# *Introduction to Tarski's problem and Model Theory*

**Abderezak Ould Houcine**

Camille Jordan Institute, University Lyon 1, France

Non Positive Curvature and the Elementary Theory of Free Groups,
Anogia, June 10, 2008

# *A little Model Theory of Groups: Formulas & Sentences*

## *Definition*

# *A little Model Theory of Groups: Formulas & Sentences*

## *Definition*

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

*Definition*

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

$$\varphi(\bar{x}) := \bigvee_{1\leq i\leq n} \Big( \bigwedge_{1\leq j\leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1\leq j\leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

$w_{ij}$ and $v_{ij}$ are words on $\{x_1, \ldots, x_n\}^{\pm 1}$.

*Definition*

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

$$\varphi(\bar{x}) := \bigvee_{1\leq i\leq n} \Big( \bigwedge_{1\leq j\leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1\leq j\leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

$w_{ij}$ and $v_{ij}$ are words on $\{x_1, \ldots, x_n\}^{\pm 1}$.

- Formula $\phi(\bar{z})$, with free variables $\bar{z}$,

*Definition*

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

$$\varphi(\bar{x}) := \bigvee_{1\leq i\leq n} \Big( \bigwedge_{1\leq j\leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1\leq j\leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

$w_{ij}$ and $v_{ij}$ are words on $\{x_1, \ldots, x_n\}^{\pm 1}$.

- Formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \varphi(x_1, y_1, \ldots, x_n, y_n, \bar{z}),$$

where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

# A little Model Theory of Groups: Formulas & Sentences

## Definition

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

$$\varphi(\bar{x}) := \bigvee_{1\leq i\leq n} \Big( \bigwedge_{1\leq j\leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1\leq j\leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

$w_{ij}$ and $v_{ij}$ are words on $\{x_1, \ldots, x_n\}^{\pm 1}$.

- Formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \varphi(x_1, y_1, \ldots, x_n, y_n, \bar{z}),$$

where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

- $\bar{g} \in G \implies \phi(\bar{g})$ is a formula with parameters from $G$.

# *A little Model Theory of Groups: Formulas & Sentences*

## *Definition*

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

$$\varphi(\bar{x}) := \bigvee_{1\leq i\leq n} \Big( \bigwedge_{1\leq j\leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1\leq j\leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

$w_{ij}$ and $v_{ij}$ are words on $\{x_1, \ldots, x_n\}^{\pm 1}$.

- Formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \varphi(x_1, y_1, \ldots, x_n, y_n, \bar{z}),$$

where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

- $\bar{g} \in G \implies \phi(\bar{g})$ is a formula with parameters from $G$.
- Sentences := formulas without free variables.

# A little Model Theory of Groups: Formulas & Sentences

## Definition

- Quantifier-free formula $\varphi(\bar{x})$, with free variables $\bar{x}$,

$$\varphi(\bar{x}) := \bigvee_{1\leq i\leq n} \Big( \bigwedge_{1\leq j\leq p_i} w_{ij}(\bar{x}) = 1 \bigwedge_{1\leq j\leq q_i} v_{ij}(\bar{x}) \neq 1 \Big),$$

$w_{ij}$ and $v_{ij}$ are words on $\{x_1, \ldots, x_n\}^{\pm 1}$.

- Formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \varphi(x_1, y_1, \ldots, x_n, y_n, \bar{z}),$$

  where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

- $\bar{g} \in G \implies \phi(\bar{g})$ is a formula with parameters from $G$.
- Sentences := formulas without free variables.
- $G \models \phi \Leftrightarrow G$ satisfies $\phi$.

**Examples:**

**Examples:**

- $\phi(z) = \forall x(x^{-1}z^{-1}xz = 1)$,

**Examples:**

- $\phi(z) = \forall x(x^{-1}z^{-1}xz = 1)$,

$$G \models \phi(g) \Leftrightarrow g \in Z(G)$$

# *A little Model Theory of Groups: Formulas & Sentences*

**Examples:**

- $\phi(z) = \forall x(x^{-1}z^{-1}xz = 1)$,

$$G \models \phi(g) \Leftrightarrow g \in Z(G)$$

- $\phi = \forall x \forall y(x^{-1}y^{-1}xy = 1)$,

**Examples:**

- $\phi(z) = \forall x(x^{-1}z^{-1}xz = 1)$,

$$G \models \phi(g) \Leftrightarrow g \in Z(G)$$

- $\phi = \forall x \forall y(x^{-1}y^{-1}xy = 1)$,

$$G \models \phi \Leftrightarrow G \text{ is abelian}$$

# *A little Model Theory of Groups: Formulas & Sentences*

**Examples:**

- $\phi(z) = \forall x(x^{-1}z^{-1}xz = 1)$,

$$G \models \phi(g) \Leftrightarrow g \in Z(G)$$

- $\phi = \forall x \forall y(x^{-1}y^{-1}xy = 1)$,

$$G \models \phi \Leftrightarrow G \text{ is abelian}$$

- $\phi_n = \forall x(x^n = 1 \Rightarrow x = 1)$, $n \geq 1$,

# *A little Model Theory of Groups: Formulas & Sentences*

**Examples:**

- $\phi(z) = \forall x(x^{-1}z^{-1}xz = 1)$,

$$G \models \phi(g) \Leftrightarrow g \in Z(G)$$

- $\phi = \forall x \forall y(x^{-1}y^{-1}xy = 1)$,

$$G \models \phi \Leftrightarrow G \text{ is abelian}$$

- $\phi_n = \forall x(x^n = 1 \Rightarrow x = 1)$, $n \geq 1$,

$$G \models \phi_n \text{ for any } n \geq 1 \Leftrightarrow G \text{ is torsion-free}$$

# *A little Model Theory of Groups: Elementary equivalence*

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

**Examples:**

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

**Examples:**

- $G$ is a group,

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

**Examples:**

- $G$ is a group,

$$Th(G) = \{\phi \text{ a sentence } | G \models \phi\},$$

$Th(G)$ is the *complete elementary theory* of $G$.

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

**Examples:**

- $G$ is a group,

$$Th(G) = \{\phi \text{ a sentence } | G \models \phi\},$$

$Th(G)$ is the *complete elementary theory* of $G$.
It is complete; that is for any sentence $\phi$ either $\phi \in Th(G)$ or $\neg\phi \in Th(G)$.

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

**Examples:**

- $G$ is a group,

$$Th(G) = \{\phi \text{ a sentence } | G \models \phi\},$$

$Th(G)$ is the *complete elementary theory* of $G$.
It is complete; that is for any sentence $\phi$ either $\phi \in Th(G)$ or $\neg\phi \in Th(G)$.

- $\Gamma = \bigcap_{G \in TFH} Th(G)$,
TFH:= class of torsion-free hyperbolic groups.

