Homogeneity of nonabelian free groups

Abderezak Ould Houcine

Université de Mons, Université Lyon 1

Vanderbilt, 19 april 2011

Contents

1 Existential homogeneity & prime models
- Definitions
- Existential homogeneity & prime models

2 Existential homogeneity in free groups
- Homogeneity in free groups

Contents

1 Existential homogeneity & prime models
- Definitions
- Existential homogeneity & prime models

2 Existential homogeneity in free groups
- Homogeneity in free groups

Homogeneity & existential homogeneity

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$.

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous*

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$,

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.
- $\mathcal{M}$ is $\exists$-*homogeneous*

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.
- $\mathcal{M}$ is $\exists$-*homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$,

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.
- $\mathcal{M}$ is $\exists$*-homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}_{\exists}(\bar{a}) = tp^{\mathcal{M}}_{\exists}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.
- $\mathcal{M}$ is $\exists$*-homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}_{\exists}(\bar{a}) = tp^{\mathcal{M}}_{\exists}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.

Remark.

Homogeneity & existential homogeneity

Let $\mathcal{M}$ be a countable model.

- Let $\bar{a} \in \mathcal{M}^n$. The type of $\bar{a}$ is defined by

$$tp^{\mathcal{M}}(\bar{a}) = \{\psi(\bar{x}) | \mathcal{M} \models \psi(\bar{a})\}.$$

- $\mathcal{M}$ is *homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.
- $\mathcal{M}$ is $\exists$*-homogeneous* $\Leftrightarrow$ for any $n \geq 1$, for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $tp^{\mathcal{M}}_{\exists}(\bar{a}) = tp^{\mathcal{M}}_{\exists}(\bar{b}) \implies \exists f \in Aut(\mathcal{M})$ s.t. $f(\bar{a}) = f(\bar{b})$.

Remark. $\exists$-homogeneity $\implies$ homogeneity.

Prime Models

Prime Models

- $\mathcal{M}$ is called *prime*

Prime Models

- $\mathcal{M}$ is called *prime* if $\mathcal{M}$ is elementary embeddable in every model of $Th(\mathcal{M})$.

The free group of rank 2

The free group of rank 2

Theorem (A. Nies, 2003)

The free group of rank 2

Theorem (A. Nies, 2003)

A free group F_2 of rank 2 is $\exists$-homogeneous and **not** *prime.*

The free group of rank 2

Theorem (A. Nies, 2003)

A free group F_2 of rank 2 is $\exists$-homogeneous and **not** *prime. In particular $Th(F_2)$ has no prime model.*

The free group of rank 2

Theorem (A. Nies, 2003)

A free group F_2 of rank 2 is $\exists$-homogeneous and **not** *prime. In particular $Th(F_2)$ has no prime model.*

The proof uses the following strong property of the free group F_2 with basis $\{a, b\}$:

The free group of rank 2

Theorem (A. Nies, 2003)

A free group F_2 of rank 2 is $\exists$-homogeneous and **not** *prime. In particular $Th(F_2)$ has no prime model.*

The proof uses the following strong property of the free group F_2 with basis $\{a, b\}$: there exists a quantifier-free formula $\varphi(x, y)$,

The free group of rank 2

Theorem (A. Nies, 2003)

A free group F_2 of rank 2 is $\exists$-homogeneous and **not** *prime. In particular $Th(F_2)$ has no prime model.*

The proof uses the following strong property of the free group F_2 with basis $\{a, b\}$: there exists a quantifier-free formula $\varphi(x, y)$, such that for any endomorphism f of F_2, if $F_2 \models \varphi(f(a), f(b))$ then f is an embedding.

The free group of rank 2

Theorem (A. Nies, 2003)

A free group F_2 of rank 2 is $\exists$-homogeneous and **not** *prime. In particular $Th(F_2)$ has no prime model.*

The proof uses the following strong property of the free group F_2 with basis $\{a, b\}$: there exists a quantifier-free formula $\varphi(x, y)$, such that for any endomorphism f of F_2, if $F_2 \models \varphi(f(a), f(b))$ then f is an embedding.

Indeed, we can take $\varphi(x, y) := [x, y] \neq 1$

Two-generated torsion-free hyperbolic groups

Two-generated torsion-free hyperbolic groups

Question:

Two-generated torsion-free hyperbolic groups

Question: What can be said about the $\exists$-homogeneity and "primeness" of two-generated torsion-free hyperbolic groups?

Two-generated torsion-free hyperbolic groups

Question: What can be said about the $\exists$-homogeneity and "primeness" of two-generated torsion-free hyperbolic groups?

Definition

A group G is said to be co-hopfian, if any injective endomorphism of G is an automorphism.

