LTTC Intensive Course : Schubert calculus on Grassmannians

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1 Introduction

Enumerative problems are an important part of Algebraic Geometry. The goal is to count the number of objects (lines, curves ...) satisfying certain incidence conditions. For example, what is the number of circles tangent to three given circles in the plane? What is the number of plane conics through 5 points? These kinds of questions have been extensively studied during the 19th century and form the bases of Schubert calculus.

This intensive course will focus on Schubert calculus on Grassmannians, which parametrise vector subspaces of a given dimension in an ambient complex vector space. It involves the beautiful combinatorics of Young tableaux, which are also related to interesting problems in representation theory and the theory of symmetric functions.

After reviewing some necessary background in Section 2, we will introduce the combinatorial setting we will be working with in Section 3. Section 4 will introduce the geometric setting, by defining Grassmannians as homogeneous spaces and projective algebraic varieties. In the next Section 5, we will prove two first results on Schubert calculus : the Pieri and Giambelli formulas. After that, the main result of Schubert calculus, the Littlewood-Richardson rule, is the subject of Section 6. Finally, we conclude in Scetion 7 by reviewing some recent developpements of Schubert calculus, including Pieri and Giambelli rules for other homogeneous spaces, as well as quantum Schubert calculus.

The content of this course is by no means new or original, and Sections 3-6 are heavily inspired from the books [Ful97] and [Man01]. However, any error would be strictly my own. Sections 3-6 are mostly self-contained, although due to time constraints, we may refer to the previously cited books for some of the proofs. However, Section 7 may require some more algebro-geometric background.

2 Background

In this section, we recall the main notions from algebraic geometry, algebraic groups and cohomology theory that will be necessary in the rest of the course. The results will not be proved. The reader wishing for more details can consult for instance [Sha94] or [Har77] for the results of Section 2.1, [Bor91], [Spr98] or [Hum75] for Section 2.3, [Ful97, Appendix B] or [Man01, Appendix A] for Section 2.2.

2.1 On projective algebraic geometry

For this subsection and in the rest of the document, the base field will always be $\mathbb{C}$.

Definition 2.1 (Basic definitions)**.** *Let V be a $\mathbb{C}$-vector space of dimension n. An* algebraic subset *X of projective space $\mathbb{P}(V)$ is a subset that is the set of zeroes of a collection of homogeneous polynomials inside the* coordinate ring *$\mathcal{S}^\bullet V^*$ of $\mathbb{P}(V)$. The* ideal *$\mathcal{I}(X) = \bigoplus_k \mathcal{I}(X)_k$ of such an algebraic subset is a homogeneous ideal in $\mathcal{S}^\bullet V^*$, where $\mathcal{I}(X)_k$ consists of the forms of degree k that vanish on X. An algebraic subset is* irreducible *if it is not the union of two proper algebraic subsets. It is then called an (embedded)* projective algebraic variety*. If X is irreducible, then $\mathcal{I}(X)$ is prime, and the graded ring $\mathcal{S}^\bullet V^*/\mathcal{I}(X)$ is the* coordinate ring *of X. An* algebraic subset *of X is the locus in X defined by a homogeneous ideal inside $\mathcal{I}(X)$.*

A basic result of algebraic geometry is Hilbert's Nullstellensatz, which describe how an algebraic subvariety and its ideal are related :

Theorem 2.2 (Nullstellensatz)**.** *If $\mathcal{I}$ is an homogeneous ideal in $\mathcal{S}^\bullet V^*$ and X is the set of zeroes of $\mathcal{I}$, then*

$$\mathcal{I}(X) = \left\{ P \in \mathcal{S}^\bullet V^* \mid \exists d \geq 1, P^d \in \mathcal{I} \right\}.$$

In particular, if $\mathcal{I}$ is prime, then $\mathcal{I}(X) = \mathcal{I}$.

2.2 On cohomology

In this subsection, we introduce Chow rings, the cup product, and Poincaré duality.

Definition 2.3 (Algebraic cycles)**.** *An* algebraic cycle *of a projective algebraic variety X is a formal finite linear combination $W = \sum_i n_i[W_i]$, where W_i is a subvariety of X, and $n_i \in \mathbb{Z}$ is the* multiplicity *of W_i. Algebraic cycles form a group $Z_*(X)$ graded by the dimension of the cycles (or $Z^*(X)$ if we grade by the codimension).*

There is an equivalence relation on $Z^*(X)$ called *rational equivalence*. To define it, we first need to recall the following :

Definition 2.4 (Rational function)**.** *A* rational function *f on a projective algebraic variety X is a regular function $f : U \to \mathbb{C}$, where U is an open dense subset of X.*

Definition 2.5 (Rational equivalence)**.** *We say that two cycles $[D], [D'] \in Z^*(X)$ of codimension 1 are* rationally equivalent*, denoted $[D] \sim [D']$, if there exists a rational function f on X such that*

$$[D] - [D'] = \operatorname{div}(f) = (f)_0 - (f)_\infty,$$

where $(f)_0$ is the locus of the zeroes of f and $(f)_\infty$ the locus of its poles.

Now if $[W]$ is any cycle of X, it is rationally equivalent to zero*, denoted $W \sim 0$, if there exists a finite collection $(Y_a)_a$ of subvarieties of X of dimension $\dim W + 1$ and rational functions f_a on Y_a such that*

$$[W] = \sum_a \operatorname{div}(f_a).$$

Finally, two cycles $[W], [W']$ are rationally equivalent *if $[W] - [W']$ is rationally equivalent to zero.*

We may now define the *intersection* of two cycles :

Definition 2.6 (Intersection)**.** *Consider X a smooth projective variety, and let $[W]$ and $[W']$ be two cycles of X of codimensions d and d' with* proper intersection*, i.e. such that $W \cap W'$ is a finite union $\bigcup_j A_j$ of subvarieties of X of codimension $d + d'$. In this case, we may define the* cup product *of $[W]$ and $[W']$ as follows :*

$$[W] \cup [W'] = \sum_j i(W \cap W'; A_j)[A_j],$$

where the number $i(W \cap W'; A_j)$ is called the intersection multiplicity of W and W' at A_j*. For the definition of intersection multiplicities, we refer to [Ful84].*

Remark. In what follows, we will always be able to assume that the cycles have *transverse intersection*, i.e. that the intersection multiplicities are 0 or 1.

We will now quotient the group $Z^i(X)$ by the equivalence relation $\sim$:

Definition 2.7 (Chow group)**.** *The* Chow groups *of a smooth projective algebraic variety X are*

$$H^i(X) := Z^i(X)/Z^i_0(X)$$

where $Z^i_0(X) \subset Z^i(X)$ is the subgroup of cycles which are rationally equivalent to 0*. We write $H^*(X) = \bigoplus_i H^i(X)$.*

The cup product equips $H^*(X)$ with a ring structure :

Theorem 2.8 (Chow ring)**.** *$H^*(X)$ is a commutative ring with respect to the cup product, called the* Chow ring*. We similarly define $H_*(X)$.*

Remark. We could have used other notions of (co)homology for this section, for instance singular (co)homology. However, in our context (homogeneous spaces), this makes no difference, hence we will indifferently write "cohomology ring" or "Chow ring".

Definition 2.9 (Fundamental class)**.** *The* fundamental cycle *of a subvariety W of a smooth projective variety X is its image $[W] \in H_*(X)$. $[W] \in H^*(X)$ is called its* fundamental class*.*

Theorem 2.10 (Poincaré duality)**.** *1. There exists a bilinear operation*

$$\cap : H^p(X) \otimes H_q(X) \to H_{q-p}(X)$$

which is compatible with the cup product.

2. The cap product with the fundamental cycle of X

$$\bullet \cap [X] : H^q(X) \to H_{n-q}$$

is an isomorphism called Poincaré duality*.*

To conclude this subsection, we introduce the notion of *cell decompositions* and give their properties :

Definition 2.11 (Cell decomposition)**.** *A* cell decomposition *of an algebraic variety X is a finite partition $X = \bigsqcup_{i \in I} C_i$, where*

1. *the* cells *C_i are isomorphic to affine spaces $\mathbb{C}^{m_i}$;*
2. *the boundary $\overline{C_i} \setminus C_i$ of a cell is itself a reunion of cells.*

An important application of cell decompositions is that they enable us to find bases of the cohomology of the variety.

Proposition 2.12. *If X is a projective algebraic variety which admits a cell decomposition $X = \bigsqcup_{i \in I} C_i$, then the fundamental classes of the closures of the cells generate the cohomology of X :*

$$H^*(X) = \bigoplus_i \mathbb{Z}\left[\overline{C_i}\right].$$

2.3 On algebraic groups

Here we recall some basic notions on algebraic groups which we will need to state the Bruhat decomposition theorem.

