Classification of contact-projective structures on supercircles

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Classification of contact-projective structures on supercircles

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Projective structures on circles were classified by Kuiper in [1]. In fact he solved the problems of the classification of the Hill equations and the orbits of the co-adjoint representation of a Virasoro algebra, solved independently by Lazutkin, Pankratov, as well as by Kirillov and Segal.

Our paper is devoted to the classification of contact-projective structures on the contact supermanifold $\mathbf{P}^{1|n}$ (of all $1|0$-dimensional subspaces of the linear symplectic space $(\mathbf{R}^{2|n}, \omega)$) and its double covering $S^{1|n}$. For $n \leqslant 3$ it is equivalent to the classification of the orbits of the co-adjoint representation of a Lie superalgebra of Neveu-Schwartz-Ramond type [2], and also the superanalogues of the Hill equation defined in [3].

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We consider the linear symplectic superspace $(\mathbf{R}^{2|n}, \omega)$, where $\omega = dp \wedge dq + \sum (d\pi^i)^2$. The supermanifold $\mathbf{P}^{1|n}$ has an induced contact structure. (All our subsequent arguments are equally valid for $S^{1|n}$.)

1. By a *contact-projective atlas* on $\mathbf{P}^{1|n}$ we mean a covering (U_j) endowed with local coordinates $(x, \xi^i)_j$ in which a contact structure is defined by the 1-form $\alpha = dx + \sum \xi^i d\xi^i$, and the attaching transformations α_{ij} are fractional-linear contact transformations (that is, $\alpha_{ij} \in \mathrm{SpO}_0(2 \mid n)$, the connected component of the supergroup of all symplectic transformations of $(\mathbf{R}^{2|n}, \omega)$). Two such atlases are said to be *equivalent* if their union is also a contact-projective atlas. An equivalence class of such atlases is called a *contact-projective structure* (c.p.s.).

2. The monodromy operator. Any c.p.s. can be defined by an atlas with a finite number N of charts such that no neighbourhood U_j lies entirely inside another. After ordering the charts so that their projections onto the support are numbered in the positive direction, we define the element $M = \Pi\alpha_{jj+1}$ of the universal covering $\mathrm{SpO}_0(2|n)$; we call this element the monodromy operator.

***Theorem* 1.** *The monodromy operator is the unique invariant of the c.p.s. with respect to the action of the supergroup of compact diffeomorphisms.*

Proof. We introduce the following concept.

3. Contact-projective connections. Any c.p.s. defines a bundle over $\mathbf{P}^{1|n}$ (and hence on its universal covering $\mathbf{R}^{1|n}$) with fibre $\mathbf{R}^*$: the coordinates (x, ξ^i) in each chart U are lifted to the coordinates (p, q, π^i) in $\mathbf{R}^* \times U$ such that $x = p/q$, $\xi^i = \pi^i/q$; the attaching functions on $\mathbf{R}^* \times (U_i \cap U_j)$ are linear transformations $\alpha_{ij} \in \mathrm{SpO}_0(2 \mid n)$. The submanifolds $(q = \text{const})$ in $\mathbf{R}^* \times U_j$ are glued to a global (horizontal) section γ of a bundle over $\mathbf{R}^{1|n}$. A generator of the group $\pi_1(\mathbf{P}^{1|n}) = \mathbf{Z}$ acts on the section as a monodromy operator.

Lemma. *The horizontal section γ is a tensor density of degree $-1/2$ on $\mathbf{R}^{1|n}$.*

This means that under the action of a contact diffeomorphism F the submanifold $q = \text{const}$ on each chart $\mathbf{R}^* \times U$ is taken to the submanifold $(qm_F^{-1/2} = \text{const})$. The function m_F is defined on U as follows. Let the contact structure be defined on U by the 1-form $\alpha = dx + \sum \xi^i d\xi^i$; then $F^*\alpha = m_F\alpha$.