Two-generated torsion-free hyperbolic groups

Question: What can be said about the $\exists$-homogeneity and "primeness" of two-generated torsion-free hyperbolic groups?

Definition

A group G is said to be co-hopfian, if any injective endomorphism of G is an automorphism.

That is a group is co-hopfian if it does not contain a subgroup isomorphic to itself.

Two-generated torsion-free hyperbolic groups

Two-generated torsion-free hyperbolic groups

Examples:

Two-generated torsion-free hyperbolic groups

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.

Two-generated torsion-free hyperbolic groups

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.
- $SL_n(\mathbb{Z})$ with $n \geq 3$ (G. Prasad, 1976).

Two-generated torsion-free hyperbolic groups

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.
- $SL_n(\mathbb{Z})$ with $n \geq 3$ (G. Prasad, 1976).
- Nonabelian freely indecomposable trosion-free hyperbolic groups (Z. Sela, 1997).

Two-generated torsion-free hyperbolic groups

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.
- $SL_n(\mathbb{Z})$ with $n \geq 3$ (G. Prasad, 1976).
- Nonabelian freely indecomposable trosion-free hyperbolic groups (Z. Sela, 1997).
- Mapping class groups of closed surfaces (N.V. Ivanov, J.D. McCarthy, 1999).

Two-generated torsion-free hyperbolic groups

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.
- $SL_n(\mathbb{Z})$ with $n \geq 3$ (G. Prasad, 1976).
- Nonabelian freely indecomposable trosion-free hyperbolic groups (Z. Sela, 1997).
- Mapping class groups of closed surfaces (N.V. Ivanov, J.D. McCarthy, 1999).
- $B_n/Z(B_n)$, where B_n is the Braid group on $n \geq 4$ strands (R.W. Bell, D. Margalit, 2005).

Two-generated torsion-free hyperbolic groups

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.
- $SL_n(\mathbb{Z})$ with $n \geq 3$ (G. Prasad, 1976).
- Nonabelian freely indecomposable trosion-free hyperbolic groups (Z. Sela, 1997).
- Mapping class groups of closed surfaces (N.V. Ivanov, J.D. McCarthy, 1999).
- $B_n/Z(B_n)$, where B_n is the Braid group on $n \geq 4$ strands (R.W. Bell, D. Margalit, 2005).
- $Out(F_n)$, where F_n is a free group of rank n (B. Farb, M. Handel, 2007).

Two-generated torsion-free hyperbolic groups

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

A group G is said to be **strongly co-hopfian**,

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

A group G is said to be **strongly co-hopfian**, *if there exists a finite subset $S \subseteq G \setminus \{1\}$*

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

A group G is said to be **strongly co-hopfian**, *if there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any endomorphism φ of G, if $1 \notin \varphi(S)$ then φ is an automorphism.*

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

A group G is said to be **strongly co-hopfian**, *if there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any endomorphism φ of G, if $1 \notin \varphi(S)$ then φ is an automorphism.*

Examples:

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

A group G is said to be **strongly co-hopfian**, *if there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any endomorphism φ of G, if $1 \notin \varphi(S)$ then φ is an automorphism.*

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.

Two-generated torsion-free hyperbolic groups

We introduce a strong form of the co-hopf property.

Definition

A group G is said to be **strongly co-hopfian**, *if there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any endomorphism φ of G, if $1 \notin \varphi(S)$ then φ is an automorphism.*

Examples:

- Finite groups, the group of the rationals $\mathbb{Q}$.
- Tarski Monster groups.

The previous notion is very interesting from the view point of model theory:

The previous notion is very interesting from the view point of model theory:

Lemma

The previous notion is very interesting from the view point of model theory:

Lemma

A finitely presented strongly co-hopfian group is $\exists$-homogeneous and prime.

The previous notion is very interesting from the view point of model theory:

Lemma

A finitely presented strongly co-hopfian group is $\exists$-homogeneous and prime.

Let $G = \langle \bar{a} | r_1(\bar{a}), r_2(\bar{a}), \ldots, r_n(\bar{a}) \rangle$ be finitely presented and strongly co-hopfian.

The previous notion is very interesting from the view point of model theory:

Lemma

A finitely presented strongly co-hopfian group is $\exists$-homogeneous and prime.

Let $G = \langle \bar{a} | r_1(\bar{a}), r_2(\bar{a}), \ldots, r_n(\bar{a}) \rangle$ be finitely presented and strongly co-hopfian. Then there exists a quantifier-free formula $\varphi(\bar{x})$ such that for any endomorphism f of G, if $G \models \varphi(f(\bar{a}))$ then f is an automorphism.