# *A little Model Theory of Groups: Elementary equivalence*

- First order theory:= set of sentences.

**Examples:**

- $G$ is a group,

$$Th(G) = \{\phi \text{ a sentence } | G \models \phi\},$$

$Th(G)$ is the *complete elementary theory* of $G$.
It is complete; that is for any sentence $\phi$ either $\phi \in Th(G)$ or $\neg\phi \in Th(G)$.

- $\Gamma = \bigcap_{G \in TFH} Th(G)$,

TFH:= class of torsion-free hyperbolic groups.
$\Gamma$ is a first order theory but it is not complete.

# *A little Model Theory of Groups: Elementary equivalence*

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

- If $H \leq G$, $H$ is an *elementary subgroup* of $G$, $H \prec G$, if for any formula $\phi(\bar{z})$, for any tuple $\bar{h}$ in $H$,

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

- If $H \leq G$, $H$ is an *elementary subgroup* of $G$, $H \prec G$, if for any formula $\phi(\bar{z})$, for any tuple $\bar{h}$ in $H$,

$$G \models \phi(\bar{h}) \Rightarrow H \models \phi(\bar{h}).$$

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

- If $H \leq G$, $H$ is an *elementary subgroup* of $G$, $H \prec G$, if for any formula $\phi(\bar{z})$, for any tuple $\bar{h}$ in $H$,

$$G \models \phi(\bar{h}) \Rightarrow H \models \phi(\bar{h}).$$

- (Tarski-Vaught Test)

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

- If $H \leq G$, $H$ is an *elementary subgroup* of $G$, $H \prec G$, if for any formula $\phi(\bar{z})$, for any tuple $\bar{h}$ in $H$,

$$G \models \phi(\bar{h}) \Rightarrow H \models \phi(\bar{h}).$$

- (Tarski-Vaught Test) If $H \leq G$, then

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

- If $H \leq G$, $H$ is an *elementary subgroup* of $G$, $H \prec G$, if for any formula $\phi(\bar{z})$, for any tuple $\bar{h}$ in $H$,

$$G \models \phi(\bar{h}) \Rightarrow H \models \phi(\bar{h}).$$

- (Tarski-Vaught Test) If $H \leq G$, then

$$H \prec G$$

# *A little Model Theory of Groups: Elementary equivalence*

- $H$ and $G$ are *elementary equivalent* $\Leftrightarrow$ $Th(G) = Th(H)$.

A stronger property:

- If $H \leq G$, $H$ is an *elementary subgroup* of $G$, $H \prec G$, if for any formula $\phi(\bar{z})$, for any tuple $\bar{h}$ in $H$,

$$G \models \phi(\bar{h}) \Rightarrow H \models \phi(\bar{h}).$$

- (Tarski-Vaught Test) If $H \leq G$, then

$$H \prec G \Leftrightarrow \left\{ \begin{array}{c} \text{for any formula } \phi(x;\bar{z}), \text{ for any } \bar{h} \in H, \\ \exists c \in G, G \models \phi(c;\bar{h}) \Rightarrow \exists c \in H, G \models \phi(c,\bar{h}) \end{array} \right.$$

# *A little Model Theory of Groups: Elementary equivalence*

# *A little Model Theory of Groups: Elementary equivalence*

Roughly speaking:

# *A little Model Theory of Groups: Elementary equivalence*

Roughly speaking:

$$Th(H) = Th(G)$$

$$\Updownarrow$$

$H$ and $G$ are indiscernible in the first order logic

# *A little Model Theory of Groups: Elementary equivalence*

Roughly speaking:

$$Th(H) = Th(G)$$

$$\Updownarrow$$

$H$ and $G$ are indiscernible in the first order logic

and

# *A little Model Theory of Groups: Elementary equivalence*

Roughly speaking:

$$Th(H) = Th(G)$$

$$\Updownarrow$$

$H$ and $G$ are indiscernible in the first order logic

and

$$H \prec G$$

$$\Updownarrow$$

$H$ is "algebraically closed" in $G$

# *Tarski's Problem*

# *Tarski's Problem*

$F_X$ the free group on $X$, $|X|$ the cardinal of $X$.

# *Tarski's Problem*

$F_X$ the free group on $X$, $|X|$ the cardinal of $X$.

Around 1945, Tarski asks:

# *Tarski's Problem*

$F_X$ the free group on $X$, $|X|$ the cardinal of $X$.

Around 1945, Tarski asks:

**Pb1** (Following Vaught): Is it true that $Th(F_X) = Th(F_Y)$, for $|X|, |Y| \geq 2$ ?

# *Tarski's Problem*

$F_X$ the free group on $X$, $|X|$ the cardinal of $X$.

Around 1945, Tarski asks:

**Pb1** (Following Vaught): Is it true that $Th(F_X) = Th(F_Y)$, for $|X|, |Y| \geq 2$ ?

**Pb2**: Is it true that $Th(F_X)$ decidable ?

# *Tarski's Problem*

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.**

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.** We use Tarski-Vaught test.

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.** We use Tarski-Vaught test.

The problem is reduced to show:

$$F_Y \models \phi(\bar{g}; b) \Rightarrow \exists b' \in F_X, F_Y \models \phi(\bar{g}; b').$$

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.** We use Tarski-Vaught test.

The problem is reduced to show:

$$F_Y \models \phi(\bar{g}; b) \Rightarrow \exists b' \in F_X, F_Y \models \phi(\bar{g}; b').$$

$F_Y \models \phi(\bar{g}; b) \Rightarrow \bar{g} \in \langle X_0 \rangle, b \in \langle X_0, Y_0 \rangle$, with $X_0 \subseteq X$, $Y_0 \subseteq Y$ and $X_0 \cap Y_0 = \emptyset$.

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.** We use Tarski-Vaught test.

The problem is reduced to show:

$$F_Y \models \phi(\bar{g}; b) \Rightarrow \exists b' \in F_X, F_Y \models \phi(\bar{g}; b').$$

$F_Y \models \phi(\bar{g}; b) \Rightarrow \bar{g} \in \langle X_0 \rangle, b \in \langle X_0, Y_0 \rangle$, with $X_0 \subseteq X$, $Y_0 \subseteq Y$ and $X_0 \cap Y_0 = \emptyset$.

Let $X_1 \subseteq X \setminus X_0$ with $|X_1| = |Y_0|$.

# *Tarski's Problem*

*Theorem 1 (Vaught, 1955)*

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.** We use Tarski-Vaught test.

The problem is reduced to show:

$$F_Y \models \phi(\bar{g}; b) \Rightarrow \exists b' \in F_X, F_Y \models \phi(\bar{g}; b').$$

$F_Y \models \phi(\bar{g}; b) \Rightarrow \bar{g} \in \langle X_0 \rangle, b \in \langle X_0, Y_0 \rangle$, with $X_0 \subseteq X$, $Y_0 \subseteq Y$ and $X_0 \cap Y_0 = \emptyset$.