Corollary. *A c.p.s. defines a $2|n$-dimensional space of tensor densities of degree $-1/2$ on $\mathbf{R}^{1|n}$. A basis of it in each chart (x, ξ) is γ, $x\gamma$, $\xi^i\gamma$, where γ is a horizontal section.*

4. We apply the homotopy method. Consider an infinitely small deformation of the c.p.s. Let $y_1, y_2, \varphi_1, \ldots, \varphi_n$ be an arbitrary basis in the corresponding space of tensor densities, and let $(y_i)_t = y_i + tz_i$, $(\varphi_j)_t = \varphi_j + t\psi_j$ be a deformation of it. It turns out that there exists an invariant differential operator on $\mathbf{R}^{1|n}$ that recovers the contact vector field defining the deformation of the c.p.s.

***Theorem* 2.** *A deformation of a c.p.s. is defined by the action of the contact vector field with contact Hamiltonian (having the meaning of tensor density of degree* -1*)*

$$h = \operatorname{Ber}\left(\begin{array}{cc|ccc} y_1 & y_2 & \varphi_1 & \dots & \varphi_n \\ z_1 & z_2 & \psi_1 & \dots & \psi_n \\ \hline D_1 y_1 & D_1 y_2 & D_1\varphi_1 & \dots & D_1\varphi_n \\ \dots & \dots & \dots & \dots & \dots \\ D_n y_1 & D_n y_2 & D_n\varphi_1 & \dots & D_n\varphi_n \end{array}\right) \Big/ \operatorname{Ber}\left(\begin{array}{cc|ccc} y_1 & y_2 & \varphi_1 & \dots & \varphi_n \\ \partial_x y_1 & \partial_x y_2 & \partial_x\varphi_1 & \dots & \partial_x\varphi_n \\ \hline D_1 y_1 & D_1 y_2 & D_1\varphi_1 & \dots & D_1\varphi_n \\ \dots & \dots & \dots & \dots & \dots \\ D_n y_1 & D_n y_2 & D_n\varphi_1 & \dots & D_n\varphi_n \end{array}\right).$$

The second Berezinian in this formula (the "Wronskian") is a constant. If the deformation preserves the monodromy, then under the action of the generator $\pi_1(\mathbf{P}^{1|n})$ *the Hamiltonian h is multiplied by* Ber $M = 1$ *and is therefore uniquely defined on* $\mathbf{P}^{1|n}$.

The proof of Theorem 2 is carried out by a straightforward calculation.

Proposition. *Two c.p.s.'s with the same monodromy operators are homotopic in the class of such c.p.s.'s.*

Theorem 1 is now proved.

Assertion. *For* $n \leqslant 3$ *a pair of c.p.s.'s defines a Hill superequation. In the chart* (z, ζ^i) *of the first c.p.s. the* $-1/2$*-densities defined by the second c.p.s. satisfy the differential equation*

$$[D_1 \dots D_n \partial_z^{2-n} + u(z, \zeta)]y = 0,$$

where $D_i = \partial_{\zeta^i} + \zeta^i \partial_z$, *and the potential* $u = S/2$, *where S is the superSchwartzian* [3]. *The potential u is an invariant of the pair of c.p.s.'s that is independent of the choice of atlases.*

***Corollary of Theorem* 1.** *Under the natural projection of the c.p.s.'s (Hill superequations) defined on* $\mathbf{P}^{1|n}$, *c.p.s.'s that are equivalent in the space of c.p.s.'s on* $\mathbf{P}^{1|2}$ *can become inequivalent, since*

$$\pi_1(\mathrm{SpO}_0(2|3)) = \mathbf{Z}\times\mathbf{Z}_2 \quad \textit{and} \quad \pi_1(\mathrm{SpO}_0(2|2)) = \mathbf{Z}\times\mathbf{Z}.$$

5. **Versal deformations.** ***Corollary of Theorem* 2.** *A versal deformation of a c.p.s. reduces to a versal deformation of the class of adjoint elements of the supergroup* $\mathrm{SpO}_0(2|n)$ *defined by the monodromy operator. This is locally the set of equivalence classes of c.p.s.'s constructed as the set of classes of adjoint elements (of orbits of the co-adjoint representation) of* $\mathrm{SpO}_0(2|n)$.

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