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Multi-Parameter Deformations of the Module of Symbols of Differential Operators

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1 Introduction

In this paper, we discuss the following general principle. Given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, one can canonically associate to $(\mathfrak{g}, V)$ a commutative associative algebra. The generators of this commutative algebra are the nontrivial cohomology classes in $H^1(\mathfrak{g}; \mathrm{End}(V))$, while the relations between the generators are encoded by elements of $H^2(\mathfrak{g}; \mathrm{End}(V))$. More precisely, the relations correspond to the obstructions for integrability of infinitesimal deformations of V.

The classical deformation theory of Lie algebras and modules over Lie algebras traditionally deals with one-parameter deformations (cf. [13, 14, 21, 24]). It is, however, natural to consider, as in other deformation theories, "multi-parameter" deformations, that is, deformations of Lie algebras over commutative algebras. This viewpoint has been adopted in [9], and the existence of the so-called miniversal deformation has been proven. A construction of miniversal deformations of Lie algebras was given in [10]. Similar methods was applied in [22, 23] to deformations of homomorphisms of some infinite-dimensional Lie algebras.

The canonical notion of miniversal deformation leads to a natural commutative algebra intrinsically associated with a Lie algebra (or with a module over a Lie algebra). This interesting algebraic characteristic deserves a further investigation.

In this paper we consider the space, $\mathcal{D}(\mathbb{R}^n)$, of linear differential operators on $\mathbb{R}^n$ viewed as a module over the Lie algebra, $\mathrm{Vect}(\mathbb{R}^n)$, of smooth vector fields on $\mathbb{R}^n$. This module structure has been recently studied in [3, 4, 5, 6, 7, 12, 16, 17, 20] (see also

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the references therein). The module of differential operators can be viewed as a deformation of the corresponding module of symbols; the general framework of the deformation theory (see, e.g., [10, 11, 13, 14, 21, 24]), therefore, relates this study to the cohomology of the Lie algebra of vector fields (cf. [7, 17]).

The main purpose of this paper is to introduce the commutative algebra associated to the $\mathrm{Vect}(\mathbb{R}^n)$-module of symbols of differential operators on $\mathbb{R}^n$. Geometrically speaking, symbols are symmetric contravariant tensor fields on $\mathbb{R}^n$, or, in other words, polynomial functions on $T^*\mathbb{R}^n$. We will describe the miniversal deformation of this module.

Let $\mathcal{F}_\lambda$ be the space of tensor densities of degree $\lambda \in \mathbb{R}$ on $\mathbb{R}^n$. The two-parameter family of $\mathrm{Vect}(\mathbb{R}^n)$-modules, $\mathcal{D}_{\lambda,\mu}$, of linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ will provide us with an important class of examples of nontrivial deformations of the module of symbols.

The first cohomology space of the Lie algebra of vector fields, classifying the infinitesimal deformations of the module of symbols has been calculated, for an arbitrary smooth manifold, in [17] (see also [3] for the details in the one-dimensional case). Of course, not for every infinitesimal deformation there exists a formal deformation containing the latter as an infinitesimal part. The obstructions are characterized in terms of Nijenhuis-Richardson products of nontrivial first cohomology classes. The main problem considered in this paper is to determine the integrability condition, that is, a necessary and sufficient condition for an infinitesimal deformation that guarantees existence of a formal deformation. We provide such a condition in the case of $\mathbb{R}^n$, where $n \geq 2$.

2 The general framework

We start with the notion of (multi-parameter) deformations over a commutative algebra. Our approach will be similar to those of [22, 23]; it corresponds to the notion of miniversal deformations [10] in a special case when one can choose a basis of the first cohomology space.

2.1 Deformations over commutative algebras

Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ (or $\mathbb{R}$) and (V, ρ) a $\mathfrak{g}$-module, where V is a vector space and ρ is a homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V). \tag{2.1}$$

Let $\mathfrak{A}$ be a commutative associative algebra with identity with fixed augmentation $\varepsilon : \mathfrak{A} \to \mathbb{C}$ such that $\varepsilon(1) = 1$. Put $\mathfrak{M} = \ker \varepsilon$ the associated maximal ideal of $\mathfrak{A}$. Following [10], we have the following definition.

Definition 2.1. A deformation of the $\mathfrak{g}$-module (V, ρ) with base $(\mathfrak{A}, \mathfrak{M})$ is a Lie algebra homomorphism

$$\widetilde{\rho} : \mathfrak{g} \longrightarrow \mathfrak{A} \otimes \mathrm{End}(V) \tag{2.2}$$

such that $(\varepsilon \otimes \mathrm{Id}) \circ \widetilde{\rho} = \rho$.

We specify the above definition to the following natural case.

Basic example 2.2. Any finitely generated commutative algebra is of the form

$$\mathfrak{A} = \mathbb{C}\left[t_1, \ldots, t_p\right]/\mathcal{R}, \tag{2.3}$$

where $\mathcal{R} \subset \mathbb{C}[t_1, \ldots, t_p]$ is an ideal, that is, the set of *relations*. Choose the natural augmentation

$$\varepsilon_0 : \mathbb{C}\left[t_1, \ldots, t_p\right] \longrightarrow \mathbb{C}, \qquad \varepsilon_0(P) = P(0), \tag{2.4}$$

and we have to assume that $\mathcal{R} \subset \ker(\varepsilon_0)$, so that ε_0 is well defined on the quotient-algebra $\mathfrak{A}$. We call $t = (t_1, \ldots, t_p)$ *parameters of deformation*.

Any deformation (2.2) is of the form

$$\widetilde{\rho} = \rho + \varphi, \tag{2.5}$$

where φ is a linear map from $\mathfrak{g}$ to $\mathrm{End}(V) \otimes \mathbb{C}[t]$ such that $(\varepsilon \otimes \mathrm{Id}) \circ \varphi = 0$.

2.2 The Maurer-Cartan equation

The expression $\widetilde{\rho}$ must satisfy the homomorphism condition, that is,

$$\widetilde{\rho}([X, Y]) = \left[\widetilde{\rho}(X), \widetilde{\rho}(Y)\right] \tag{2.6}$$

for every $X, Y \in \mathfrak{g}$. Note that the bracket in the right-hand side stands for the standard commutator in $\mathrm{End}(V)$ extended to $\mathfrak{A} \otimes \mathrm{End}(V)$.

The standard Chevalley-Eilenberg differential (see [11]) retains, in the case of linear maps from $\mathfrak{g}$ to $\mathfrak{A} \otimes \mathrm{End}(V)$, to the following formula. Given a linear map $a : \mathfrak{g} \to \mathfrak{A} \otimes \mathrm{End}(V)$, its differential δa is the bilinear skew-symmetric map

$$\delta a(X, Y) = a\big([X, Y]\big) - \big[\rho(X), a(Y)\big] + \big[\rho(Y), a(X)\big]. \tag{2.7}$$

The standard cup-product of linear maps $a, b : \mathfrak{g} \to \mathfrak{A} \otimes \mathrm{End}(V)$ is a bilinear map $[[a, b]] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{A} \otimes \mathrm{End}(V)$ defined by

$$[[a, b]](X, Y) = -\big[a(X), b(Y)\big] + \big[a(Y), b(X)\big]. \tag{2.8}$$

It is also called the Nijenhuis-Richardson product [21].

