Journal of Algebra 323 (2010) 1649–1664

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Simple graded commutative algebras

Sophie Morier-Genoud [a], Valentin Ovsienko [b,*]

[a] Université Paris Diderot Paris 7, UFR de mathématiques case 7012, 75205 Paris Cedex 13, France
[b] CNRS, Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France

ARTICLE INFO

Article history:
Received 4 May 2009
Available online 15 January 2010
Communicated by Vera Serganova

Keywords:
Graded commutative algebra
Clifford algebras

ABSTRACT

We study the notion of Γ-graded commutative algebra for an arbitrary abelian group Γ. The main examples are the Clifford algebras already treated in H. Albuquerque and S. Majid (2002) [2]. We prove that the Clifford algebras are the only simple finite-dimensional associative graded commutative algebras over $\mathbb{R}$ or $\mathbb{C}$. Our approach also leads to non-associative graded commutative algebras extending the Clifford algebras.

1. Introduction and the main theorems

Let $(\Gamma, +)$ be an abelian group and $\langle\,,\rangle : \Gamma \times \Gamma \to \mathbb{Z}_2$ a bilinear map. An algebra $\mathcal{A}$ is called a Γ-graded commutative (or Γ-commutative for short) if $\mathcal{A}$ is Γ-graded in the usual sense:

$$\mathcal{A} = \bigoplus_{\gamma \in \Gamma} \mathcal{A}_\gamma$$

such that $\mathcal{A}_\gamma \cdot \mathcal{A}_{\gamma'} \subset \mathcal{A}_{\gamma+\gamma'}$ and for all homogeneous elements $a, b \in \mathcal{A}$, one has

$$ab = (-1)^{\langle \bar{a}, \bar{b} \rangle} ba. \tag{1}$$

where $\bar{a}$ and $\bar{b}$ are the corresponding degrees.

If $\Gamma = \mathbb{Z}_2$ and $\langle\,,\rangle$ is the standard product, then the above definition coincides with that of "supercommutative algebra". The main examples of associative supercommutative algebras are: the algebras of differential forms on manifolds or, more generally, algebras of functions on supermanifolds. These algebras cannot be simple (i.e., they always contain a non-trivial proper ideal).

* Corresponding author.
E-mail addresses: sophiemg@math.jussieu.fr (S. Morier-Genoud), ovsienko@math.univ-lyon1.fr (V. Ovsienko).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.01.004

Study of Γ-commutative algebras is a quite new subject. We cite here the pioneering works of Lychagin and Albuquerque and Majid, [11,1], where the classical algebra of quaternions $\mathbb{H}$ is understood as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-commutative algebra. This result was generalized for the Clifford algebras in [2]. In [4] a more general notion of β-commutative algebra is considered and the structure of simple algebras is completely determined. An application of graded commutative algebras to deformation quantization is recently proposed in [8].

In this paper we also consider the quaternion algebra and the Clifford algebras as graded commutative algebras. Our grading is slightly different from that of [1,2]. This difference concerns essentially the parity of the elements.

The first example

The starting point of this work is the following observation, see [14]. The quaternion algebra $\mathbb{H}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-commutative in the following sense. Associate the "triple degree" to the standard basis elements of $\mathbb{H}$:

$$\begin{aligned} \bar{\varepsilon} &= (0,0,0), \\ \bar{i} &= (0,1,1), \\ \bar{j} &= (1,0,1), \\ \bar{k} &= (1,1,0), \end{aligned} \tag{2}$$

where ε denotes the unit. The usual product of quaternions then satisfies the condition (1), where $\langle\, , \rangle$ is the usual *scalar product* of 3-vectors. Indeed, $\langle \bar{i}, \bar{j} \rangle = 1$ and similarly for k, so that i, j and k anticommute with each other. But, $\langle \bar{i}, \bar{i} \rangle = 0$, so that i, j, k commute with themselves.

In fact, the first two components in (2) contain the full information of the grading of $\mathbb{H}$. By "forgetting" the third component, one obtains a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on the quaternions:

$$\bar{1} = (0,0), \qquad \bar{i} = (0,1), \qquad \bar{j} = (1,0), \qquad \bar{k} = (1,1)$$

which is nothing but the Albuquerque–Majid grading. However, the bilinear map $\langle\, , \rangle$ in no longer the scalar product. (It has to be replaced by the determinant of the 2×2-matrix formed by the degrees of the elements.) The elements i and j are odd in this grading while k is even. We think that it is more natural if these elements have the same parity.

The classification

We are interested in *simple* Γ-commutative algebras. Recall that an algebra is called simple if it has no proper (two-sided) ideal. In the usual commutative associative case, a simple algebra (over $\mathbb{R}$ or $\mathbb{C}$) is necessarily a division algebra. The associative division algebras are classified by (a particular case of) the classical Frobenius theorem. Classification of simple Γ-commutative algebras can therefore be understood as an analog of the Frobenius theorem.

We obtain a classification of simple associative Γ-graded commutative algebras over $\mathbb{R}$ and $\mathbb{C}$. The following theorem is the main result of this paper.

Theorem 1. *Every finite-dimensional simple associative Γ-commutative algebra over $\mathbb{C}$ or over $\mathbb{R}$ is isomorphic to a Clifford algebra.*

The well-known classification of simple Clifford algebras (cf. [16]) readily gives a complete list:

(1) The algebras $\mathrm{Cl}_{2m}(\mathbb{C})$ ($\cong \mathrm{M}_{2^m}(\mathbb{C})$) are the only simple associative Γ-commutative algebras over $\mathbb{C}$.

(2) The real Clifford algebras $\mathrm{Cl}_{p,q}$ with $p-q\neq 4k+1$ and the algebras $\mathrm{Cl}_{2m}(\mathbb{C})$ viewed as algebras over $\mathbb{R}$ are the only real simple associative Γ-commutative algebras.

Note that the Clifford algebras $\mathrm{Cl}_{2m+1}(\mathbb{C})$ and $\mathrm{Cl}_{p,q}$ with $p-q=4k+1$ are of course not simple; however these algebras are graded-simple.

As a consequence of the above classification, we have a quite amazing statement.

Corollary 1.1. *The associative algebra* M_n *of* $n\times n$*-matrices over* $\mathbb{R}$ *or* $\mathbb{C}$*, can be realized as* Γ*-commutative algebra for some* Γ*, if and only if* $n=2^m$.

