Homogeneity and algebraic closure in free groups

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Equations and First-order properties in Groups

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Remark. $\exists$-homogeneity $\implies$ homogeneity.

Prime Models

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- $\mathcal{M}$ is called *prime* if $\mathcal{M}$ is elementary embeddable in every model of $Th(\mathcal{M})$.

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Indeed, we can take $\varphi(x, y) := [x, y] \neq 1$

Two-generated torsion-free hyperbolic groups

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That is a group is co-hopfian if it does not contain a subgroup isomorphic to itself.

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- $Out(F_n)$, where F_n is a free group of rank n (B. Farb, M. Handel, 2007).

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A non-free two-generated torsion-free hyperbolic group is strongly co-hopfian.

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Let F be a free group and let $\bar{a} = (a_1, \ldots, a_m)$ be a tuple from F. We say that $\bar{a}$ is a power of a primitive element if there exist integers $p_1, \ldots, p_m$ and a primitive element u such that $a_i = u^{p_i}$ for all i.

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- *the tuple $\bar{a}$ has the same existential type as a power of a primitive element;*
- *there exists an e.c. subgroup $E(\bar{a})$ (resp. $E(\bar{b})$) containing P and $\bar{a}$ (resp. $\bar{b}$) and an isomorphism $\sigma : E(\bar{a}) \to E(\bar{b})$ fixing pointwise P and sending $\bar{a}$ to $\bar{b}$.*

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Lemma

Let G be a finitely generated equationally noetherian group. Let P be a subset of G. Then there exists a finite subset $P_0 \subseteq P$ such that for any endomorphism f of G, if f fixes pointwise P_0 then f fixes pointwise P.

Hence: $\exists f \in Aut(F|P)$ s.t. $f(\bar{a}) = f(\bar{b}) \Leftrightarrow \exists f \in Aut(F)$ s.t. $f(\bar{a}P_0) = f(\bar{b}P_0)$.

Sketch of proof of Theorem 3

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Hence: $\exists f \in Aut(F|P)$ s.t. $f(\bar{a}) = f(\bar{b}) \Leftrightarrow \exists f \in Aut(F)$ s.t. $f(\bar{a}P_0) = f(\bar{b}P_0)$.
Note that : If $tp_\exists(\bar{a}|P) = tp_\exists(\bar{b}|P)$ then $tp_\exists(\bar{a}|P_0) = tp_\exists(\bar{b}|P_0)$ and $tp_\exists(\bar{a}P_0) = tp_\exists(\bar{b}P_0)$.

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Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2).

Sketch of proof of Theorem 3

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Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2). We say that $(\bar{a}, \bar{b})$ is existentially rigid, if there is no nontrivial free decomposition $F_1 = A * B$ such that A contains a tuple $\bar{c}$ with $tp_{\exists}^{F_1}(\bar{a}) \subseteq tp_{\exists}^{A}(\bar{c}) \subseteq tp_{\exists}^{F_2}(\bar{b})$.

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Then there exists a quantifier-free formula $\varphi(\bar{x}, \bar{y})$, such that $F_1 \models \varphi(\bar{a}, \bar{s})$ and such that for any $f \in Hom(F_1|\bar{a}, F_2|\bar{b})$, if $F_2 \models \varphi(\bar{b}, f(\bar{s}))$ then f is an embedding.

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Let F_1 and F_2 be nonabelian free groups of finite rank and $\bar{a}$ (resp. $\bar{b}$) a tuple from F_1 (resp. F_2) such that $tp_{\exists}^{F_1}(\bar{a}) = tp_{\exists}^{F_2}(\bar{b})$.

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Let F_1 and F_2 be nonabelian free groups of finite rank and $\bar{a}$ (resp. $\bar{b}$) a tuple from F_1 (resp. F_2) such that $tp_{\exists}^{F_1}(\bar{a}) = tp_{\exists}^{F_2}(\bar{b})$. Suppose that $(\bar{a}, \bar{b})$ is existentially rigid.

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Suppose that $(\bar{a}, \bar{b})$ is existentially rigid.
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Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp_{\exists}^{F}(\bar{a}) = tp_{\exists}^{F}(\bar{b})$.

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Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp_\exists^F(\bar{a}) = tp_\exists^C(\bar{c})$.

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By Propsition (2), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{c}, F|\bar{a})$ (resp. $h_2 \in Hom(C|\bar{c}, F|\bar{b})$) such that $h_1(C)$ (resp. $h_2(C)$) is an e.c. subgroup of F.

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Suppose that $\bar{c}$ is not a power of a primitive element. By setting $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$, we have $h_2 \circ h_1^{-1} : E(\bar{a}) \rightarrow E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$. $\square$

Homogeneity in free groups

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Let F be a nonabelian free group of finite rank. For any tuples $\bar{a}, \bar{b} \in F^n$ and for any subset $P \subseteq F$, if $tp^F(\bar{a}|P) = tp^F(\bar{b}|P)$ then there exists an automorphism of F fixing pointwise P and sending $\bar{a}$ to $\bar{b}$.

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The above theorem is also proved by Perin and Sklinos.