Definition 2.13 (Algebraic group)**.** *An* algebraic group *is a group which is isomorphic to an algebraic variety, and such that all group operations (multiplication, inverse) are morphisms of algebraic varieties. In the sequel we will always consider* linear algebraic groups*, which are algebraic groups that are isomorphic to an algebraic subgroup of* $\mathrm{GL}_N(\mathbb{C})$ *for some* N*.*

Example. $\mathrm{GL}_N(\mathbb{C})$ itself is of course a (linear) algebraic group.

Definition 2.14 (Homogeneous spaces)**.** *A* homogeneous space *for an algebraic group* G *is a non-empty algebraic variety* X *endowed with a transitive action of* G *which is also a morphism of algebraic varieties.*

Example. Projective space $\mathbb{P}(V)$ is homogeneous under the action of $\mathrm{GL}(V)$.

Definition 2.15 (Borel subgroup)**.** *A* Borel subgroup B *of an algebraic group* G *is a maximal solvable algebraic subgroup of* G*.*

Example. If $G = \mathrm{GL}_N(\mathbb{C})$, the subgroup of invertible upper-triangular matrices is a Borel subgroup of G.

Definition 2.16 (Tori)**.** *A* torus *inside an algebraic group* G *is an abelian subgroup of* G*. It is isomorphic as an algebraic group to* $(\mathbb{C}^*)^N$ *for some* N*. A* maximal torus *is a torus that is maximal among abelian subgroups of* G*.*

Example. The (invertible) diagonal matrices inside $G = \mathrm{GL}_N(\mathbb{C})$ form a maximal torus.

Proposition 2.17. *All Borel subgroups of an algebraic group* G *are conjugate : if* B, B' *are two Borel subgroups of* G*, then there exists* $g \in G$ *such that* $B' = gBg^{-1}$*.*

Definition 2.18 (Weyl group)**.** *The* Weyl group *associated to an algebraic group* G *and a Borel subgroup* B *of* G *is*

$$W = N_G(T)/Z_G(T),$$

where T *is the maximal torus of* B*,* $N_G(T) = \{g \in G \mid gT = Tg\}$ *is the normalizer of* T *in* G*, and* $Z_G(T) = \{g \in G \mid tg = gt \; \forall t \in T\}$ *is its centralizer.*

Remark. The Weyl group is a finite group and its isomorphism class does not depend on the choice of a Borel subgroup.

Example. The Weyl group of $\mathrm{GL}_N(\mathbb{C})$ is the symmetric group $\mathfrak{S}_N$.

Definition 2.19 (Parabolic subgroup)**.** *A* parabolic subgroup P *of an algebraic group* G *is an algebraic subgroup which contains a Borel subgroup.*

Proposition 2.20. *An algebraic subgroup* $P \subset G$ *is parabolic if and only if the quotient* G/P *is a projective algebraic variety.*

Example. The maximal parabolic subgroups of $\mathrm{GL}_N(\mathbb{C})$ containing the Borel subgroup of upper-triangular matrices are of the form

$$\begin{pmatrix} \mathrm{GL}_k(\mathbb{C}) & \mathrm{M}_{k,N-k}(\mathbb{C}) \\ 0 & \mathrm{GL}_{N-k}(\mathbb{C}) \end{pmatrix}$$

for $1 \leq k \leq N-1$.

Remark. Such a quotient G/P is homogeneous for the G-action given by left multiplication. It is called a *complete homogeneous space.*

Example. Projective space $\mathbb{P}(V)$ is a complete homogeneous space.

We now introduce the *Bruhat decomposition* of a complete homogeneous spaces $X = G/P$. In the special case of the Grassmannian, we will prove in Prop. 4.14 that it is a cell decomposition.

Proposition 2.21 (Bruhat decomposition)**.** *Let $X = G/P$ be a complete homogeneous space, B a Borel subgroup of G, W its Weyl group, and W_P the Weyl group of P. Then*

$$X = \bigsqcup_{w \in W/W_P} BwP/P.$$

Definition 2.22 (Schubert cells, Schubert varieties)**.** *The sets $C_w := BwP/P$ for $w \in W/W_P$ are the* Schubert cells *of X, and their closures $X_w = \overline{BwP/P}$ are its* Schubert varieties*.*

3 Symmetric polynomials

This section introduces the combinatorial background we will need for our study of Schubert calculus on the Grassmannian. We define the ring of symmetric polynomials, of which the cohomology ring of the Grassmannian will be a quotient (cf Cor. 5.4).

3.1 Definition and examples

Definition 3.1. *A* symmetric polynomial *in the variables $x_1, \dots, x_m$ is a polynomial P in the variables $x_1, \dots, x_m$ with integer coefficients such that for all $w \in \mathfrak{S}_m$:*

$$P(x_{w(1)}, \dots, x_{w(m)}) = P(x_1, \dots, x_m).$$

Symmetric polynomials in $x_1, \dots, x_m$ form a ring denoted by Λ_m.

Here are two fundamental examples of symmetric polynomials :

Definition 3.2. *An* elementary symmetric polynomial *in $x_1, \dots, x_m$ is a polynomial of the form*

$$e_k = \sum_{1 \leq i_1 < \dots < i_k \leq m} x_{i_1} \dots x_{i_k}$$

for $1 \leq k \leq m$. Similarly, a complete symmetric polynomial *is a polynomial of the form*

$$h_l = \sum_{1 \leq i_1 \leq \dots \leq i_l \leq m} x_{i_1} \dots x_{i_k}$$

for $k \geq 1$.

Example. Suppose $m = 3$. Then

$$e_1 = x_1 + x_2 + x_3, \quad e_2 = x_1x_2 + x_2x_3 + x_3x_1, \quad e_3 = x_1x_2x_3$$

and

$$h_1 = x_1 + x_2 + x_3, \quad h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1.$$

To define another family of symmetric polynomials, the *monomial symmetric polynomials*, we will need the following :

Definition 3.3. *A* partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ *is a (possibly empty) finite sequence of decreasing positive integers. The non-negative integer* $\ell(\lambda)$ *is called the* length *of the partition. The* λ_i*'s are called its* parts*, and the sum* $|\lambda|$ *of all parts is its* weight*.*

A partition can be represented by a combinatorial object called a Young diagram*, where a partition is drawn as a set of boxes. The diagram consists of* $\ell(\lambda)$ *rows of* λ_i *boxes (drawn from top to bottom) as in Fig. 1.*

We say that a partition λ *is* contained in an $m \times k$ rectangle *if* $\ell(\lambda) \leq m$ *and* $\lambda_1 \leq k$*. Finally, a partition* λ contains *a partition* μ*, denoted* $\lambda \supset \mu$*, if* $\lambda_i \geq \mu_i$ *for all* i*.*

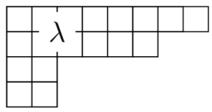

Figure 1: The Young diagram of the partition $\lambda = (8, 6, 2, 2)$

Now :

Definition 3.4. *If* α *is a* m*-uple of non-negative integers, we write* $x^\alpha := x_1^{\alpha_1} \ldots x_m^{\alpha_m}$*. Then if* λ *is a partition with at most* m *parts, the* monomial symmetric polynomial m_λ *is*

$$m_\lambda = \sum_{\alpha \in \mathfrak{S}(\lambda)} x^\alpha$$

where $\mathfrak{S}(\lambda)$ *is the set of all (distinct)* m*-uples obtained from* λ *by permutation. Note that even if* $\ell(\lambda) < m$*, we still see* λ *as an* m*-uple in the definition of* $\mathfrak{S}(\lambda)$ *(completing it by the relevant number of zeroes).*

Example. Suppose $m = 3$. Then

$$m_{2,2} = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2, \quad m_{2,1} = x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + x_2 x_3^2 + x_3^2 x_1 + x_3 x_1^2.$$

Monomial symmetric functions m_λ, with λ having at most m parts, constitute an additive basis of Λ_m. This can be proved inductively by defining a total order on monomials x^λ. Moreover :

Theorem 3.5.

$$\Lambda_m = \mathbb{Z}[e_1, \ldots, e_m] = \mathbb{Z}[h_1, \ldots, h_m]$$

This is the fundamental theorem of symmetric functions. The proof can be found in [Bou72].

Finally, we will introduce *Schur polynomials*, which will play the same role than Schubert classes of the Grassmannian :

Definition 3.6 (Schur polynomial)**.** *Write, for any* m*-uple* α

$$a_\alpha = \sum_{w \in \mathfrak{S}_m} \epsilon(w) x^{w(\alpha)}.$$

Let $\delta = (m-1, m-2, \ldots, 1)$ *be the smallest decreasing partition. Then for any partition* λ *with at most* m *parts, the* Schur polynomial s_λ *is*

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

Proposition 3.7. *For any partition λ with at most m parts :*

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j+m-j})_{1\leq i,j\leq m}.$$

Proof. The proposition is proved by induction on m, developping the determinant with respect to the last row and noticing that

$$\{w \in \mathfrak{S}_m \mid w(m) = j\} \cong \mathfrak{S}_{m-1}$$

for any $1 \leq j \leq m$. ☐

Moreover, it is easy to show that $a_\delta = \det(x_i^{m-j})_{1\leq i,j\leq m}$ is the Vandermonde determinant, hence

$$a_\delta = \prod_{1\leq i<j\leq m} (x_i - x_j)$$

is the basic alternating polynomial in m variables. It follows that any other alternating polynomial, in particular $a_{\lambda+\delta}$ for any partition λ with at most m parts, is divisible by a_δ, and that the quotient is a symmetric polynomial. This proves that the Schur polynomials are indeed symmetric polynomials in the variables $x_1, \dots, x_m$.

