CSA-Groups and bad Groups

Abderezak OULD HOUCINE

Definitions

• A subgroup H of a group G is *malnormal* in G, if $H \cap H^x = 1$ for every $x \in G \setminus H$.

• A *CSA-group* ("Conjugately Separated Abelian") is a group in which every maximal abelian subgroup is malnormal.

An alternative definition of CSA-groups is the following: a group G is a CSA-group if and only if the centralizer of every nontrivial element of G is abelian and selfnormalizing.

The class of CSA-groups is universal and axiomatized by the following sentences:

$$\text{(i)}\ \forall x \forall y \forall z\ (x \neq 1 \wedge [x,y] = 1 \wedge [x,z] = 1) \Rightarrow [y,z] = 1.$$
$$\text{(ii)}\ \forall x \forall y\ (x \neq 1 \wedge [x,x^y] = 1) \Rightarrow [x,y] = 1.$$

A CSA-group with an *involution*, i.e. an element of order 2, is abelian. We denote by *CSA*-group* any CSA-group without involutions.

Free groups are principal examples of nonabelian CSA-groups.

But the class of CSA-groups contains also several interesting groups:

- torsion-free hyperbolic groups,
- groups acting freely on Λ-tree, and
- groups which are models of the universal theory of nonabelian free groups.

A bad group is a connected nonsoluble group of finite Morley rank all of whose proper connected definable subgroups are nilpotent.

Simple bad groups are hypothetical counterexamples to a well-known conjecture in model theory:

Cherlin-Zil'ber Conjecture: An infinite simple group of finite Morley rank is a linear algebraic group over an algebraically closed field.

The point of interest for us in the talk is: a simple bad group of finite Morley rank 3 is a (nonabelian) CSA-group.

Conversely we have:

Proposition 1 *A nonabelian CSA-group of finite Morley rank has a definable nonabelian simple subgroup (which is bad) all of whose proper definable subgroups are abelian.*

Proof

By the descending chain condition, G has a nonabelian definable subgroup H all of whose proper definable subgroups are abelian. Since H is nonabelian and CSA it contains no nontrivial abelian normal proper subgroup. Thus H is simple and bad. □

Thus if there exists a nonabelian CSA-group of finite Morley rank then there exists a bad group.

B.Poizat has suggested to study existentially closed CSA*-groups and to see if they are a good candidates to be bad.

Theorem 2 (*with E.Jaligot*) *Let G be an existentially closed CSA*-group. Then G is simple and divisible and if C is a maximal abelian subgroup of G, then $G = \cup_{g \in G} C^g$.*

Thus an existentially closed CSA*-group has some properties of simple bad groups.

Is there an existentially closed CSA*-group of finite Morley rank?

The answer is negative and we have:

Theorem 3 *An existentially closed CSA*-group is not ω-stable.*

The proof is based on the following general theorem:

Theorem 4 *Every countable CSA*-group embeds into a finitely generated one having, up to conjugacy, the same maximal abelian subgroups, except maybe the infinite cyclic ones.*

Before giving the sketch of the proof of this theorem we deduce the following which gives a proof of Theorem 3:

Corollary 5 *There exist $2^{\aleph_0}$ finitely generated CSA*-groups. In particular there exist $2^{\aleph_0}$ countable existentially closed CSA*-groups. Furthermore if G is existentially closed CSA*-group then $Th(G)$ has $2^{\aleph_0}$ types over $\emptyset$.*

Proof of corollary 5
For $X \subseteq \mathbb{N} \setminus \{0,1\}$, let $G(X) = *_{n \in X} \mathbb{Z}^n$. Then:

• the maximal abelian subgroups of $G(X)$, except maybe the infinite cyclic ones, are isomorphic to $\mathbb{Z}^n$, with n in X.

• $G(X)$ is a CSA*-group.

Thus, by Theorem 4, $G(X)$ embeds into a finitely generated CSA*-group $F(X)$ having, up to isomorphism, the same maximal abelian subgroups, except maybe the infinite cyclic ones. Thus $F(X) \not\cong F(Y)$ whenever $Y \subseteq \mathbb{N} \setminus \{0,1\}$ and $Y \neq X$. It follows that there exist $2^{\aleph_0}$ pairwise nonisomorphic finitely generated CSA*-groups of the form $F(X)$.

Thus there exist $2^{\aleph_0}$ countable existentially closed CSA*-groups.

