Combination for equationally noetherian groups

Abderezak Ould Houcine

Universit¬édeMons, Universit¬éLyon1

Model Theory of Groups
CIRM, 22 november 2011

Contents

1 Equationally noetherian groups
- Definition
- Groups acting on trees
- Equivalent definitions of equational noetherianty

2 The context
- The context from a nonstandard view point
- The context
- Limit spaces
- Actions of strict restricted Γ-limit groups
- Actions of bounded restricted Γ-limit groups

3 Shortening quotients
- Construction
- Shortening quotients

4 The final step : DCC & the factorisation property

Let G be a group. Let $\bar{x} = \{x_1, \ldots, x_n\}$ be a finite set of variables.
Let $G\langle\bar{x}\rangle$ to be the free product $G * \langle x_1, \ldots, x_n|\rangle$.
Let $S \subseteq G\langle\bar{x}\rangle$.The set

$$V_G(S) = \{\bar{g} \in G^n | w(\bar{g}) = 1, \forall w \in S\}$$

is called the *algebraic variety* defined by S.

Definition 1 (Baumslag, Myasnikov, Remeslennikov)

A group G is called equationally noetherian if for any $n \geq 1$, for any set $S \subseteq G\langle\bar{x}\rangle$, there exists a finite subset S_0 such that $V_G(S) = V_G(S_0)$. □

- Finitely generated abelian-by-nilpotent groups (Bryant, 1977)
- Free groups (Guba, 1986). More generally linear groups (and abelian groups) (Baumslag, Myasnikov, Remeslennikov, 1999)
- Torsion-free hyperbolic groups (Sela, 2002)
- Torsion-free groups which are hyperbolic relative to a collection of free abelian subgroups (Groves, 2005)
- Hyperbolic groups (Reinfeldt and Weidmann, 2010)
- The free product of two equationally noetherian groups is equationally noetherian (Sela, 2010)

Definition 4

Let $\mathcal{A} = (A_i)_{i \in I}$ be a familly of groups. We say that $\mathcal{A}$ is uniformly equationally noetherian if :

- *Each A_i is equationally noetherian*
- *For any $n \geq 1$, for any system of equations $S \subseteq \langle x_1, \ldots, x_n | \rangle$, there exists a finite subsystem $S_0 \subseteq S$ such that*

$$V_{A_i}(S) = V_{A_i}(S_0) \text{ for any } i \in I.$$

Main Theorem

Let Γ be a group acting on simplicial tree X. Suppose that

- *The familly of stabilizers of edges is boundedly finite*
- *The familly of stabilizers of vertices is uniformly equationally noetherian*

Then Γ is equationally noetherian.

Remark. The second condition is a necessary condition.

Corollary 1

Let Γ be a group acting on a simplicial tree X. Suppose that X/G is finite, stabilizers of edges are finite and stabilizers of vertices are equationally noetherian. Then Γ is equationally noetherian.

We get also a general version of equational noetherianty of free products.

Corollary 2

Let $\mathcal{A} = (A_i)_{i \in I}$ be a familly of groups. Then the free product $*_{i \in I} A_i$ is equationally noetherian if and only if $\mathcal{A}$ is uniformly equationally noetherian.

Example: If $(A)_{i \in I}$ is a familly of equationally noetherian groups such that the set $\{Th_\forall(A_i) : i \in \mathbb{N}\}$ is finite then $*_{i \in I} A_i$ is equationally noetherian.

Lemma 1

A group is equationally noetherian if and only if all of its countable subgroups are equationally noetherian.

Proof. An application of L¬*öwenheim*√*êSkolemtheorem*. □

Γ-limit groups

Definition 2

- *A Γ-group G is a group with a fixed embedded copy of Γ. A Γ-homomorphism between two Γ-groups G_1 and G_2 is a homomorphism identifying images of Γ (pointwise).*
- *Let μ be a nonprincipal ultrafilter over $\mathbb{N}$. Let Γ^μ be a nonprincipal ultrapower of Γ. A restricted Γ-limit group L is a subgroup of Γ^μ such that there exists a finite set A such that L is generated by $\Gamma \cup A$. It has a natural structure of a Γ-group.*
- *For a sequence $\hat{f} = (f_n : \Gamma\langle\bar{x}\rangle \to \Gamma)_{n\in\mathbb{N}}$ of Γ-homomorphisms, we let $\ker_\mu(\hat{f}) = \{g \in \Gamma\langle\bar{x}\rangle | \mu(\{n | f_n(g) = 1\}) = 1\}$.*

Lemma 3

Suppose that Γ is countable. A group L is a restricted Γ-limit group if and anly if there exists a sequence $\hat{f} = (f_n : \Gamma\langle\bar{x}\rangle \to \Gamma)_{n\in\mathbb{N}}$ of Γ-homomorphisms such that $L = \Gamma\langle\bar{x}\rangle/ker_\mu(\hat{f})$. □

Remark.