Note that gradings on the algebras of matrices and more generally on associative algebras, but without the commutativity assumption, is an important subject, see [5,6,10] and references therein.

Universality of $(\mathbb{Z}_2)^n$*-grading*

It turns out that the $(\mathbb{Z}_2)^n$-grading is the most general, and $\langle\,,\rangle$ can always be reduced to the scalar product.

Theorem 2.

(i) *If the abelian group* Γ *is finitely generated, then for an arbitrary* Γ*-commutative algebra* $\mathcal{A}$*, there exists* n *such that* $\mathcal{A}$ *is* $(\mathbb{Z}_2)^n$*-commutative.*
(ii) *The bilinear map* $\langle\,,\rangle$ *can be chosen as the usual scalar product.*

This theorem is proved in Section 3. This is the main tool for our classification of simple associative Γ-commutative algebras.

Non-associative extensions of Clifford algebras

Similarly to the grading of the quaternion algebra (2), all the elements of Clifford algebras in our grading correspond only to the even elements of $(\mathbb{Z}_2)^{n+1}$. It is therefore interesting to ask the following question. *Given a Clifford algebra, is there a natural larger algebra that contains this Clifford algebra as an even part?*

We will show (see Corollary 2.4) that such an extension of a Clifford algebra cannot be associative. In Section 5, we construct simple $(\mathbb{Z}_2)^{n+1}$-commutative *non-associative* algebras $\mathcal{A}$ such that $\mathcal{A}^0=\mathrm{Cl}_n$. We require one additional condition: existence of at least one *odd derivation*. This means, the even part $\mathcal{A}_0$ and the odd part $\mathcal{A}_1$ are not separated and can be "mixed up".

We have a complete classification only in the simplest case of $\mathbb{Z}_2$-commutative algebras. We obtain exactly two 3-dimensional algebras. One of these algebras has the basis $\{\varepsilon; a, b\}$, where $\bar{\varepsilon}=0$ and $\bar{a}=\bar{b}=1$, satisfying the relations

$$\begin{aligned} &\varepsilon\varepsilon=\varepsilon,\\ &\varepsilon a=\frac{1}{2}a,\qquad \varepsilon b=\frac{1}{2}b,\\ &ab=\varepsilon. \end{aligned}\tag{3}$$

This algebra is called *tiny Kaplansky superalgebra*, see [12] and also [7] and denoted K_3. It was rediscovered in [15] (under the name asl_2) and further studied in [13]. The corresponding algebra of derivations is the simple Lie superalgebra osp(1|2). We believe that the natural extension of Clifford algebras with non-trivial odd part are the algebras:

$$\mathrm{Cl}_n \otimes_{\mathbb{C}} K_3, \qquad \mathrm{Cl}_{p,q} \otimes_{\mathbb{R}} K_3$$

in the complex and in the real case, respectively.

We will however construct another series of extensions of Clifford algebras, that have different properties. For instance, these algebras have the unit element.

We hope that some of the new algebras constructed in this paper may be of interest for mathematical physics.

2. Preliminary results

We start with simple results and observations.

2.1. Clifford algebra is indeed commutative

A real Clifford algebra $\mathrm{Cl}_{p,q}$ is an associative algebra with unit ε and $n = p + q$ generators $\alpha_1, \dots, \alpha_n$ subject to the relations

$$\alpha_i \alpha_j = -\alpha_j \alpha_i, \quad i \neq j, \qquad \alpha_i^2 = \begin{cases} \varepsilon, & 1 \leqslant i \leqslant p, \\ -\varepsilon, & p < i \leqslant n. \end{cases}$$

The complex Clifford algebra $\mathrm{Cl}_n = \mathrm{Cl}_{p,q} \otimes \mathbb{C}$ can be defined by the same formulæ, but one can always choose the generators in such a way that $\alpha_i^2 = \varepsilon$ for all i.

Let us show that n-generated Clifford algebras are $(\mathbb{Z}_2)^{n+1}$-commutative. The construction is the same in the real and complex cases. We assign the following degree to every basis element

$$\begin{aligned} \overline{\alpha_1} &= (1, 0, 0, \dots, 0, 1), \\ \overline{\alpha_2} &= (0, 1, 0, \dots, 0, 1), \\ &\dots \\ \overline{\alpha_n} &= (0, 0, 0 \dots, 1, 1). \end{aligned} \tag{4}$$

One then has $\langle \alpha_i, \alpha_j \rangle = 1$ so that the anticommuting generators α_i and α_j become commuting in the $(\mathbb{Z}_2)^{n+1}$-grading sense. Furthermore, two monomials $\alpha_{i_1} \cdots \alpha_{i_k}$ and $\alpha_{j_1} \cdots \alpha_{j_\ell}$ commute if and only if either k or ℓ is even. It worth noticing that every monomial is homogeneous and there is a one-to-one correspondence between the monomial basis of the Clifford algebra and the even elements of $(\mathbb{Z}_2)^{n+1}$.

We notice that the degrees of elements in (4) are purely even. We will show in Section 2.2 that this is not a coincidence.

Remark 2.1. A $(\mathbb{Z}_2)^n$-graded commutative structure of the Clifford algebras was defined in [2]: the degree of a generator α_i is the n-vector $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 stays at the i-th position. As in the case of the quaternion algebra, our $(\mathbb{Z}_2)^{n+1}$-grading is equivalent to that of Albuquerque–Majid. However, the symmetric bilinear map of [2] is different of ours, namely

$$\beta(\gamma, \gamma') = \sum_i \gamma_i \gamma_i' + \Bigl(\sum_i \gamma_i\Bigr)\Bigl(\sum_j \gamma_j'\Bigr),$$

instead of the scalar product. We will show in the next section that an n-generated Clifford algebra cannot be realized as $(\mathbb{Z}_2)^k$-commutative algebra with $k < n+1$, provided the bilinear map $\langle\, , \rangle$ is the scalar product.

Let us also mention that the classical $\mathbb{Z}_2$-grading of the Clifford algebras, see [3] can be obtained from our grading by projection to the last $\mathbb{Z}_2$-component in $(\mathbb{Z}_2)^{n+1}$. Indeed, degree 1 is then assigned to every generator.