Example. Suppose $m = 3$. Then

$$s_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1, \quad s_{2,1,1} = (x_1 + x_2 + x_3)x_1x_2x_3.$$

Schur polynomials are another basis of the ring of symmetric functions :

Proposition 3.8. *Schur polynomials (s_λ), where λ spans partitions with at most m parts, generate Λ_m.*

Idea of proof. Consider the lexicographic order on monomials. A Schur polynomial s_λ has higher-order term x^λ, same as m_λ. ☐

We can write complete and elementary symmetric polynomials as Schur polynomials, as follows

$$e_k = s_{1^k}, \quad h_k = s_k,$$

where we denote by 1^k the partition with k parts equal to 1. In the next subsection, we give a rule for multiplying a Schur polynomial with an elementary or complete polynomial.

3.2 The combinatorial Pieri formula

We will first need the following :

Notation. Consider a partition λ and an integer $p \geq 1$. We write $\mu \in \lambda \otimes p$ if μ is obtained by adding p boxes to λ so that no two are in the same *column*. Similarly, for $r \geq 1$ we write $\mu \in \lambda \otimes 1^r$ if μ is obtained by adding r boxes to λ so that no two are in the same *row*.

We may now state the Pieri formula for Schur functions :

Theorem 3.9.

$$e_r s_\lambda = \sum_{\mu\in\lambda\otimes 1^r} s_\mu \quad \textit{and} \quad h_p s_\lambda = \sum_{\mu\in\lambda\otimes p} s_\mu.$$

Proof. We have

$$e_r a_{\lambda+\delta} = \sum_{w\in\mathfrak{S}_m} \sum_{1\le i_1<\dots<i_r\le m} \epsilon(w) x^{w(\lambda+\delta)} x_{w(i_1)}\dots x_{w(i_r)}$$

Since $a_{\alpha+\delta}$ is non-zero if and only if α is a partition, it follows that

$$e_r a_{\lambda+\delta} = \sum_{\substack{\beta\in\{0,1\}^m\\ |\beta|=r}} a_{\lambda+\beta+\delta}.$$

This proves that is s_μ appears in the product $e_r s_\lambda$, then $\mu_i \in \{\lambda_i, \lambda_i+1\}$ for all i, hence no two added boxes are in the same row.

Similarly

$$h_p a_{\lambda+\delta} = \sum_{|\beta|=p} a_{\lambda+\beta+\delta}.$$

If μ is obtained from λ by adding p boxes, and that two of these boxes are in the same column, then there exists β of weight p and i such that $\mu-\lambda=\beta$, β contributes non-trivially to the sum, and $\beta_i > \lambda_i - \lambda_{i+1}$. Now we define another sequence γ of the same weight p by setting

$$\gamma_j = \begin{cases} \beta_{i+1} - (\lambda_i - \lambda_{i+1}+1) & \text{if } j=i\ ; \\ \beta_i + (\lambda_i - \lambda_{i+1}+1) & \text{if } j=i+1\ ; \\ \beta_j & \text{otherwise.} \end{cases}$$

Then $a_{\lambda+\gamma+\delta}$ also contributes to the sum, and $a_{\lambda+\gamma+\delta} = -a_{\lambda+\beta+\delta}$. Hence both terms cancel in the sum. As a consequence, such an s_μ does not appear in the product $h_p s_\lambda$. $\square$

Example. Suppose $m=3$. Then the product $s_2 s_{2,1,1}$ is

$$s_2 s_{2,1,1} = s_{4,1,1} + s_{3,2,1} + s_{3,1,1,1} + s_{2,2,1,1},$$

as shown in Fig. 2.

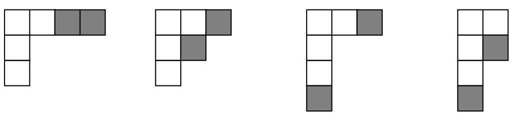

Figure 2: Partitions $\mu \in (2,1,1)\otimes 2$

In the next subsection, we will explain how to write a Schur polynomial as a polynomial in the elementary / complete symmetric polynomials.

3.3 The Jacobi-Trudi formulas

Definition 3.10 (Transpose partition). *Let λ be a partition. We denote by λ^* the partition obtained from λ by exchamging the role of rows and columns.*

We may now state the result :

Theorem 3.11 (Jacobi-Trudi formulas). *If λ is a partition with at most m parts, then*

$$s_\lambda = \det(h_{\lambda_i+j-i})_{1\le j\le m} \quad \textit{and} \quad s_{\lambda^*} = \det(e_{\lambda_i+j-i})_{1\le j\le m}.$$

Proof. We only prove the first identity. The other is proved in a similar fashion. Write $l := \ell(\lambda)$. Then

$$\det(h_{\lambda_i+j-i})_{1\leq j\leq m} = \det(h_{\lambda_i+j-i})_{1\leq j\leq l}.$$

We will use induction on l. Decomposing the determinant with respect to the last column, we obtain

$$\sum_{i=1}^{l}(-1)^{l-i}h_{\lambda_i+l-i}s_{\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1}-1,\dots,\lambda_l-1}.$$

Using Pieri formula, each summand looks like

$$\sum_{\mu\in\mathcal{P}_i} s_\mu + \sum_{\mu\in\mathcal{P}_{i+1}} s_\mu,$$

where

$$\mathcal{P}_i = \{\mu \text{ partition of weight } |\lambda| \mid \lambda_j \leq \mu_j \leq \lambda_{j-1}\ \forall j < i, \lambda_{j+1} - 1 \leq \mu_j \leq \lambda_j - 1\ \forall j \geq i\}.$$

We then see that most terms in the alternating sum cancel, which inductively gives the first formula. □

Together with the Pieri formula 3.9, the Jacobi-Trudi formula enables us to compute the product of any two Schur functions. However, as can be seen in the following exercise, this is lengthy and tedious :

Exercise 3.1. Compute $s_{2,1}^2$ in Λ_3.

We will come back to symmetric functions when studying the Littlewood-Richardson rule in Section 6, but for now, we switch to geometry.

4 Grassmannians

We now turn to our may object of study, the Grassmannian. In this section, we first define the Grassmannian as a homogeneous space (Subsection 4.1). We then study its embedding into projective space (Subsection 4.2), describe its Schubert varieties (Subsection 4.3) and the additive structure of its cohomology ring.

4.1 The Grassmannian as a homogeneous space

Definition 4.1 (Grassmannian)**.** *Let V be a $\mathbb{C}$-vector space of dimension n, with basis $(e_1,\dots,e_n)$, and $1 \leq m \leq n$ be an integer. The* Grassmannian of m-planes in V *is the set*

$$\mathrm{Gr}(m,V) := \{\Sigma \subset V \mid \dim\Sigma = m\}.$$

Remark.

1. The Grassmannians $\mathrm{Gr}(m,V)$ and $\mathrm{Gr}(n-m,V)$ are canonically isomorphic.
2. Projective space $\mathbb{P}(V)$ is the Grassmannian $\mathrm{Gr}(1,V)$.
3. Up to isomorphism, $\mathrm{Gr}(m,V)$ does not depend on V, but only on its dimension n. So we will often denote it by $\mathrm{Gr}(m,n)$.

The Grassmannian $\mathrm{Gr}(m,n)$ is endowed with an action of the general linear group $G := \mathrm{GL}_n(\mathbb{C})$, given by

$$\begin{array}{ccccc} \mathrm{GL}_n(\mathbb{C}) & \times & \mathrm{Gr}(m,n) & \to & \mathrm{Gr}(m,n) \\ & (g,\Sigma) & & \mapsto & \mathrm{Im}(g_{|\Sigma}) \end{array}$$

Consider the following subgroup of G :

$$P := \left\{ g \in G \mid g_{|\mathrm{Vect}(e_1,\dots,e_m)} \subset \mathrm{Vect}(e_1,\dots,e_m) \right\}.$$

It is a parabolic subgroup of G, and clearly

Proposition 4.2. *The Grassmannian* $\mathrm{Gr}(m,n)$ *is isomorphic to the complete homogeneous space* $\mathrm{GL}_n(\mathbb{C})/P$.