- To get the same equivalence for a general Γ, one may replace $\mathbb{N}$ be an adequate set I and adequate ultrafilter μ related to the cardinality of Γ.
- The Γ-homomorphism $\hat{f} : \Gamma\langle\bar{x}\rangle \to \Gamma^\mu$ defined by

$$\hat{f}(g) = (f_n(g)|n \in \mathbb{N})$$

is a Γ-homomorphism whose kernel is $\ker_\mu(\hat{f})$.

Definition 3

Let H be a finitely generated Γ-group. Let $K(H)$ be the set of tuples (f, L) where $f : H \to L$ is a Γ-epimorphism and L is a restricted Γ-limit group

- *We define an equivalence relation on $K(H)$ by $(f_1, L_2) \equiv (f_2, L_2)$ if there exists a Γ-isomorphism $\tau : L_1 \to L_2$ such that $f_2 = f_1 \circ \tau$.*
- *On $\mathcal{K}(H) = K(H)/\equiv$, we define a partial order by $(f_1, L_1) > (f_2, L_2)$ if there exists a non-injective Γ-epimorphism $\tau : L_1 \to L_2$ such that $f_2 = f_1 \circ \tau$.*

Lemma 4

If Γ is countable then $\mathcal{K}(H)$ has maximal elements. □

Lemma 5

Suppose that Γ is countable. The following properties are equivalent.
(1) *Γ is equationally noetherian.*
(2) *The following properties are satisfied :*
(*i*) *For any finitely generated Γ-group H, $\mathcal{K}(H)$ has only finitely many maximal elements.*
(*ii*) *Any descending chain of restricted Γ-limit groups is stationary.* □

Remark. Again, to get the same properties for a general Γ, one may replace $\mathbb{N}$ be an adequate set I and adequate ultrafilter μ related to the cardinality of Γ.

Let Γ be a group acting on a simplicial tree X. Let μ be a nonprincipal ultrafilter on $\mathbb{N}$. Let Γ^μ be the ultrapower of Γ and X^μ be the ultrapower of X, relative to μ. The distance of X^μ will be denoted d^μ, it taks its values in the ultrapower $\mathbb{Z}^\mu$. Thus (X^μ, d^μ) is a $\mathbb{Z}^\mu$-tree on which Γ^μ acts by isometries and without inversions in the obvious way.
For every $g \in \Gamma^\mu$, we let

$$\ell^\mu(g) = \inf_{x \in X^\mu} d^\mu(x, gx),$$

the translation length. Since X^μ is $\mathbb{Z}^\mu$-tree, the above length function is well-defined and there exists $e \in X^\mu$ such that $\ell^\mu(g) = d(e, ge)$.

Definition 5

Let L be a finitely generated subgroup of Γ^μ (i.e. L is a "pure" Γ-limit group).
(2) *We say that L is bounded, if for any $g \in L$, $\ell^\mu(g) \in \mathbb{Z}$.*
(3) *We say that L is strict, if there exists $g \in L$ such that $\ell^\mu(g) \in \mathbb{Z}^\mu \setminus \mathbb{Z}$.* □

Remark. If Γ is free and X is the Cayley graph of Γ, then "strict" in the previous definition coincides with usual one.

Let Γ be a group acting on a simplicial tree X satisfying conditions given in the main theorem. Since such conditions are inheritad by passing to subgroups, and by Lemma 1, we may assume that Γ is countable.
The action of Γ on X is in fact $(N, 1)$-acylindrical in the sense of Delzant and Weidmann, where N is the uniform bound on the cardinality of stabilizers of edges.
To show that Γ is equationally noetherian, by Lemma 5, one must show that

- For every finitely generated Γ-group H, $\mathcal{K}(H)$ has only finitely many maximal elements
- Any descending chain of restricted Γ-limit groups is stationary

We will discuss in the sequel only the second property.

Let $\hat{f} = (f_i : \Gamma\langle\bar{x}\rangle \to \Gamma)_{i\in\mathbb{N}}$ be a sequence of Γ-homomorphisms. We denote by $\Gamma(\hat{f})$ the corresponding restricted Γ-limit group $\Gamma\langle\bar{x}\rangle/\ker_\mu(\hat{f})$.
Let $e \in X$ be a fixed point and let $\gamma \in \Gamma$ be a fixed hyperbolic element. For every $i \in \mathbb{N}$, we let

$$l_i = \max\{d(e, f_i(x_j)e), d(e, \gamma e) | 1 \leq j \leq p\} \geq d(e, \gamma e) > 0$$

and we set $\hat{l} = (l_i)_{i\in\mathbb{N}}$.

