Presentation of the Kervaire Conjecture

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Abstract

The goal of this preprint is to present the Kervaire Conjecture.

The Kervaire conjecture can be stated as follows : *If G is a non trivial group then the group $(G * \mathbb{Z})/(r)$ is non trivial where $r \in G * \mathbb{Z}$ and (r) is the normal closure of r in $G * \mathbb{Z}$.*

By imposing some restrictions on the group G, the problem became more flexible and some works goes in this direction. For example the Kervaire Conjecture is true for the following classes of groups:

• Locally residually finite groups.(O.S. Rothaus, 1977 [1]).

• Locally indecable groups. (J. Howie, 1981 [2]). Recall that a group G is said locally indecable if every finitely generated subgroup has an infinite cyclic quotient. In fact in this case we have a more general result which can be stated as follows: *if $(G_i)_{i \in \lambda}$ is a sequence of non trivial locally indecable groups and $r \in F = *_i G_i$, $r \notin \bigcup G_i$, then for every i the natural homomorphism $\pi_i : G_i \to (F/(r))$ is an embedding, where (r) is the normal closure of r in F.*

• Torsion-free groups. (A. Klyachko, 1993 [3]).

Another approach is to impose restrictions on the relator r. The most common condition come from small cancellation theory. For example if $r = s^m$ with $m \geq 7$ then the symmetrezed set generated by r satisfies the condition $C'(1/6)$ and the Kervaire conjecture can be deduced from small cancellation theory. Gonzalez-Acuna and Short [4] proved the case $m = 6$ and Howie proved the case $m = 4, m = 5$ and with Brodskii and Duncan some part of the case $m = 3$. In general the case $m = 2$ is remains open.

We denote by $G(r)$ the group $(G * \mathbb{Z})/(r)$, and by G^* the group $G * \mathbb{Z}$. For any group G and for any element g in G we denote by $(g)_G$ the normal closure of g in G. In the rest of this presentation we write $\mathbb{Z} = \langle t | \rangle$. If $r \in G * \mathbb{Z}$, then $r = g_0.t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$, put $\exp(r) = \sum_{i=1}^{i=n} \varepsilon_i$. We call the elements g_i the **coefficient** of r.

We see that if $\exp(r) \neq \pm 1$ then $G(r)$ has a non trivial quotient. Therefore the only case which present difficulty is the case $\exp(r) = \pm 1$, and by taking r^{-1} in place of r, we can treat only the case $\exp(r) = 1$. We denote by π_1 the natural homomorphism from G to $G(r)$, and by π_2 the natural homomorphism from $\mathbb{Z}$ to $G(r)$.
We have the following simple proposition:

Proposition 0.1 *The following properties are equivalent:*
(1) *The Kervaire conjecture is true.*
(2) *If G is non trivial then π_1 is an embedding or $\pi_2(t) \neq 1$.*
(3) *If G is non trivial and $exp(r) = 1$ then π_1 is an embedding.*
(4) *If G is non trivial and $exp(r) = 1$ then the natural homomorphism $\varphi_1 : G \to F(r)$ is an embedding, where $F(r) = \langle G * \mathbb{Z} \mid [r, G] = [r, \mathbb{Z}] = 1 \rangle$.*

Proof
(1) $\Rightarrow$ (2). Suppose that the Kervaire Conjecture is true. Let G be a non trivial group. G is embeddable in a simple group say H. Let $r \in G * \mathbb{Z}$, and suppose that there is an non trivial element $g_0 \in G$ such that $g_0 \in (r)_{G^*}$. Then $g_0 \in (r)_{H^*}$; thus $g_0 \in H \cap (r)_{H^*}$. Since H is simple we have $H \cap (r)_{H^*} = H$. Therefore we have $H \subseteq (r)_{H^*}$.