4.2 Projective embedding

We will prove that the Grassmannian $\mathrm{Gr}(m,V)$ is a subvariety of projective space $\mathbb{P}(\bigwedge^m V)$, via an explicit embedding :

Definition 4.3 (Plücker embedding)**.** *Consider the map*

$$\Phi : \begin{array}{ccc} \mathrm{Gr}(m,V) & \to & \mathbb{P}(\bigwedge^m V) \\ \Sigma = \mathrm{Vect}(v_1,\dots,v_m) & \mapsto & [v_1 \wedge \cdots \wedge v_m] \end{array}$$

The map Φ is well-defined. Indeed, if $(w_1,\dots,w_m)$ is another basis of Σ, write

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} = P \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix},$$

where $P \in \mathrm{GL}_m(\mathbb{C})$ is an invertible matrix. Then $w_1 \wedge \cdots \wedge w_k = \det(P) v_1 \wedge \cdots \wedge v_k$. Moreover, the map Φ is also an *embedding*, since

$$\Sigma = \{ w \in V \mid w \wedge \Phi(\Sigma) = 0 \}.$$

We will now prove that $\mathrm{Gr}(m,V)$ is a subvariety of $\mathbb{P}(\bigwedge^m V)$ given by a homogeneous ideal generated by the *Plücker relations.*

An element $\Sigma = \mathrm{Vect}(v_1,\dots,v_m) \in \mathrm{Gr}(m,V)$ can be represented is the basis $(e_1,\dots,e_n)$ of V by an $m \times n$ matrix $A = (a_{i,j})$ by writing $v_i = \sum_{j=1}^n a_{i,j} e_j$. This matrix A depends on the choice of a basis $(v_1,\dots,v_m)$ for Σ ; however, its $m \times m$ minors will only depend on it by the same scalar coefficient :

Definition 4.4 (Plücker coordinates)**.** *Let* $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ *be integers. We define the* Plücker coordinate $P_{j_1,j_2,\dots,j_m}(\Sigma)$ of Σ *as the* $m \times m$ *minor of* A *corresponding to the columns* $j_1 < j_2 < \cdots < j_m$. *As an element of* $\mathbb{P}(\bigwedge^m V)$, $(P_{1,2,\dots,m}(\Sigma) : \cdots : P_{n-m+1,n-m+2,\dots,n}(\Sigma))$ *only depends on* Σ *(and not on the choice of a basis* $(v_1,\dots,v_m)$*). We extend this definition to arbitrary* m*-uplets of integers in* $\{1,\dots,n\}$ *by setting*

$$P_{j_1,j_2,\dots,j_m} = 0$$

if there exists $r \neq s$ *such that* $j_r = j_s$*, and*

$$P_{j_1,j_2,\dots,j_m} = \epsilon(\sigma) P_{j_{\sigma(1)},j_{\sigma(2)},\dots,j_{\sigma(m)}},$$

where $\sigma \in \mathfrak{S}_m$ *is a permutation such that* $j_{\sigma(1)} < j_{\sigma(2)} < \cdots < j_{\sigma(m)}$.

We now define a set of quadratic relations in the Plücker coordinates :

Definition 4.5 (Plücker relations)**.** *Let $i_1, \dots, i_m, j_1, \dots, j_m$ be integers between* 1 *and* n *and* l *an integer between* 1 *and* m*. The associated* Plücker relation $\mathcal{R}_{i_1,\dots,i_m}^{j_1,\dots,j_m}(l)$ *is*

$$\sum_{\sigma \in \mathfrak{S}/\mathfrak{S}_1 \times \mathfrak{S}_2} \epsilon(\sigma) P_{i_1,\dots,i_{l-1},\sigma(i_l),\dots,\sigma(i_m)} P_{\sigma(j_1),\dots,\sigma(j_l),j_{l+1},\dots,j_m} = 0 \tag{1}$$

where $\mathfrak{S}$ *(resp.* $\mathfrak{S}_1$ *;* $\mathfrak{S}_2$*) is the group of permutations of the integers* $i_l, \dots, i_m, j_1, \dots, j_l$ *(resp.* $i_1, \dots, i_l$ *;* $j_1, \dots, j_l$*).*

Proposition 4.6. *The Plücker relations* (1) *hold on* $\mathrm{Gr}(m, V)$ *for all integers* $i_1, \dots, i_m$*,* $j_1, \dots, j_m$ *between* 1 *and* n *and* l *between* 1 *and* m*.*

Proof. Let c_k for $1 \leq k \leq m-1$ be fixed vectors of $\mathbb{C}^m$. The $m+1$-linear form which to $(b_{i_l}, \dots, b_{i_m}, b_{j_1}, \dots, b_{j_l})$ associates :

$$\sum_{\sigma \in \mathfrak{S}/\mathfrak{S}_1 \times \mathfrak{S}_2} \epsilon(\sigma) \det(c_1, \dots, c_{l-1}, b_{\sigma(i_l)}, \dots, b_{\sigma(i_m)}) \det(b_{\sigma(j_1)}, \dots, b_{\sigma(j_l)}, c_l, \dots, c_{m-1})$$

is alternating. Since $\bigwedge^{m+1} \mathbb{C}^m = 0$, it follows that it is identically zero.

Now to prove the proposition, consider $\Sigma \in \mathrm{Gr}(m, V)$, denote by A this associated matrix in the basis $(e_1, \dots, e_n)$ of V and set

$$c_k = \begin{cases} \mathrm{Col}(A, i_k) & \text{if } 1 \leq k \leq l-1 \text{ ;} \\ \mathrm{Col}(A, j_{k+1}) & \text{if } 1 \leq k \leq m-1. \end{cases}$$

The Plücker relation (1) is then the consequence of the previous remark. □

Exercise 4.1. Describe the Plücker relations for $\mathrm{Gr}(2, 4)$.

Now denote by $\mathcal{I}$ the homogeneous ideal of $\mathbb{C}[P_{1,2,\dots,m}, \dots, P_{n-m+1,n-m+2,\dots,n}]$ generated by the quadratic Plücker relations.

Theorem 4.7. *The Grassmannian* $\mathrm{Gr}(m, V)$ *is the algebraic subvariety of* $\mathbb{P}(\bigwedge^m V)$ *defined by the homogeneous ideal* $\mathcal{I}$*.*

Proof. Prop. 4.6 implies that the coordinates of any point on $\mathrm{Gr}(m, V)$ satisfy the Plücker relations.

Now consider a point X in $\mathbb{P}(\bigwedge^m V)$ whose coordinates satisfy the relations. Fix $J = (1 \leq j_1 < j_2 < \cdots < j_m \leq n)$ such that the coordinate X_J of X on $e_{j_1} \wedge \cdots \wedge e_{j_m}$ is non-zero. For simplicity, assume $X_J = 1$. We define an $m \times n$ matrix A by setting :

$$a_{r,s} = X_{j_1,\dots,j_{r-1},s,j_{r+1},\dots,j_m}$$

for all $1 \leq r \leq m$, $1 \leq s \leq n$. Denote by W the image of the map $A : \mathbb{C}^m \to V$. It has dimension m. Indeed, it is clear that the submatrix of A corresponding to the columns $j \in J$ is the identity matrix.

We claim that W and X have the same coordinates in $\mathbb{P}(V)$, which will mean that $X = K \in \mathrm{Gr}(m, V)$. We have already seen that $P_J(W) = 1$. Moreover, if I has $m-1$ entries in common with J, i.e. I is obtained by replacing j_r with s, then the corresponding submatrix is diagonal, with all diagonal entries equal to 1, except for one equal to $a_{r,s}$.

Hence $P_I(W) = a_{r,s} = X_I$. The other cases are obtained by (descending) induction on $|I \cap J|$ using the Plücker relations (1).

We are now left with proving that if $f \in \mathcal{S}^\bullet(\bigwedge^m V)$ is a polynomial in the Plücker coordinates which vanishes on $\mathrm{Gr}(m, V)$, then $f \in \mathcal{I}$. This is a consequence of the Nullstellensatz (see Thm. 2.2) and of the fact that the ideal $\mathcal{I}$ is prime. The proof of this last statement is the subject of Ex. 4.2. It can also be found in [Man01]. $\square$

Exercise 4.2. Prove that the ideal $\mathcal{I}$ generated by Plücker relations is prime.

4.3 Schubert varieties

It follows from Prop. 4.2 and Section 2.3 that the Grassmannian $\mathrm{Gr}(m, n)$ admits the following Bruhat decomposition :

Proposition 4.8.

$$\mathrm{Gr}(m,n) = \bigsqcup_{w \in \mathfrak{S}_n/\mathfrak{S}_m \times \mathfrak{S}_{n-m}} BwP/P$$

where B is the Borel subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathbb{C})$.