2.2. Commutativity, simplicity and zero divisors

Unless we specify the ground field, the results of this section hold over $\mathbb{C}$ or $\mathbb{R}$. A simple associative commutative algebra is a division algebra. Indeed, for every $a \in \mathcal{A}$, the set

$$\{b \in \mathcal{A} \mid ab = 0\} \tag{5}$$

is a (two-sided) ideal.

A simple Γ-commutative associative algebra can have zero divisors. Consider for example the complexified quaternion algebra $\mathbb{H} \otimes \mathbb{C} \cong \mathrm{M}_2(\mathbb{C})$. This is of course a simple $(\mathbb{Z}_2)^3$-commutative algebra. It has elements α such that $\alpha^2 = \varepsilon$. One then obviously obtains: $(\varepsilon + \alpha)(\varepsilon - \alpha) = 0$.

The following lemma shows however that there is a property of simple Γ-commutative associative algebras similar to the usual commutative case.

Lemma 2.2. *Every homogeneous element of a simple associative Γ-commutative algebra is not a left or right zero divisor.*

Proof. If a is a homogeneous element then for every $c \in \mathcal{A}$, there exists $\tilde{c} \in \mathcal{A}$ such that $ac = \tilde{c}a$. Indeed, writing c as a sum of homogeneous terms one obtains:

$$c = \sum_{\gamma \in \Gamma} c_\gamma, \qquad \tilde{c} = \sum_{\gamma \in \Gamma} (-1)^{\bar{a}\bar{\gamma}} c_\gamma.$$

It follows that the set (5) is again a two-sided ideal. By simplicity of $\mathcal{A}$ this ideal has to be trivial. □

Definition 2.3. Let us introduce the *parity function*

$$p(a) = \langle \bar{a}, \bar{a} \rangle, \tag{6}$$

where $a, b \in \mathcal{A}$ are homogeneous. A Γ-commutative algebra is then split as a vector space into $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, where $\mathcal{A}^0$ and $\mathcal{A}^1$ are generated by even and odd elements, respectively.

One thus obtains a $\mathbb{Z}_2$-grading on $\mathcal{A}$, that however, does not mean $\mathbb{Z}_2$-commutativity of $\mathcal{A}$. In particular, the even subspace $\mathcal{A}^0$ is a subalgebra of $\mathcal{A}$ which is not commutative in general.

Lemma 2.2 has several important corollaries.

Corollary 2.4. *Every simple associative Γ-commutative algebra $\mathcal{A}$ is even, that is, $\mathcal{A} = \mathcal{A}^0$.*

Indeed, an odd element $a \in \mathcal{A}^1$ anticommutes with itself, so that $a^2 = 0$. But this is impossible by Lemma 2.2. It follows that the odd part of $\mathcal{A}$ is trivial, that is, $\mathcal{A}^1 = \{0\}$.

Denote by $\mathcal{A}_0$ the subalgebra of $\mathcal{A}$ consisting of homogeneous elements of degree $0 \in \Gamma$. Another corollary of Lemma 2.2 is as follows.

Corollary 2.5. *Let $\mathcal{A}$ be a simple associative Γ-commutative algebra, then*:

(i) *In the complex case, one has $\mathcal{A}_0 = \mathbb{C}$.*
(ii) *In the real case, $\mathcal{A}_0 = \mathbb{R}$ or $\mathbb{C}$ (viewed as an $\mathbb{R}$-algebra).*

Indeed, the space $\mathcal{A}_0$ is a commutative associative division algebra. By the classical Frobenius theorem, $\mathcal{A}_0 = \mathbb{R}$ or $\mathbb{C}$ in the real case and $\mathcal{A}_0 = \mathbb{C}$ in the complex case.

A strengthened version of the above corollary in the complex case is as follows.

Corollary 2.6. *If $\mathcal{A}$ is a simple complex associative $(\mathbb{Z}_2)^n$-commutative algebra and $\mathcal{A}_\gamma$ is a non-zero homogeneous component of $\mathcal{A}$, then $\dim \mathcal{A}_\gamma = 1$. Furthermore, there exists $\alpha \in \mathcal{A}_\gamma$ such that $\alpha^2 = \varepsilon$.*

Indeed, for every $a, b \in \mathcal{A}_\gamma$, the product ab belongs to $\mathcal{A}_0$ since $\Gamma = (\mathbb{Z}_2)^n$. This product is different from zero (Lemma 2.2). It follows that every homogeneous component is at most one-dimensional.

As a first application of the above statements, we can now easily see that a complex Clifford algebra with n generators cannot be realized as a $(\mathbb{Z}_2)^k$-commutative algebra with $k < n + 1$. Indeed, the dimension of the Clifford algebra is equal to 2^n and there are exactly 2^n even elements in $(\mathbb{Z}_2)^{n+1}$. Our claim then follows from Corollary 2.6.

3. Reducing the abelian group

In this section we prove Theorem 2.

3.1. Universality of $(\mathbb{Z}_2)^n$

Let Γ be a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, one can write Γ as a direct product

$$\Gamma = \mathbb{Z}^n \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m},$$

where $n_i = p_i^{k_i}$ and where $p_1, \ldots, p_m$ are not necessarily distinct prime numbers. Assume that there is a non-trivial bilinear map $\langle\,,\rangle : \Gamma \times \Gamma \to \mathbb{Z}_2$.

In the case where $\Gamma = \Gamma' \times \Gamma''$, if $\langle y, x\rangle = 0$ for all $x \in \Gamma'$ and $y \in \Gamma$, then the component Γ' is not significative in the Γ-grading of $\mathcal{A}$, i.e., $\mathcal{A}$ is Γ''-commutative.

Let us choose one of the components, $\Gamma' = \mathbb{Z}$ or $\Gamma' = \mathbb{Z}_{n_i}$. We can assume that there exists an element $x \in \Gamma$, such that the map

$$\varphi_x : \Gamma' \to \mathbb{Z}_2,$$

defined on Γ' by $\langle\,, x\rangle$ defines a non-trivial group homomorphism.

In the case where $\Gamma' = \mathbb{Z}$, the only non-trivial homomorphism is defined by $\varphi(0) = 0$ and $\varphi(1) = 1$. Therefore, one can replace the component $\Gamma' = \mathbb{Z}$ of Γ by $\Gamma' = \mathbb{Z}_2$. The algebra $\mathcal{A}$ remains Γ-commutative.