Let $X_1 \subseteq X \setminus X_0$ with $|X_1| = |Y_0|$.

$f : Y \to Y, f(X \setminus X_1) = X \setminus X_1, f(Y_0) = X_1, f(X_1) = f(Y_0)$, and leaves the rest unchanged, is an automorphism and $f(\bar{g}) = \bar{g}$.

# *Tarski's Problem*

**Theorem 1 (Vaught, 1955)**

*If $X \subseteq Y$, $X$ is infinite, then $F_X$ is an elementary subgroup of $F_Y$.*

**Proof.** We use Tarski-Vaught test.

The problem is reduced to show:

$$F_Y \models \phi(\bar{g}; b) \Rightarrow \exists b' \in F_X, F_Y \models \phi(\bar{g}; b').$$

$F_Y \models \phi(\bar{g}; b) \Rightarrow \bar{g} \in \langle X_0 \rangle, b \in \langle X_0, Y_0 \rangle$, with $X_0 \subseteq X$, $Y_0 \subseteq Y$ and $X_0 \cap Y_0 = \emptyset$.

Let $X_1 \subseteq X \setminus X_0$ with $|X_1| = |Y_0|$.

$f : Y \to Y, f(X \setminus X_1) = X \setminus X_1, f(Y_0) = X_1, f(X_1) = f(Y_0)$, and leaves the rest unchanged, is an automorphism and $f(\bar{g}) = \bar{g}$.

Hence $F_Y \models \phi(\bar{g}; f(b)), f(b) \in F_X$. □

# *Tarski's Problem*

# *Tarski's Problem*

What about free "object" in other varieties ?

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.
  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.

  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.

  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.

  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.

  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).

- $N_n^c :=$ the free nilpotent group of class $c$ and rank $n$.

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.
  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.
  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).
- $N_n^c :=$ the free nilpotent group of class $c$ and rank $n$.
  $Th(N_c^n) = Th(N_{c'}^{n'})$ if and only if $n = n'$ and $c = c'$.

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.
  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.
  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).
- $N_n^c :=$ the free nilpotent group of class $c$ and rank $n$.
  $Th(N_c^n) = Th(N_{c'}^{n'})$ if and only if $n = n'$ and $c = c'$.
  for $c \geq 2, n \geq 2$, $Th(N_c^n)$ is undecidable (Mal'cev, 1960).

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.
  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.
  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).
- $N_n^c :=$ the free nilpotent group of class $c$ and rank $n$.
  $Th(N_c^n) = Th(N_{c'}^{n'})$ if and only if $n = n'$ and $c = c'$.
  for $c \geq 2, n \geq 2$, $Th(N_c^n)$ is undecidable (Mal'cev, 1960).
- $S_n^c :=$ the free soluble group of class $c$ and rank $n$.

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.
  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.
  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).
- $N_n^c :=$ the free nilpotent group of class $c$ and rank $n$.
  $Th(N_c^n) = Th(N_{c'}^{n'})$ if and only if $n = n'$ and $c = c'$.
  for $c \geq 2, n \geq 2$, $Th(N_c^n)$ is undecidable (Mal'cev, 1960).
- $S_n^c :=$ the free soluble group of class $c$ and rank $n$.
  $Th(S_n^c) = Th(S_{n'}^c)$ if and only if $n = n'$ and $c = c'$.

# *Tarski's Problem*

What about free "object" in other varieties ?

- $\mathbb{Z}_n :=$ the free abelian group of rank $n$.
  $Th(\mathbb{Z}_n) = Th(\mathbb{Z}_m)$ if and only if $n = m$.
  $Th(\mathbb{Z}_n)$ is decidable (Szmielew, 1955).
- $N_n^c :=$ the free nilpotent group of class $c$ and rank $n$.
  $Th(N_c^n) = Th(N_{c'}^{n'})$ if and only if $n = n'$ and $c = c'$.
  for $c \geq 2, n \geq 2$, $Th(N_c^n)$ is undecidable (Mal'cev, 1960).
- $S_n^c :=$ the free soluble group of class $c$ and rank $n$.
  $Th(S_n^c) = Th(S_{n'}^c)$ if and only if $n = n'$ and $c = c'$.
  for $c, n \geq 2$, $Th(S_n^c)$ is undecidable (Mal'cev, 1960).

# *Tarski's Problem*

# *Tarski's Problem*

**Remark**.

# *Tarski's Problem*

**Remark**. Theorem 1 is still true for free object of varieties.

# *Tarski's Problem*

**Remark**. Theorem 1 is still true for free object of varieties. In particular we have:

$$N_X^c := \text{the free nilpotent group of class } c, \text{ on } X$$

$$X \subseteq Y, X \text{ infinite } \Rightarrow N_X^c \prec N_Y^c,$$

$$S_X^c := \text{the free soluble group of class } c, \text{ on } X$$

$$X \subseteq Y, X \text{ infinite } \Rightarrow S_X^c \prec S_Y^c.$$

# *Tarski's Problem: Universal theory & Equations*

## *Definition*

# *Tarski's Problem: Universal theory & Equations*

*Definition*

- an universal formula $\phi(\bar{z})$, with free variables $\bar{z}$,

# *Tarski's Problem: Universal theory & Equations*

*Definition*

- an universal formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\forall \bar{x} \varphi(\bar{x}, \bar{z}),$$

where $\varphi(\bar{x}, \bar{z})$ is a quantifier-free formula.

# *Tarski's Problem: Universal theory & Equations*

*Definition*

- an universal formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\forall \bar{x} \varphi(\bar{x}, \bar{z}),$$

  where $\varphi(\bar{x}, \bar{z})$ is a quantifier-free formula.
- universal sentence:= universal formula without free variables.

# *Tarski's Problem: Universal theory & Equations*

*Definition*

- an universal formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\forall \bar{x} \varphi(\bar{x}, \bar{z}),$$

  where $\varphi(\bar{x}, \bar{z})$ is a quantifier-free formula.
- universal sentence:= universal formula without free variables.
- Let $G$ be a group.

# *Tarski's Problem: Universal theory & Equations*

*Definition*

- an universal formula $\phi(\bar{z})$, with free variables $\bar{z}$,

$$\forall \bar{x} \varphi(\bar{x}, \bar{z}),$$

  where $\varphi(\bar{x}, \bar{z})$ is a quantifier-free formula.
- universal sentence:= universal formula without free variables.
- Let $G$ be a group. The universal theory of $G$,

$$Th_{\forall}(G) := \text{the set of universal sentences true in } G.$$

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

$|X| = \omega$, $F_X$ is embeddable in $F_2$.