Definition 4.9 (Schubert cells, Schubert varieties)**.** *The sets $C_w := BwP/P$ for $w \in \mathfrak{S}_n/\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ are the* Schubert cells *of* $\mathrm{Gr}(m, n)$*, and their closures $X_w = \overline{BwP/P}$ are its* Schubert varieties*. Note that if we choose a different Borel subgroup, we will also call the cells and varieties it defines Schubert cells or varieties.*

We will now describe another set of indices for Schubert cells and Schubert varieties :

Proposition 4.10. *The set of permutations $w \in \mathfrak{S}_n/\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ is in 1:1 correspondence with the set of partitions contained in an $m \times (n-m)$ rectangle.*

Proof. Let us denote both sets by $\Pi_{m,n}$ and $Y_{m,n}$ respectively. The map

$$\begin{array}{ccl} Y_{m,n} & \to & \Pi_{m,n} \\ \lambda & \mapsto & (1+\lambda_m, 2+\lambda_{m-1}, \ldots, m+\lambda_1; \text{remaining integers in increasing order}) \end{array}$$

is a bijection. $\square$

This means we can index now Schubert varieties by partitions instead of permutations.

Finally, let us introduce a more geometric way of representing Schubert varieties :

Definition 4.11. *A* complete flag *of V is a sequence of nested subspaces*

$$F_\bullet = 0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = V,$$

where $\dim F_i = i$.

Remark. The stabilizer of any complete flag is a Borel subgroup of $\mathrm{GL}(V)$.

To a complete flag $F_\bullet$ and integers $\underline{p} = (1 \le p_1 < p_2 < \cdots < p_m \le n)$, we associate a subvariety of $\mathrm{Gr}(m, V)$ as follows

$$X_{\underline{p}}(F_\bullet) = \left\{ \Sigma \in \mathrm{Gr}(m,V) \mid \dim(\Sigma \cap F_{p_j}) \ge j \ \forall 1 \le j \le m \right\}.$$

We say that $X_{\underline{p}}(F_\bullet)$ is *defined by incidence conditions* with respect to the complete flag $F_\bullet$. We denote by $I_{m,n}$ the set of all multi-indices $\underline{p} = (1 \le p_1 < p_2 < \cdots < p_m \le n)$.

Proposition 4.12. *There is a bijection*

$$\begin{array}{ccc} Y_{m,n} & \to & I_{m,n} \\ \lambda & \mapsto & \underline{p} \end{array}$$

where $p_j = n - m + j - \lambda_j$ for all $1 \leq j \leq m$.

Hence we can also index these varieties by partitions contained in an $m \times (n-m)$ rectangle. We can now compare them with Schubert varieties :

Proposition 4.13. *If we denote by $E_\bullet$ the flag defined by $E_i = \mathrm{Vect}(e_1, \dots, e_i)$ for all i, then*

$$X_w = X_\lambda(E_\bullet),$$

where w and λ are related by the bijection of Prop. 4.10.

It follows that Schubert varieties can be defined by incidence conditions. Moreover, complete flags are in 1:1 correspondence with Borel subgroups of $\mathrm{GL}_n(\mathbb{C})$ as follows.

Let B' be a Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$. As seen in Section 2.3, all Borel subgroups are conjugate, so $B' = gBg^{-1}$ for some $g \in \mathrm{GL}_n(\mathbb{C})$. We may then define a flag $F_\bullet^{B'}$ by setting

$$F_i^{B'} = \mathrm{Vect}(g(e_1), \dots, g(e_i)).$$

This flag is stabilized by B'. Conversely, if $F_\bullet$ is a complete flag, then its stabilizer is a Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$. Hence varieties defined by incidence conditions are Schubert varieties.

Schubert varieties and Schubert cells satisfy the following :

Proposition 4.14.

1. *A Schubert variety X_λ is an algebraic subvariety of codimension $|\lambda|$ of $\mathrm{Gr}(m, V)$.*
2. *A Schubert cell C_λ is an affine space $\mathbb{C}^{m(n-m)-|\lambda|}$.*
3. *$X_\lambda = \bigsqcup_{\mu \supset \lambda} C_\mu$.*
4. *$X_\lambda \supset X_\mu$ if and only if $\lambda \subset \mu$.*

Exercise 4.3. Prove Prop. 4.14.

4.4 Schubert classes and the cohomology ring

Prop. 4.14 shows that the Bruhat decomposition of the Grassmannian is a *cell decomposition* (cf. Def. 2.11). The cohomology classes of Schubert varieties are called *Schubert classes* :

Definition 4.15 (Schubert class)**.** *The fundamental class $[X_\lambda]$ of a Schubert variety X_λ is called a* Schubert class *and denoted by σ_λ. It is an element of $H^{\ell(\lambda)}(\mathrm{Gr}(m,n))$.*

Prop. 2.12 implies that Schubert classes generate the cohomology of the Grassmannian :

Proposition 4.16. *As a $\mathbb{Z}$-module, the cohomology $H^*(\mathrm{Gr}(m,n))$ is generated by Schubert classes :*

$$H^*(\mathrm{Gr}(m,V)) = \bigoplus_{\lambda \in R_{m,n}} \mathbb{Z}\sigma_\lambda.$$

Since the Grassmannian is a smooth algebraic variety (as can be seen by looking at local coordinates), Thm. 2.8 implies that its cohomology is endowed with a ring structure given by the *cup product* $\cup$. The goal of the next section is to study this ring structure.

5 The Pieri and Giambelli rules

The goal of this section is to give a formula for the cup product of a Schubert class by a *special Schubert class*, called the *Pieri rule*, and a formula that expresses a Schubert class in terms of special Schubert classes, called the *Giambelli rule*.

Definition 5.1 (Special Schubert class). *A partition $\lambda = (p)$ for $1 \leq p \leq n-m$ is called a* row partition. *Similarly, a partition $\mu = (1, \ldots, 1)$ is called a* column partition. *Both are called* special partitions, *and the associated Schubert classes $\sigma_\lambda := \sigma_p$ and $\sigma_\mu := \sigma_{1^r}$ are called* special Schubert classes.

In Subsection 5.2, we will explain how to write a product $\sigma_p \cup \sigma_\lambda$ as a sum of Schubert classes, while in Subsection 5.3, we will show how to write a Schubert class σ_λ in terms of the σ_p's (and respectively for the σ_{1^r}'s). But first, we describe how Poincaré duality works inside the cohomology ring of the Grassmannian.

5.1 Poincaré duality

If X is a smooth algebraic variety of dimension N, we have seen in Subsection 2.2 that there is an isomorphism $H_k(X) \cong H^{N-k}(X)$, called Poincaré duality. Moreover, we have an isomorphism $H_k(X) \cong H^k(X)$, hence finally

$$H^k(X) \cong H^{N-k}(X).$$

Explicitely, if $(\gamma_i)_{i \in I}$ is a basis of $H^*(X)$, then the Poincaré dual basis $(\gamma_i^\vee)_{i \in I}$ is given by

$$\int_{[X]} \gamma_i \cup \alpha = \begin{cases} 1 & \text{if } \alpha = \gamma_i^\vee, \\ 0 & \text{otherwise.} \end{cases}$$

Now we see that the Schubert basis of $H^*(\mathrm{Gr}(m, V))$ is Poincaré self-dual :

Proposition 5.2. *Let λ and μ be two partitions contained in an $m \times (n-m)$ rectangle and such that $|\lambda| + |\mu| = m(n-m)$. Then*

$$\sigma_\lambda \cup \sigma_\mu = \begin{cases} \sigma_{pt} & \textit{if } \mu \neq \lambda^\vee, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is geometric and uses the description of Schubert varieties by incidence conditions with respect to complete flags (see Prop. 4.13 and above). Let E be the standard flag of V and E^- be the *opposite* flag, i.e.

$$E_i^- = \mathrm{Vect}(e_n, e_{n-1}, \ldots, e_{n+1-i}).$$

The flags E and E^- being in general position (since $\dim(E_i \cap E_j^-) = 0$ if $i + j \leq n$ and $\dim(E_i \cap E_j^-) = i + j - n$ otherwise), it follows that the cup product can be computed by looking at the intersection

$$X_\lambda(E_\bullet) \cap X_\mu(E_\bullet^-).$$

If Σ is an element of this intersection, then

$$\dim(\Sigma \cap E_{n-m+i-\lambda_i} \cap E^-_{n+1-i-\mu_{m+1-i}}) \geq 1,$$

hence $\dim(E_{n-m+i-\lambda_i} \cap E^-_{n+1-i-\mu_{m+1-i}}) \geq 1$, which implies that $\lambda_i + \mu_{m+1-i} \leq n-m$ for all $1 \leq i \leq m$. By definition of the dual partition, this means that $\mu \subset \lambda^\vee$. Since $|\lambda| + |\mu| = m(n-m)$, it follows that $X_\lambda(E_\bullet) \cap X_\mu(E^-_\bullet) \neq 0$ implies $\mu = \lambda^\vee$.