Put $\varphi = \widetilde{\rho} - \rho$ as in (2.5), we easily check that condition (2.6) reads

$$\delta\varphi + \frac{1}{2}[[\varphi, \varphi]] = 0. \tag{2.9}$$

This is the Maurer-Cartan equation (also called the deformation equation, cf. [21]). Although it is equivalent to (2.6), it is useful to relate the deformations (2.5) with the cohomology theory.

2.3 Equivalent deformations

Following [10], we introduce the notion of equivalence of deformations of $\mathfrak{g}$-modules.

Definition 2.3. Two deformations $\widetilde{\rho}$ and $\widetilde{\rho}'$ with the same base $(\mathfrak{A}, \mathfrak{M})$ are called *equivalent* if there exists an inner automorphism ψ of the Lie algebra $\mathfrak{A} \otimes \mathrm{End}(V)$ such that

$$\psi \circ \widetilde{\rho} = \widetilde{\rho}' \tag{2.10}$$

which is compatible with the augmentation, that is, $(\varepsilon \otimes \mathrm{Id}) \circ \psi = \varepsilon \otimes \mathrm{Id}$.

Note that this definition corresponds to the standard definition of equivalence for 1-parameter deformations with $\mathfrak{A} = \mathbb{C}[t]$ (see, e.g., [21]).

2.4 Infinitesimal deformations and the first cohomology

A deformation (2.2) is called an *infinitesimal deformation* if $\mathfrak{M}^2 = 0$ in $\mathfrak{A}$. In the example of Section 2.1, this means that $\mathfrak{A}$ is the algebra of first-order polynomials.

Given an arbitrary deformation (2.2) with base $(\mathfrak{A}, \mathfrak{M})$, one defines an infinitesimal deformation canonically associated to the given one. Consider the quotient-algebra $\mathfrak{A}_1 = \mathfrak{A}/\mathfrak{M}^2$, then there is a natural projection (the "push out" in the sense of [10]) of the initial deformation to a deformation with the base $(\mathfrak{A}_1, \mathfrak{M}/\mathfrak{M}^2)$.

In particular, if the deformation is written in the form (2.5), then the associated infinitesimal deformation is of the form

$$\widetilde{\rho}_1 = \rho + \varphi_1 \quad \text{with } \varphi_1 = t_1 c_1 + \cdots + t_p c_p, \tag{2.11}$$

where $(t_1, \ldots, t_p) \bmod(\mathfrak{M}^2)$ are the parameters (i.e., satisfying the relations $t_i t_j = 0$ for $i, j = 1, \ldots, p$).

Equation (2.9) implies that each linear map $c_i : \mathfrak{g} \to \mathrm{End}(V)$ is a 1-*cocycle* (cf. [11]). Furthermore, if $\widetilde{\rho}$ and $\widetilde{\rho}'$ are equivalent deformations, then the corresponding cocycles in the infinitesimal deformations are cohomologous, namely $c_i = c'_i + \delta A_i$ for some $A_1, \ldots, A_p \in \mathrm{End}(V)$.

Therefore, an infinitesimal deformation is defined, up to equivalence, by the cohomology classes $\bar{c}_1, \ldots, \bar{c}_p$ in $H^1(\mathfrak{g}; \mathrm{End}(V))$.

2.5 Miniversal deformations and integrability conditions

The aim of this section is to link the approach of [10] with the classical Nijenhuis-Richardson theory [21]. Following [10], we have the following definition.

Definition 2.4. (i) We call a deformation $\widetilde{\rho}$ with base $(\mathfrak{A}, \mathfrak{M})$ *versal* if for any deformation $\widetilde{\rho}'$ with base $(\mathfrak{A}', \mathfrak{M}')$ there is a homomorphism $\psi : \mathfrak{A} \to \mathfrak{A}'$ satisfying $\psi(1) = 1$ and $\varepsilon' \circ \psi = \varepsilon$ such that

$$\widetilde{\rho}' = (\psi \otimes \mathrm{Id}) \circ \widetilde{\rho}. \tag{2.12}$$

(ii) A versal deformation $\widetilde{\rho}$ is called *miniversal* if for any infinitesimal deformation $\widetilde{\rho}'$, the above homomorphism ψ is unique.

An explicit construction of miniversal deformations of Lie algebras was suggested in [10]. A similar construction can be applied to deformations of modules. We will define a commutative algebra of the form (2.3) with augmentation (2.4) and a miniversal deformation as in our basic example of Section 2.1.

First, we put $p = \dim H^1(\mathfrak{g}; \mathrm{End}(V))$ and choose a basis $\bar{c}_1, \ldots, \bar{c}_p$ of the space $H^1(\mathfrak{g}; \mathrm{End}(V))$; we define an infinitesimal deformation of the form (2.11). In order to construct a miniversal deformation, we define a sequence of commutative algebras

$\mathfrak{A}_m = \mathbb{C}[t_1, \ldots, t_p]/\mathcal{R}_m$ as a series of extensions

$$0 \longrightarrow \mathcal{R}_{m-1}/\mathcal{R}_m \longrightarrow \mathfrak{A}_m \longrightarrow \mathfrak{A}_{m-1} \longrightarrow 0 \tag{2.13}$$

and construct a sequence of deformations of the form (2.5).

Each ideal $\mathcal{R}_m$ is of the form

$$\mathcal{R}_m = \langle \mathcal{R}'_m, \mathbb{C}_{m+1}[t] \rangle, \tag{2.14}$$

where $\mathcal{R}'_m$ is a set of homogeneous relations of order $\leq m$ and $\mathbb{C}_{m+1}[t]$ the ideal of all polynomials of valuation $\geq m+1$. In other words, any monomial of degree $\geq m+1$ in $\mathfrak{A}_m$ vanishes. We thus construct an ascending chain of ideals

$$\mathcal{R}'_1 \subset \cdots \subset \mathcal{R}'_{m-1} \subset \mathcal{R}'_m \subset \cdots, \tag{2.15}$$

where $\mathcal{R}'_1 = 0$.

Assume, by induction, that we have already constructed the first $m-1$ algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_{m-1}$ and corresponding deformations. We will now construct the algebra $(\mathfrak{A}_m, \varepsilon)$ and a deformation with this base.