In the case where $\Gamma' = \mathbb{Z}_{n_i}$, a non-trivial homomorphism exists if and only if n_i is even. We may then assume that $n_i = 2^{k_i}$. However, in this case, again the only non-trivial homomorphism is defined by $\varphi(0) = 0$ and $\varphi(1) = 1$ so that one replaces the component $\Gamma' = \mathbb{Z}_{2^{k_i}}$ of Γ by $\Gamma' = \mathbb{Z}_2$ without loss of Γ-commutativity.

The first part of Theorem 2 is proved.

3.2. Universality of the scalar product

Let us now prove the second part of Theorem 2, namely that the standard scalar product is the only relevant bilinear map from Γ to $\mathbb{Z}_2$.

Given an arbitrary bilinear symmetric form $\beta : (\mathbb{Z}_2)^n \times (\mathbb{Z}_2)^n \to \mathbb{Z}_2$, we will show that there exist an integer $N \leqslant 2n$ and an abelian group homomorphism

$$\sigma : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^N$$

such that

$$\big\langle \sigma(x), \sigma(y) \big\rangle = \beta(x, y),$$

for all $x, y \in (\mathbb{Z}_2)^n$ and where $\langle\, ,\rangle$ is the standard scalar product on $(\mathbb{Z}_2)^N \times (\mathbb{Z}_2)^N$.

Consider the standard basic elements of $\mathbb{Z}_2^n$:

$$\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0),$$

where all the entries are zero except the i-th entry that is equal to one. The form β is completely determined by the numbers

$$\beta_{i,j} := \beta(\varepsilon_i, \varepsilon_j), \quad 1 \leqslant i, j \leqslant n.$$

We first construct a family of vectors σ_i in $(\mathbb{Z}_2)^n$ that have the following property

$$\langle \sigma_i, \sigma_j \rangle = \beta_{i,j}, \quad \text{for all } i \neq j.$$

The explicit formula for σ_i is:

$$\begin{aligned}
\sigma_1 &= (1, 0, 0, \dots, 0), \\
\sigma_2 &= (\beta_{12}, 1, 0, \dots, 0), \\
\sigma_3 &= (\beta_{13}, \beta_{23} - \beta_{12}\beta_{13}, 1, \dots, 0), \\
&\ \ \vdots \\
\sigma_n &= (\beta_{1n}, \beta_{2n} - \beta_{12}\beta_{1n}, \dots, \dots, 1).
\end{aligned}$$

This construction does not guarantee that $\langle \sigma_i, \sigma_i \rangle = \beta_{i,i}$. However, by adding at most n columns, we can obviously satisfy these identities as well.

Theorem 2 is proved.

Since now on, we will assume that $\Gamma = (\mathbb{Z}_2)^n$; this is the only group relevant for the notion of Γ-commutative algebra. We will also assume that the bilinear map $\langle\, ,\rangle$ is the usual scalar product.

4. Completing the classification

In this section we prove Theorem 1.

4.1. The complex case

Let $\mathcal{A}$ be a simple associative $(\mathbb{Z}_2)^n$-commutative algebra over $\mathbb{C}$. We will be considering minimal sets of homogeneous generators of $\mathcal{A}$. By definition, two homogeneous elements $\alpha, \beta \in \mathcal{A}$ either commute or anticommute. We thus can organize the generators in two sets

$$\{\alpha_1, \dots, \alpha_p\} \cup \{\beta_1, \dots, \beta_q\}$$

where the subset $\{\alpha_1, \dots, \alpha_p\}$ is the biggest subset of pairwise anticommutative generators: $\alpha_i\alpha_j = -\alpha_j\alpha_i$, while each generator β_i commutes with at least one generator α_j. We can assume $p \geqslant 2$ otherwise the algebra $\mathcal{A}$ is commutative and so it is $\mathbb{C}$ itself (by Frobenius theorem).

Among the minimal sets of homogeneous generators we choose one with the greatest p. If $q = 0$ then $\mathcal{A}$ is exactly Cl_p. Suppose that $q > 0$.

Lemma 4.1. *The generators $\{\beta_1, \dots, \beta_q\}$ can be chosen in such a way that every β_i commutes with exactly one element in $\{\alpha_1, \dots, \alpha_p\}$.*

Proof. Suppose β_i anticommutes with $\alpha_1, \dots, \alpha_s$ and commutes with $\alpha_{s+1}, \dots, \alpha_p$. Changing β_i to $\tilde{\beta}_i := \alpha_{s+1}\alpha_{s+2}\beta_i$, one obtains a new generator that anticommutes with $\alpha_1, \dots, \alpha_{s+2}$ and commutes with $\alpha_{s+3}, \dots, \alpha_p$. Repeating this procedure, we can change β_i to a new generator that commutes with at most one of the α_j's.

By the assumption of maximality of p, the generator β_i has to commute with at least one of the α_j's. $\square$

Lemma 4.2. *The generators $\{\beta_1, \dots, \beta_q\}$ can be chosen commuting with each other and with the same generator α_p.*

Proof. The construction of such a set of generators will be obtained by induction. Consider β_1 and β_2. By Lemma 4.1, they both commutes with one of the α-generators. Let β_1 commutes with α_{p_1} and β_2 commutes with α_{p_2}. There are four cases:

(1) β_1 commutes with β_2 and $\alpha_{p_1} = \alpha_{p_2}$,
(2) β_1 anticommutes with β_2 and $\alpha_{p_1} = \alpha_{p_2}$,
(3) β_1 commutes with β_2 and $\alpha_{p_1} \neq \alpha_{p_2}$,
(4) β_1 anticommutes with β_2 and $\alpha_{p_1} \neq \alpha_{p_2}$.

Let us show that we can always choose the generators in such a way that the case (1) holds. Indeed, in the case (2), we get a bigger set

$$\{\alpha_1, \dots, \alpha_p\} \setminus \{\alpha_{p_1}\} \cup \{\beta_1, \beta_2\}$$

of pairwise anticommutative generators. Therefore, case (2) is not possible.

In the case (3), we replace β_2 by $\tilde{\beta}_2 := \beta_1\beta_2\alpha_{p_2}$. The new generator $\tilde{\beta}_2$ anticommutes with all the α_j's, where $j \neq p_1$, commutes with α_{p_1} and anticommutes with β_1. Thus, we got back to case (2), that is a contradiction.