Finally, if $\mu = \lambda^\vee$, we see that

$$X_\lambda(E_\bullet) \cap X_\mu(E^-_\bullet) = \left\{ \bigoplus_{i=1}^m \mathbb{C} e_{n-m+i-\lambda_i} \right\}.$$

□

5.2 The Pieri rule

We will now use Prop. 5.2 to prove :

Theorem 5.3 (Pieri rule for $\mathrm{Gr}(m,V)$)**.** *1. The cup product of a special Schubert class σ_p $(1 \leq p \leq n-m)$ with a Schubert class σ_λ is given by*

$$\sigma_p \cup \sigma_\lambda = \sum_{\substack{\mu \subset m\times(n-m) \\ \mu \in \lambda \otimes p}} \sigma_\mu .$$

2. Similarly, if $1 \leq r \leq m$:

$$\sigma_r \cup \sigma_\lambda = \sum_{\substack{\mu \subset m\times(n-m) \\ \mu \in \lambda \otimes 1^r}} \sigma_\mu .$$

Proof. Prop. 5.2 means that if $|\lambda| + |\nu| + p = m(n-m) - p$, then the theorem is equivalent to

$$\sigma_\lambda \cup \sigma_\nu \cup \sigma_p = \begin{cases} 1 & \text{if } n-m-\lambda_m \geq \nu_1 \geq n-m-\lambda_{m-1} \geq \cdots \geq n-m-\lambda_1 \geq \nu_m \\ 0 & \text{otherwise.} \end{cases}$$

Notice also that we may assume that $\lambda_i + \nu_{m+1-i} \leq n-m$ for all $1 \leq i \leq m$, otherwise $\sigma_\lambda \cup \sigma_\nu = 0$. Write

$$W_i := E_{n-m+i-\lambda_i} \cap E^-_{n+1-i-\nu_{m+1-i}}.$$

The cup product above is non-zero if and only if the W_i's are direct summands. Write

$$W := \sum_{i=1}^m W_i = \bigcap_{i=1}^m (E_{n-m+i-\lambda_i} + E^-_{n-i-\nu_{m-i}})$$

If $\Sigma \in X_\lambda(E_\bullet) \cap X_\nu(E^-_\bullet)$, then $\dim(\Sigma \cap E_{n-m+i-\lambda_i}) \geq i$ and $\dim(\Sigma \cap E^-_{n+1-i-\nu_{m+1-i}}) \geq m+1-i$. Hence $\Sigma \subset E_{n-m+i-\lambda_i} + E^-_{n-i-\nu_{m-i}}$ for all $1 \leq i \leq m$. Indeed, either $E_{n-m+i-\lambda_i} + E^-_{n-i-\nu_{m-i}} = V$, or the sum is direct. In this case

$$\dim(\Sigma \cap (E_{n-m+i-\lambda_i} \oplus E^-_{n-i-\nu_{m-i}})) \geq i + (m-i) = m,$$

hence we still have $\Sigma \subset E_{n-m+i-\lambda_i} + E^-_{n-i-\nu_{m-i}}$. It now follows that $\Sigma \subset W$.

We now consider a linear subspace L of dimension $n-m+1-p$ of V. The variety

$$X_p(L) = \{\Sigma \in \mathrm{Gr}(m,V) \mid \dim(\Sigma \cap L) \geq 1\}$$

is a special Schubert variety with cohomology class σ_p.

If the W_i's are not direct summands, then $\dim W \leq m+p-1$, hence there exists a linear subspace L of dimension $n-m+1-p$ such that $L \cap W = 0$. In this case

$$X_\lambda(E_\bullet) \cap X_\nu(E_\bullet^-) \cap X_p(L) = \emptyset.$$

Otherwise, for a generic L, the intersection $L \cap W$ has dimension 1. Let $l \in V$ generate this intersection, and write

$$l = \sum_{i=1}^{m} w_i$$

with $w_i \in W_i$. The w_i have to be in W, which means that they form a basis. Hence the triple intersection $X_\lambda(E_\bullet) \cap X_\nu(E_\bullet^-) \cap X_p(L)$ is reduced to a point. Since it is transverse, this concludes the proof. $\square$

An important corollary of the Pieri rule is that we can now define a ring homomorphism from the ring of symmetric functions to the Chow ring of the Grassmannian :

Corollary 5.4. *The map*

$$\begin{array}{rccc} \Theta_{m,n} : & \Lambda_m & \to & H^*(\mathrm{Gr}(m,n)) \\ & s_\lambda & \mapsto & \sigma_\lambda, \end{array}$$

where we set $\sigma_\lambda = 0$ if λ is not contained in an $m \times (n-m)$ rectangle, is a surjective ring homomorphism.

It particular, this implies that the special Schubert classes generate $H^*(\mathrm{Gr}(m,n))$:

Proposition 5.5. *Special Schubert classes σ_p with $1 \leq p \leq n-m$ (resp. σ_{1^r} with $1 \leq r \leq m$) are multiplicative generators of the cohomology of the Grassmannian.*

Proof. The result is a consequence of Cor. 5.4 and Thm. 3.5, using the fact that $\Theta_{m,n}(e_r) = \sigma_{1^r}$ and $\Theta_{m,n}(h_p) = \sigma_p$. $\square$

In the next subsection, Cor. 5.4 and the Jacobi-Trudi formulas will enable us to write any Schubert class as a polynomial in special Schubert classes.

5.3 The Giambelli rule

Theorem 5.6 (The Giambelli rule)**.** *Any Schubert class $\sigma_\lambda \in H^*(\mathrm{Gr}(m,n))$ can be expressed as*

$$\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{1\leq i,j\leq m},$$

where we set $\sigma_p = 1$ if $p = 0$ and $\sigma_p = 0$ if $p < 0$ or $p > n-m$.

Using the Pieri rule 5.3 and the above Giambelli rule, it is possible to compute any cup product $\sigma_\lambda \cup \sigma_\mu$, by first expressing σ_μ in terms of special classes, and then computing the product of σ_λ with each of these special classes. However, this approach is neither practical nor effective. In the next section, we will introduce a combinatorial formula for computing such products directly.

6 The Littlewood-Richardson rule

In this section, we will prove a positive combinatorial formula for the product of two Schubert classes. It will be a consequence of a similar formula for the product of two Schur functions, using Cor. 5.4. The proof will rely heavily on the combinatorics of Young tableaux, which we introduce in the next subsection.

Due to time restrictions, we will not be able to prove all intermediate results in their full generality. However, we will provide references at each step, and illustrate the results by detailed examples.

6.1 Young tableaux

Definition 6.1 (Young tableaux). *A* semi-standard Young tableau T *is the data of a Young diagram, together with a numbering of each box by positive integers, so that numbers are non-decreasing along rows (from left to right) and increasing along columns (from top to bottom), as in Fig. 3. Such a tableau is* standard *if it is numbered by successive integers starting from* 1*, each appearing only once. The* shape $\lambda(T)$ *of a Young tableau is its support partition, and its* weight $\mu(T)$ *is defined as*

$$\mu(T)_i = \#\{\textit{entries of } T \textit{ equal to } i\}$$

1	1	2	2	3
2	3	4		
3				
5				

Figure 3: A semi-standard (but not standard) Young tableau

Example. The tableau T of Fig. 3 has shape $\lambda(T) = (5,3,1,1)$ and weight $\mu(T) = (2,3,3,1,1)$.

6.2 The Knuth correspondence

Theorem 6.2 (Knuth correspondence). *There exists a bijective correspondence between matrices A with non-negative integer entries, and pairs (S,T) of semi-standard Young tableaux of the same shape. Under this correspondence, the column sums (resp. the row sums) of A are given by the weight of T (resp. of S).*

This result can be proved constructively by a method due to Fulton. Here we introduce the method and illustrate it by an example. For a complete proof, we refer to [Ful97, Chap. 4].

Consider a matrix A with non-negative entries, and view it as a stack of boxes, where in the cell (i,j), we put $a_{i,j}$ numbered balls, disposed from northwest (NW) to south-east (SE). We number these balls starting with 1 so that, for each ball, if the maximal number indexing a ball to the NW of this ball is i, then the ball is numbered with $i+1$.

Example. If

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

then the associated picture is

$$\begin{array}{|c|c|c|c|}\hline & & & (a) \\ \hline 1 & 2 & 3 & \\ \hline & 3 & & 4 \\ \hline (b) & & & \\ \hline \end{array}$$

where

$$(a) = \begin{array}{cc} 1 & \\ & 2 \end{array} \text{ and } (b) = \begin{array}{cc} 2 & \\ & 3 \end{array}$$

By construction, for all i, the balls numbered i form a SW to NE chain $(i_1, j_1) \to \cdots \to (i_p, j_p)$. We construct a new picture by replacing these chains by the chain $(i_1, j_2) \to \cdots \to (i_{p-1}, j_p)$. Doing this for all i and renumbering in the same way as for A, we obtain a picture which is associated to a matrix with non-negative entries. We denote this matrix by ∂A.

Example. In our example

$$\partial A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

We then iterate this operation. After a finite number of steps k, we will obtain $\partial^k A = 0$, at which point we stop.