# *Tarski's Problem: Universal theory & Equations*

$|X| = \omega$, $F_X$ is embeddable in $F_2$.
Indeed: $F_2 = \langle a, b | \rangle$, the subgroup $\langle a^{-n} b a^n ; i \in \mathbb{N} \rangle$ is isomorphic to $F_\omega$.

# *Tarski's Problem: Universal theory & Equations*

$|X| = \omega$, $F_X$ is embeddable in $F_2$.
Indeed: $F_2 = \langle a, b | \rangle$, the subgroup $\langle a^{-n} b a^n ; i \in \mathbb{N} \rangle$ is isomorphic to $F_\omega$.
**Consequence** (combining with Theorem 1):

# *Tarski's Problem: Universal theory & Equations*

$|X| = \omega$, $F_X$ is embeddable in $F_2$.
Indeed: $F_2 = \langle a, b | \rangle$, the subgroup $\langle a^{-n} b a^n ; i \in \mathbb{N} \rangle$ is isomorphic to $F_\omega$.
**Consequence** (combining with Theorem 1):
$Th_\forall(F_X) = Th_\forall(F_Y)$, for $|X|, |Y| \geq 2$.

# *Tarski's Problem: Universal theory & Equations*

$|X| = \omega$, $F_X$ is embeddable in $F_2$.
Indeed: $F_2 = \langle a, b | \rangle$, the subgroup $\langle a^{-n} b a^n ; i \in \mathbb{N} \rangle$ is isomorphic to $F_\omega$.
**Consequence** (combining with Theorem 1):
$Th_\forall(F_X) = Th_\forall(F_Y)$, for $|X|, |Y| \geq 2$.

$\Longrightarrow$ nonabelian free groups have the same universal theory.

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

Vaught made the (unpublished) test problem:

# *Tarski's Problem: Universal theory & Equations*

Vaught made the (unpublished) test problem:

$$\text{Is it true that } F_X \models \forall x \forall y \forall z (x^2 y^2 z^2 = 1 \Rightarrow xy = yx)?$$

**Theorem 2 (Lyndon, 1959)**

*The answer is yes.* □

# *Tarski's Problem: Universal theory & Equations*

Vaught made the (unpublished) test problem:

$$\text{Is it true that } F_X \models \forall x \forall y \forall z (x^2 y^2 z^2 = 1 \Rightarrow xy = yx)?$$

*Theorem 2 (Lyndon, 1959)*

*The answer is yes.* □

**Consequence**: the surface group with presentation

$$\langle x, y, z | x^2 y^2 z^2 = 1 \rangle$$

does not satisfy the universal theory of nonabelian free groups.

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

**Lemma 3 (Mal'cev, 1962)**

# *Tarski's Problem: Universal theory & Equations*

**Lemma 3 (Mal'cev, 1962)**

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$.*

# *Tarski's Problem: Universal theory & Equations*

**Lemma 3 (Mal'cev, 1962)**

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$. Then*

$$F \models \forall x \forall y (x^2 a x^2 a^{-1} (yby b^{-1})^{-2} = 1 \Leftrightarrow x = 1 \wedge y = 1). \quad \square$$

# *Tarski's Problem: Universal theory & Equations*

*Lemma 3 (Mal'cev, 1962)*

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$. Then*

$$F \models \forall x \forall y (x^2 a x^2 a^{-1} (yby b^{-1})^{-2} = 1 \Leftrightarrow x = 1 \wedge y = 1). \quad \square$$

**Consequence**: *A finite system of equations in a nonabelian free group is (effectively) equivalent to a single equation.*

# *Tarski's Problem: Universal theory & Equations*

*Lemma 3 (Mal'cev, 1962)*

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$. Then*

$$F \models \forall x \forall y (x^2 a x^2 a^{-1} (yby b^{-1})^{-2} = 1 \Leftrightarrow x = 1 \wedge y = 1). \quad \square$$

**Consequence**: *A finite system of equations in a nonabelian free group is (effectively) equivalent to a single equation.*

**Remark.** Lemma 3 holds also in nonabelian models of the univeral theory of nonabelian free groups (Kharlampovich and Myasnikov, 1998).

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

**Lemma 4 (Guervich)**

# *Tarski's Problem: Universal theory & Equations*

**Lemma 4 (Guervich)**

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$.*

# *Tarski's Problem: Universal theory & Equations*

**Lemma 4 (Guervich)**

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$. Then*

$$F \models \forall x \forall y([x, y^a] = 1 \wedge [x, y^b] = 1 \wedge [x, y^{ab}] = 1 \Leftrightarrow x = 1 \vee y = 1).$$

□

# *Tarski's Problem: Universal theory & Equations*

**Lemma 4 (Guervich)**

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$. Then*

$$F \models \forall x \forall y([x, y^a] = 1 \wedge [x, y^b] = 1 \wedge [x, y^{ab}] = 1 \Leftrightarrow x = 1 \vee y = 1).$$

□

**Consequence**: *The disjunction of equations in a nonabelian free group is (effectively) equivalent to a finite system of equations.*

# *Tarski's Problem: Universal theory & Equations*

*Lemma 4 (Guervich)*

*Let $F$ be a free group and $a, b \in F$ such that $[a, b] \neq 1$. Then*

$$F \models \forall x \forall y([x, y^a] = 1 \wedge [x, y^b] = 1 \wedge [x, y^{ab}] = 1 \Leftrightarrow x = 1 \vee y = 1).$$

□

**Consequence**: *The disjunction of equations in a nonabelian free group is (effectively) equivalent to a finite system of equations.*

**Remark.** Lemma 4 holds also in nonabelian models of the universal theory of nonabelian free groups (Kharlampovich and Myasnikov, 1998).

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

**Conclusion**:

Let $G$ be a nonabelian model of the universal theory of nonabelian free groups.

# *Tarski's Problem: Universal theory & Equations*

**Conclusion**:
Let $G$ be a nonabelian model of the universal theory of nonabelian free groups.

- A quantifier-free formula $\varphi(\bar{x})$ is (effectivelly) equivalent to a formula of the form

$$\bigvee_{1 \leq i \leq n} (w_i(\bar{x}, a, b) = 1 \wedge v_i(\bar{x}, a, b) \neq 1)$$

# *Tarski's Problem: Universal theory & Equations*

**Conclusion**:
Let $G$ be a nonabelian model of the universal theory of nonabelian free groups.

- A quantifier-free formula $\varphi(\bar{x})$ is (effectivelly) equivalent to a formula of the form

$$\bigvee_{1 \leq i \leq n} (w_i(\bar{x}, a, b) = 1 \wedge v_i(\bar{x}, a, b) \neq 1)$$