In the case (4), we replace β_1 by $\tilde{\beta}_2 := \beta_1\beta_2\alpha_{p_2}$. The generator $\tilde{\beta}_2$ anticommutes with all the α_k, $k \neq p_1$, commutes with α_{p_1} and commutes with β_1.

Thus, we obtain a set of generators satisfying the case (1). This provides the base of induction.

Suppose that ℓ generators $\{\beta_1, \dots, \beta_\ell\}$ pairwise commute and commute with the same generator α_p. Consider an extra generator $\beta_{\ell+1}$ that commutes with $\alpha_{p'}$. Replacing $\beta_{\ell+1}$ by $\tilde{\beta}_{\ell+1} := \alpha_p\alpha_{p'}\beta_{\ell+1}$, one obtains a generator commuting with α_p and anticommuting with all the α_j's, for $j \neq p$. If there is a β_i, for $i \leqslant \ell$ such that β_i and $\tilde{\beta}_{\ell+1}$ anticommute, then we get a bigger set of pairwise anticommuting generators:

$$\{\alpha_1, \dots, \alpha_{p-1}, \beta_i, \tilde{\beta}_{\ell+1}\}$$

that contradicts the maximality of p.

In conclusion, we have constructed a set $\{\beta_1, \dots, \beta_{\ell+1}\}$ of pairwise commuting elements that commute with the same element α_p and anticommute with the rest of α_j's. $\square$

We now choose a set of generators $\{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\}$ as in Lemma 4.2. In addition, we can normalize the generators by $\alpha_i^2 = \beta_i^2 = \varepsilon$ for all i. Indeed, by Theorem 2 we assume $\Gamma = (\mathbb{Z}_2)^n$ and then use Corollary 2.6.

Lemma 4.3. *If the number q of commuting generators is greater than zero, then the algebra $\mathcal{A}$ cannot be simple.*

Proof. First, the space $(\alpha_p + \beta_1)\mathcal{A}$ is an ideal of $\mathcal{A}$. Indeed, this space is clearly a left ideal. It is also a right ideal because α_p and β_1 commute or anticommute with any generator of $\mathcal{A}$ simultaneously. Therefore,

$$(\alpha_p + \beta_1)c = \tilde{c}(\alpha_p + \beta_1).$$

It remains to show that $(\alpha_p + \beta_1)\mathcal{A}$ is a proper ideal. If $\mathcal{A} = (\alpha_p + \beta_1)\mathcal{A}$ then we can write

$$\varepsilon = (\alpha_p + \beta_1)a,$$

for some $a \in \mathcal{A}$. Multiplying this equality by $(\alpha_p - \beta_1)$, we get

$$(\alpha_p - \beta_1) = (\alpha_p - \beta_1)(\alpha_p + \beta_1)a.$$

But we have

$$(\alpha_p - \beta_1)(\alpha_p + \beta_1) = \alpha_p^2 - \beta_1^2 = \varepsilon - \varepsilon = 0$$

so we deduce $\alpha_p - \beta_1 = 0$. This is not possible because the set of generators is chosen minimal so that α_p and β_1 are different. □

Combining Lemmas 4.1–4.3, we conclude that $\mathcal{A}$ is generated by homogeneous anticommuting generators. Theorem 1 is proved in the complex case.

4.2. The real case

In the case of a real simple associative Γ-commutative algebra $\mathcal{A}$, the component of degree $0 \in \Gamma$ can be one- or two-dimensional:

$$\mathcal{A}_0 = \mathbb{R} \quad \text{or} \quad \mathbb{C},$$

see Corollary 2.5.

- Case $\mathcal{A}_0 = \mathbb{R}$.

We proceed in a similar way as in the complex case. Lemma 4.2 still holds. The proof of Lemma 4.3 is however false because we cannot assume that $\alpha_p^2 = \beta_1^2 = \varepsilon$. We can only assume $\alpha_i^2 = \pm\varepsilon$ and $\beta_i^2 = \pm\varepsilon$ for all i (we again use Corollary 2.6).

Choose a system of homogeneous generators

$$\{\alpha_1, \ldots, \alpha_p\} \cup \{\beta_1, \ldots, \beta_q\}$$

such that:

(1) The system is minimal.
(2) The generators $\{\alpha_1, \dots, \alpha_p\}$ pairwise anticommute and the number p of anticommuting generators is maximal (within the minimal sets of homogeneous generators).
(3) The elements $\{\beta_1, \dots, \beta_q\}$ pairwise commute, they also commute with α_p and anticommute with $\alpha_1, \dots, \alpha_{p-1}$.

The existence of such a system of generators is guaranteed by Lemma 4.2.

Lemma 4.4. *The number q of commuting generators is zero.*

Proof. If $\alpha_p^2 = \beta_1^2$ then, as in the complex case, the element $\alpha_p + \beta_1$ generates a proper ideal. We will assume that

$$\alpha_p^2 = -\varepsilon, \qquad \beta_1^2 = \varepsilon.$$

If $q \geqslant 2$ then among $\alpha_p, \beta_1, \dots, \beta_q$ at least two elements have same square. The sum of these two elements again generates a proper ideal. We therefore have the last possibility: $q = 1$.

If p is even, then we replace β_1 by

$$\tilde{\beta}_1 = \beta_1 \alpha_1 \cdots \alpha_{p-1}.$$

It is then easy to check that $\tilde{\beta}_1$ anticommutes with all the α_i's. Therefore we have obtained a system of $p+1$ pairwise anticommuting generators $\{\alpha_1, \dots, \alpha_p, \tilde{\beta}_1\}$. This is a contradiction with the maximality of p.