Example. In our example

$$\partial^2 A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad , \quad \partial^3 A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \partial^4 A = 0.$$

Finally, we denote by $p_{i,j}$ (resp. $q_{i,j}$) the smallest index of a column (resp. of a row) of the ball matrix associated to $\partial^{i-1} A$ in which the integer j appears. We can now define two semi-standard tableaux $P(A)$ (resp. $Q(A)$) so that the cell (i, j) is numbered with the index $p_{i,j}$ (resp. $q_{i,j}$).

Example. In our example :

$$P(A) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad \text{and} \quad Q(A) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

We check that the weights of these tableaux correspondend indeed to the column and row sums of A.

6.3 The plactic ring

We first define a non-commutative ring which will be associated to the ring of symmetric polynomials

Definition 6.3 (Plactic ring)**.** *The* plactic ring $\mathcal{P}_m$ *is the ring of polynomials in m non-commutative variables $x_1, \dots, x_m$ with integer coefficients, subject to the* Knuth relations

$$\begin{aligned} x_i x_k x_j &\sim x_k x_i x_j \quad \text{if } i \leq j < k, \\ x_j x_k x_i &\sim x_j x_i x_k \quad \text{if } i < j \leq k. \end{aligned}$$

Now if T is a semi-standard Young tableau, we may associate to it an element of the plactic ring as follows :

Definition 6.4 (Word of a tableau)**.** *If T is a Young tableau, we denote by $m(T)$ the element of the plactic ring obtained by reading its entries from bottom to top and from left to right.*

Example. If

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} \\ \cline{1-3} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{5} \\ \cline{1-1} \end{array}$$

then $m(T) = x_5 x_3 x_2 x_3 x_4 x_1 x_1 x_2 x_2 x_3$.

Theorem 6.5. *Each Knuth equivalence class contains the word of a unique tableau. As a consequence, it follows that the plactic ring has a set of tableau as a basis over the integers.*

Proof. See [Man01, Prop. 1.5.11] □

In the plactic ring, products are easy to compute. We will use the correspondence between tableaux and elements of the plactic ring to compute the product of two tableaux. In the next subsection, we describe an algorithm which achieves this.

6.4 The jeu de taquin

Definition 6.6 (Skew tableau)**.** *Let $\lambda \supset \mu$ be two partitions. The complement $\lambda \setminus \mu$ of μ inside λ is called a* skew partition*. Any semi-standard numbering of $\lambda \setminus \mu$ is called a* skew tableau.

Jeu de taquin is an algorithm which associates to a skew tableau a Knuth-equivalent semi-standard Young tableau. It is defined as follows :

1. Choose a corner of the skew tableau, and denote by x (resp. y) the index of the cell just below (resp. to the right) of this corner.
2. If $x \leq y$, slide x in into the place of the corner. Otherwise slide y.
3. Repeat the procedure until there is no corner left and the tableau obtained is semi-standard.

Example. Consider the skew tableau

$$\begin{array}{cccc} \cline{3-5} & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \cline{2-5} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \cline{1-4} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \cline{1-2} \end{array}$$

Jeu de taquin gives

$$\begin{array}{ccccc} & \bullet & 1 & 1 & 2 \\ & 1 & 2 & 3 & \\ 2 & 2 & & & \end{array} \to \begin{array}{ccccc} & 1 & 1 & 1 & 2 \\ & \bullet & 2 & 3 & \\ 2 & 2 & & & \end{array} \to \begin{array}{ccccc} & 1 & 1 & 1 & 2 \\ \bullet & 2 & 2 & 3 & \\ 2 & & & & \end{array} \to \begin{array}{ccccc} \bullet & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 & \end{array}$$

and then

$$\begin{array}{ccccc} 1 & \bullet & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 & \end{array} \to \begin{array}{ccccc} 1 & 1 & \bullet & 1 & 2 \\ 2 & 2 & 2 & 3 & \end{array} \to \begin{array}{ccccc} 1 & 1 & 1 & \bullet & 2 \\ 2 & 2 & 2 & 3 & \end{array} \to \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 \end{array}$$

Using Knuth relations it is easy to check that

Proposition 6.7. *The jeu de taquin is compatible with Knuth equivalence.*

It follows from this that the tableau obtained from a skew tableau by jeu de taquin does not depend on the choice of a corner made at each step.

Thanks to jeu de taquin, we may now define the product of two Young tableaux T_1 and T_2 by constructing a skew tableau $T_{1,2}$ as follows

$$T_{1,2} = \begin{array}{c|c} & T_2 \\ \hline T_1 & \end{array}$$

and then turning it into a tableau using jeu de taquin.

Example. Let us compute the product of

$$T_1 = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & & \end{array} \quad \text{and} \quad T_2 = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 & \\ 3 & & \end{array}$$

We start from

$$T_{1,2} = \begin{array}{cccccc} & & & 1 & 2 & 2 \\ & & & 2 & 3 & \\ & & \bullet & 3 & & \\ 1 & 2 & 2 & & & \\ 2 & 3 & & & & \\ 3 & & & & & \end{array}$$

Skipping easy steps, jeu de taquin gives

$$\begin{array}{cccccc} & & & 1 & 2 & 2 \\ & & \bullet & 2 & 3 & \\ 1 & 2 & 2 & 3 & & \\ 2 & 3 & & & & \\ 3 & & & & & \end{array} \to \begin{array}{cccccc} & & \bullet & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 & 3 & \\ 2 & 3 & 3 & & & \\ 3 & & & & & \end{array} \to \begin{array}{ccccc} & \bullet & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & & \\ 3 & & & & \end{array} \to \begin{array}{ccccc} & 1 & \bullet & 2 & 2 \\ 1 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & & \\ 3 & & & & \end{array}$$

and then

$$\begin{array}{ccccc} & 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 & \\ 2 & 3 & 3 & & \\ 3 & & & & \end{array} \to \begin{array}{ccccc} & 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 & \\ 2 & 3 & 3 & & \\ 3 & & & & \end{array} \to \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & \\ 3 & & & \end{array}$$

The last tableau is the product.

We will now state a multiplication rule for Schur functions called the *Littlewood-Richardson rule.*

6.5 The Littelwood-Richardson rule

Proposition 6.8. *Let T and T' be two semi-standard tableaux with shapes μ and μ'. Then there exists a one to one correspondence between pairs (S, S') of tableaux with shapes λ, μ and product T' on one side, and skew tableaux Q with shape μ/λ Knuth-equivalent to T on the other side.*

Proof. See [Man01, 1.5.20] □

Corollary 6.9. *In the plactic ring, a product of Schur functions decomposes as a sum of Schur functions.*

Definition 6.10 (Yamanouchi words)**.** *A word $x_1 \dots x_r$ is* Yamanouchi *if for all s and i, the subword $x_s \dots x_r$ has at least as many i's as $(i+1)$'s.*

Definition 6.11 (Canonical tableau)**.** *A tableau is* canonical *if each of its cells is labeled with its row index.*

Proposition 6.12. *A word is Yamanouchi if and only if its associated tableau is canonical.*

Proof. The Yamanouchi property is preserved by Knuth equivalence. Morever, suppose that the word $m(T)$ of a tableau T is Yamanouchi. Look at the first row of T and denote by $x_s \dots x_r$ the associated word. It contains at least as many ones than twos. Suppose it contains at least one two. Then $x_r = 2$, which contradicts the Yamanouchi. So the first row of T is labeled only by ones, and similarly, its second row is labeled only by twos, ... □

Theorem 6.13 (Combinatorial Littelwood-Richardson rule)**.** *The coefficient $c^{\nu}_{\lambda\mu}$ of s_ν in the product $s_\lambda s_\mu$ is equal to the number of skew tableaux with shape ν/μ and weight λ, for which the associated word is Yamanouchi.*

Theorem 6.14 (Geometric Littelwood-Richardson rule)**.** *The cup product of Schubert classes $\sigma_\lambda \cup \sigma_\mu$ inside the cohomology of the Grassmannian* $\mathrm{Gr}(m,n)$ *is given by*

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subset m \times (n-m)} c^{\nu}_{\lambda\mu} \sigma_\nu.$$

Proof. The theorem is the consequence of Cor. 5.4 and Thm. 6.13. □

Example. The product $\sigma_{2,1} \cup \sigma_{2,1}$ inside $\mathrm{Gr}(3,7)$ is $\sigma_{4,2} + \sigma_{4,1,1} + \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}$.

7 Generalisations

In this last section, we briefly introduce two current generalisations of Schubert calculus on the Grassmannian : Schubert calculus on other complete homogeneous spaces and quantum Schubert calculus. Except for quantum Schubert calculus on the Grassmannian, we will mainly give references to the results which are known in this context. This is pretty much a current area of research, and there is much left to study. Unlike other sections, this section requires some more background on algebraic geometry and algebraic groups.