Suppose that $t \in (r)_{G^*}$, then $t \in (r)_{H^*}$, thus we have $H^* \subseteq (r)_{H^*}$. Then $H(r)$ is trivial; this contradict the Kervaire conjecture.

(2) $\Rightarrow$ (3). Suppose that $\exp(r) = 1$; then $t \in (r)_{G^*}$ thus by (2) π_1 is an embedding.

(3) $\Rightarrow$ (4). It is not difficult to see that there is a natural homomorphism $\varphi : F(r) \to G(r)$ such that $\varphi(\varphi_1(g)) = \pi_1(g)$ for every $g \in G$. Since $\pi_1(g) \neq 1$ for every $g \in G$, $g \neq 1$, we have $\varphi_1(g) \neq 1$ for every $g \in G$, $g \neq 1$.

(4) $\Rightarrow$ (1). Now suppose that the enounced property is true. Let K be the subgroup of $F(r)$ generated by $\varphi_1(G)$ and by $r^{-1}t$. If $r = g_0.t^{\varepsilon_1} g_2 \cdots t^{\varepsilon_n} g_n$ then we see that :

$$g_0.(r^{-1}t)^{\varepsilon_1} g_1 \cdots (r^{-1}t)^{\varepsilon_n} g_n = r^{-exp(r)} g_0.t^{\varepsilon_1} g_2 \cdots t^{\varepsilon_n} g_n = 1$$

Therefore there is an homomorphism $\varphi : \hat{G}(r) \to K$, such that $\varphi(\pi_1(g)) = \varphi_1(g)$ for every $g \in G$. Now since for every $g \in G - \{1\}$, $\varphi_1(g) \neq 1$, we have the result.
□

The following example shows that we can not have always π_1 an embedding : let G be a group containing two non trivial elements a, b of different finite order $m, n; m \neq n$ and let $r = at^{-1}bt$, then we see that π_1 is not an embedding. But in this case we have $\exp(r) = 0$ and the Kervaire Conjecture is obviously true. This instigate to introduce a strong version of the Kervaire Conjecture. Let say that r is nonsingular if $\exp(r) \neq 0$. Then the **strong Kervaire Conjecture** is : *if r is nonsingular then π_1 is an embedding.*

The above problem was asked by R.C.Lyndon in [5].

There is another conjecture called Scott-Wiegold Conjecture which can be stated as follows : The group $((\mathbb{Z}_n * \mathbb{Z}_m * \mathbb{Z}_s)/(r))$ is not trivial, where (r) is the normal closure of r in $(\mathbb{Z}_n * \mathbb{Z}_m * \mathbb{Z}_s)$. This Conjecture was solved positively by J. Howie in [6](2002).

Now we state an topological conjecture which is equivalent to the Kervaire conjecture. Fico and Ramirez has recentely proved that the following conjecture is equivalent to the Kervaire conjecture:

Conjecture $\mathbb{Z}$ If F is a compact orientable non-seperaing surface properly embedded in the Knot exterior E then $\pi_1(E/F)$ (the fundamental group of the quotient space) is infinite cyclic.

We will not discuss about the above Conjecture in this preprint. We will concentrate our exposition on brief presentation of the method used by Klyachko to prove the Kervaire Conjecture for torsion-free groups. In fact he has proved the following strong version:

Theorem 0.2 [3] *If G is a non trivial group, and $exp(r) = 1$, and every coefficient of r is of infinite order then π_1 is an embedding.*

Let H be a subgroup of G and let $g \in G$. We say that g is **free relative** to H if the subgroup $\langle g, H\rangle$ of G generated by g and H is naturally the free product $\langle g\rangle * H$.

