real tree. Suppose that X can be covered by a family of closed subtrees $\mathcal{Y} = (Y_i)_{i \in I}$ which satisfy:

- any segment of X is covered by finitely many subtrees of $\mathcal{Y}$,
- for $i \neq j$, $Y_i \cap Y_j$ contains at most one point,
- the family $\mathcal{Y}$ is invariant under the action of G.

Such a family is called a **transverse-covering**.


Y_4
x
Y_3
y
Y_2
Y_1

Then one can associate in a canonical way a simplicial tree T on which G acts and which confers to (G, X) a structure of a "graph of actions":



- The set of vertices $V(T)$ of T is the union $V_0(T) \cup V_1(T)$ where $V_0(T)$ is the set of subtrees of $\mathcal{Y}$ and $V_1(T)$ is the set of points x lying in the intersection of distinct subtrees of $\mathcal{Y}$,
- the set of positively oriented edges $E^+(T)$ is formed by tuples (x, Y_i), $x \in V_1(T)$, $Y_i \in V_0(T)$ such that $x \in Y_i$.

Then stabilizers of vertices of T are either of the form $Stab(x)$ where x lies in the intersection of distinct subtrees of $\mathcal{Y}$ or of the form $St(Y_i) =$ the *global* stabilizer of Y_i.

The action of $St(Y_i)$ on Y_i is called a **vertex action**.

Theorem 3 (Rips-Sela 94, Sela 98, Guirardel 2007)

Let G be a finitely generated freely indecomposable group acting nontrivially and minimally on a real tree X with abelian arc stabilizers and trivial tripod stabilizers.

Then X has a transverse-covering $\mathcal{X} = (X_v)_{v \in V}$ where each vertex action is either

- *of surface type*: the action of $ST(X_V)$ on X_v is dual to an arational measured foliation on a surface with boundary;
- *of axial type*: X_v is a line and $St(X_v)$ acts on X with dense orbits;
- *of simplicial type*: X_v is a simplicial tree and the action of $St(X_v)$ on X_v is simplicial.

Ultrapowers of metric spaces

Let (X, d) be a metric space and μ a non principal ultrafilter on I. In a similar way, one defines the ultrapower X^μ of X by setting, on X^I

$$\hat{x} = (x_i)_{i\in I} \sim \hat{y} = (y_i)_{i\in I} \Leftrightarrow \mu(\{i \in I \mid x_i = y_i\}) = 1,$$

and $X^\mu = X^I/\sim$.

A natural $\mathbb{R}^\mu$-distance is defined on X^μ:

$$d^\mu(\hat{x}, \hat{y}) = (d(x_i, y_i) \mid i \in I)/\sim .$$

If X is a real-tree, then X^μ is an $\mathbb{R}^\mu$-tree.

Actions

Let G be a group acting (by isometries) on a metric space (X, d). Then G^μ acts on X^μ in a natural way :

$$((g_i)_{i\in I}/\sim)\cdot((x_i)_{i\in I}/\sim) = (g_i\cdot x_i)_{i\in I}/\sim$$

The asymptotic cone from a nonstandard viewpoint

Given a nontrivial finite tuple $\bar{a}$ from F^μ and a point $\hat{e} \in X^\mu$ we rescal the distance of X^μ by $|\bar{a}|$ and we denote the new distance by d_1^μ, namely $d_1^\mu = d^\mu/|\bar{a}|$. We let

$$\hat{X}(\bar{a}) = \{\hat{x} \in X^\mu | \exists m_{\hat{x}} \in \mathbb{R} \text{ such that } d_1^\mu(\hat{x}, \hat{e}) \leq m_{\hat{x}}\}.$$

We endow $\hat{X}(\bar{a})$ with the pseudodistance

$$D(\hat{x}, \hat{y}) = st(d_1^\mu(\hat{x}, \hat{y})),$$

where here $st(r)$ denotes the standard part of the real r. We have $D(\hat{x}, \hat{y}) = 0$ if and only if $d_1^\mu(\hat{x}, \hat{y})$ is infinitisimal. We let d_μ to be the distance induced by D and we let $(Con_{\hat{e}}^\mu(\bar{a}), d_\mu)$ to be the corresponding metric space. We denote by $[\hat{x}]$, where $\hat{x} \in \hat{X}(\bar{a})$, the equivalence class of $\hat{x}$.