7.1 Quantum Schubert calculus on Grassmannians

The *quantum cohomology ring* of a smooth complex projective variety X is a deformation of its cohomology ring. While the cohomology ring of X encodes how subvarieties intersect, its quantum cohomology ring encodes how they are linked by rational curves. More precisely, the quantum cohomology ring is constructed from so-called *Gromov-Witten invariants*, which, in good cases such as for homogeneous spaces, count the number of rational curves meeting three given subvarieties.

In those cases, the *Gromov-Witten invariant of degree* $d \geq 0$ associated to three cohomology classes $\gamma_1, \gamma_2, \gamma_3$ representing generic subvarieties Y_1, Y_2, Y_3, is equal to the number of rational curves of degree d meeting Y_1, Y_2, Y_3 if this number is finite, or to zero otherwise.

Now the quantum cohomology ring $QH^*(X)$ of X is defined as follows. If γ_1 and γ_2 are two cohomology classes on X, their *quantum product* $\gamma_1 \star \gamma_2$ is given by

$$\gamma_1 \star \gamma_2 = \sum_{d=0}^{\infty} q^d \sum_{\gamma_3} I_\beta\left(\gamma_1, \gamma_2, \gamma_3^\vee\right) \gamma_3,$$

where γ_3 runs over a basis of $H^*(X)$ and $\gamma_3^\vee$ runs over the Poincaré dual basis.

We will now describe the main ideas of a very elegant proof due to Buch [Buc03] of quantum Pieri and Giambelli rules for the Grassmannian $\mathrm{Gr}(m,n)$. Before that, quantum cohomology of the Grassmannian had been studied by Witten [Wit95] and Bertram [Ber97]. This proof uses the so-called *quantum-to-classical principle*, which states that in some cases, the quantum product can be computed form the usual cup product on an auxiliary homogeneous space.

The idea of the quantum-to-classical principle is to associate two vector spaces to a rational curve in a homogeneous space : its *kernel* and its *span*. For $\mathrm{Gr}(m,n)$, they are defined as follows :

Definition 7.1 (Kernel and span)**.** *Let C be a rational curve of degree d in* $\mathrm{Gr}(m,n)$*. Points of C then represent vector subspaces of dimension m inside $\mathbb{C}^n$. We call* kernel *of C, denoted by* $\mathrm{Ker}(C)$*, the biggest vector space contained in all those spaces. Similarly, we call* span *of C the smallest vector space containing all those spaces, and we denote it by* $\mathrm{Span}(C)$*.*

The dimension of the kernel and span is bounded :

$$\dim \mathrm{Ker} C \geq m - d \text{ and } \dim \mathrm{Span} C \leq m + d.$$

Incidence conditions on C can be rewritten as incidence conditions on $\mathrm{Ker}(C)$ and $\mathrm{Span}(C)$. For instance, if we denote by $\hat{\lambda}$ the partition obtained by removing the first d columns of a partition λ – i.e. by setting $\hat{\lambda}_i = \max(\lambda_i - d, 0)$, we get :

Proposition 7.2 ([Buc03])**.** *Let $C \subset \mathrm{Gr}(m,n)$ be a rational curve of degree $d \leq n - m$, W be a vector space of dimension $m + d$ containing* $\mathrm{Span}(C)$ *and $F_\bullet$ be a complete flag. If a partition λ is such that $C \cap X_\lambda(F_\bullet) \neq \emptyset$, then W belongs to the Schubert variety $X_{\hat{\lambda}}(F_\bullet)$ of* $\mathrm{Gr}(m+d,n)$*.*

The quantum-to-classical principle rephrases the Gromov-Witten invariants of $\mathrm{Gr}(m,n)$ in terms of classical intersection theory on Schubert varieties on the two-step flag variety $\mathrm{F}(m-d, m+d; n)$ (a point of this flag variety corresponding to a pair kernel-span associated to a rational curve), or even, as in the above Prop. 7.2, in the Grassmannian

$\mathrm{Gr}(m+d,n)$. The computations can then be done using the classical Pieri rule 5.3 and Giambelli rule 5.6.

In this fashion, Buch obtains the following rules, which were originally proved by [Ber97] using more complicated methods.

Theorem 7.3 (Quantum Pieri rule)**.** *Let $\lambda \subset m\times(n-m)$ be a partition and $1 \leq p \leq n-m$ be an integer. Then*

$$\sigma_p \star \sigma_\lambda = \sum_\mu \sigma_\mu + q \sum_\nu \sigma_\nu,$$

where the first sum is given by the classical Pieri formula, and in the second sum, ν runs over all partitions ν obtained by removing $n-p$ boxes from λ in such a way that :

$$\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \nu_2 \geq \cdots \geq \lambda_m - 1 \geq \nu_m \geq 0.$$

Example. In $\mathrm{Gr}(3,6)$, we have $\sigma_1 \star \sigma_{3,2,1} = \sigma_{3,3,1} + \sigma_{3,2,2} + q\sigma_1$.

Unexpectedly, the quantum Giambelli rule is identical to the usual Giambelli rule :

Theorem 7.4 (Quantum Giambelli rule)**.**

$$\sigma_\lambda = \det\left(\sigma_{\lambda_i+j-i}\right)_{1\leq i,j\leq m},$$

where the product is the quantum product.

To conclude the section, we mention another application of the quantum-to-classical principle [Cos09], for computing the quantum Littelwood-Richardson rule.

7.2 Schubert calculus for other homogeneous spaces

In this section, we review quickly some results concerning Schubert calculus on other homogeneous spaces. Indeed, it follows from Bruhat decomposition 2.21 that every complete homogeneous space has a cohomology generated by Schubert classes, hence possesses a Schubert calculus.

Orthogonal and symplectic Grassmannians. A Pieri rule has been proved by Pragacz and Ratajski (cf [PR96] and [PR03]). More recent results are mentioned in the next subsection.

Flag varieties. Let V be a $\mathbb{C}$-vector space of dimension N and $0 < r_1 < \cdots < r_k < N$ be an increasing sequence of integers. A *flag of V of type* $(r_1,\ldots,r_k)$ is an increasing sequence $0 \subset V_1 \subset \cdots \subset V_k \subset V$ of subspaces of V such that $\dim V_i = r_i$ for all i. If $k = N-1$, $0 \subset V_1 \subset \cdots \subset V_{N-1} \subset V$ is a complete flag.

The set $F(r_1,\ldots,r_k;V)$ of flags of V of type $(r_1,\ldots,r_k)$ only depends on the dimension of V, and has the structure of a smooth projective variety. It is called a *flag variety.* It is homogeneous under $\mathrm{GL}(N)$. More precisely

$$F(r_1,\ldots,r_k;N) \cong \mathrm{GL}(N)/P_{r_1,\ldots,r_k},$$

where $P_{r_1,\ldots,r_k}$ is associated to the vertices $r_1,\ldots,r_k$ of the Dynkin diagram of $\mathrm{GL}(N)$.

Coskun proved in [Cos09] a Littelwood-Richardson rule for (type A) two-step flag varieties, and he exposed and preliminary version for the general case in [Cos].

7.3 Quantum Schubert calculus on other homogeneous spaces

Here we review some results concerning the quantum cohomology of other homogeneous spaces ; we refer to [Tam07] for more details.

Orthogonal and symplectic Grassmannians. Schubert classes of orthogonal Grassmannians are indexed by so-called *k-strict partitions*. Buch, Kresch and Tamvakis proved a quantum Pieri rule (cf [BKT09]), and a quantum Giambelli rule (cf [BKT08]). Another quantum Pieri rule is stated in [LL], but without the corresponding Giambelli rule.

Minuscule and adjoint homogeneous spaces. Let G be an algebraic group, and P be a maximal parabolic of G associated to a fundamental weight ω. A fundamental weight ω is *minuscule* if $|\langle\omega,\check{\alpha}\rangle| \leq 1$ for any root α. If ω is minuscule, then $X = G/P$ is also said to be minuscule. Finally, $X = G/P$ is called *adjoint* if P is the parabolic associated to the longest root, and *co-adjoint* if P is associated to the longest short root. A table describing all those types of varieties can be found in [CP09].

In [CMP08], [CMP07] and [CMP10], using a quantum-to-classical principle, Chaput, Manivel and Perrin study the quantum cohomology ring of minuscule varieties. In [CP09], Chaput and Perrin extend part of these results to (co)-adjoint varieties.

Flag varieties. In [GK95], Givental and Kim gave a presentation of the quantum cohomology of flag varieties. In [CF99], Ciocan-Fontanine gave quantum Pieri and Giambelli formulas ; other references can be found in [FW04].

Generalised flag varieties. The quantum cohomology of generalised flag varieties is not known in general. However, in [FW04], Fulton and Woodward proved a quantum Chevalley formula for all G/P, i.e a formula for the quantum product of a Schubert class by a Schubert divisor.

An unpublished result of Peterson [Pet] relates the quantum cohomology of G/P and the homology of the associated affine Grassmannian. A proof can be found in [LS10]. This result allows to express Gromov-Witten invariants of G/P in terms of those of G/B (cf [Woo05]). It has been used by Leung and Li in [LL10].

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