Let G be an existentially closed CSA*-group. Since the class of CSA*-groups has JEP then $Th_\forall(G)$ is true in any CSA*-group. Therefore every finitely generated CSA*-group is embeddable in a model of $Th(G)$. Since there exists $2^{\aleph_0}$ finitely generated CSA*-group, then $Th(G)$ has $2^{\aleph_0}$ types over $\emptyset$. □

The proof of Theorem 5 is based on combinatorial tools:

Definition 6

• *An HNN-extension $G^* = \langle G,t \mid A^t = B\rangle$ is said to be* separated *if $A \cap B^x = 1$ for every $x \in G$.*

Theorem 7

Let $G^* = \langle G,t \mid A^t = B\rangle$ be a separated HNN-extension of a CSA-group G where A and B are malnormal subgroups of G. Then G^* is a CSA*-group and G^* and G have, up to conjugacy in G^*, the same maximal abelian subgroups, except maybe the infinite cyclic ones.* □

Theorem 8 *Let G be a group and $\{g_i \mid i \in 2\mathbb{N}+1\}$ be an infinite set of elements of G such that $g_i^{-1} \neq g_j$ whenever $i,j \in 2\mathbb{N}+1$. Let $F = G * \langle a,b \mid \rangle$ and $K = \langle g_i b^{-i} a^i b^i \mid i \in 2\mathbb{N}+1\rangle$. Then K is a free group, with basis the set of indicated elements, and K is* **malnormal** *in F.* □

Sketch of proof of Theorem 4

Let $X = 2\mathbb{N}+1$. As G has no involutions, it has by Zorn's Lemma a generating set $\{g_i \mid i \in X\}$ such that $g_i^{-1} \neq g_j$ for every $i,j \in X$. Consider the free product

$$F = G * \langle a \mid \rangle * \langle b \mid \rangle * \langle c \mid \rangle.$$

Consider the subgroups

$$D = \langle c^i b^{-i} a^i b^i \mid i \in X \rangle \text{ and } E = \langle g_i a^{-i} b^i a^i \mid i \in X \rangle$$

of F. It is clear that D and E are free over their indicated generators and that the function $\varphi \ : \ c^i b^{-i} a^i b^i \mapsto g_i a^{-i} b^i a^i$ $(i \in X)$ extends to an isomorphism from D to E. Consider the HNN-extension

$$F^* = \langle F,t \mid t^{-1} d t = \varphi(d) \ , \ d \in D \rangle.$$

We claim that the supergroup F^* of G has the required properties. First F^* is finitely generated by $\{a,b,c,t\}$. By Theorem 0.7, it is sufficient to prove that the HNN-extension F^* of F is separated and that D and E are malnormal in F. We have:

- F^* is finitely generated by $\{a,b,c,t\}$,
- By the Kurosh Subgroup Theorem, F and G have, up to conjugacy, the same maximal abelian subgroups, except maybe the infinite cyclic ones.
- By Theorem 8, D and E are malnormal in F.
- F^* is separated.

Therefore F^* have the desired properties. $\square$

The next natural step is to see if existentially closed CSA*-groups are at least supertable and this was also asked by Frank Wagner.

But the answer is also negative:

Theorem 9 *Let G be an existentially closed CSA*-group. Then G is not λ-stable for every cardinal λ which satisfies $\lambda^{\aleph_0} = 2^\lambda$. Thus G is not superstable.* □

Universal theory of nonabelian free groups

When G is an existentially closed CSA-group, the obstacle to being ω-stable was the fact that $Th_{\forall}(G)$ has $2^{\aleph_0}$ finitely generated models.

Since the models of the universal theory of nonabelian free groups are a CSA-groups and their theories have at most countable finitely generated models, it became natural to ask if it has a nonabelian superstable model.

The answer is also negative:

Theorem 10 [1] *A superstable model of the universal theory of nonabelian free groups is abelian.*

We sketch the proof.
First we need a generalization of proposition 1 to superstable case:

1. This theorem was proved first by Mustafin and Poizat

Theorem 11 *A superstable nonabelian CSA-group has a definable infinite simple subgroup all of whose proper definable subgroups are abelian.* □

The proof of that theorem is more complicated of that in the case of finite Morley rank.

We let Γ to be the universal theory of nonabelian free groups. We need also the following:

Theorem 12

- *(D.I.Fouxe-Rabinovitch) A locally-free group is not simple*[2].
- Γ *has an existentially closed locally-free model.* □

Sketch of proof of Theorem 10

Suppose towards a contradiction that there exists a nonabelian superstable model of Γ. Then by Theorem 11 there exists a nonabelian simple superstable model H of Γ. Now if $G \equiv H$ then G is nonabelian and simple. Let K be an existentially closed locally-free model of Γ. Then $Th_\forall(K) = Th_\forall(H)$ and thus K is embeddable in a model $G \equiv H$, (thus G is simple). Since K is e.c. then K is simple, contradiction with Theorem 12. □

2. In my talk, I have attributed this Theorem to Mal'cev. I apologize for this error.