Finally, we assume that p is odd. Let us introduce the elements

$$\tilde{\alpha} = \alpha_1 \cdots \alpha_p,$$
$$\tilde{\beta} = \alpha_1 \cdots \alpha_{p-1} \beta_1.$$

It is easy to check that $\tilde{\alpha}$ and $\tilde{\beta}$ both commute with all the generators α_i's and with β_1. Furthermore,

$$\tilde{\alpha}^2 = (\alpha_1 \cdots \alpha_{p-1})^2 \alpha_p^2,$$
$$\tilde{\beta}^2 = (\alpha_1 \cdots \alpha_{p-1})^2 \beta_1^2.$$

Since we are in the case of $\alpha_p^2 \neq \beta_1^2$, we have $\tilde{\alpha}^2 \neq \tilde{\beta}^2$. Assume without loss of generality that $\tilde{\alpha}^2 = \varepsilon$. The space $(\varepsilon + \tilde{\alpha})\mathcal{A}$ is a (two-sided) ideal of $\mathcal{A}$ because ε and $\tilde{\alpha}$ commute with all the generators of $\mathcal{A}$. In addition,

$$(\varepsilon - \tilde{\alpha})(\varepsilon + \tilde{\alpha}) = \varepsilon - \tilde{\alpha}^2 = 0$$

implies that ε does not belong to $(\varepsilon + \tilde{\alpha})\mathcal{A}$ so that the ideal is proper. $\square$

In conclusion, the existence of an element β_1 leads to contradictions. Consequently, the algebra $\mathcal{A}$ is generated by pairwise anticommutative generators. Therefore, $\mathcal{A}$ is isomorphic to a real Clifford algebra. Theorem 1 is proved in the case $\mathcal{A}_0 = \mathbb{R}$.

- Case $\mathcal{A}_0 = \mathbb{C}$.

Since the zero component of $\mathcal{A}$ is two-dimensional, one has the following statement.

Lemma 4.5. *Every non-trivial homogeneous component $\mathcal{A}_\gamma$ is 2-dimensional and contains elements α_+ and α_- such that*

$$(\alpha_+)^2 = \varepsilon, \qquad (\alpha_-)^2 = -\varepsilon.$$

Proof. Denote the basis of $\mathcal{A}_0$ by $\{\varepsilon, i\}$. A non-zero component $\mathcal{A}_\gamma$ is at least two-dimensional since for every $\alpha \in \mathcal{A}_\gamma$, the element $i\alpha$ is linearly independent with α. Indeed, if $\lambda\alpha + \mu i\alpha = 0$, with $\lambda, \mu \in \mathbb{R}$, then $(\lambda + \mu i)\alpha = 0$, thus, by Lemma 2.2, $\lambda + \mu i = 0$, so that $\lambda = \mu = 0$. Furthermore, combining α and $i\alpha$, one easily finds two elements $\alpha_+, \alpha_- \in \mathcal{A}_\gamma$ such that $(\alpha_+)^2 = \varepsilon$ and $(\alpha_-)^2 = -\varepsilon$.

It remains to prove that $\dim \mathcal{A}_\gamma = 2$. Suppose $\dim \mathcal{A}_\gamma \geqslant 3$. That there exists $\beta \in \mathcal{A}_\gamma$ linearly independent with α and $i\alpha$. Since $\alpha\beta \in \mathcal{A}_0$, there is a linear combination $\lambda\varepsilon + \mu i + \nu\alpha\beta = 0$. Multiplying this equation by α (and assuming without loss of generality $\alpha^2 = \varepsilon$), one has:

$$\lambda\alpha + \mu i\alpha + \nu\beta = 0.$$

Hence a contradiction. $\square$

The algebra $\mathcal{A}$ is therefore a $\mathbb{C}$-algebra. The end of the proof of Theorem 1 in the case where $\mathcal{A}_0 = \mathbb{C}$ is exactly the same as in the complex case.

Theorem 1 is now completely proved.

5. Non-associative extensions of the Clifford algebras

In this section, we construct simple $(\mathbb{Z}_2)^n$-commutative algebras extending the Clifford algebras. The algebras we construct contain the Clifford algebras as even parts. According to Corollary 2.4, such algebras cannot be associative.

Recall that, if $\mathcal{A}$ is a Γ-commutative algebra, then the space $\mathrm{End}(\mathcal{A})$ is naturally Γ-graded. A homogeneous linear map $T \in \mathrm{End}(\mathcal{A})$ is called a derivation of $\mathcal{A}$ if for all homogeneous $a, b \in \mathcal{A}$ one has

$$T(ab) = T(a)b + (-1)^{\langle \bar{T}, \bar{a} \rangle} aT(b). \tag{7}$$

This formula then extends by linearity for arbitrary T and a, b. The space $\mathrm{Der}(\mathcal{A})$ of all derivations of $\mathcal{A}$ is a Γ-graded Lie algebra.

We will restrict our considerations to the case of $(\mathbb{Z}_2)^n$-commutative algebras that have non-trivial odd derivations. This means we assume that there exists a derivation T exchanging $\mathcal{A}^0$ and A^1. We think that this assumption is quite natural since in this case the two parts of $\mathcal{A}$ are not separated from each other.

5.1. Classification in the $\mathbb{Z}_2$-graded case

Let us start with the simplest case of $\mathbb{Z}_2$-commutative algebras. The following statement provides a classification of such algebras.

Proposition 3. *There exist exactly two simple $\mathbb{Z}_2$-commutative algebras $\mathcal{A}$ with the following two properties*:

- *The even part $\mathcal{A}^0 = \mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.*
- *There exists a non-trivial odd derivation T of $\mathcal{A}$.*

Proof. Denote by ε the unit element of $\mathcal{A}^0$. There are two different possibilities:

(1) $T(\varepsilon) = 0$, i.e., $T|_{\mathcal{A}^0} = 0$.
(2) $T(\varepsilon) = a \neq 0$.

In the case (1), there exists an odd element a such that $T(a) = \varepsilon$. One then has

$$\mathcal{A}^1 = \mathbb{K}a \oplus \ker T.$$

Moreover, $\varepsilon b = 0$, for all $b \in \ker T$. Indeed, $0 = T(ab) = T(a)b = \varepsilon b$. One has $\varepsilon a = \lambda a + b$, for some $\lambda \in \mathbb{K}$ and $b \in \ker T$. Applying (7) one gets

$$T(\varepsilon a) = T(\varepsilon)a + \varepsilon T(a) = \varepsilon.$$

On the other hand,

$$T(\varepsilon a) = T(\lambda a + b) = \lambda\varepsilon.$$

Therefore, $\lambda = 1$ and $\varepsilon(a + b) = a + b$. Replacing a by $a + b$, one gets $\varepsilon a = a$.

It follows that: (a) if the space $\ker T$ is non-trivial, then the space spanned by ε and a is a proper ideal; (b) if $\ker T$ is trivial, then $\mathbb{K}a$ is a proper ideal. This is a contradiction with the simplicity assumption, therefore the case (1) cannot occur.

Consider the case (2) where $T(\varepsilon) = a \neq 0$. From $T(\varepsilon) = T(\varepsilon\varepsilon) = 2\varepsilon T(\varepsilon)$ one deduces $\varepsilon a = \frac{1}{2}a$. Applying T, one obtains $T(a) = 0$.