The previous notion is very interesting from the view point of model theory:

Lemma

A finitely presented strongly co-hopfian group is $\exists$-homogeneous and prime.

Let $G = \langle \bar{a} | r_1(\bar{a}), r_2(\bar{a}), \ldots, r_n(\bar{a}) \rangle$ be finitely presented and strongly co-hopfian. Then there exists a quantifier-free formula $\varphi(\bar{x})$ such that for any endomorphism f of G, if $G \models \varphi(f(\bar{a}))$ then f is an automorphism. A property analogous to that of the free group of rank 2.

Two-generated torsion-free hyperbolic groups

Two-generated torsion-free hyperbolic groups

Theorem 1

Two-generated torsion-free hyperbolic groups

Theorem 1

A non-free two-generated torsion-free hyperbolic group is strongly co-hopfian.

Two-generated torsion-free hyperbolic groups

Two-generated torsion-free hyperbolic groups

Since torsion-free hyperbolic groups are finitely presented, we conclude by the previous Lemma:

Two-generated torsion-free hyperbolic groups

Since torsion-free hyperbolic groups are finitely presented, we conclude by the previous Lemma:

Theorem 2

Two-generated torsion-free hyperbolic groups

Since torsion-free hyperbolic groups are finitely presented, we conclude by the previous Lemma:

Theorem 2

A non-free two-generated torsion-free hyperbolic group is $\exists$-homogeneous and prime.

Sketch of proof of Theorem 2

Sketch of proof of Theorem 2

Let Γ be a non-free two-generated trosion-free hyperbolic group.

Sketch of proof of Theorem 2

Let Γ be a non-free two-generated trosion-free hyperbolic group.

- There exists a sequence of subroups $\Gamma = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_n$ satisfying the following properties:

Sketch of proof of Theorem 2

Let Γ be a non-free two-generated trosion-free hyperbolic group.

- There exists a sequence of subroups $\Gamma = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_n$ satisfying the following properties:

  (i) Each Γ_i is two-generated, hyperbolic and quasiconvex;

Sketch of proof of Theorem 2

Let Γ be a non-free two-generated trosion-free hyperbolic group.

- There exists a sequence of subroups $\Gamma = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_n$ satisfying the following properties:

  (i) Each Γ_i is two-generated, hyperbolic and quasiconvex;
  (ii) $\Gamma_i = \langle \Gamma_{i+1}, t | A^t = B \rangle$, where A and B are a nontrivial malnormal cyclic subgroups of Γ_{i+1};

Sketch of proof of Theorem 2

Let Γ be a non-free two-generated trosion-free hyperbolic group.

- There exists a sequence of subroups $\Gamma = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_n$ satisfying the following properties:

  (i) Each Γ_i is two-generated, hyperbolic and quasiconvex;
  (ii) $\Gamma_i = \langle \Gamma_{i+1}, t | A^t = B \rangle$, where A and B are a nontrivial malnormal cyclic subgroups of Γ_{i+1};
  (iii) Γ_n is a rigid subgroup of Γ.

Sketch of proof of Theorem 1

Sketch of proof of Theorem 1

- Γ_n is Γ-determined;

Sketch of proof of Theorem 1

- Γ_n is Γ-determined; that is, there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any homomorphism $\varphi : \Gamma_n \to \Gamma$, if $1 \notin \varphi(S)$ then φ is an embedding.

Sketch of proof of Theorem 1

- Γ_n is Γ-determined; that is, there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any homomorphism $\varphi : \Gamma_n \to \Gamma$, if $1 \notin \varphi(S)$ then φ is an embedding.
- Let φ be an endomorphism of Γ such that $1 \notin \varphi(S)$. Then the restriction of φ to every Γ_i is an automorphism of Γ_i.

Sketch of proof of Theorem 1

- Γ_n is Γ-determined; that is, there exists a finite subset $S \subseteq G \setminus \{1\}$ such that for any homomorphism $\varphi : \Gamma_n \to \Gamma$, if $1 \notin \varphi(S)$ then φ is an embedding.
- Let φ be an endomorphism of Γ such that $1 \notin \varphi(S)$. Then the restriction of φ to every Γ_i is an automorphism of Γ_i. In particular φ is an automorphism of Γ. $\square$

Quasi-axiomatizable groups

Quasi-axiomatizable groups

Definition (A. Nies)

Quasi-axiomatizable groups

Definition (A. Nies)

A finitely generated group G is said to be Quasi-Axiomatizable

Quasi-axiomatizable groups

Definition (A. Nies)

A finitely generated group G is said to be Quasi-Axiomatizable if any finitely generated group which is elementary equivalent to G is isomorphic to G.