*Definition*

A positive formula $\phi(\bar{z})$, with free variables $\bar{z}$, is a formula

# *Tarski's Problem: Universal theory & Equations*

**Conclusion**:
Let $G$ be a nonabelian model of the universal theory of nonabelian free groups.

- A quantifier-free formula $\varphi(\bar{x})$ is (effectivelly) equivalent to a formula of the form

$$\bigvee_{1 \leq i \leq n} (w_i(\bar{x}, a, b) = 1 \wedge v_i(\bar{x}, a, b) \neq 1)$$

*Definition*

A positive formula $\phi(\bar{z})$, with free variables $\bar{z}$, is a formula

$$\phi(\bar{z}) := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \bigvee_{1 \leq i \leq n} \Big( \bigwedge_{1 \leq j \leq p_i} w_{ij}(\bar{x}, \bar{y}, \bar{z}) = 1 \Big)$$

# *Tarski's Problem: Universal theory & Equations*

- A positive formula $\varphi(\bar{x})$ is (effectivelly) equivalent to a formula of the form

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n (w(\bar{x}, \bar{y}, \bar{z}; a, b) = 1)$$

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

**Theorem 5 (Following Kharlampovich&Myasnikov, Merzljakov's Theorem, 1966)**

# *Tarski's Problem: Universal theory & Equations*

**Theorem 5 (Following Kharlampovich&Myasnikov, Merzljakov's Theorem, 1966)**

*Let $F$ be a nonabelian free group.*

# *Tarski's Problem: Universal theory & Equations*

**Theorem 5 (Following Kharlampovich&Myasnikov, Merzljakov's Theorem, 1966)**

*Let $F$ be a nonabelian free group. If*

$$F \models \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \; w(x_1, y_1, \ldots, x_n, y_n; \bar{a}) = 1,$$

# *Tarski's Problem: Universal theory & Equations*

**Theorem 5 (Following Kharlampovich&Myasnikov, Merzljakov's Theorem, 1966)**

*Let $F$ be a nonabelian free group. If*

$$F \models \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \; w(x_1, y_1, \ldots, x_n, y_n; \bar{a}) = 1,$$

*then, there exist words, with parameters from $F$, $v_1(x_1), v_2(x_1, x_2), \ldots, v_n(x_1, x_2, \ldots, x_n)$, such that*

# *Tarski's Problem: Universal theory & Equations*

**Theorem 5 (Following Kharlampovich&Myasnikov, Merzljakov's Theorem, 1966)**

*Let $F$ be a nonabelian free group. If*

$$F \models \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \; w(x_1, y_1, \ldots, x_n, y_n; \bar{a}) = 1,$$

*then, there exist words, with parameters from $F$, $v_1(x_1), v_2(x_1, x_2), \ldots, v_n(x_1, x_2, \ldots, x_n)$, such that*

$$F[x_1, \ldots, x_n] \models w\big(x_1, v_1(x_1), \ldots, x_n, v(x_1, \ldots, x_n); \bar{a}\big) = 1.$$

*Here $F[x_1, \ldots, x_n] = F * \langle x_1, \ldots, x_n | \rangle$.*

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

What about decidability ?

# *Tarski's Problem: Universal theory & Equations*

What about decidability ?

**Theorem 6 (Makanin)**

# *Tarski's Problem: Universal theory & Equations*

What about decidability ?

**Theorem 6 (Makanin)**

(1) *There is a an algorithm for recognizing the solvability of an arbitrary equation in a free group (1982).*

# *Tarski's Problem: Universal theory & Equations*

What about decidability ?

**Theorem 6 (Makanin)**

(1) *There is a an algorithm for recognizing the solvability of an arbitrary equation in a free group (1982).*
(2) *The universal and the positive theory of a nonabelian free group is decidable (1985).*

# *Tarski's Problem: Universal theory & Equations*

What about decidability ?

*Theorem 6 (Makanin)*

(1) *There is a an algorithm for recognizing the solvability of an arbitrary equation in a free group (1982).*
(2) *The universal and the positive theory of a nonabelian free group is decidable (1985).*

**Remark**. The decidability of the positive theory follows easilly from (1) and Merzljakov's Theorem.

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

The first step to understand the elementary theory of a free group is to understand the set of solutions of equations.

# *Tarski's Problem: Universal theory & Equations*

The first step to understand the elementary theory of a free group is to understand the set of solutions of equations.

**_Theorem 7 (Lorents 1963, Appel 1968, Chiswell & Remeslennikov 2000)_**

*The set of solutions of a system of equations, with* **one** *variable, in a free group, is a finite union of cosets of centralizers.* □

# *Tarski's Problem: Universal theory & Equations*

The first step to understand the elementary theory of a free group is to understand the set of solutions of equations.

*Theorem 7 (Lorents 1963, Appel 1968, Chiswell & Remeslennikov 2000)*

*The set of solutions of a system of equations, with* **one** *variable, in a free group, is a finite union of cosets of centralizers.* □

**Remark**. Lorents has anounced the theorem without proof and the proof of Appel contains a gap.

# *Tarski's Problem: Universal theory & Equations*

# *Tarski's Problem: Universal theory & Equations*

**Theorem 8**

# *Tarski's Problem: Universal theory & Equations*

*Theorem 8*

*The set of solutions of a system of equations, with* **one** *variable, in a torsion-free 2-free hyperbolic group, is a finite union of cosets of centralizers.* □

# *Tarski's Problem: Equations & Sela's approach*

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:
Let $w(\bar{x}) = 1$ be an equation (to simplify without parameters).

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:
Let $w(\bar{x}) = 1$ be an equation (to simplify without parameters). Let $H_w = \langle \bar{x} | w(\bar{x}) = 1 \rangle$.

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:
Let $w(\bar{x}) = 1$ be an equation (to simplify without parameters). Let $H_w = \langle \bar{x} | w(\bar{x}) = 1 \rangle$. Let $G$ be a group.

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:
Let $w(\bar{x}) = 1$ be an equation (to simplify without parameters). Let $H_w = \langle \bar{x} | w(\bar{x}) = 1 \rangle$. Let $G$ be a group.

- If $\bar{g} \in G$ such that $G \models w(\bar{g}) = 1$, then there exists a homomorphism $f : H_w \to G$ such that $f(\bar{x}) = \bar{g}$.

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:
Let $w(\bar{x}) = 1$ be an equation (to simplify without parameters). Let $H_w = \langle \bar{x} | w(\bar{x}) = 1 \rangle$. Let $G$ be a group.

- If $\bar{g} \in G$ such that $G \models w(\bar{g}) = 1$, then there exists a homomorphism $f : H_w \to G$ such that $f(\bar{x}) = \bar{g}$.
- Conversely, if $f : H_w \to G$ is an homomorphism, then $f(\bar{x})$ is a solution of $w(\bar{x}) = 1$.

# *Tarski's Problem: Equations & Sela's approach*

**Summary of Sela's approach**:
Let $w(\bar{x}) = 1$ be an equation (to simplify without parameters). Let $H_w = \langle \bar{x} | w(\bar{x}) = 1 \rangle$. Let $G$ be a group.

