Embeddings in finitely presented groups which preserve the center

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where $P(X)$ is a set of words over $X^{\pm 1}$, is **recursively presented** if $P(X)$ is recursively enumerable.

G it said to have a **soluble word problem** if the set $\{w | w = 1 \text{ in } G\}$ is recursive.

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Consequences

• A countable recursively presented group is embeddable in a finitely presented group.
• A countable abelian group is embeddable in a finitely presented group. In fact there exists a finitely presented group which embeds every countable abelian group, because every countable abelian group is embbedable in the group

$$\mathbb{Q}^{(\aleph_0)} \oplus (\oplus_{i\in\mathbb{N}}\mathbb{Z}[\pi_i^{\infty}]^{(\aleph_0)}),$$

which has a soluble word problem.

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There are a number of important embedding theorems of finitely generated groups into finitely presented groups, due to Baumslag, Birget, Olshanskii, Rips, Sapir.

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Theorem II *A countable abelian group is the center of a finitely presented group with* **soluble word problem** *if and only if it has a soluble word problem.*

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(1) *If G is recursively presented and $Z(G)$ is recursively enumerable in G, then we can take K recursively presented.*

(2) *If G has a soluble word problem and $Z(G)$ has a generalized soluble word problem in G then we can take K with soluble word problem.*

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For every $i \in \mathbb{N}$ put

$$w_i = a_i^{-1}.(x_1 y)^{80i+1}(x_2 y)(x_1 y)^{80i+2}$$
$$(x_2 y)(x_1 y)^{80i+3}(x_2 y) \cdots (x_1 y)^{80i+80}(x_2 y).$$

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And $Z(K/(R)) = \pi(Z(K)) = \pi(Z(G))$. $\square$

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Let $\{a_1, \ldots ,a_n\}$ be a generating set of G. The group G_0 is generated by the set $\{a_1, \ldots ,a_n,z\}$.
Hence there is an isomorphism $\nu : \mathbb{F}_X/R \cong G_0$, where $\mathbb{F}_X$ is the free non-abelian group of rank $n+1$ with basis $X = \{x_1, \ldots ,x_n,x_{n+1}\}$, and R is the normal closure of the presentation (which is recursively enumerable) of G_0 and ν satisfies: $\nu(\overline{x_i}) = a_i$, for $i = 1, \ldots ,n$ and $\nu(\overline{x_{n+1}}) = z$ where $\overline{x_i}$ is the class of x_i relative to the subgroup R.

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Without loss of generality we can assume that $x_1, \ldots ,x_n,x_{n+1}$ are included among the generating symbols of the given presentation of H.

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Define a homomorphism : $\phi : L \to G_0$ by $\phi(w) = \overline{w}$ and $\phi(d^{-1}wd) = 1$.(Since the two definitions agree on the amalgamated part, ϕ is well-defined.)

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Consider the group $H \times G_0$.
Define a map $\psi : L \times Z(G_0) \to H \times G_0$ by

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Hence we can form the HNN-extension:

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$$\Gamma = K *_G G \times \langle t|\rangle.$$

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Let

$$r = (s^{-1}z) \cdot t \cdot z \cdot t^2 \cdot z \cdot t^3 \cdot z \cdots z \cdot t^{80}$$

Let N be the normal closure of $\{r\}$ in Γ.

Then $\overline{\Gamma} = \Gamma/N$ satisfies:

- $\overline{\Gamma}$ is finitely presented,
- $Z(\overline{\Gamma}) = Z(G_0) = Z(G)$.