Developing (2.9), we obtain the following equation of order m:

$$\delta\varphi_m = -\frac{1}{2} \sum_{i+j=m} [[\varphi_i, \varphi_j]] \tag{2.16}$$

with indeterminate φ_m. The right-hand side of (2.16) is a 2-cocycle with coefficients that are homogeneous polynomials of degree m in t. The cohomology class of this cocycle is

$$\overline{[[\varphi_i, \varphi_j]]} \in H^2(\mathfrak{g}; \mathrm{End}(V)) \otimes \mathbb{C}[t]/\mathcal{R}'_{m-1}. \tag{2.17}$$

It is an *obstruction* for existence of solutions of (2.16). We choose $\mathcal{R}'_m$ as a minimal ideal of $\mathbb{C}[t]$ such that the image of the above obstruction vanishes after projection

$$\mathbb{C}[t]/\mathcal{R}'_{m-1} \longrightarrow \mathbb{C}[t]/\mathcal{R}'_m. \tag{2.18}$$

Equation (2.16) then has a solution over the commutative algebra $\mathfrak{A}_m = \mathbb{C}[t]/\mathcal{R}_m$; this solution is not unique and is defined up to an arbitrary 1-cocycle.

Example 2.5. The second-order term in (2.16) is

$$\delta\varphi_2(t) = -\frac{1}{2}[[\varphi_1(t), \varphi_1(t)]]. \tag{2.19}$$

The cohomology class of $[[\varphi_1(t), \varphi_1(t)]]$ is, therefore, an obstruction to the existence of the second-order term $\varphi_2(t)$. It is an element of $H^2(\mathfrak{g}; \mathrm{End}(V)) \otimes \mathbb{C}[t]$, where polynomial coefficients are homogeneous second-order polynomials in t. For existence of $\varphi_2(t)$, it is necessary and sufficient that these obstructions vanish. We thus obtain second-order relations for the parameters $t_1, \ldots, t_p$.

Finally, taking the inductive limit $\mathfrak{A} = \varinjlim \mathfrak{A}_m$, we obtain a deformation with base $(\mathfrak{A}, \mathfrak{M})$. We call the generators of the ideal $\mathcal{R} = \varinjlim \mathcal{R}_m$ the *necessary and sufficient conditions for integrability of the infinitesimal deformation* (2.11).

Proposition 2.6. The constructed deformation is miniversal and does not depend on the choice of solutions of (2.16). □

The above construction follows the construction, from [10], of a versal deformation of Lie algebras. Although, for the sake of simplicity, we do not use here the language of Harrison cohomology describing the extensions (2.13). The proof that the constructed deformation is indeed miniversal, is similar to those of [10].

3 Deformations of $\mathbb{Z}$-graded modules

Up to now, we were considering only deformations with a finite number of parameters. However, following [10], we will take into consideration the case of graded modules with infinitely many independent parameters of deformation.

Consider a module (V, ρ) splitted into a direct sum of $\mathfrak{g}$-modules

$$V = \bigoplus_{k \in \mathbb{Z}} V_k. \tag{3.1}$$

Suppose that for some values $i \in \mathbb{Z}$ there exist nontrivial cocycles c_i on $\mathfrak{g}$ with values in $\mathrm{End}(V)$ such that, for all $X \in \mathfrak{g}$ we have

$$c_i(X)|_{V_k} \subset V_{k-i}. \tag{3.2}$$

Assume, furthermore, that there is a deformation of the form

$$\widetilde{\rho}(\tau) = \rho + \sum_{i \in \mathbb{Z}} \tau_i c_i + \left(\tau^2\right), \tag{3.3}$$

where τ_i are the *free* parameters, that is, the parameters generate the free commutative algebra $\mathbb{C}[\tau_i]$.

The following construction is meant to use the “extra degrees of freedom” related to the decomposition (3.1). We will introduce new parameters t_i^k with $k \in \mathbb{Z}$. Consider each cocycle $c_i^k : \mathfrak{g} \to \mathrm{Hom}(V_k, V_{k-i})$ defined by the restriction

$$c_i^k(X) := c_i(X)|_{V_k} \tag{3.4}$$

as independent.

Proposition 3.1. There exists a formal deformation of the form

$$\widetilde{\rho}(t) = \rho + \sum_{i,k\in\mathbb{Z}} t_i^k c_i^k + \left(t^2\right), \tag{3.5}$$

where t_i^k are generators of the commutative algebra $\mathbb{C}[t]/\mathcal{R}$, where the ideal $\mathcal{R}$ is generated by the following relations:

$$t_i^{k-j} t_j^k = t_i^k t_j^{k-i} \quad \forall\, i,\ j,\ k \in \mathbb{Z}. \tag{3.6}$$

□

Proof. The original deformation (3.3) satisfies the Maurer-Cartan equation (2.9). In each order m, (2.16) for the deformation (3.3) has a solution

$$\varphi_m(\tau) \in \mathrm{Hom}\left(\mathfrak{g}; \mathrm{End}(V)\right) \otimes \mathbb{C}[\tau] \tag{3.7}$$

which is a homogeneous polynomial in τ of degree m. Replacing in $\varphi_m(\tau)|_{V_k}$ each monomial $\tau_{i_1} \cdots \tau_{i_{m-1}} \tau_{i_m}$ by $t_{i_1}^{k-i_2-\cdots-i_m} \cdots t_{i_{m-1}}^{k-i_m} t_{i_m}^k$, we obviously get a solution $\varphi_m(t)$ of (2.16). ■

4 The main results

In this section we define our main object: the space of symmetric contravariant tensor fields on $\mathbb{R}^n$, and describe the versal deformation of the natural structure of $\mathrm{Vect}(\mathbb{R}^n)$-module on this space. In other words, we deform the Lie derivative of symmetric contravariant tensor fields.

4.1 The space of symbols

Consider the Lie algebra $\mathrm{Vect}(\mathbb{R}^n)$ of smooth vector fields on $\mathbb{R}^n$ and the space $\mathcal{S}$ of smooth symmetric contravariant tensor fields on $\mathbb{R}^n$. The space $\mathcal{S}$ is naturally isomorphic to the space of functions on $T^*\mathbb{R}^n$ polynomial on fibers. Clearly, $\mathcal{S}$ has a structure of a Poisson algebra with natural graduation

$$\mathcal{S} = \bigoplus_{k=0}^{\infty} \mathcal{S}_k, \tag{4.1}$$

where $\mathcal{S}_k$ is the space of kth order tensor fields.

The space $\mathcal{S}$ is a $\mathrm{Vect}(\mathbb{R}^n)$-module since $\mathrm{Vect}(\mathbb{R}^n) \subset \mathcal{S}$. In Darboux coordinates, the action of $X \in \mathrm{Vect}(\mathbb{R}^n)$ on $\mathcal{S}$ is given by the Hamiltonian vector field[1]

$$L_X = \frac{\partial X}{\partial \xi_i}\frac{\partial}{\partial x^i} - \frac{\partial X}{\partial x^i}\frac{\partial}{\partial \xi_i}, \tag{4.2}$$

which is nothing but the Lie derivative of tensor fields.

The aim of this paper is to study multi-parameter formal deformations of this module. We will restrict our considerations to the multi-parameter formal deformations which are *differentiable*, that is, each term in the formal series (2.5) is supposed to be a differential operator on $\mathcal{S}$.