Lemma 5.1. *The product in $\mathcal{A}$ restricted to $\mathcal{A}^1$, that is, $\mathcal{A}^1 \times \mathcal{A}^1 \to \mathbb{K}\varepsilon$ is a non-degenerate bilinear skewsymmetric form.*

Proof. Suppose there exists $c \in \mathcal{A}^1$ such that $cb = 0$ for all $b \in \mathcal{A}^1$. We will show that we also have $\varepsilon c = 0$. Consequently the element c generates a two-sided ideal that contradicts the assumption of simplicity.

One has $T(c) = \mu\varepsilon$ for some μ. On the one hand,

$$T(c)a = \mu\varepsilon a = \frac{\mu}{2}a.$$

On the other hand,

$$T(c)a = T(ca) - cT(a) = 0.$$

Therefore one obtains $\mu = 0$, i.e., $T(c) = 0$.

We can find $b \in \mathcal{A}^1$ such that $ba = \varepsilon$. Indeed, if $ba = 0$ for all $b \in \mathcal{A}^1$ then $\mathbb{C}a$ is an ideal. From $\varepsilon a = \frac{1}{2}a$ and

$$T(ba) = T(\varepsilon) = a = T(b)a - bT(a) = T(b)a$$

one deduces $T(b) = 2\varepsilon$. Now, from

$$T(bc) = 0 = T(b)c - bT(c) = 2\varepsilon c$$

one gets $\varepsilon c = 0$. $\square$

Let us denote the bilinear form from Lemma 5.1 by ω.

Lemma 5.2. *The space $\mathcal{A}^1$ is 2-dimensional.*

Proof. The dimension of $\mathcal{A}^1$ is even and there exists a basis $\{a_1, \dots, a_m, b_1, \dots, b_m\}$ such that $a_i b_j = \delta_{i,j}\varepsilon$. Moreover, we can assume $a_1 = a$.

For all $b \in \ker T$ one has $T(ab) = T(a)b - aT(b) = 0$, since we have proved $T(a) = 0$. So necessarily $ab = 0$. This implies: $\ker T \subset (\mathbb{K}a)^{\perp\omega}$. Since the dimensions of these spaces are the same we have

$$\ker T = (\mathbb{K}a)^{\perp\omega} = \langle a_1, \dots, a_m, b_2, \dots b_m\rangle$$

If $m > 1$ then we have two vectors a_2, b_2 such that

$$T(a_2 b_2) = T(a_2)b_2 - a_2 T(b_2) = 0.$$

But $a_2 b_2 = \varepsilon$ and $T(\varepsilon) \neq 0$. This is a contradiction. $\square$

So far we have shown that $\mathcal{A}$ is $1|2$-dimensional and has a basis $\{\varepsilon; a, b\}$ such that

$$\varepsilon a = \frac{1}{2}a, \qquad ab = \varepsilon.$$

We want now to determine the product εb.

In general,

$$\varepsilon b = \lambda a + \mu b, \qquad T(b) = \nu b.$$

Applying the derivation T to the expression for ab, one obtains

$$a = T(\varepsilon) = T(ab) = T(a)b - aT(b) = -a\nu\varepsilon,$$

that implies $\nu = -2$. Applying T to the above expression for εb, one has

$$-2\mu\varepsilon = T(\varepsilon b) = T(\varepsilon)b + \varepsilon T(b) = ab + \varepsilon(-2\varepsilon),$$

that gives $\mu = \frac{1}{2}$. The parameter λ cannot be found from the above equations.

One obtains the most general formula for the product of the basis elements:

$$\begin{aligned}
&\varepsilon\varepsilon = \varepsilon, \\
&\varepsilon a = \frac{1}{2}a, \qquad \varepsilon b = \frac{1}{2}b + \lambda a, \\
&ab = \varepsilon.
\end{aligned}$$

To complete the proof, one observes that all the algebras corresponding to $\lambda \neq 0$ are isomorphic to each other but not isomorphic to the algebra corresponding to $\lambda = 0$. One therefor has exactly two non-isomorphic algebras.

Proposition 3 is proved. $\square$

If $\lambda = 0$, we recognize the algebra $K_3(\mathbb{K})$ (see formula (3)). This algebra is particularly interesting. This is the only simple Z_2-commutative algebra that have the Lie superalgebra $\mathrm{osp}(1|2)$ as the algebra of derivations, see [15] and also [13]. The algebra of derivations in the case $\lambda \neq 0$ is of dimension $1|1$.

Table 1

$\cdot$	a	a_i	a_j	a_k	b	b_i	b_j	b_k
2ε	a	a_i	a_j	a_k	b	b_i	b_j	b_k
$2i$	a_i	$-a$	a_k	$-a_j$	b_i	$-b$	b_k	$-b_j$
$2j$	a_j	$-a_k$	$-a$	a_i	b_j	$-b_k$	$-b$	b_i
$2k$	a_k	a_j	$-a_i$	$-a$	b_k	b_j	$-b_i$	$-b$
a	0	0	0	0	ε	i	j	k
a_i	0	0	0	0	i	$-\varepsilon$	k	$-j$
a_j	0	0	0	0	j	$-k$	$-\varepsilon$	i
a_k	0	0	0	0	k	j	$-i$	$-\varepsilon$

5.2. Extended Clifford algebras

We associate to every Clifford algebra a simple $(\mathbb{Z}_2)^{n+1}$-commutative algebra of dimension $2^n|2^{n+1}$. It is defined by the tensor product with K_3. More precisely, we define the "extended Clifford algebras"

$$\mathcal{A} = \mathrm{Cl}_n \otimes_{\mathbb{C}} K_3(\mathbb{C}), \quad \text{and} \quad \mathcal{A} = \mathrm{Cl}_{p,q} \otimes_{\mathbb{R}} K_3(\mathbb{R}) \tag{8}$$

in the complex and in the real case, respectively.

