Algebraic Geometry over Groups
&
Model-theoretic ranks

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Model Theory and Algebra

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Topological Preliminaries

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Proposition 2 Let $f : X \to Y$ be a continuous surjective function. Then $\mathrm{Dim}(Y) \leq \mathrm{Dim}(X)$. □

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- If X is noetherian then X has only finitely many irreducible components.

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for any $Y \subseteq X$,

$$\dim(Y) = \text{ sup } \{\dim(Z) \mid Z \text{ is an irreducible component of } Y\ \}.$$

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Proposition 3 $\dim(X) \leq \mathrm{Dim}(X)$. □

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X is a discreet space $\Rightarrow \dim(X) = 0$, $\mathrm{Dim}(X) = \infty$.

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closed subsets of G^n: $\bigcap (V(S_1) \cup \ldots \cup V(S_n))$:= intersection of finite unions of algebraic sets.

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G is equationally noetherian

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- finitely generated abelian-by-nilpotent groups (Bryant, 1977),
- torsion-free hyperbolic groups, (Sela, 2002),
- torsion-free relatively hyperbolic groups to some collection of free abelian subgroups, (D. Groves, 2005).

Algebraic Geometry over Groups

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Theorem (Baumaslag, Myasnikov, Remeslennikov, 1998)

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$\forall\, n \geq 1$, the Zariski topology on G^n is noetherian.

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Theorem (Myasnikov, Remeslennikov) If G is equationally noetherian then

$$G = G^\circ \Leftrightarrow G \text{ is irreducible}$$

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consequence:

$G = G^\circ \Leftrightarrow G$ has no constructible subgroup of finite index

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Furthermore if $|G/H|$ is finite then $\forall\ n \geq 1$

$$\mathrm{Dim}(G^n) \leq \mathrm{Dim}_{G^n}(H^n) \otimes \mathrm{Dim}((G/H)^n).$$

□

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Corollary 1 (Baumslag-Myasnikov-Roman'kov, 1997) If G is equationally noetherian and $H \leq G$, H normal and closed, then G/H is equationally noetherian. □

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Corollary 2 (Baumslag-Myasnikov-Roman'kov,1997) If $H \leq G$, H normal and closed, G/H is equationally noetherian and $|G/H|$ is finite, then G is equationally noetherian. □

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is not true if (G/H) is infinite.

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is not true if (G/H) is infinite. Baumslag, Myasnikov and Roman'kov have given an example of a group G such that $G/Z(G)$ is equationally noetherian and G is not.

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Proposition 4

$$\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathrm{rk}(\mathcal{M}) \leq \mathrm{rk}(\mathcal{N}).$$

In particular if $\mathcal{N}$ is ω-stable then $\mathrm{rk}(\mathcal{M})$ is an ordinal. □

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- if $deg(X) = 1$ then
  $stab_G(X) = \bigcup_{1 \leq i \leq n} (\bigcap_{1 \leq j \leq m} a_{i,j} X^{-1}), \qquad a_{i,j} \in X.$

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Theorem 3 Let G such that $\mathrm{rk}(G) < \infty$. Let $X \subseteq G$, q-f-definable and generic.

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$$G = a_1 X \cup \cdots \cup a_n X.$$

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Main Theorem Let G equationally noetherian and $\mathrm{rk}(G) < \infty$. Let $H \leq G$ such that $\mathrm{rk}(H) = \mathrm{rk}(G)$. If G is connected then $\overline{H} = G$. In particular H is connected.

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Proof

$$\mathrm{rk}(H) = \mathrm{rk}(G) \ \& \ \mathrm{rk}(H) \leq \mathrm{rk}(\overline{H}) \leq \mathrm{rk}(G)$$

$$\Rightarrow \overline{H} \text{ generic}$$

$$\Rightarrow G = a_1\overline{H} \cup \cdots \cup a_n\overline{H}$$

$$\Rightarrow G = \overline{H}$$

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- $\mathcal{M} \preceq_\exists \mathcal{M} * \mathcal{K}$, for any model $\mathcal{K}$ of the universal theory of free groups, in particuliar for $\mathcal{K} = \mathbb{Z}$,

Application

Models of the universal theory of free groups

Theorem 5 $\mathcal{M} \leq \mathcal{N}$, $\mathcal{M}$ non-abelian, then

- $\mathcal{M}$ is connected,
- $\overline{\mathcal{M}} = \mathcal{N}$,
- $\mathcal{M} \preceq_\exists \mathcal{M} * \mathcal{K}$, for any model $\mathcal{K}$ of the universal theory of free groups, in particuliar for $\mathcal{K} = \mathbb{Z}$,
- any q-f-definable (or constructible) proper subgroup of $\mathcal{M}$ is abelian.

END OF THE HISTORY

Thank you for your attention

Many thanks for the great hospitality