4.2 Description of the infinitesimal deformations

According to the general framework, we need an information about the space of the first cohomology of $\mathrm{Vect}(\mathbb{R}^n)$ with coefficients in $\mathrm{End}(\mathcal{S})$ in order to describe the infinitesimal deformations. The module $\mathrm{End}(\mathcal{S})$ is decomposed as follows:

$$\mathrm{End}(\mathcal{S}) = \bigoplus_{k,\ell} \mathrm{Hom}\left(\mathcal{S}_k, \mathcal{S}_\ell\right). \tag{4.3}$$

To study the $\mathrm{Vect}(\mathbb{R}^n)$-cohomology with coefficients in $\mathrm{End}(\mathcal{S})$ it then suffices to consider the cohomology with coefficients in each module $\mathrm{Hom}(\mathcal{S}_k, \mathcal{S}_\ell)$. We will, furthermore, restrict ourself to the subspace $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell) \subset \mathrm{Hom}(\mathcal{S}_k, \mathcal{S}_\ell)$ given by differential operators from $\mathcal{S}_k$ to $\mathcal{S}_\ell$.

The space of first cohomology of the Lie algebra of vector fields with coefficients in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ has been calculated, for an arbitrary manifold M of $\dim M \geq 2$, in [17]. We recall here the result in the case $M = \mathbb{R}^n$

$$H^1\left(\mathrm{Vect}\left(\mathbb{R}^n\right); \mathcal{D}\left(\mathcal{S}_k, \mathcal{S}_\ell\right)\right) = \begin{cases} \mathbb{R}, & \text{if } k-\ell = 0,\ k-\ell = 1 \text{ and } \ell \neq 0,\ k-\ell = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

[1]Here and below the sum over repeated indices is understood.

We have, therefore, infinitely many nontrivial cohomology classes generating an infinitesimal deformation of the $\mathrm{Vect}(\mathbb{R}^n)$-module $\mathcal{S}$.

We give the explicit formulæ for corresponding 1-cocycles.

(a) For all $k \geq 0$, there is a 1-cocycle with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_k)$ that associates to $X \in \mathrm{Vect}(\mathbb{R}^n)$ the operator of multiplication by the function

$$c_0(X) = \mathrm{Div}(X). \tag{4.5}$$

(b) For all $k \geq 2$, there is a 1-cocycle with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-1})$ given by

$$c_1(X) = \frac{\partial^2 X}{\partial x^i \partial x^j} \frac{\partial^2}{\partial \xi_i \partial \xi_j}. \tag{4.6}$$

Remark 4.1. More geometrically, this cocycle can be written as the Lie derivative of the (flat) connection on $\mathbb{R}^n$, namely, $c_1(X) = L_X(\nabla)$.

(c) For all $k \geq 2$, there is a 1-cocycle with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-2})$ given by

$$c_2(X) = \frac{\partial^3 X}{\partial x^i \partial x^j \partial x^l} \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_l} - 3 \frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l} \frac{\partial^2}{\partial \xi_i \partial \xi_j} \frac{\partial}{\partial x^l}. \tag{4.7}$$

Remark 4.2. This cocycle is related to the famous Moyal product, namely for $P \in \mathcal{S}_k$, $c_2(X)(P)$ coincides with the third-order term in the Moyal product of X and P.

As in Section 3, we will use the notation

$$c_i^k = c_i|_{\mathcal{S}_k}, \quad i = 0, 1, 2, \tag{4.8}$$

and deal with independent cocycles c_0^k, c_1^k, c_2^k.

4.3 Integrability conditions

According to the results of [17] (see Section 4.2), the infinitesimal deformations of the Lie derivative (4.2) are of the form $\widetilde{\rho}(X) = L_X + \varphi_1(X)$ with

$$\varphi_1 = \sum_{0 \leq k < \infty} t_0^k c_0^k + \sum_{2 \leq k < \infty} \left(t_1^k c_1^k + t_2^k c_2^k \right), \tag{4.9}$$

where the symbols t_0^k, t_1^k, t_2^k stand for independent parameters. We, therefore, have to deal with infinitesimal deformations with infinite number of parameters.

We formulate the main result of this paper.

Theorem 4.3. The following relations:

(a) one series of second-order relation $R_2^k(t)$:

$$t_1^k t_2^{k-1} - t_1^{k-2} t_2^k = 0, \quad k \geq 4, \tag{4.10}$$

(b) two series of third-order relations, namely $R_3^k(t)$:

$$\left(t_0^k - t_0^{k-1}\right) t_1^k t_2^{k-1} = 0, \quad k \geq 3, \tag{4.11}$$

and $\widetilde{R}_3^k(t)$:

$$\left(t_0^k - t_0^{k-2}\right) t_2^k t_2^{k-2} = 0, \quad k \geq 4, \tag{4.12}$$

are necessary and sufficient for the integrability of the infinitesimal deformation (4.9). □

In other words, the base of the miniversal deformation of the $\mathrm{Vect}(\mathbb{R}^n)$-module $\mathcal{S}$ is the commutative algebra $\mathfrak{A} = \mathbb{C}[t]/\mathcal{R}$, where the ideal $\mathcal{R}$ is generated by the relations (4.10), (4.11), and (4.12).

The proof that the relations (4.10), (4.11), and (4.12) are necessary is just a result of a straightforward computation; it will be given in Section 6. The proof that these conditions are sufficient will be based on the existence of an important class of deformations corresponding to the $\mathrm{Vect}(\mathbb{R}^n)$-modules of differential operators.

The following statement is a corollary of Theorem 4.3.

Proposition 4.4. An infinitesimal deformation (4.9) with additional series of relations: $t_2^k = 0$ for all k, is integrable without any condition on t_0^k and t_1^k. □

5 Module of differential operators

Consider the space $\mathcal{D}$ of linear differential operators on $\mathbb{R}^n$. It is isomorphic to $\mathcal{S}$ as a vector space, but its structure as a $\mathrm{Vect}(\mathbb{R}^n)$-module is quite different. In this section we interpret $\mathcal{D}$ as a deformation of the $\mathrm{Vect}(\mathbb{R}^n)$-module $\mathcal{S}$.

5.1 Lie derivative of differential operators

The composition of differential operators is defined by

$$A \circ B = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k A}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} \frac{\partial^k B}{\partial x^{i_1} \cdots \partial x^{i_k}}. \tag{5.1}$$

Of course, since A is a polynomial in ξ, there are only a finite number of terms in this sum. There is a filtration of the associative algebra $\mathcal{D}$

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^r \subset \cdots, \tag{5.2}$$

where $\mathcal{D}^r$ is the space of rth-order differential operators (isomorphic to $\bigoplus_{i \le r} \mathcal{S}_i$ as a vector space). We have $\mathcal{S} = \operatorname{gr} \mathcal{D}$ as well as an associative algebra and as a Lie algebra. The space $\mathcal{S}$ is usually called the space of symbols associated to $\mathcal{D}$.