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Theorem 5

A nonabelian free factor of a free group of finite rank is an elementary subgroup. □

Sketch of proof of Theorem 4

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Let $\varphi(\bar{x})$ be a formula. Then there exists a boolean combination of $\exists\forall$-forumulas $\phi(\bar{x})$, such that for any nonabelian free group F of finite rank, one has $F \models \forall\bar{x}(\varphi(\bar{x}) \Leftrightarrow \phi(\bar{x}))$. □

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We notice, in particular, that if $\bar{a}, \bar{b} \in F^n$ such that $tp^F_{\exists\forall}(\bar{a}) = tp^F_{\exists\forall}(\bar{b})$, then $tp^F(\bar{a}) = tp^F(\bar{b})$.

Sketch of proof of Theorem 4

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Theorem 7 (C. Perin, 2008)

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Theorem 8 (A. Pillay, 2009)

Let F be a nonabelian free group of finite rank and $u, v \in F$ such that $tp^F(u) = tp^F(v)$. If u is primitive, then v is primitive. □

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Let F be nonabelian free group of finite rank and let $\bar{a}$ be a tuple of F. We say that $\bar{a}$ is rigid if there is no nontrivial free decomposition $F = A * B$ such that A contains a tuple $\bar{c}$ with $tp^{F_1}_{\exists\forall}(\bar{a}) = tp^{A}_{\exists\forall}(\bar{c})$.

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Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2) such that $tp^{F_1}_{\exists\forall}(\bar{a}) = tp^{F_2}_{\exists\forall}(\bar{b})$.

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Then either $rk(F_1) = 2$ and $\bar{a}$ is a power of a primitive element, or there exists an embedding $h \in Hom(F_1|\bar{a}, F_2|\bar{b})$ such that $h(F_1) \preceq_{\exists\forall} F_2$.

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Then $\bar{c}$ is rigid.
By Propsition (3), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{a}, F|\bar{b})$ (resp. $h_2 \in Hom(C|\bar{a}, F|\bar{b})$) such that $h_1(C) \preceq_{\exists\forall} F$ (resp. $h_2(C) \preceq_{\exists\forall} F$).

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Then $\bar{c}$ is rigid.
By Propsition (3), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{a}, F|\bar{b})$ (resp. $h_2 \in Hom(C|\bar{a}, F|\bar{b})$) such that $h_1(C) \preceq_{\exists\forall} F$ (resp. $h_2(C) \preceq_{\exists\forall} F$).

If $\bar{c}$ is a power of a primitive element then the result follows from Theorem 8.

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We have $h_2 \circ h_1^{-1} : E(\bar{a}) \to E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$.

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Since $E(\bar{a}) \preceq_{\exists\forall} F$ (resp. $E(\bar{b}) \preceq_{\exists\forall} F$) we get by Theorem 7 that $E(\bar{a}) \preceq F$ (resp. $E(\bar{b}) \preceq F$).

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By Theorem 7, $E(\bar{a})$ (resp. $E(\bar{b})$) is a free factor of F.
Therefore $h_2 \circ h_1^{-1}$ can be extended to an isomorphism of F. $\square$

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Definition

Let G be a group and A a subset of G.

- *The algebraic closure of A, denoted $acl_G(A)$, is the set of elements $g \in G$ such that there exists a formula $\phi(x)$ with parameters from A such that $G \models \phi(g)$ and $\phi(G)$ is finite.*
- *The definable closure of A, denoted $dcl_G(A)$, is the set of elements $g \in G$ such that there exists a formula $\phi(x)$ with parameters from A such that $G \models \phi(g)$ and $\phi(G)$ is a singleton.*

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- If $\Gamma = \Gamma_1 * \Gamma_2$ and $A \leq \Gamma_1$ then $acl(A) \leq \Gamma_1$. Similarly for $dcl(A)$. Hence, we may assume that Γ is freely A-indecomposable.
- If A is abelian then $acl(A) = dcl(A) = C_\Gamma(A)$. Hence , we may assume that A is nonabelian.

Theorem 9

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Let Γ be a torsion-free hyperbolic group and $A \leq \Gamma$ where A is nonabelian.
Then Γ can be constructed from $acl(A)$ by a finite sequence of free products and HNN-extensions along cyclic subgroups.
In particular, $acl(A)$ is finitely generated, quasiconvex and hyperbolic. □

Acl and the JSJ-decomposition

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Let F be a nonabelian free group of finite rank and let $A \leq F$ be a nonabelian subgroup. Then:

- *$dcl(A)$ is a free factor of $acl(A)$.*

Acl and the JSJ-decomposition

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Let F be a nonabelian free group of finite rank and let $A \leq F$ be a nonabelian subgroup. Then:

- $dcl(A)$ *is a free factor of* $acl(A)$. *In particular, if* $rk(F) = 2$ *then* $acl(A) = dcl(A)$.

Acl and the JSJ-decomposition

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Let F be a nonabelian free group of finite rank and let $A \leq F$ be a nonabelian subgroup. Then:

- $dcl(A)$ *is a free factor of* $acl(A)$*. In particular, if* $rk(F) = 2$ *then* $acl(A) = dcl(A)$*.*
- $acl(A)$ *is exactly the vertex group containing* A *in the cyclic JSJ-decomposition of* F *relative to* A*.*

A counterexample

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Theorem 11

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Let A_0 be a finite set (possibly empty)

A counterexample

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