Quasi-axiomatizable groups

Definition (A. Nies)

A finitely generated group G is said to be Quasi-Axiomatizable if any finitely generated group which is elementary equivalent to G is isomorphic to G.

Question (A. Nies):

Quasi-axiomatizable groups

Definition (A. Nies)

A finitely generated group G is said to be Quasi-Axiomatizable if any finitely generated group which is elementary equivalent to G is isomorphic to G.

Question (A. Nies): Is there a f.g. group which is prime but not QA?

Quasi-axiomatizable groups

Quasi-axiomatizable groups

Using Sela's work on the elementary theory of torsion-free hyperbolic groups, we have

Quasi-axiomatizable groups

Using Sela's work on the elementary theory of torsion-free hyperbolic groups, we have

Theorem

Quasi-axiomatizable groups

Using Sela's work on the elementary theory of torsion-free hyperbolic groups, we have

Theorem

Let Γ be a two-generated trosion-free hyperbolic group.

Quasi-axiomatizable groups

Using Sela's work on the elementary theory of torsion-free hyperbolic groups, we have

Theorem

Let Γ be a two-generated trosion-free hyperbolic group. Then Γ is elementary equivalent to $\Gamma * \mathbb{Z}$.

Quasi-axiomatizable groups

Using Sela's work on the elementary theory of torsion-free hyperbolic groups, we have

Theorem

Let Γ be a two-generated trosion-free hyperbolic group. Then Γ is elementary equivalent to $\Gamma * \mathbb{Z}$.

Hence

Quasi-axiomatizable groups

Using Sela's work on the elementary theory of torsion-free hyperbolic groups, we have

Theorem

Let Γ be a two-generated trosion-free hyperbolic group. Then Γ is elementary equivalent to $\Gamma * \mathbb{Z}$.

Hence if Γ is non-free then it is prime but not QA.

Contents

1 Existential homogeneity & prime models
- Definitions
- Existential homogeneity & prime models

2 Existential homogeneity in free groups
- Homogeneity in free groups

Question : What can be said about existential homogeneity in free groups?

Question : What can be said about existential homogeneity in free groups?
Recall that:

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Let $\mathcal{M}$ be a model and $\mathcal{N}$ a submodel of $\mathcal{M}$.

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Let $\mathcal{M}$ be a model and $\mathcal{N}$ a submodel of $\mathcal{M}$. The model $\mathcal{N}$ is said to be existentially closed (abbreviated e.c.) in $\mathcal{M}$,

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Let $\mathcal{M}$ be a model and $\mathcal{N}$ a submodel of $\mathcal{M}$. The model $\mathcal{N}$ is said to be existentially closed (abbreviated e.c.) in $\mathcal{M}$, if for any existential formula φ with parameters from $\mathcal{N}$, if $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$.

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Let $\mathcal{M}$ be a model and $\mathcal{N}$ a submodel of $\mathcal{M}$. The model $\mathcal{N}$ is said to be existentially closed (abbreviated e.c.) in $\mathcal{M}$, if for any existential formula φ with parameters from $\mathcal{N}$, if $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$.

Definition

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Let $\mathcal{M}$ be a model and $\mathcal{N}$ a submodel of $\mathcal{M}$. The model $\mathcal{N}$ is said to be existentially closed (abbreviated e.c.) in $\mathcal{M}$, if for any existential formula φ with parameters from $\mathcal{N}$, if $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$.

Definition

Let F be a free group and let $\bar{a} = (a_1, \ldots, a_m)$ be a tuple from F.

Question : What can be said about existential homogeneity in free groups?
Recall that:

Definition

Let $\mathcal{M}$ be a model and $\mathcal{N}$ a submodel of $\mathcal{M}$. The model $\mathcal{N}$ is said to be existentially closed (abbreviated e.c.) in $\mathcal{M}$, if for any existential formula φ with parameters from $\mathcal{N}$, if $\mathcal{M} \models \varphi$, then $\mathcal{N} \models \varphi$.

Definition

Let F be a free group and let $\bar{a} = (a_1, \ldots, a_m)$ be a tuple from F.We say that $\bar{a}$ is a power of a primitive element if there exist integers $p_1, \ldots, p_m$ and a primitive element u such that $a_i = u^{p_i}$ for all i.

Theorem 3

Theorem 3

Let F be a nonabelian free group of finite rank.

Theorem 3

Let F be a nonabelian free group of finite rank. Let $\bar{a}, \bar{b} \in F^n$ and $P \subseteq F$ such that $tp^F_\exists(\bar{a}|P) = tp^F_\exists(\bar{b}|P)$.