The space $\mathcal{D}$ is a $\mathrm{Vect}(\mathbb{R}^n)$-module since $\mathrm{Vect}(\mathbb{R}^n)$ is a Lie subalgebra of $\mathcal{D}$. Moreover, there is a family of embeddings $\mathrm{Vect}(\mathbb{R}^n) \hookrightarrow \mathcal{D}$ depending on a parameter $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) given by

$$i^\lambda : X \longmapsto X + \lambda \operatorname{Div}(X), \tag{5.3}$$

where $X \in \mathrm{Vect}(\mathbb{R}^n)$ and $\operatorname{Div}(X)$ is the divergence with respect to the standard volume form on $\mathbb{R}^n$. This defines a one-parameter family of $\mathrm{Vect}(\mathbb{R}^n)$-module structures on the space $\mathcal{D}$. More generally, one can define a two-parameter family of $\mathrm{Vect}(\mathbb{R}^n)$-modules on $\mathcal{D}$ by

$$\mathcal{L}_X^{\lambda,\mu}(A) = i^\mu(X) \circ A - A \circ i^\lambda(X). \tag{5.4}$$

These modules are denoted $\mathcal{D}_{\lambda,\mu}$.

Remark 5.1. From the geometrical viewpoint, the module $\mathcal{D}_{\lambda,\mu}$ is the space of differential operators acting on the space of tensor densities (cf. [4, 5, 7, 17]); the first-order differential operator $i^\lambda(X)$ is a Lie derivative of tensor densities of degree λ.

Lemma 5.2. The explicit formula of the $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_{\lambda,\mu}$ is

$$\begin{aligned}\mathcal{L}_X^{\lambda,\mu} &= L_X + (\mu - \lambda) \operatorname{Div}(X) \\ &\quad - \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{\partial^k X}{\partial x^{i_1} \cdots \partial x^{i_k}} \frac{\partial^k}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} + k\lambda \frac{\partial^{k-1} \operatorname{Div}(X)}{\partial x^{i_1} \cdots \partial x^{i_{k-1}}} \frac{\partial^{k-1}}{\partial \xi_{i_1} \cdots \partial \xi_{i_{k-1}}} \right),\end{aligned} \tag{5.5}$$

where L_X is as in (4.2). □

Proof. This formula readily follows from (5.1). ■

5.2 The Weyl symbols

Consider the operator D on $\mathcal{D}_{\lambda,\mu}$ given by

$$D = \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i} \tag{5.6}$$

that extends the divergence of vector fields to the space of symmetric contravariant tensor fields. Recall that the linear map

$$\exp(\lambda D) : \mathcal{D} \longrightarrow \mathcal{S} \tag{5.7}$$

defines the famous Weyl symbol of a differential operator (see [1]). Note that the parameter in this formula is usually interpreted in terms of the Planck constant, namely $\lambda = i\hbar/2$.

Lemma 5.3. The action (5.5) becomes after the transformation (5.7) as follows: the action $\widetilde{\mathcal{L}}^{\lambda,\mu}$ is of the form

$$\widetilde{\mathcal{L}}_X^{\lambda,\mu} = L_X + \tau_0 c_0(X) + \tau_1 c_1(X) + \tau_2 c_2(X) + \sum_{m\geq 3} L_m(X) \tag{5.8}$$

with

$$\tau_0 = \mu - \lambda, \qquad \tau_1 = \lambda - \frac{1}{2}, \qquad \tau_2 = \lambda(\lambda - 1), \tag{5.9}$$

where $L_m(X)$ are the terms with the degree shift m, that is, for the operators from $\mathcal{S}_k$ to $\mathcal{S}_\ell$ with $\ell - k = m$. □

Proof. By definition, $\widetilde{\mathcal{L}}_X^{\lambda,\mu} = \exp(-\lambda D) \circ \mathcal{L}_X^{\lambda,\mu} \circ \exp(\lambda D)$, a straightforward computation then yields (5.8) and (5.9). ■

This new expression of the $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_{\lambda,\mu}$ allows us to consider this module as a deformation of $\mathcal{S}$.

5.3 Module of differential operators as a deformation

The existence of modules $\mathcal{D}_{\lambda,\mu}$ allow us to construct a big class of formal deformations. The idea is to consider the parameters τ_0, τ_1, τ_2 as independent parameters using the fact that the expressions (5.9) do not satisfy any nontrivial homogeneous relation.

Lemma 5.4. There exists a deformation of the form (5.8) with base $\mathfrak{A} = \mathbb{C}[\tau_0, \tau_1, \tau_2]$ (i.e., the parameters τ_0, τ_1, τ_2 are independent). □

Proof. We use the existence of modules $\mathcal{D}_{\lambda,\mu}$. Each term L_m in (5.8) polynomially depends on τ_0, τ_1, τ_2. The operator $\widetilde{\mathcal{L}}_X^{\lambda,\mu}$ defines a $\mathrm{Vect}(\mathbb{R}^n)$-action and, so, satisfies the homomorphism condition (2.6). A term of degree of schift m in (2.6) is again a polynomial in τ_0, τ_1, τ_2, more precisely, a sum of the terms

$$\tau_0^{m_0}\tau_1^{m_1}\tau_2^{m_2}, \quad \text{where} \quad m_1 + 2m_2 = m, \tag{5.10}$$

with operator coefficients. But, all the monomials (5.10) with τ_0, τ_1, τ_2 given by (5.9) are, obviously, linearly independent and, so, (2.6) has to be satisfied independently of the operator coefficients of all monomials (5.10). These conditions are therefore independent of τ_0, τ_1, τ_2. ■

Applying the construction from Section 3 to obtain a formal deformation with the infinitesimal part of the form (4.9), we then obtain the following intermediate result.

Proposition 5.5. The following relations:

$$t_1^k t_2^{k-1} - t_1^{k-2} t_2^k = 0, \quad k \geq 4, \tag{5.11}$$

$$\left(t_0^k - t_0^{k-1}\right) t_1^k = 0, \quad k \geq 3, \tag{5.12}$$

$$\left(t_0^k - t_0^{k-2}\right) t_2^k = 0, \quad k \geq 4, \tag{5.13}$$

are sufficient for integrability of the infinitesimal deformation (4.9). □

Proof. Conditions (5.11), (5.12), and (5.13) coincide with conditions (3.6) of Proposition 3.1 that are sufficient for integrability. ■

Remark 5.6. Conditions (5.12) and (5.13) are stronger than (4.11) and (4.12), respectively. Therefore, the ideal generated by these polynomials in (5.11), (5.12), and (5.13) is bigger than the one generated by $R_2(t), R_3(t)$, and $R_3'(t)$. The deformation naturally related to the modules of differential operators is not a versal deformation of the module $\mathcal{S}$ in the sense of [10].

6 Affine and projective invariance

An important problem of the deformation theory is the problem of *invariance* with respect to the action of a Lie group or a Lie algebra of symmetries.