- If $\bar{g} \in G$ such that $G \models w(\bar{g}) = 1$, then there exists a homomorphism $f : H_w \to G$ such that $f(\bar{x}) = \bar{g}$.
- Conversely, if $f : H_w \to G$ is an homomorphism, then $f(\bar{x})$ is a solution of $w(\bar{x}) = 1$.

$\Longrightarrow$ correspondance between solutions of $w(\bar{x}) = 1$ and $\mathrm{Hom}(H_w, G)$.

# *Tarski's Problem: Equations & Sela's approach*

# *Tarski's Problem: Equations & Sela's approach*

If $G$ has a "good" action on a "good" space (like free groups and torsion-free hyperbolic groups), then one can understand set of solutions of equations.

# *Tarski's Problem: Equations & Sela's approach*

If $G$ has a "good" action on a "good" space (like free groups and torsion-free hyperbolic groups), then one can understand set of solutions of equations.
Let $F$ be a free group and $f_n : H \to F$ be a "good" sequence of homomorphisms.

# *Tarski's Problem: Equations & Sela's approach*

If $G$ has a "good" action on a "good" space (like free groups and torsion-free hyperbolic groups), then one can understand set of solutions of equations.
Let $F$ be a free group and $f_n : H \to F$ be a "good" sequence of homomorphisms.
$\Longrightarrow$ we get a sequence of actions of $H$ on the Cayley graph of $F$.

# *Tarski's Problem: Equations & Sela's approach*

If $G$ has a "good" action on a "good" space (like free groups and torsion-free hyperbolic groups), then one can understand set of solutions of equations.
Let $F$ be a free group and $f_n : H \to F$ be a "good" sequence of homomorphisms.
$\Longrightarrow$ we get a sequence of actions of $H$ on the Cayley graph of $F$.
$\Longrightarrow$ one can extracts a subsequence converging in the Gromov-Hausdorff topology to an action on a reel tree.

# *Tarski's Problem: Equations & Sela's approach*

If $G$ has a "good" action on a "good" space (like free groups and torsion-free hyperbolic groups), then one can understand set of solutions of equations.
Let $F$ be a free group and $f_n : H \to F$ be a "good" sequence of homomorphisms.
$\Longrightarrow$ we get a sequence of actions of $H$ on the Cayley graph of $F$.
$\Longrightarrow$ one can extracts a subsequence converging in the Gromov-Hausdorff topology to an action on a reel tree.
$\Longrightarrow$ a "good" action of $H$ on a reel tree.

# *Tarski's Problem: Equations & Sela's approach*

If $G$ has a "good" action on a "good" space (like free groups and torsion-free hyperbolic groups), then one can understand set of solutions of equations.
Let $F$ be a free group and $f_n : H \to F$ be a "good" sequence of homomorphisms.
$\Longrightarrow$ we get a sequence of actions of $H$ on the Cayley graph of $F$.
$\Longrightarrow$ one can extracts a subsequence converging in the Gromov-Hausdorff topology to an action on a reel tree.
$\Longrightarrow$ a "good" action of $H$ on a reel tree.
$\Longrightarrow$ a beautiful structure of $H$(modulo the kernel of the action) and $\mathrm{Hom}(H, F)$.

# *Tarski's Problem: Equations & Sela's approach*

# *Tarski's Problem: Equations & Sela's approach*

Recall that:

# *Tarski's Problem: Equations & Sela's approach*

Recall that:

*Definition*

Let $F$ be a free group and $G$ a group.

# *Tarski's Problem: Equations & Sela's approach*

Recall that:

*Definition*

Let $F$ be a free group and $G$ a group.

- A sequence of homomorphisms $f_n : G \to F$ is convergent if for any $g \in G$, one of the following sets is finite

$$S_1(g) = \{n | f_n(g) = 1\}, \quad S_2(g) = \{n | f_n(g) \neq 1\}.$$

# *Tarski's Problem: Equations & Sela's approach*

Recall that:

*Definition*

Let $F$ be a free group and $G$ a group.

- A sequence of homomorphisms $f_n : G \to F$ is convergent if for any $g \in G$, one of the following sets is finite

$$S_1(g) = \{n | f_n(g) = 1\}, \quad S_2(g) = \{n | f_n(g) \neq 1\}.$$

- $\ker_\infty(f_n) = \{g \in G | S_2(g) \text{ is finite}\}$.

# *Tarski's Problem: Equations & Sela's approach*

Recall that:

*Definition*

Let $F$ be a free group and $G$ a group.

- A sequence of homomorphisms $f_n : G \to F$ is convergent if for any $g \in G$, one of the following sets is finite

$$S_1(g) = \{n | f_n(g) = 1\}, \quad S_2(g) = \{n | f_n(g) \neq 1\}.$$

- $\ker_\infty(f_n) = \{g \in G | S_2(g) \text{ is finite}\}$.
- $H$ is a *limit group*, if $H = G/ker_\infty(f_n)$ for some group $G$ and a convergent sequence $f_n : G \to F$.

# *Tarski's Problem: Equations & Sela's approach*

# *Tarski's Problem: Equations & Sela's approach*

**Link with the universal theory**:

# *Tarski's Problem: Equations & Sela's approach*

**Link with the universal theory**: the following properties are equivalent:

# *Tarski's Problem: Equations & Sela's approach*

**Link with the universal theory**: the following properties are equivalent:
(1) $H$ is a limit group,

# *Tarski's Problem: Equations & Sela's approach*

**Link with the universal theory**: the following properties are equivalent:

(1) $H$ is a limit group,

(2) $H$ is $\omega$-residually free,

# *Tarski's Problem: Equations & Sela's approach*

**Link with the universal theory**: the following properties are equivalent:

(1) $H$ is a limit group,

(2) $H$ is $\omega$-residually free,

(3) $H$ is a model of the universal theory of a free group.

# *Tarski's Problem: Quantifier elimination*

# *Tarski's Problem: Quantifier elimination*

*Definition*

Let $T$ be a theory and $S$ a set of formulas.

# *Tarski's Problem: Quantifier elimination*

*Definition*

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$,

# *Tarski's Problem: Quantifier elimination*

*Definition*

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

# *Tarski's Problem: Quantifier elimination*

**Definition**

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

$T \models \phi$ means $M \models T \Rightarrow M \models \phi$.