Bounded Hulls

Let $\bar{a} = (\hat{a}_1, \ldots, \hat{a}_p)$ be a finite tuple of elements of G^μ and let $\hat{e} \in X^\mu$. We set

$$|\bar{a}|_{\hat{e}} = \sum_{1 \leq i \leq p} d^\mu(\hat{e}, \hat{a}_i \hat{e}).$$

Definition 3

The *Bounded Hull* of $\bar{a}$ (with respect to $\hat{e}$) is the *subgroup* of G^μ defined by

$$B_{\hat{e}}(\bar{a}) = \{\hat{g} \in G^\mu \mid \exists n \in \mathbb{N} \text{ such that } |\hat{g}|_{\hat{e}} \leq n|\bar{a}|_{\hat{e}}\}.$$

Free groups

Let F be a nonabelian free group. Let (X, d) be its metric Cayley graph (i.e. the metric realization of its combinatorial Cayley graph T) with respect to a fixed basis. Then (X, d) is a real tree on which F acts freely and by isometries. This property transfers to the action of G^μ on X^μ.

Geometric properties

Lemma 1

Let $\bar{a}$ be a nontrivial finite tuple from F^μ and $\hat{e} \in X^\mu$. Then $Con^\mu_{\hat{e}}(\bar{a})$ is a real tree and by defining for $\hat{g} \in B_{\hat{e}}(\bar{a})$ and $[\hat{x}] \in Con^\mu_{\hat{e}}(\bar{a})$, where $\hat{x} \in \hat{X}(\bar{a})$,

$$\hat{g}[\hat{x}] = [\hat{g}\hat{x}]$$

we get an action of $B_{\hat{e}}(\bar{a})$ on $Con^\mu_{\hat{e}}(\bar{a})$ by isometries satisfying the following properties:

(i) the action is transitive;
(ii) stabilizers of arcs are abelian and stabilizers of tripods are trivial;
(iii) for any $\hat{g} \in B_{\hat{e}}(\bar{a})$, $\hat{g}$ fixes $\hat{e}$ if and only if $\hat{g} \preceq \bar{a}$.

Notation: $\hat{g} \preceq \bar{a} \Leftrightarrow \forall n, n|\hat{g}|_{\hat{e}} \leq |\bar{a}|_{\hat{e}}$.

Geometric properties

Lemma 2

Let $\bar{a}$ be a nontrivial finite tuple from F^μ and $\hat{e} \in X^\mu$. Let M be a subgroup of $B_{\hat{e}}(\bar{a})$ containing $\bar{a}$. If one of the following conditions holds

(1) $\hat{e} \in A_{\hat{a}_1} \cap \cdots \cap A_{\hat{a}_p}$;
(2) $St(\hat{e}) \cap M$ is nonabelian;

then the action of M on $Con^\mu_{\hat{e}}(\bar{a})$ is nontrivial and $\hat{e}$ belongs to the minimal M-invariant subtree of $Con^\mu_{\hat{e}}(\bar{a})$.

Geometric properties

Lemma 3

Let $\bar{a}$ be a nontrivial finite tuple from F^μ and $\hat{e} \in X^\mu$. Then the action of $B_{\hat{e}}(\bar{a})$ on $Con^\mu_{\hat{e}}(\bar{a})$ is nontrivial.

Geometric properties

Theorem 4

Let $\bar{a}$ be a nontrivial finite tuple from F^μ and let $\hat{e} \in X^\mu$. Let H be a subgroup of $B_{\hat{e}}(\bar{a})$, which is locally freely-indecomposable and whose action by restriction on $Con^\mu_{\hat{e}}(\bar{a})$ is nontrivial. Let X be the minimal H-invariant subtree of $Con^\mu_{\hat{e}}(\bar{a})$.

Then X has a transverse-covering $\mathcal{X} = (X_v)_{v \in V}$ where each vertex action is either of surface type, of axial type or of simplicial type (as in Theorem 3).

Geometric properties

Let X be a real tree.

- A *tree-covering* of X is a covering $\mathcal{X} = (X_v)_{v \in V}$ of X by closed subtrees such that for $u \neq v$, $X_u \cap X_v$ contains at most one point.
- If G acts on X then a *tree-covering* of X with respect to the action of G is a tree-covering $\mathcal{X} = (X_v)_{v \in V}$ which is G-invariant.