We will consider the following two Lie algebras: the algebra of infinitesimal affine transformations $\mathrm{aff}(n,\mathbb{R}) = \mathrm{gl}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ and the algebra of infinitesimal projective transformations, $\mathrm{sl}(n+1,\mathbb{R})$, spanned by the vector fields on $\mathbb{R}^n$

$$X_i = \frac{\partial}{\partial x^i}, \qquad X_{ij} = x^i \frac{\partial}{\partial x^j}, \qquad \bar{X}_i = x^i \mathcal{E}, \tag{6.1}$$

where

$$\mathcal{E} = x^i \frac{\partial}{\partial x^i}. \tag{6.2}$$

The Lie subalgebras $\mathrm{gl}(n,\mathbb{R})$ and $\mathrm{aff}(n,\mathbb{R})$ of $\mathrm{sl}(n+1,\mathbb{R})$ are generated by X_{ij} and $\{X_i, X_{ij}\}$, respectively.

Note that the cocycles (4.5), (4.6), and (4.7) are $\mathrm{aff}(n,\mathbb{R})$-invariant. Moreover, the cocycles (4.6) and (4.7) can be written in an $\mathrm{sl}(n+1,\mathbb{R})$-invariant form (cf. [17]).

6.1 Affinely invariant differential operators

Classification of linear operators on $\mathcal{S}$ invariant with respect to $\mathrm{aff}(n,\mathbb{R})$ goes back to the beginning of the theory of invariants (see [25]).

The algebra of the $\mathrm{aff}(n,\mathbb{R})$-invariant operators on $\mathcal{S}$ is generated by the divergence operator D given by (5.6) and the Euler operator

$$E = \xi_i \frac{\partial}{\partial \xi_i}. \tag{6.3}$$

In particular, for $k \geq \ell$, any $\mathrm{aff}(n,\mathbb{R})$-invariant operator from $\mathcal{S}_k$ to $\mathcal{S}_\ell$ is proportional to $D^{k-\ell}$ and, for $k < \ell$, there are no nonzero operators.

Similarly, a bilinear $\mathrm{aff}(n,\mathbb{R})$-invariant operator $A : \mathrm{Vect}(\mathbb{R}^n) \otimes \mathcal{S}_k \to \mathcal{S}_\ell$ is given by

$$\begin{aligned} A = \sum_{0\leq t\leq s} \Bigg(& \alpha_t \frac{\partial^{s-t} X}{\partial x^{i_1} \cdots \partial x^{i_{s-t}}} \frac{\partial^t}{\partial x^{i_{s-t+1}} \cdots \partial x^{i_s}} \frac{\partial^s}{\partial \xi_{i_1} \cdots \partial \xi_{i_s}} \\ & + \beta_t \frac{\partial^{s-t+1} X}{\partial x^{i_1} \cdots \partial x^{i_{s-t}} \partial \xi_{i_1}} \frac{\partial^t}{\partial x^{i_{s-t+1}} \cdots \partial x^{i_s}} \frac{\partial^{s-1}}{\partial \xi_{i_2} \cdots \partial \xi_{i_s}} \\ & + \gamma_t \frac{\partial^{s-t} X}{\partial x^{i_1} \cdots \partial x^{i_{s-t-1}} \partial \xi_{i_{s-t}}} \frac{\partial^{t+1}}{\partial x^{i_{s-t}} \cdots \partial x^{i_s}} \frac{\partial^{s-1}}{\partial \xi_{i_1} \cdots \partial \widehat{\xi}_{i_{s-t}} \cdots \partial \xi_{i_s}} \Bigg), \end{aligned} \tag{6.4}$$

where $s = k - \ell + 1$ and α_t, β_t, and γ_t are arbitrary constants.

Remark 6.1. The cocycles (4.5), (4.6), and (4.7) are precisely of the form (6.4) and, therefore, affinely invariant.

6.2 Affinely invariant deformations

The following statement will be an important ingredient of the proof of Theorem 4.3.

Theorem 6.2. Every deformation of the $\mathrm{Vect}(\mathbb{R}^n)$-module $\mathcal{S}$ is equivalent to some $\mathrm{aff}(n)$-invariant deformation. □

Proof. Given a deformation of the $\mathrm{Vect}(\mathbb{R}^n)$-module $\mathcal{S}$ written in the form (2.5), the operator φ is of the form

$$\varphi = \sum_{m \geq 1} \varphi_m. \tag{6.5}$$

Assume, by induction, that the operators $\varphi_1, \ldots, \varphi_{k-1}$ are $\mathrm{aff}(n)$-invariant and each term φ_m is a homogeneous polynomial in t of order m. The chosen deformation provides a solution φ_k of (2.16); we show that there is always another solution φ'_k which is $\mathrm{aff}(n)$-invariant.

The solution φ_k can be chosen homogeneous in t of degree k, such that

$$\varphi_k|_{\mathcal{S}_m} \subset \mathcal{S}_{m-k}. \tag{6.6}$$

Indeed, the coboundary operator δ and the cup product (2.8) preserve the homogeneity and, by induction, one can always choose the right-hand side of (2.16) homogeneous of degree k.

The cup-product of two $\mathrm{aff}(n)$-invariant linear maps is a bilinear map which is also $\mathrm{aff}(n)$-invariant. Hence, the right-hand side of (2.16) is $\mathrm{aff}(n)$-invariant. Take $X \in \mathrm{aff}(n)$, then $L_X(\delta\varphi_m) = 0$. Since δ commutes with Lie derivative, this means that the map

$$L_X(\varphi_m) \in \mathrm{Hom}\left(\mathrm{Vect}\left(\mathbb{R}^n\right), \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})\right) \tag{6.7}$$

is a 1-*cocycle* on $\mathrm{Vect}(\mathbb{R}^n)$ for all $X \in \mathrm{aff}(n)$.

This 1-cocycle is a coboundary for $m \geq 3$ (see [17] and Section 4.2), and therefore there exists an element $\gamma_X \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ depending on $X \in \mathrm{aff}(n)$, such that

$$L_X(\varphi_m) = \delta(\gamma_X). \tag{6.8}$$

This defines a linear map $\gamma : \mathrm{aff}(n) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ by

$$\gamma(X) = \gamma_X. \tag{6.9}$$

Lemma 6.3. The linear map (6.9) is a 1-cocycle on the Lie algebra[2] $\mathrm{aff}(n)$. □

Proof. By definition,

$$d\gamma(X,Y) = L_X(\gamma_Y) - L_Y(\gamma_X) - \gamma_{[X,Y]} \tag{6.10}$$

for all $X, Y \in \mathrm{aff}(n)$. Since, tautologically, $(L_X L_Y - L_Y L_X - L_{[X,Y]})\varphi_m = 0$, we have

$$\delta\big(L_X(\gamma_Y) - L_Y(\gamma_X) - \gamma_{[X,Y]}\big) = 0. \tag{6.11}$$

Therefore, the operator $d\gamma(X,Y) \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ has to be a 0-cocycle with respect to the $\mathrm{Vect}(\mathbb{R}^n)$-cohomology, or, in other words, $\mathrm{Vect}(\mathbb{R}^n)$-invariant. This implies $d\gamma(X,Y) = 0$ for all $X, Y \in \mathrm{aff}(n)$.

