Groups of Finite Quantifier-free Rank

Abderezak Ould Houcine

In a preceding work [1], we have studied existentially closed (abbreviated e.c.) CSA-groups. This was motivated by a question of Poizat whenever an e.c. CSA-group is a bad group and by the fact that a bad group of Morley rank 3 is a CSA-group. We showed in [3] that these groups are not superstable. This is due to the existence of a lot of quantifier-free types.

One defines a quantifier-free rank (abbreviated QF-rank), by analogy with the Morley rank restricted to quantifier-free formulas (abbreviated QF-formulas).

Definition. Let $\mathcal{M}$ be a model and let $X \subseteq \mathcal{M}$ be a QF-definable set. We define $QF(X) = RC(X)_{\mathcal{M}^*}$, where $\mathcal{M}^*$ is an $\aleph_0$-saturated elementary extension of $\mathcal{M}$ and $RC(X)_{\mathcal{M}^*}$ is the Cantor rank of X in $\mathcal{M}^*$ relatively to the boolean algebra of QF-definable subsets. We define $QF(\mathcal{M}) = QF(x = x)$.

It follows that a subgroup of a group of finite Morley rank has a finite QF-rank. So in any tentative of construction of groups of finite Morley rank from classes of finitely generated groups, it is necessary to take finitely generated groups of finite QF-rank.

As linear groups are subgroups of groups of finite Morley rank, they have a finite QF-rank and this gives a model-theoretic context in which we can study them.

We discuss here some results about groups of finite QF-rank [2]. First we have the following important tool.

Theorem 1. *Let $\mathcal{M}$ be a model of ordinal QF-rank. Let $\phi(\bar{x})$ and $\psi(\bar{x}, \bar{y})$ be a QF-formulas. Then there exists a finite sequence $\bar{a}_{ij} \in \phi(\mathcal{M})$, such that the set*

$$\{\bar{b} \in \mathcal{M} \mid QF(\phi(\bar{x}) \wedge \psi(\bar{x}, \bar{b})) = QF(\phi)\},$$

is QF-definable in $\mathcal{M}$ by $\bigvee_{i=1}^{i=n} (\bigwedge_{j=1}^{j=m} \psi(\bar{a}_{ij}, \bar{y}))$. □

As in the context of groups of finite Morley rank one introduces: stabilizer relatively to the action of G on the set of QF-1-types, QF-generic sets, QF-connectedness and so on ...

Some results are generalizable but by using methods which are different from those in the finite Morley rank case. For instance, for any QF-1-type p, $stab(p)$ is QF-definable and if X is a QF-generic set then a finite number of translates of X cover G. As an application we have:

Theorem 2. *Let $\mathcal{M}$ be a nonabelian model of the universal theory of free groups. Then $\mathcal{M}$ is QF-connected, $\mathcal{M}$ is e.c. in $\mathcal{M} * \mathbb{Z}$, and every QF-definable proper subgroup of $\mathcal{M}$ is abelian.* □

Here $\mathcal{M} * \mathbb{Z}$ denotes the free product of $\mathcal{M}$ and $\mathbb{Z}$. The fact that $\mathcal{M}$ is QF-connected and $\mathcal{M}$ is e.c. in $\mathcal{M} * \mathbb{Z}$ can be deduced from [4, 5].

In that context the following questions are open.

Question 3. *If $\mathcal{M}$ is a nonabelian model of the universal theory of free groups, is $\mathcal{M} \preceq \mathcal{M} * \mathbb{Z}$?*

Question 4. *If $\mathcal{M}$ is a nonabelian model of the universal theory of free groups such that $\mathcal{M} \preceq \mathcal{M} * \mathbb{Z}$, is any proper definable subgroup of $\mathcal{M}$ abelian ?*

The answer to the question 3 is positive when $\mathcal{M}$ is finitely generated and this was pointed out to me by Sela. The answer to the question 4 is positive if $\mathcal{M}$ is a nonabelian free group and this was anounced by Bestvina and Feighn.

In dealing with finitely generated groups of finite QF-rank the following question arises naturally.

Question 5. *Is there any alternative for groups of finite QF-rank which looks like the Tits alternative?*

First one looks at locally (solvable-by-finite) groups.

Theorem 6.

• *A locally (solvable-by-finite) group of finite QF-rank is either locally finite or has a nontrivial normal abelian subgroup.*

• *A locally solvable group of finite QF-rank is solvable.* □

It follows that the solvable radical $R(G)$ is solvable (which is in fact definable by equations), and if G is locally (solvable-by-finite) then $G/R(G)$ is locally finite. We have also that if $H \leq G$ is a QF-definable subgroup then $N_G(H)$ is QF-definable. This generalizes known properties of linear groups.