Definition 6

$\Gamma(\hat{f})$ is called bounded if $\lim_\mu \hat{l} < \infty$ and it is called strict if $\lim_\mu \hat{l} = \infty$.

Let (X, d) be a metric space. We let

(1) $\hat{l} = (l_i)_{i\in\mathbb{N}}$ to be a sequence of non negative real numbers.

(2) $\hat{e} = (e_i)_{i\in\mathbb{N}}$ to be a sequence of X.

We let

$$X_\infty = \{\hat{x} \in X^{\mathbb{N}} | \exists C \geq 0 \text{ such that } \mu(\{n \in \mathbb{N} | d(e_n, x_n) \leq C l_n\}) = 1\}.$$

We define an equivalence relation $\approx$ on X_∞ by

$$\hat{x} \approx \hat{y} \text{ if and only if } \lim_\mu \frac{d(x_n, y_n)}{l_n} = 0.$$

We call the quotient space $X_\infty/\approx$ with the metric

$$d_\mu(\hat{x}, \hat{y}) = \lim_\mu \frac{d(x_n, y_n)}{l_n}$$

the *limit space* of X with respect to μ, to the base points $(e_n)_{n\in\mathbb{N}}$ and to the sequence of scalars $(l_n)_{n\in\mathbb{N}}$ and we denote it by $C^\mu(X, \hat{e}, \hat{l})$.

Definition 7

The space $C^{\mu}(X, \hat{e}, \hat{l})$ is called the asymptotic cone of X, with respect to μ, $\hat{e}$ and $\hat{l}$ if $\lim_{\mu} l_n = \infty$; and in that case we denote it by $Con^{\mu}(X, \hat{e}, \hat{l})$. Otherwise, i.e when $\lim_{\mu} l_n < \infty$, we call $C^{\mu}(X, \hat{e}, \hat{l})$ the bounded limit space of X with respect to μ, $\hat{e}$ and $\hat{l}$. □

Lemma 6

Let (X, d), μ, $\hat{e}$ and $\hat{l}$ as above and denote by C the limit space $C^{\mu}(X, \hat{e}, \hat{l})$.
(1) *If X is a real tree then so is C.*
(2) *Suppose that $0 < l = \lim_{\mu} l_n < \infty$. If X is a simplicial tree then so is C. If for any $n, m \in \mathbb{N}$, $e_n = e_m$, then the map $\alpha : X \to C$ defined by $\alpha(x) = (x_n = x)_{n \in \mathbb{N}}$ is an $(1/l)$-isometric embedding.*

Let N be the uniform bound on the cardinality of stabilizers of edges.

Proposition 1

If $\Gamma(\hat{f})$ is strict then it acts on the asymptotic cone $C^{\mu}(X, e, \hat{l})$ such that :

- *(non degenerate) arc stabilizers are (N-finite)-by-(torsion-free abelian) and (non degenerate) tripod stabilizers are N-finite.*
- *Γ fixes the base point e.*
- *If Γ is not (N-finite)-by-abelian-by-finite then the action of $\Gamma(\hat{f})$ is non trivial.*
- *Also other technical properties ....*

Using Guirardel's version of Rips and Sela decompositions, either

(1) $\Gamma(\hat{f})$ splits over an N-finite group (relative to Γ), or

(2) $\Gamma(\hat{f})$ has a graph of actions $(G_v, T_v)_{v \in V}$ (relative to Γ) such that each vertex group action is one of the following type :

- simplicial type : T_v is simplicial and the action of G_v on T_v is simplicial.
- axial type : T_v is a line and G_v acts on T_v with dense orbits and the image of G_v in $Isom(\mathbb{R})$ is finitely generated.
- surface (or Seifert) type : the action of G_v on T_v has kernel N_v, which is N-finite, and the faithful action of G_v/N_v on T_v is dual to an arational measured foliation on a closed 2-orbifold with boundary.

To deal with case (1) one must study splitting (relative to Γ) over N-finite groups.

Definition 9

Let L be a countable group which is finitely generated relative to a subgroup Γ. We say that L is N-one-ended relative to Γ, if L does not admit any weakly reduced splitting relative to L, where every edge stabilizer is N-finite. □

Theorem 1

Let L be a countable group which is finitely generated relative to a subgroup Γ. Let $\mathcal{C}$ be the class of finite groups of cardinality at most N. Then there exists a weakly reduced $(\mathcal{C}, \Gamma)$-splitting of L, that we call the relative Dunwoody decomposition, satisfying the following properties:
(1) *The vertex group containing Γ is finitely generated and N-one-ended relative to Γ.*
(2) *The other vertices groups are finitely generated and N-one-ended (relative to the trivial group).* □

So, we can restrict our attention to N-one-ended restricted Γ-limit groups.