Theorem 3

Let F be a nonabelian free group of finite rank. Let $\bar{a}, \bar{b} \in F^n$ and $P \subseteq F$ such that $tp^F_\exists(\bar{a}|P) = tp^F_\exists(\bar{b}|P)$. Then one of the following cases holds:

Theorem 3

Let F be a nonabelian free group of finite rank. Let $\bar{a}, \bar{b} \in F^n$ and $P \subseteq F$ such that $tp^F_\exists(\bar{a}|P) = tp^F_\exists(\bar{b}|P)$. Then one of the following cases holds:

- *the tuple $\bar{a}$ has the same existential type as a power of a primitive element;*

Theorem 3

Let F be a nonabelian free group of finite rank. Let $\bar{a}, \bar{b} \in F^n$ and $P \subseteq F$ such that $tp^F_\exists(\bar{a}|P) = tp^F_\exists(\bar{b}|P)$. Then one of the following cases holds:

- *the tuple $\bar{a}$ has the same existential type as a power of a primitive element;*
- *there exists an e.c. subgroup $E(\bar{a})$ (resp. $E(\bar{b})$) containing P and $\bar{a}$ (resp. $\bar{b}$) and an isomorphism $\sigma : E(\bar{a}) \to E(\bar{b})$ fixing pointwise P and sending $\bar{a}$ to $\bar{b}$.*

Sketch of proof of Theorem 3

Sketch of proof of Theorem 3

We eliminate first parameters.

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*;

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Let G be a finitely generated equationally noetherian group.

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Let G be a finitely generated equationally noetherian group. Let P be a subset of G.

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Let G be a finitely generated equationally noetherian group. Let P be a subset of G. Then there exists a finite subset $P_0 \subseteq P$ such that for any endomorphism f of G, if f fixes P_0 pointwise then f fixes P pointwise.

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Let G be a finitely generated equationally noetherian group. Let P be a subset of G. Then there exists a finite subset $P_0 \subseteq P$ such that for any endomorphism f of G, if f fixes P_0 pointwise then f fixes P pointwise.

Hence:

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Let G be a finitely generated equationally noetherian group. Let P be a subset of G. Then there exists a finite subset $P_0 \subseteq P$ such that for any endomorphism f of G, if f fixes P_0 pointwise then f fixes P pointwise.

Hence: $\exists f \in Aut(F|P)$ s.t. $f(\bar{a}) = f(\bar{b}) \Leftrightarrow \exists f \in Aut(F)$ s.t. $f(\bar{a}P_0) = f(\bar{b}P_0)$.

Sketch of proof of Theorem 3

We eliminate first parameters. Recall that a free group is *equationally noetherian*; that is any system of equations in finitely many variable is equivalent to a finite subsystem.

Lemma

Let G be a finitely generated equationally noetherian group. Let P be a subset of G. Then there exists a finite subset $P_0 \subseteq P$ such that for any endomorphism f of G, if f fixes P_0 pointwise then f fixes P pointwise.

Hence: $\exists f \in Aut(F|P)$ s.t. $f(\bar{a}) = f(\bar{b}) \Leftrightarrow \exists f \in Aut(F)$ s.t. $f(\bar{a}P_0) = f(\bar{b}P_0)$.
Note that : If $tp_\exists(\bar{a}|P) = tp_\exists(\bar{b}|P)$ then $tp_\exists(\bar{a}|P_0) = tp_\exists(\bar{b}|P_0)$ and $tp_\exists(\bar{a}P_0) = tp_\exists(\bar{b}P_0)$.

Sketch of proof of Theorem 3

Sketch of proof of Theorem 3

Definition

Sketch of proof of Theorem 3

Definition

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2).

Sketch of proof of Theorem 3

Definition

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2). We say that $(\bar{a}, \bar{b})$ is existentially rigid, if there is no nontrivial free decomposition $F_1 = A * B$ such that A contains a tuple $\bar{c}$ with $tp_{\exists}^{F_1}(\bar{a}) \subseteq tp_{\exists}^{A}(\bar{c}) \subseteq tp_{\exists}^{F_2}(\bar{b})$.

Sketch of proof of Theorem 3

Sketch of proof of Theorem 3

Proposition (1)

Sketch of proof of Theorem 3

Proposition (1)

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2).

Sketch of proof of Theorem 3

Proposition (1)

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2). Suppose that $(\bar{a}, \bar{b})$ is existentially rigid and let $\bar{s}$ be a basis of F_1.

Sketch of proof of Theorem 3

Proposition (1)

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2). Suppose that $(\bar{a}, \bar{b})$ is existentially rigid and let $\bar{s}$ be a basis of F_1.
Then there exists a quantifier-free formula $\varphi(\bar{x}, \bar{y})$, such that $F_1 \models \varphi(\bar{a}, \bar{s})$ and such that for any $f \in Hom(F_1|\bar{a}, F_2|\bar{b})$, if $F_2 \models \varphi(\bar{b}, f(\bar{s}))$ then f is an embedding.