Theorem 5

Let $\bar{a}$ be a nontrivial finite tuple from F^μ and let $\hat{e} \in X^\mu$. Then $Con^\mu_{\hat{e}}(\bar{a})$ has a tree-covering $\mathcal{X} = (X_v)_{v \in V}$, with respect to the action of $B_{\hat{e}}(\bar{p})$, where each component-action is either

(1) of pseudo-simplicial type: $X_v = \overline{Y}$ and for any finitely generated subgroup $A \leq St(X_v)$, the action of A on the minimal A-invariant subtree $Y_A \subseteq Y$ is simplicial;
(2) of axial type: X_v is a line and $St(X_v)$ acts on X_v with dense orbits;
(3) of surface type: the action of $St(X_v)$ on X_v is dual to an arational measured foliation on a surface with boundary;
(4) of exotic type.

The existential case

Theorem 6

For any nontrivial finite tuple $\bar{p}$ from F^μ and $\hat{e} \in X^\mu$, $B_{\hat{e}}(\bar{p})$ is existentially closed in F^μ.
That is, for any *existential* formula $\exists \bar{y} \phi(\bar{x}; \bar{y})$, where here $\phi(\bar{x}; \bar{y})$ is quantifier-free, for any $\bar{a} \in B_{\hat{e}}(\bar{p})$, if $F^\mu \models \exists \bar{y} \phi(\bar{a}; \bar{y})$ then $B_{\hat{e}}(\bar{p}) \models \exists \bar{y} \phi(\bar{a}; \bar{y})$

The existential case

Sketch of the proof.

Let $\varphi(\bar{x};\bar{y})$ be a quantifier-free formula such that for some tuple $\bar{c}$ from $B_{\hat{e}}(\bar{p})$ we have $F^{\mu} \models \exists \bar{y} \varphi(\bar{c};\bar{y})$.

We show that for any $\bar{s} \in F^{\mu}$ such that $F^{\mu} \models \varphi(\bar{c},\bar{s})$, there exists $\bar{s}' \in F^{\mu}$ and an epimorphism $\tau : D(\bar{c})(\bar{s}) \to D(\bar{c})(\bar{s}')$ satisfying:

- τ is the identity on $D(\bar{c})$;
- $D(\bar{c})(\bar{s}') \models \varphi(\bar{c};\bar{s}')$;
- if $\bar{c} \preceq \bar{s}'$ then τ is proper.

Hence a descending chain $D(\bar{c})(\bar{s}_1) \to D(\bar{c})(\bar{s}_2) \to \cdots$, which must stops after finitely many steps (DCC on limit groups). Thus there exists s_n such that $F^{\mu} \models \varphi(\bar{c},\bar{s}_n)$ and $\bar{c} \not\preceq \bar{s}_n$. Hence $\bar{s}_n \in B_{\hat{e}}(\bar{p})$. □

The existential case

Let $\bar{a} \in F^\mu$ and $\phi(\bar{x}; \bar{y})$ be a formula such that $F^\mu \models \exists \bar{y} \phi(\bar{a}; \bar{y})$. Then there exists $\bar{s} \in F^\mu$ such that $|\bar{s}|_{\hat{e}}$ is minimal among all tuples $\bar{s}'$ such that $F^\mu \models \phi(\bar{a}; \bar{s}')$.

Such a realization is called a **minimal realization**.

Corollary 1

Any minimal realization of a quantifier-free formula $\phi(\bar{a}; \bar{y})$ belongs to $B_{\hat{e}}(\bar{a})$.

The existential case

Corollary 2 (Uniformity)

Let $\phi(\bar{x};\bar{y})$ be a quantifier-free formula. Then there exists a natural number n_ϕ such that for any tuple $\bar{a}$ from F, if $F \models \phi(\bar{a};\bar{b})$ for any tuple $\bar{b}$ from the ball of radius $n_\phi|\bar{a}|$, then $F \models \forall\bar{y}\phi(\bar{a};\bar{y})$.

The global case

Theorem (in progress!)

For any nontrivial finite tuple $\bar{p}$ from F^μ and $\hat{e} \in X^\mu$, $B_{\hat{e}}(\bar{p})$ is an elementary subgroup of F^μ.