The even part $\mathcal{A}^0$ of each of these algebras coincides with the corresponding Clifford algebra: we identify $\alpha \otimes \varepsilon$ with α. The odd part is twice bigger; every odd element is of the form

$$x = \alpha \otimes a + \beta \otimes b,$$

where α, β are elements of the Clifford algebra and a, b are the basis elements of K_3. The $(\mathbb{Z}_2)^{n+1}$-grading on $\mathcal{A}^1$ is defined by

$$\overline{\alpha \otimes a} = \bar{\alpha} + (1, 1, \dots, 1), \qquad \overline{\beta \otimes b} = \bar{\beta} + (1, 1, \dots, 1).$$

The product in $\mathcal{A}$ is given by

$$\begin{aligned} \gamma(\alpha \otimes a + \beta \otimes b) &= \frac{1}{2}(\gamma\alpha \otimes a + \gamma\beta \otimes b), \\ (\alpha \otimes a + \beta \otimes b)\big(\alpha' \otimes a + \beta' \otimes b\big) &= \alpha\beta' - \alpha'\beta, \end{aligned} \tag{9}$$

where $\gamma \in \mathcal{A}_0$. Note that $\frac{1}{2}$ appearing in (9) is crucial (a similar formula without $\frac{1}{2}$ leads to an algebra with no non-trivial odd derivations).

Example 5.3. Let us describe with more details the extended algebra of quaternions $\mathcal{A} = \mathbb{H} \otimes_{\mathbb{R}} K_3(\mathbb{R})$. This algebra is of dimension 4|8. The even part $\mathcal{A}^0 = \mathbb{H}$ is spanned by $\{\varepsilon, i, j, k\}$; the odd part $\mathcal{A}^1$ has the basis $\{a, a_i, a_j, a_k, b, b_i, b_j, b_k\}$ and the multiplication is given by Table 1. This table can be completed using the multiplication of quaternions and the graded-commutativity. For instance, we get

$$a_i i = i a_i = -\frac{1}{2} a, \qquad a_j i = -i a_j = -\frac{1}{2} a_k,$$

for the products of even and odd elements. For the products of odd elements with each other we have:

$$ab = -ba = \varepsilon, \qquad ab_i = -b_i a = i, \qquad a_i b_j = b_j a_i = k,$$

etc.

Remark 5.4. The extended quaternion algebra $\mathbb{H} \otimes_{\mathbb{R}} K_3(\mathbb{R})$ has an interesting resemblance to the octonion algebra. It worth mentioning that the octonion algebra itself cannot be realized as a Γ-commutative algebra (cf. [14]). We cite [9] for a complete classification of group gradings on the octonion algebra without the commutativity requirement.

One can show that the constructed algebras (8) satisfy cubic identities. For instance, the odd elements satisfy the graded Jacobi identity. This property is not too far from the associativity of Clifford algebras. We think that these algebras are the only possible extensions of Clifford algebras satisfying cubic identity.

The extended Clifford algebras have large algebras of derivations. The following statement can be checked by a straightforward calculation.

Proposition 4. *The algebras of derivations of the extended Clifford algebras are the following $(\mathbb{Z}_2)^{n+1}$-graded Lie algebras*:

$$\mathrm{Der}(\mathcal{A}) = \mathrm{Cl}_n \otimes \mathrm{osp}(1|2), \quad \textit{or} \quad \mathrm{Der}(\mathcal{A}) = \mathrm{Cl}_{p,q} \otimes \mathrm{osp}(1|2),$$

respectively.

It is quite remarkable that, for every two elements $a, b \in \mathcal{A}$, there exists $T \in \mathrm{Der}(\mathcal{A})$ such that $T(a) = T(b)$. We conjecture that the defined algebras are the only simple $(\mathbb{Z}_2)^{n+1}$-commutative algebras satisfying this property.

5.3. Further examples

Let us show more examples of simple $(\mathbb{Z}_2)^{n+1}$-commutative algebras that contain the Clifford algebras as even part. These algebras are $2^n|2^n$-dimensional. A nice property of these new algebras is that they have the unit element. However, their algebra of derivations is too small.

In the simplest case $\mathcal{A}_0 = \mathbb{H}$, the basis of the algebra is: $\{\varepsilon, i, j, k; a, a_i, a_j, a_k\}$. The non-trivial odd derivation is as follows:

$$T_{(1,1,1)} : \varepsilon, i, j, k \mapsto 0, a_i, a_j, a_k,$$

$$T_{(1,1,1)} : a, a_i, a_j, a_k, \mapsto \varepsilon, 0, 0, 0.$$

The complete multiplication table is Table 2, where $\lambda, \mu, \nu \in \mathbb{C}$ are parameters. The obtained algebras are isomorphic if and only if the corresponding parameters are proportional. One thus have a two-parameter family of algebras parametrized by $\mathbb{CP}^2/\tau$, where τ is the action of the cyclic group $\mathbb{Z}_3$ of coordinate permutation.

It is easy to check that the algebra of derivations $\mathrm{Der}(\mathcal{A})$ does not depend on λ, μ, ν. This algebra is $1|1$-dimensional, it has one even generator T_0 and one odd generator T_1 satisfying the commutation relations $[T_0, T_1] = T_1$ and $[T_1, T_1] = 0$.

It is of course very easy to define an analogous construction for an arbitrary Clifford algebra. We will not dwell on it here.

Table 2

$\cdot$	ε	i	j	k	a	a_i	a_j	a_k
ε	ε	i	j	k	a	a_i	a_j	a_k
i	i	$-\varepsilon$	k	$-j$	λa_i	0	$\frac{1}{2}a_k$	$-\frac{1}{2}a_j$
j	j	$-k$	$-\varepsilon$	i	μa_j	$-\frac{1}{2}a_k$	0	$\frac{1}{2}a_i$
k	k	j	$-i$	$-\varepsilon$	νa_k	$\frac{1}{2}a_j$	$-\frac{1}{2}a_i$	0
a	a	λa_i	μa_j	νa_k	0	i	j	k
a_i	a_i	0	$-\frac{1}{2}a_k$	$\frac{1}{2}a_j$	$-i$	0	0	0
a_j	a_j	$\frac{1}{2}a_k$	0	$-\frac{1}{2}a_i$	$-j$	0	0	0
a_k	a_k	$-\frac{1}{2}a_j$	$\frac{1}{2}a_i$	0	$-k$	0	0	0

Acknowledgments

The main part of this work was done at the Mathematisches Forschungsinstitut Oberwolfach (MFO) during a *Research in Pairs* stay from April 5 to April 18 2009. We are grateful to MFO for hospitality. We are pleased to thank Yu. Bahturin, A. Elduque and D. Leites for enlightening discussions.

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