The proof of Klyachko of the above theorem pass by the following theorem:

Theorem 0.3 *Let H, H' be two isomorphic subgroups of a group Γ under the isomorphism $\phi : H \to H'$. Let $(a_i)_{i=0,r}$, $(b_i)_{i=0,r}$ two sequences of elements of Γ such that :*

(1) *For every i, a_i and b_i are of infinite order.*

(2) *For every i, a_i is free relative to H and b_i is free relative to H'.*

Let c be an arbitrary element of Γ. Then the system of equations :

$$(b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t)ct = 1,$$
$$\phi(h) = h^t, \quad h \in H,$$

has a solution over Γ. □

Let G be a group and consider the homomorphism $\exp : G * \langle t \mid \rangle :\to \mathbb{Z}$. Let Ker be the kernel of exp. Any element k of Ker has an expression $k = g_1^{t^{\sigma_1}} \cdots g_r^{t^{\sigma_r}}$, $\sigma_i \neq \sigma_{i+1}$. Define $min(k) = min\{\sigma_i\}$ and $max(k) = max\{\sigma_i\}$. For an integer m let:

$$H_m = \langle k \in Ker \ ; \ min(k) \geq 0, max(k) \leq m-2\rangle$$
$$H'_m = \langle k \in Ker \ ; \ min(k) \geq 1, max(k) \leq m-1\rangle$$
$$J_m = \langle k \in Ker \ ; \ min(k) \geq 0, max(k) \leq m-1\rangle$$

And the following subsets :

$$X_m = \{k \in Ker \ ; \ min(k) = 0, max(k) \leq m-1\}$$
$$Y_m = \{k \in Ker \ ; \ min(k) \geq 0, max(k) = m\}$$

Then we have the following lemma:

Lemma 0.4 *Let $r \in G * \langle t \rangle$ satisfy $exp(r) = 1$, and every coefficient of r is of infinite order. Then after conjugation, r can be written as a product:*

$$(b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t)ct,$$

where for some m we have: for every $i = 0, \ldots, r$, $a_i \in Y_m$, $b_i \in X_m$, a_i, b_i are of infinite order, and a_i is free relative to H_m and b_i is free relative to H'_m, and $c \in J$. □

Now we are going to deduce theorem 0.1, from theorem 0.2 and lemma 0.3. By the above lemma we can assume that $r(t) = (b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t)ct$, $a_i \in Y_m$, $b_i \in X_m$, a_i, b_i are of infinite order, and a_i is free relative to H_m and b_i is free relative to H'_m, and $c \in J$. We need to think of each a_i, b_i, c as functions of t and for clarity we shall introduce a new variable s, and we consider: $w(s,t) = (b_0(t)a_0(t)^s b_1(t)a_1(t)^s \cdots b_r(t)a_r(t)^s)cs$.

Let $\Gamma = G * \langle t \rangle$. Now we see that H_m and H'_m are isomorphic under the isomorphism $\phi(x) = x^t$. By theorem 0.2., Γ embeds in

$$\tilde{\Gamma} = \langle \Gamma, s \mid w(s,t) = e, h^s = h^t, h \in H \rangle.$$

Therefore we have: for every $g \in G$, $g \in H$ and we have: $g^s = g^t$. Therefore we have: $a_i(t) = a_i(s)$ and finally we have :

$$w(s,s) = (b_0(s)a_0(s)^s b_1(s)a_1(s)^s \cdots b_r(s)a_r(s)^s)cs = r(s) = 1.$$

Therefore there is an extension of G in which $r(x) = 1$ has a solution. This prove the theorem 0.1.

Klyachko has also showen others results of the same kind :

Theorem 0.5 (Free product) *Let A, B be groups and suppose that each factor of $u \in A * B - A$ has infinite order. Then the natural homomorphism $A \rightarrow \langle A * B \mid [A, u] = 1 \rangle$ is an embedding.*

Theorem 0.6 (HNN extension) *Let H, H' be two isomrphic subgroup of the group A under ϕ $H \rightarrow H'$. Let B a group and let $w \in A * B - A$ have a torsion free factors. Then the natural map : $A \rightarrow \langle A * B \mid w^{-1}hw = \phi(h), h \in H \rangle$ is an embedding.*

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