Sketch of proof of Theorem 3

Sketch of proof of Theorem 3

Proposition (2)

Sketch of proof of Theorem 3

Proposition (2)

Let F_1 and F_2 be nonabelian free groups of finite rank and $\bar{a}$ (resp. $\bar{b}$) a tuple from F_1 (resp. F_2) such that $tp_{\exists}^{F_1}(\bar{a}) = tp_{\exists}^{F_2}(\bar{b})$.

Sketch of proof of Theorem 3

Proposition (2)

Let F_1 and F_2 be nonabelian free groups of finite rank and $\bar{a}$ (resp. $\bar{b}$) a tuple from F_1 (resp. F_2) such that $tp_{\exists}^{F_1}(\bar{a}) = tp_{\exists}^{F_2}(\bar{b})$. Suppose that $(\bar{a}, \bar{b})$ is existentially rigid.

Sketch of proof of Theorem 3

Proposition (2)

Let F_1 and F_2 be nonabelian free groups of finite rank and $\bar{a}$ (resp. $\bar{b}$) a tuple from F_1 (resp. F_2) such that $tp_{\exists}^{F_1}(\bar{a}) = tp_{\exists}^{F_2}(\bar{b})$.
Suppose that $(\bar{a}, \bar{b})$ is existentially rigid.
Then either $rk(F_1) = 2$ and $\bar{a}$ is a power of a primitive element, or there exists an embedding $h \in Hom(F_1|\bar{a}, F_2|\bar{b})$ such that $h(F_1)$ is an e.c. subgroup of F_2.

Sketch of proof of Theorem 3

Sketch of proof of Theorem 3

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F_\exists(\bar{a}) = tp^F_\exists(\bar{b})$.

Sketch of proof of Theorem 3

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F_\exists(\bar{a}) = tp^F_\exists(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F_\exists(\bar{a}) = tp^C_\exists(\bar{c})$.

Sketch of proof of Theorem 3

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F_\exists(\bar{a}) = tp^F_\exists(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F_\exists(\bar{a}) = tp^C_\exists(\bar{c})$.
Then $(\bar{c}, \bar{a})$ and $(\bar{c}, \bar{b})$ are existentially rigid.

Sketch of proof of Theorem 3

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F_\exists(\bar{a}) = tp^F_\exists(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F_\exists(\bar{a}) = tp^C_\exists(\bar{c})$.
Then $(\bar{c}, \bar{a})$ and $(\bar{c}, \bar{b})$ are existentially rigid.
By Propsition (2), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{c}, F|\bar{a})$ (resp. $h_2 \in Hom(C|\bar{c}, F|\bar{b})$) such that $h_1(C)$ (resp. $h_2(C)$) is an e.c. subgroup of F.

Sketch of proof of Theorem 3

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F_\exists(\bar{a}) = tp^F_\exists(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F_\exists(\bar{a}) = tp^C_\exists(\bar{c})$.
Then $(\bar{c}, \bar{a})$ and $(\bar{c}, \bar{b})$ are existentially rigid.
By Propsition (2), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{c}, F|\bar{a})$ (resp. $h_2 \in Hom(C|\bar{c}, F|\bar{b})$) such that $h_1(C)$ (resp. $h_2(C)$) is an e.c. subgroup of F.
Suppose that $\bar{c}$ is not a power of a primitive element.

Sketch of proof of Theorem 3

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F_\exists(\bar{a}) = tp^F_\exists(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F_\exists(\bar{a}) = tp^C_\exists(\bar{c})$.
Then $(\bar{c}, \bar{a})$ and $(\bar{c}, \bar{b})$ are existentially rigid.
By Propsition (2), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{c}, F|\bar{a})$ (resp. $h_2 \in Hom(C|\bar{c}, F|\bar{b})$) such that $h_1(C)$ (resp. $h_2(C)$) is an e.c. subgroup of F.
Suppose that $\bar{c}$ is not a power of a primitive element. By setting $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$, we have $h_2 \circ h_1^{-1} : E(\bar{a}) \to E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$. □

Theorem 4

Theorem 4

Let F be a nonabelian free group of finite rank.

Theorem 4

Let F be a nonabelian free group of finite rank. For any tuples $\bar{a}, \bar{b} \in F^n$ and for any subset $P \subseteq F$, if $tp^F(\bar{a}|P) = tp^F(\bar{b}|P)$ then there exists an automorphism of F fixing pointwise P and sending $\bar{a}$ to $\bar{b}$.