# *Tarski's Problem: Quantifier elimination*

**Definition**

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

$T \models \phi$ means $M \models T \Rightarrow M \models \phi$.
A formula $\phi(\bar{z})$ is $\forall\exists$-formula if it is of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \bar{z})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

# *Tarski's Problem: Quantifier elimination*

**Definition**

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

$T \models \phi$ means $M \models T \Rightarrow M \models \phi$.
A formula $\phi(\bar{z})$ is $\forall\exists$-formula if it is of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \bar{z})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

**Theorem 9 (Sela)**

# *Tarski's Problem: Quantifier elimination*

**Definition**

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

$T \models \phi$ means $M \models T \Rightarrow M \models \phi$.
A formula $\phi(\bar{z})$ is $\forall\exists$-formula if it is of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \bar{z})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

**Theorem 9 (Sela)**

*Let $F$ be a nonabelian free group of finite rank.*

# *Tarski's Problem: Quantifier elimination*

**Definition**

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

$T \models \phi$ means $M \models T \Rightarrow M \models \phi$.
A formula $\phi(\bar{z})$ is $\forall\exists$-formula if it is of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \bar{z})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

**Theorem 9 (Sela)**

*Let $F$ be a nonabelian free group of finite rank. Then $Th(F)$ has quantifier elimination down to $\forall\exists$-formulas.*

# *Tarski's Problem: Quantifier elimination*

**Definition**

Let $T$ be a theory and $S$ a set of formulas.
T has *quantifier elimination* down to $S$ if for any formula $\phi(\bar{x})$, there exists a boolean combination of formulas from $S$, $\varphi(\bar{x})$, such that $T \models \forall \bar{x}(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$.

$T \models \phi$ means $M \models T \Rightarrow M \models \phi$.
A formula $\phi(\bar{z})$ is $\forall\exists$-formula if it is of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}, \bar{z})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is a quantifier-free formula.

**Theorem 9 (Sela)**

*Let $F$ be a nonabelian free group of finite rank. Then $Th(F)$ has quantifier elimination down to $\forall\exists$-formulas.*
*Furtheremore, for any formula $\phi(\bar{x})$, there exists a boolean combination of $\forall\exists$-formulas, $\varphi(\bar{x})$, such that:*
*$F \models \forall x(\phi(\bar{x}) \Leftrightarrow \varphi(\bar{x}))$, for any free nonabelian group $F$.*

# *Tarski's Problem: the solution*

# *Tarski's Problem: the solution*

**Theorem 10 (Kharlampovich&Myasnikov, Independently Sela)**

$F_n \prec F_m$ for $m \geq n \geq 2$. □

# *Tarski's Problem: the solution*

**Theorem 10 (Kharlampovich&Myasnikov, Independently Sela)**

$F_n \prec F_m$ for $m \geq n \geq 2$. □

**Theorem 11 (Kharlampovich&Myasnikov)**

$Th(F_n)$ is decidable for $n \geq 2$. □

# *Tarski's Problem: the solution*

**Theorem 10 (Kharlampovich&Myasnikov, Independently Sela)**

$F_n \prec F_m$ *for* $m \geq n \geq 2$. □

**Theorem 11 (Kharlampovich&Myasnikov)**

$Th(F_n)$ *is decidable for* $n \geq 2$. □

**Consequence**:

$F_X \prec F_Y$ for $X \subseteq Y, |X| \geq 2$.

$Th(F_X)$ is decidable.

# *More Model Theory: Stability*

# *More Model Theory: Stability*

## *Definition*

# *More Model Theory: Stability*

## *Definition*

- Let $T$ be a theory and $\phi(\bar{x}, \bar{y})$ a formula.

# *More Model Theory: Stability*

## *Definition*

- Let $T$ be a theory and $\phi(\bar{x}, \bar{y})$ a formula. $\phi(\bar{x})$ has the *order property*,

# *More Model Theory: Stability*

## *Definition*

- Let $T$ be a theory and $\phi(\bar{x}, \bar{y})$ a formula.
  $\phi(\bar{x})$ has the *order property*, if for any $n \in \mathbb{N}$, there exists a model $M \models T$ and a sequence $(\bar{a}_i, \bar{b}_i), 0 \leq i \leq n$, in $M$ such that:

# *More Model Theory: Stability*

## *Definition*

- Let $T$ be a theory and $\phi(\bar{x}, \bar{y})$ a formula.
  $\phi(\bar{x})$ has the *order property*, if for any $n \in \mathbb{N}$, there exists a model $M \models T$ and a sequence $(\bar{a}_i, \bar{b}_i), 0 \leq i \leq n$, in $M$ such that:

$$M \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j.$$

# *More Model Theory: Stability*

## *Definition*

- Let $T$ be a theory and $\phi(\bar{x}, \bar{y})$ a formula.
  $\phi(\bar{x})$ has the *order property*, if for any $n \in \mathbb{N}$, there exists a model $M \models T$ and a sequence $(\bar{a}_i, \bar{b}_i), 0 \leq i \leq n$, in $M$ such that:
  $$M \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j.$$

- $T$ is *stable*, if whenever $\phi(\bar{x}, \bar{y})$ is a formula such that $T \cup \{\exists \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})\}$ has a model, $\phi(\bar{x}, \bar{y})$ is without the order property.

# *More Model Theory: Stability*

# *More Model Theory: Stability*

Other notions of "stability":

# *More Model Theory: Stability*

Other notions of "stability":

Finite Morley rank $\subseteq$ *$\omega$-stability* $\subseteq$ *superstability* $\subseteq$ *stability*.

# *More Model Theory: Stability*

Other notions of "stability":

$$\text{Finite Morley rank} \subseteq \textit{$\omega$-stability} \subseteq \textit{superstability} \subseteq \textit{stability}.$$

$$\left.\begin{array}{c}\text{Finite Morley rank}\\ \textit{$\omega$-stability}\\ \textit{superstability}\end{array}\right\} \Longrightarrow \quad \text{a good abstract notion of dimension.}$$

# *More Model Theory: Stability*

# *More Model Theory: Stability*

**Examples:**

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.
- Free groups of infinite rank in the variety of nilpotent groups of class $c$ of exponent $p^n$, $p$ is a prime $> c$, are $\omega$-stable with infinite Morley rank (Baudisch).

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.
- Free groups of infinite rank in the variety of nilpotent groups of class $c$ of exponent $p^n$, $p$ is a prime $> c$, are $\omega$-stable with infinite Morley rank (Baudisch).
- Abelian-by-finite groups are stable, $\mathbb{Z}$ is superstable but not $\omega$-stable.

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.
- Free groups of infinite rank in the variety of nilpotent groups of class $c$ of exponent $p^n$, $p$ is a prime $> c$, are $\omega$-stable with infinite Morley rank (Baudisch).
- Abelian-by-finite groups are stable, $\mathbb{Z}$ is superstable but not $\omega$-stable.
- Nonabelian free groups are not superstable (Gibone, Poizat).

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.
- Free groups of infinite rank in the variety of nilpotent groups of class $c$ of exponent $p^n$, $p$ is a prime $> c$, are $\omega$-stable with infinite Morley rank (Baudisch).
- Abelian-by-finite groups are stable, $\mathbb{Z}$ is superstable but not $\omega$-stable.
- Nonabelian free groups are not superstable (Gibone, Poizat). More generally a superstable model of the universal theory of nonabelian free groups is abelian (Mustafin-Poizat, independently Ould Houcine).