Proposition 2

Suppose that $\Gamma(\hat{f})$ *is bounded. Then :*
(1) $C^{\mu}(X, \hat{l})$ *is a simplicial tree with a natural* $(1/l)$*-isometric embedding* $i : X \to C^{\mu}(X, \hat{l})$.
(2) $\Gamma(\hat{f})$ *acts on* $C^{\mu}(X, \hat{l})$ *in such a way that for any point* $\hat{x}$, $St(\hat{x})$ *is a model of the universal theory of* $\cap_{n\in\mathbb{N}} St(x_n)$ *and the stabilizer of any edge is N-finite.*

Important consequence : If $\Gamma(\hat{f})$ is bounded then it has a graph of groups decomposition such that

- stabilizers of edges are N-finite.
- a stabilizer $G_{\hat{v}}$ of a vertex $\hat{v}$ is one of the following type :
  - either it is finitely generated and satisfies the universal theory $\cap_{v \in V(X)} Th_{\forall}(St(v))$
  - or there exists a point $v \in V(X)$ such that G_v contains $St(v)$ and it is a $St(v)$-limit group.

The factorization property

Definition 10

Let $\Gamma(\hat{f})$ be a restricted Γ-limit group. We say that $\Gamma(\hat{f})$ has the factorization property if there exists a set $I \subseteq \mathbb{N}$ such that $\mu(I) = 1$ and for any $n \in I$ there exists a Γ-homomorphism $\sigma_n : \Gamma(\hat{f}) \to \Gamma$ such that $f_n = \sigma_n \circ \hat{f}$.

Proposition 3

If $\Gamma(\hat{f})$ is bounded then it has the factorization property. □

Definition 11

Let L be a finitely generated Γ-group. A modular filtration of L is a sequence $MF = (L_n, \varphi_n)_{n\in\mathbb{N}}$, where

(1) *L_n is a finitely generated Γ-group,*
(2) *$\varphi_n : L_n \to L_{n+1}$ is a Γ-homomorphism,*

such that the following properties hold

(i) L is the direct limit of the sequence $(\varphi_n : L_n \to L_{n+1})_{n\in\mathbb{N}}$.
(ii) Let $\hat{\varphi}_n : L_n \to L$ be the natural homomorphism. For any $\sigma \in Mod_\Gamma(L)$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, there exists $\hat{\sigma}_n \in End_\Gamma(L_n)$ such that $\varphi_n \circ \hat{\sigma}_n = \sigma \circ \hat{\varphi}_n$.

Remark We note that we may always assume that $L_0 = \Gamma[\bar{x}]$. We will assume that in the rest of this section. We denote also by $\phi_n : L_0 \to L_n$ the natural homomorphism.

Lemma 7

Any finitely generated Γ-group has a modular filtration. □

Definition 12

Let L be a restricted Γ-limit group and let $MF = (L_n, \varphi_n)_{n\in\mathbb{N}}$ be a modular filtration.
(1) *Let $\hat{f} = (f_n : L_n \to \Gamma)_{n\in\mathbb{N}}$ be a sequence of Γ-homomorphisms. We say that $\hat{f}$ is short, if for any $n \in \mathbb{N}$, for any $\tau \in End_\Gamma(L_n)$*

$$\max_{1\leq i\leq p} d(e, f_n \circ \tau \circ \phi_n(x_i)e) \leq \max_{1\leq i\leq p} d(e, f_n \circ \phi_n(x_i)e).$$

(2) *Let $\hat{f} = (f_n : L_n \to \Gamma)_{n\in\mathbb{N}}$ be a short sequence of Γ-homomorphisms. Let $\hat{h} = (h_n = f_n \circ \phi_n : \Gamma[\bar{x}] \to \Gamma)$ be the associated sequence of Γ-homomorphisms. Suppose that $\hat{h}$ is convergente and that the associated restricted Γ-limit group $\Gamma(\hat{h})$ is strict. Then $\Gamma(\hat{h})$ is called a shortening quotient of L.* □

Theorem 2

Any shortening quotient of a N-one-ended (relative to Γ) restricted Γ-limit group is a proper quotient. □

Suppose that there exists an infinite descending chain $L_1 > L_2 > \cdots$ of restricted Γ-limit groups that does not stabilize. Then one can construct a restricted Γ-limit group $\Gamma(\hat{f})$ satisfying:

- $\Gamma(\hat{f})$ does not satisfy the factorization property
- andy descending chain $\Gamma(\hat{f}) > L_1 > L_2 \cdots$ of restricted Γ-limit groups terminates.

Then by taking a sequence $\Gamma(\hat{f}) > L_1 > L_2 \cdots$ of shortening quotients terminates. Hence the last group is a bounded restricted Γ-limit group and thus has the factorization property. Hence $\Gamma(\hat{f})$ has also the factorization property, a contradiction.