Lemma 6.3 is proved. ■

Every cocycle $\gamma : \mathrm{aff}(n) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ is of the form

$$\gamma_X = \kappa\, \mathrm{tr}(X)\, D^m + L_X(A), \tag{6.12}$$

where κ is a constant, $\mathrm{tr}(X)$ is the trace of X in the standard matrix representation of $\mathrm{aff}(n)$ and D is the operator (5.6); the second summand is a coboundary with an arbitrary $A \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ (see [15]).

We now substitute expression (6.12) into (6.8) and prove that the constant κ has to be zero.

Lemma 6.4. A cocycle $\gamma : \mathrm{aff}(n) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ satisfying (6.8) is a coboundary. □

Proof. Choose $X = \mathcal{E}$ given by (6.2) and substitute the general form (6.12) of γ into relation (6.8). This leads to the equation

$$L_{\mathcal{E}}(\varphi) = \kappa' \delta\big(D^m\big) \tag{6.13}$$

for a linear map $\varphi : \mathrm{Vect}(\mathbb{R}) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$ and some $\kappa' \in \mathbb{R}$.

In other words, for all $X \in \mathrm{Vect}(\mathbb{R}^n)$, we have

$$\big[L_{\mathcal{E}}, \varphi(X)\big] - \varphi\big([\mathcal{E}, X]\big) = k'\big[L_X, D^m\big], \tag{6.14}$$

where L_X is the Lie derivative on $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})$, so that, $L_{\mathcal{E}} = \mathcal{E} - E$. Equation (6.14),

[2] Note that we deal here with cohomology of $\mathrm{aff}(n)$, and not of $\mathrm{Vect}(M)$. We will denote by d the corresponding coboundary operator.

therefore, leads to

$$\left[\mathcal{E}, \varphi(X)\right] - \varphi\left([\mathcal{E}, X]\right) - \left[\,E, \varphi(X)\right] = k'\left[L_X, D^m\,\right]. \tag{6.15}$$

From translation equivariance, one can assume that the map φ is with constant coefficients, more precisely

$$\varphi(X) = \sum \varphi^{i_1,\dots,i_t} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_s}}(X) \frac{\partial}{\partial x^{i_{s+1}}} \cdots \frac{\partial}{\partial x^{i_t}}, \tag{6.16}$$

where coefficients $\varphi^{i_1,\dots,i_t}$ do not depend on x. Each homogeneous term in the left hand side of (6.15) is then proportional to itself with coefficient $m - t$; but, the right-hand side of (6.15) is homogeneous of degree m in x. Therefore, the only solution of (6.15) corresponds to $\kappa' = 0$.

Lemma 6.4 is proved. ■

We proved that $\kappa = 0$ in (6.12) so that γ is a coboundary, then we put $\widetilde{\varphi_m} = \varphi_m - \delta A$. This is a new solution of the equation (2.16) which is aff(n)-invariant.

Theorem 6.2 is proved. ■

6.3 Projectively invariant deformations

Consider the deformations invariant with respect to the full algebra of projective symmetries.

Proposition 6.5. Every integrable infinitesimal deformation of the Vect$(\mathbb{R}^n)$-module $\mathcal{S}$ generated by the cocycles (4.6) and (4.7) corresponds to some sl$(n+1)$-invariant deformation. □

Proof. The restriction of cohomology classes of cocycles c_1 and c_2 to sl$(n+1)$ vanish. In other words, these classes can be represented by the unique projectively invariant cocycles cohomologous to c_1 and c_2 (see [17]). The proof of Proposition 6.5 is analogous to that of Theorem 6.2 but simpler since $H^1(\mathrm{sl}(n+1); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-m})) = 0$ (see [15]) and the analogous of Lemma 6.4 is trivially held. ■

7 Proof of the main theorem

The proof contains two parts. First, we show by a straightforward computation that conditions (4.10), (4.11), and (4.12) are necessary. Second, we use the existence of the deformation constructed in the preceding section to prove that these conditions are, indeed, sufficient.

7.1 Cup-products of the nontrivial 1-cocycles

We will need to calculate the cup-products of the nontrivial 1-cocycles c_0, c_1, c_2. Obviously, $[[c_0, c_0]] = [[c_0, c_1]] = 0$ as 2-cocycles. One, furthermore, has a 2-cocycles

$$\begin{aligned}
\big[[c_1, c_1]\big](X, Y) &= -4\frac{\partial^2 X}{\partial x^i \partial x^j}\frac{\partial^3 Y}{\partial x^l \partial x^m \partial \xi_i}\frac{\partial^3}{\partial \xi_j \partial \xi_l \partial \xi_m} - (X \leftrightarrow Y),\\
\big[[c_0, c_2]\big](X, Y) &= 3\frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_i}\frac{\partial^3 Y}{\partial x^l \partial x^m \partial \xi_j}\frac{\partial^2}{\partial \xi_l \partial \xi_m} - (X \leftrightarrow Y),\\
\big[[c_1, c_2]\big](X, Y) &= -2\frac{\partial^2 X}{\partial x^i \partial x^j}\frac{\partial^4 Y}{\partial x^l \partial x^m \partial x^p \partial \xi_i}\frac{\partial^4}{\partial \xi_j \partial \xi_l \partial \xi_m \partial \xi_p},\\
&\quad + 6\frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l}\frac{\partial^3 Y}{\partial x^l \partial x^m \partial x^p}\frac{\partial^4}{\partial \xi_i \partial \xi_j \partial \xi_m \partial \xi_p}\\
&\quad - 6\frac{\partial^4 X}{\partial x^i \partial x^j \partial x^l \partial \xi_m}\frac{\partial^3 Y}{\partial x^m \partial x^p \partial \xi_l}\frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_p}\\
&\quad - (X \leftrightarrow Y),\\
\frac{1}{2}\big[[c_2, c_2]\big](X, Y) &= -6\frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l}\frac{\partial^4 Y}{\partial x^m \partial x^p \partial x^q \partial \xi_i}\frac{\partial}{\partial x^l}\frac{\partial^4}{\partial \xi_j \partial \xi_m \partial \xi_p \partial \xi_q}\\
&\quad + 9\frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l}\frac{\partial^4 Y}{\partial x^l \partial x^m \partial x^p \partial \xi_q}\frac{\partial}{\partial x^q}\frac{\partial^4}{\partial \xi_i \partial \xi_j \partial \xi_m \partial \xi_p}\\
&\quad + 3\frac{\partial^3 X}{\partial x^i \partial x^j \partial x^l}\frac{\partial^4 Y}{\partial x^m \partial x^p \partial x^q \partial \xi_i}\frac{\partial^5}{\partial \xi_j \partial \xi_l \partial \xi_m \partial \xi_p \partial \xi_q}\\
&\quad - 3\frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l}\frac{\partial^4 Y}{\partial x^l \partial x^m \partial x^p \partial x^q}\frac{\partial^5}{\partial \xi_i \partial \xi_j \partial \xi_m \partial \xi_p \partial \xi_q}\\
&\quad - 6\frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l}\frac{\partial^5 Y}{\partial x^l \partial x^m \partial x^p \partial x^q \partial \xi_i}\frac{\partial^4}{\partial \xi_j \partial \xi_m \partial \xi_p \partial \xi_q}\\
&\quad - (X \leftrightarrow Y)
\end{aligned} \tag{7.1}$$

as well as $[[c_2, c_0]] = [[c_0, c_2]]$ and $[[c_2, c_1]] = [[c_1, c_2]]$.