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.
- Free groups of infinite rank in the variety of nilpotent groups of class $c$ of exponent $p^n$, $p$ is a prime $> c$, are $\omega$-stable with infinite Morley rank (Baudisch).
- Abelian-by-finite groups are stable, $\mathbb{Z}$ is superstable but not $\omega$-stable.
- Nonabelian free groups are not superstable (Gibone, Poizat). More generally a superstable model of the universal theory of nonabelian free groups is abelian (Mustafin-Poizat, independently Ould Houcine).
- Non-cyclic torsion-free hyperbolic groups are not superstable (Ould Houcine).

# *More Model Theory: Stability*

**Examples:**

- Algebraic groups over algerbraically closed fields are of finite Morley rank, $\mathbb{Q}$ has a finite Morley rank.
- Free groups of infinite rank in the variety of nilpotent groups of class $c$ of exponent $p^n$, $p$ is a prime $> c$, are $\omega$-stable with infinite Morley rank (Baudisch).
- Abelian-by-finite groups are stable, $\mathbb{Z}$ is superstable but not $\omega$-stable.
- Nonabelian free groups are not superstable (Gibone, Poizat). More generally a superstable model of the universal theory of nonabelian free groups is abelian (Mustafin-Poizat, independently Ould Houcine).
- Non-cyclic torsion-free hyperbolic groups are not superstable (Ould Houcine).
- A group acting nontrivially and without inversions on a simplicial tree is not $\omega$-stable (Ould Houcine).

# *More Model Theory: Stability*

# *More Model Theory: Stability*

Equivalent properties of stability.

# *More Model Theory: Stability*

Equivalent properties of stability.

*Definition*

Let $M$ be a group and $A$ a subset of $M$.

# *More Model Theory: Stability*

Equivalent properties of stability.

*Definition*

Let $M$ be a group and $A$ a subset of $M$.

- A sequence $(b_i : i \in I)$, *is indiscernible over* $A$ ,

# *More Model Theory: Stability*

Equivalent properties of stability.

*Definition*

Let $M$ be a group and $A$ a subset of $M$.

- A sequence $(b_i : i \in I)$, *is indiscernible over $A$* , if for any formula with parameters from $A$, $\phi(x_1, \ldots, x_n)$, and $i_1, \ldots, i_n$, $j_1, \ldots, j_n$, we have

$$M \models \phi(b_{i_1}, \ldots, b_{i_n}) \Leftrightarrow M \models \phi(b_{j_1}, \ldots, b_{j_n}).$$

- A sequence $(b_i : i \in I)$, $I$ ordered, *is order-indiscernible over $A$* ,

# *More Model Theory: Stability*

Equivalent properties of stability.

*Definition*

Let $M$ be a group and $A$ a subset of $M$.

- A sequence $(b_i : i \in I)$, *is indiscernible over $A$* , if for any formula with parameters from $A$, $\phi(x_1, \ldots, x_n)$, and $i_1, \ldots, i_n$, $j_1, \ldots, j_n$, we have

$$M \models \phi(b_{i_1}, \ldots, b_{i_n}) \Leftrightarrow M \models \phi(b_{j_1}, \ldots, b_{j_n}).$$

- A sequence $(b_i : i \in I)$, $I$ ordered, *is order-indiscernible over $A$* , if for any formula with parameters from $A$, $\phi(x_1, \ldots, x_n)$, and $i_1 < i_2 < \cdots < i_n$, $j_1 < \cdots < j_n$, we have

# *More Model Theory: Stability*

Equivalent properties of stability.

*Definition*

Let $M$ be a group and $A$ a subset of $M$.

- A sequence $(b_i : i \in I)$, *is indiscernible over $A$* , if for any formula with parameters from $A$, $\phi(x_1, \ldots, x_n)$, and $i_1, \ldots, i_n$, $j_1, \ldots, j_n$, we have

$$M \models \phi(b_{i_1}, \ldots, b_{i_n}) \Leftrightarrow M \models \phi(b_{j_1}, \ldots, b_{j_n}).$$

- A sequence $(b_i : i \in I)$, $I$ ordered, *is order-indiscernible over $A$* , if for any formula with parameters from $A$, $\phi(x_1, \ldots, x_n)$, and $i_1 < i_2 < \cdots < i_n$, $j_1 < \cdots < j_n$, we have

$$M \models \phi(b_{i_1}, \ldots, b_{i_n}) \Leftrightarrow M \models \phi(b_{j_1}, \ldots, b_{j_n}).$$

# *More Model Theory: Stability*

# *More Model Theory: Stability*

**Example:**

**Example:**
Let $F$ be the free group on $X = (x_i | i \in \mathbb{N})$.

**Example:**
Let $F$ be the free group on $X = (x_i | i \in \mathbb{N})$. Then $(x_i | i \in \mathbb{N})$ is indiscernible in $F$(over $\emptyset$).

**Example:**
Let $F$ be the free group on $X = (x_i | i \in \mathbb{N})$. Then $(x_i | i \in \mathbb{N})$ is indiscernible in $F$(over $\emptyset$). Indeed any permutation of $X$ unduce an isomorphism, and isomorphisms preserve satisfaction of formulas.

# *More Model Theory: Stability*

# *More Model Theory: Stability*

The following properties are equivallent:

# *More Model Theory: Stability*

The following properties are equivallent:

- $T$ is stable,

# *More Model Theory: Stability*

The following properties are equivallent:

- $T$ is stable,
- in any model of $T$, any order-indiscernible sequence is indiscernible,

# *More Model Theory: Stability*

The following properties are equivallent:

- $T$ is stable,
- in any model of $T$, any order-indiscernible sequence is indiscernible,
- there exists a notion of independence in models of $T$ with good properties.

# *More Model Theory: Stability*

# *More Model Theory: Stability*

**Theorem 12 (Sela)**

*A trosion-free hyperbolic group is stable.* □

# *More Model Theory: Stability*

*Theorem 12 (Sela)*

*A trosion-free hyperbolic group is stable.* □

**Consequence**: existence of a good notion of independence in groups elementary equivalent to a trosion-free hyperbolic group.

# *More Model Theory: Stability*

# *More Model Theory: Stability*

**Theorem 13**

*Let $G$ be a group. Suppose that $G$ is stable and $G \prec G * \mathbb{Z}$. Then $G$ is connected and any positive formula is equivalent to an existential positive one.* □

*Theorem 13*

*Let $G$ be a group. Suppose that $G$ is stable and $G \prec G * \mathbb{Z}$. Then $G$ is connected and any positive formula is equivalent to an existential positive one.* □

# THE END