Under additional hypotheses one can get an alternative. Recall that a group is called locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index. For instance locally residually finite groups are locally graded as well as linear groups.

Theorem 7. *Let G be a locally graded group of finite QF-rank. Then either G is solvable-by-finite or there is an elementary extension $G \preceq K$ such that K contains a free semigroup.* □

In the context of p-groups we have.

Theorem 8. *A p-group of finite QF-rank is either nilpotent-by-finite or has a series $G_0 \trianglelefteq G_1 \leq G$, where G_1 is QF-definable, G_0 is equationally-definable in G_1, and G_1/G_0 is a nonabelian CSA-group. In particular a bad p-group interprets a CSA bad p-group.* □

Question 9. *Is there a CSA p-group of finite QF-rank ?*

As noticed above the study of groups of finite QF-rank was motivated by the understanding of subgroups of groups of finite Morley rank and to get several information in order to use an amalgamation method.

One looks initially at Fraïssé limits of classes of finitely generated groups having the amalgmation property. Let $\mathcal{K}$ be a countable class of finitely generated groups having the hereditary property (abbriviated HP) and the amalgamation property

(abbriviated AP). Since one can amalgamate over the trivial group, $\mathcal{K}$ has the joint embedding property and by Fraïssé's theorem, $\mathcal{K}$ has an ultrahomogenous Fraïssé limit which will be denoted by $\widehat{\mathcal{K}}$.

Definition. Le $\mathcal{K}$ be a class of groups.

(1) $\mathcal{K}$ is said to be *closed under free products* if whenever $A, B \in \mathcal{K}$, $A * B \in \mathcal{K}$.

(2) $\mathcal{K}$ is said to be *closed under HNN-extensions* if whenever $A \in \mathcal{K}$ such that A contains two isomorphic finitely generated subgroups A_1, A_2 then the HNN-extension $A^* = \langle A, t | A_1^t = A_2 \rangle$ is also in $\mathcal{K}$.

Theorem 10. *Let $\mathcal{K}$ be a countable class of finitely generated groups having HP and AP.*

(1) *Suppose that $\mathcal{K}$ satisfies: if $A, B \in \mathcal{K}$ then $A * B \models Th_\forall(\widehat{\mathcal{K}})$. Then $\widehat{\mathcal{K}}$ is not superstable.*

(2) *If $\mathcal{K}$ is closed under free products, then $\widehat{\mathcal{K}}$ is simple and not superstable.*

(3) *If $\mathcal{K}$ is closed under HNN-extensions, then $\widehat{\mathcal{K}}$ is simple and unstable.* □

We notice that every e.c. group is the Fraïssé limit of its age.

References

[1] E. Jaligot, A. Ould Houcine, *Existentially closed CSA-groups*, Journal of Algebra, **280** (2004), 772–796.

[2] A. Ould Houcine, *Algebraic geometry over groups and quantifier-free rank*, work in progress.

[3] A. Ould Houcine, *On superstable CSA-groups*, to appear in Annals of Pure and Applied Logic.

[4] O. Kharlampovich, A. Myasnikov, *Implicit function theorem over free groups*, Journal of Algebra, **290** (2005), 1–203.

[5] G. Baumslag, A. Myasnikov, V. N. Remeslennikov, *Algebraic geometry over groups I: Algebraic sets and ideal theory*, Journal of Algebra, **219** (1999), 16–79.

Topological Dynamics and Model Theory

Ludomir Newelski

Assume G is a group definable in a sufficiently saturated first-order structure M. $\mathfrak{C}$ denotes a monster model.

The investigation of combinatorial properties of coverings of groups by countably many type-definable sets hinted that there should be a notion of a "large" subset of a group, generalizing that of a generic subset, present in the stable context [1]. Consequently, in [2] I proposed a notion of a weak generic set and type (in a group).

Namely, we say that a set $U \subseteq G$ is *(left) generic* if some finitely many left translates of U cover G. We say that U is *weak generic* if for some non-generic $V \subseteq G$ we have that $U \cup V$ is generic. The notion of a weak generic type is defined accordingly. Unlike generic types, complete weak generic types always exist.

This notion was subsequently used to show some combinatorial properties of countable coverings of groups by type-definable sets. Eventually it turned out