Lemma 7.1. The 2-cocycle $[[c_1, c_1]]$ is a coboundary. □

Proof. Consider a 1-cochain $b \in C^1(\mathrm{Vect}(\mathbb{R}^n); \mathcal{D}(\mathcal{S}))$ given by

$$b(X) = \frac{\partial^3 X}{\partial x^i \partial x^j \partial x^k}\frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k}. \tag{7.2}$$

It can be easily checked that

$$\big[[c_1, c_1]\big] = -\frac{4}{3}\delta(b) \tag{7.3}$$

so that the cohomology class of the 2-cocycle $[[c_1, c_1]]$ vanishes. ■

7.2 Computing the integrability conditions

The Maurer-Cartan equation (2.16) in the second order reads

$$\delta\varphi_2(t)|_{\mathcal{S}_k} = -\frac{1}{2}\sum_{i,j} t_i^{k-j} t_j^k \left[\left[c_i^{k-j}, c_j^k\right]\right]. \tag{7.4}$$

The second-order term of a formal deformation is $\varphi_2(t)(X) = \sum t_i^k t_j^l \varphi_{ij}^{kl}(X)$, where $\varphi_{ij}^{kl}(X)$ are differential operators on $\mathcal{S}$ of the form (6.4) homogeneous with respect to the partial derivatives in x and in ξ of degree $i+j+2$. Tedious but direct computation yields

$$\begin{aligned}
\varphi_2(t)(X) = {} & \alpha_3^k \frac{\partial^3 X}{\partial x^i \partial x^j \partial x^l} \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_l} + \beta_3^k \frac{\partial^4 X}{\partial x^i \partial x^j \partial x^l \partial \xi_i} \frac{\partial^2}{\partial \xi_j \partial \xi_l} \\
& + \alpha_4^k \frac{\partial^4 X}{\partial x^i \partial x^j \partial x^l \partial x^m} \frac{\partial^4}{\partial \xi_i \partial \xi_j \partial \xi_l \partial \xi_m} + \gamma_3^k \frac{\partial^4 X}{\partial x^i \partial x^j \partial x^l \partial \xi_m} \frac{\partial^4}{\partial x^m \partial \xi_i \partial \xi_j \partial \xi_l} \\
& + \alpha_5^k \frac{\partial^5 X}{\partial x^i \partial x^j \partial x^l \partial x^m \partial x^p} \frac{\partial^5}{\partial \xi_i \partial \xi_j \partial \xi_l \partial \xi_m \partial \xi_p} \\
& + \gamma_4^k \frac{\partial^5 X}{\partial x^i \partial x^j \partial x^l \partial x^m \partial \xi_p} \frac{\partial^4}{\partial x^p \partial \xi_i \partial \xi_j \partial \xi_l \partial \xi_m},
\end{aligned} \tag{7.5}$$

where the coefficients $\alpha_s^k, \beta_s^k, \gamma_s^k$ are quadratic polynomials in t_i^k satisfying the following system:

$$\begin{aligned}
3\alpha_3^k &= -2t_1^{k-1} t_1^k, \\
\beta_3^k &= 3t_0^k t_2^k, \\
4\alpha_4^k + \gamma_3^k &= -2t_1^{k-2} t_2^k, \\
2\alpha_4^k &= -2t_2^{k-1} t_1^k, \\
\gamma_3^k &= 2t_2^{k-1} t_1^k, \\
10\alpha_5^k &= -3t_2^{k-2} t_2^k, \\
2\gamma_4^k &= 3t_2^{k-2} t_2^k.
\end{aligned} \tag{7.6}$$

This system has (a unique) solution if and only if condition (4.10) is satisfied. This proves that condition (4.10) is, indeed, necessary for the existence of the second order term $\varphi_2(t)$.

The proof, that the relations (4.11) and (4.12) are necessary for integrability of infinitesimal deformations, is analogous but much longer since one has to consider the third-order terms in (2.16).

7.3 The conditions of integrability are sufficient

We prove that conditions (4.10), (4.11), and (4.12) are sufficient.

Suppose that there is a condition of integrability of order m, that is, a relation $R_m(t) = 0$, where $R_m(t)$ is a homogeneous polynomial of degree m in t_0^k, t_1^k, t_2^k. We have to prove that the polynomial $R_m(t)$ belongs to the ideal, $\mathcal{R}$, generated by the relations (4.10), (4.11), and (4.12).

Proposition 5.5 implies that $R_m(t)$ belongs to the ideal generated by the polynomials in (5.11), (5.12), and (5.13). Therefore, $R_m(t)$ is split into a sum

$$R_m(t) = R_{m,1}(t) + R_{m,2}(t) + R_{m,3}(t) \tag{7.7}$$

of polynomials divisible by (5.11), (5.12), and (5.13), respectively.

The polynomial $R_{m,1}(t)$ already belongs to $\mathcal{R}$.

Consider the second term $R_{m,2}(t)$. We can assume that $m \geq 3$, since the only second-order condition is (4.10), see Section 6. The relation $[[c_0, c_1]] = 0$ implies that each monomial in $R_m(t)$ has to contain some parameter t_2^ℓ as a multiple (cf. Proposition 4.4). By assumption, the polynomial $R_{m,2}(t)$ is a multiple of $(t_0^k - t_0^{k-1})t_1^k$ for some k. But, modulo the relation (4.10), any expression of the form $(t_0^k - t_0^{k-1})t_1^k \cdots t_2^\ell$ is divisible by (4.12) and therefore $R_{m,2}(t)$ belongs to $\mathcal{R}$.

Consider the third term $R_{m,3}(t)$. Since the Nijenhuis-Richardson product $[[c_0, c_2]]$ commutes with c_0, then $R_{m,3}(t)$ has to contain the terms of the form

$$\left(t_0^k - t_0^{k-2}\right)t_2^k \cdots t_1^\ell \quad \text{or} \quad \left(t_0^k - t_0^{k-2}\right)t_2^k \cdots t_2^\ell. \tag{7.8}$$

But, using the relation (4.10) we readily get that these terms are divisible by (4.11) and (4.12), respectively and, therefore, belong to $\mathcal{R}$.

Theorem 4.3 is proved.

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