Theorem 4

Let F be a nonabelian free group of finite rank. For any tuples $\bar{a}, \bar{b} \in F^n$ and for any subset $P \subseteq F$, if $tp^F(\bar{a}|P) = tp^F(\bar{b}|P)$ then there exists an automorphism of F fixing pointwise P and sending $\bar{a}$ to $\bar{b}$.

The above theorem is also proved by Perin and Sklinos.

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

Z. Sela and O. Kharlampovich and A. Myasnikov show that nonabelian free groups have the same elementary theory, and in fact the following more explicit description.

Sketch of proof of Theorem 4

Z. Sela and O. Kharlampovich and A. Myasnikov show that nonabelian free groups have the same elementary theory, and in fact the following more explicit description.

Theorem

Sketch of proof of Theorem 4

Z. Sela and O. Kharlampovich and A. Myasnikov show that nonabelian free groups have the same elementary theory, and in fact the following more explicit description.

Theorem

A nonabelian free factor of a free group of finite rank is an elementary subgroup. □

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

They show also the following quantifier-elimination result.

Sketch of proof of Theorem 4

They show also the following quantifier-elimination result.

Theorem

Sketch of proof of Theorem 4

They show also the following quantifier-elimination result.

Theorem

Let $\varphi(\bar{x})$ be a formula.

Sketch of proof of Theorem 4

They show also the following quantifier-elimination result.

Theorem

Let $\varphi(\bar{x})$ be a formula. Then there exists a boolean combination of $\exists\forall$-forumulas $\phi(\bar{x})$, such that for any nonabelian free group F of finite rank, one has $F \models \forall\bar{x}(\varphi(\bar{x}) \Leftrightarrow \phi(\bar{x}))$. □

Sketch of proof of Theorem 4

They show also the following quantifier-elimination result.

Theorem

Let $\varphi(\bar{x})$ be a formula. Then there exists a boolean combination of $\exists\forall$-forumulas $\phi(\bar{x})$, such that for any nonabelian free group F of finite rank, one has $F \models \forall\bar{x}(\varphi(\bar{x}) \Leftrightarrow \phi(\bar{x}))$. □

We notice, in particular, that if $\bar{a}, \bar{b} \in F^n$ such that $tp^F_{\exists\forall}(\bar{a}) = tp^F_{\exists\forall}(\bar{b})$, then $tp^F(\bar{a}) = tp^F(\bar{b})$.

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

Theorem (C. Perin, 2008)

Sketch of proof of Theorem 4

Theorem (C. Perin, 2008)

An elementary subgroup of a free group of finite rank is a free factor. □

Sketch of proof of Theorem 4

Theorem (C. Perin, 2008)

An elementary subgroup of a free group of finite rank is a free factor. □

Theorem (A. Pillay, 2009)

Sketch of proof of Theorem 4

Theorem (C. Perin, 2008)

An elementary subgroup of a free group of finite rank is a free factor. □

Theorem (A. Pillay, 2009)

Let F be a nonabelian free group of finite rank and $u, v \in F$ such that $tp^F(u) = tp^F(v)$.

Sketch of proof of Theorem 4

Theorem (C. Perin, 2008)

An elementary subgroup of a free group of finite rank is a free factor. □

Theorem (A. Pillay, 2009)

Let F be a nonabelian free group of finite rank and $u, v \in F$ such that $tp^F(u) = tp^F(v)$. If u is primitive, then v is primitive. □

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

Definition

Sketch of proof of Theorem 4

Definition

Let F be nonabelian free group of finite rank and let $\bar{a}$ be a tuple of F.

Sketch of proof of Theorem 4

Definition

Let F be nonabelian free group of finite rank and let $\bar{a}$ be a tuple of F. We say that $\bar{a}$ is rigid if there is no nontrivial free decomposition $F = A * B$ such that A contains a tuple $\bar{c}$ with $tp^{F_1}_{\exists\forall}(\bar{a}) = tp^{A}_{\exists\forall}(\bar{c})$.

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

Proposition (3)

Sketch of proof of Theorem 4

Proposition (3)

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2) such that $tp^{F_1}_{\exists\forall}(\bar{a}) = tp^{F_2}_{\exists\forall}(\bar{b})$.

Sketch of proof of Theorem 4

Proposition (3)

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2) such that $tp^{F_1}_{\exists\forall}(\bar{a}) = tp^{F_2}_{\exists\forall}(\bar{b})$. Suppose that $\bar{a}$ is rigid.

Sketch of proof of Theorem 4

Proposition (3)

Let F_1 and F_2 be nonabelian free groups of finite rank and let $\bar{a}$ (resp. $\bar{b}$) be a tuple from F_1 (resp. F_2) such that $tp^{F_1}_{\exists\forall}(\bar{a}) = tp^{F_2}_{\exists\forall}(\bar{b})$. Suppose that $\bar{a}$ is rigid.
Then either $rk(F_1) = 2$ and $\bar{a}$ is a power of a primitive element, or there exists an embedding $h \in Hom(F_1|\bar{a}, F_2|\bar{b})$ such that $h(F_1) \preceq_{\exists\forall} F_2$.

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F(\bar{a}) = tp^F(\bar{b})$.

Sketch of proof of Theorem 4

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F(\bar{a}) = tp^F(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F(\bar{a}) = tp^C(\bar{c})$.

Sketch of proof of Theorem 4

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F(\bar{a}) = tp^F(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F(\bar{a}) = tp^C(\bar{c})$.
Then $\bar{c}$ is rigid.

Sketch of proof of Theorem 4

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F(\bar{a}) = tp^F(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F(\bar{a}) = tp^C(\bar{c})$.
Then $\bar{c}$ is rigid.
By Propsition (3), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{a}, F|\bar{b})$ (resp. $h_2 \in Hom(C|\bar{a}, F|\bar{b})$) such that $h_1(C) \preceq_{\exists\forall} F$ (resp. $h_2(C) \preceq_{\exists\forall} F$).

Sketch of proof of Theorem 4

Let F be a free group of finite rank and let $\bar{a}, \bar{b} \in F^n$ s.t. $tp^F(\bar{a}) = tp^F(\bar{b})$.
Let C be the smallest free factor of F such that C contains $\bar{c}$ with $tp^F(\bar{a}) = tp^C(\bar{c})$.
Then $\bar{c}$ is rigid.
By Propsition (3), either $rk(C) = 2$ and $\bar{c}$ is a power of a primitive element or there exists an embedding $h_1 \in Hom(C|\bar{a}, F|\bar{b})$ (resp. $h_2 \in Hom(C|\bar{a}, F|\bar{b})$) such that $h_1(C) \preceq_{\exists\forall} F$ (resp. $h_2(C) \preceq_{\exists\forall} F$).

If $\bar{c}$ is a power of a primitive element then the result follows from the previous theorem.

Sketch of proof of Theorem 4

Sketch of proof of Theorem 4

Suppose that $\bar{c}$ is not a power of a primitive element.

Sketch of proof of Theorem 4

Suppose that $\bar{c}$ is not a power of a primitive element. Set $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$.

Sketch of proof of Theorem 4

Suppose that $\bar{c}$ is not a power of a primitive element. Set $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$.
We have $h_2 \circ h_1^{-1} : E(\bar{a}) \to E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$.

Sketch of proof of Theorem 4

Suppose that $\bar{c}$ is not a power of a primitive element. Set $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$.
We have $h_2 \circ h_1^{-1} : E(\bar{a}) \to E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$.
Since $E(\bar{a}) \preceq_{\exists\forall} F$ (resp. $E(\bar{b}) \preceq_{\exists\forall} F$) we get by a previous theorem that $E(\bar{a}) \preceq F$ (resp. $E(\bar{b}) \preceq F$).

Sketch of proof of Theorem 4

Suppose that $\bar{c}$ is not a power of a primitive element. Set $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$.
We have $h_2 \circ h_1^{-1} : E(\bar{a}) \to E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$.
Since $E(\bar{a}) \preceq_{\exists\forall} F$ (resp. $E(\bar{b}) \preceq_{\exists\forall} F$) we get by a previous theorem that $E(\bar{a}) \preceq F$ (resp. $E(\bar{b}) \preceq F$).
Hence, $E(\bar{a})$ (resp. $E(\bar{b})$) is a free factor of F.

Sketch of proof of Theorem 4

Suppose that $\bar{c}$ is not a power of a primitive element. Set $E(\bar{a}) = h_1(C)$ and $E(\bar{b}) = h_2(C)$.
We have $h_2 \circ h_1^{-1} : E(\bar{a}) \to E(\bar{b})$ is an isomorphism with $h_2 \circ h_1^{-1}(\bar{a}) = \bar{b}$.
Since $E(\bar{a}) \preceq_{\exists\forall} F$ (resp. $E(\bar{b}) \preceq_{\exists\forall} F$) we get by a previous theorem that $E(\bar{a}) \preceq F$ (resp. $E(\bar{b}) \preceq F$).
Hence, $E(\bar{a})$ (resp. $E(\bar{b})$) is a free factor of F.
Therefore $h_2 \circ h_1^{-1}$ can be extended to an isomorphism of F. □