ON LANDAU-GINZBURG MODELS FOR QUADRICS AND FLAT SECTIONS OF DUBROVIN CONNECTIONS

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Abstract. This paper proves a version of mirror symmetry expressing the (small) Dubrovin connection for even-dimensional quadrics in terms of a mirror-dual Landau-Ginzburg model on the complement of an anticanonical divisor in a dual quadric. We go into greater depth for all quadrics, even and odd, treating them as a series starting with Q_3 and $Q_4 = Gr_2(4)$. This turns out to work very naturally after restricting to a particular torus, and leads to a combinatorial model for the superpotential in terms of a quiver, in the vein of those proposed by Batyrev, Ciocan-Fontanine, Kim and van Straten for Grassmannians in the 1990's. The Laurent polynomial superpotentials form a single series, despite the fact that our mirrors of even quadrics are defined on dual quadrics, while the mirror to an odd quadric is naturally defined on a projective space. We use this combinatorial description to compute the constant term of the J-function.

Contents

1. Introduction

Suppose X is a smooth projective complex Fano variety X of dimension N. Starting from X as the 'A-model,' Dubrovin constructed a flat connection on a trivial bundle with fiber $H^*(X, \mathbb{C})$, using Gromov-Witten invariants of X. One incarnation of mirror symmetry reproduces the same connection via a Gauss-Manin system on a 'B-model'.

In our setting X will always have Picard rank 1 and the base of the trivial bundle on the A-side can be taken to be the two-dimensional complex torus $\mathbb{C}^*_q \times \mathbb{C}^*_\hbar$ with coordinates q and $\hbar$. The Dubrovin connection is flat and therefore defines a D-module M_A, where $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$. The B-model for X as above is a *Landau-Ginzburg model*, which is a pair $(\check{X}, W)$ consisting of an affine algebraic variety $\check{X}$ over $\mathbb{C}$ and a regular function $W_q : \check{X} \to \mathbb{C}$ called the *superpotential*. This data gives rise to a Gauss-Manin system, via a kind of twisted N-th (algebraic) de Rham cohomology. Namely one defines the D-module

$$M_B = \Omega^N(\check{X}, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}])/(d - \frac{1}{\hbar}dW \wedge \_\ )\Omega^{N-1}(\check{X}, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}])$$

where $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$, which is intended to recover the Dubrovin connection of X. One of the central problems of mirror symmetry is how to construct the LG model $(\check{X}, W)$ given X. In the case of quadrics there are two direct approaches. One of them is the approach due to Hori and Vafa [HV00], which applies to Fano hypersurfaces in projective space. The other approach is via [Rie08], which applies to homogeneous spaces G/P.

When X is a hypersurface in a complex projective space, its conjectured Landau-Ginzburg model, the 'Hori-Vafa mirror', is a torus together with a Laurent polynomial in N variables [HV00], [Prz09, Rmk. 19]. In the case of an N-dimensional quadric Q_N the LG model of Hori and Vafa is $L_q : (\mathbb{C}^*)^N \to \mathbb{C}$ where

$$L_q = Y_1 + Y_2 + \ldots + Y_{N-1} + \frac{(Y_N + q)^2}{Y_1 Y_2 \cdots Y_N}. \tag{1}$$

Note that this expression is indeed equivalent to the original Hori-Vafa model (see [Prz09]).

On the other hand the smooth quadric Q_N may also be identified with the homogeneous space $\mathrm{SO}_{N+2}(\mathbb{C})/P$. Here we think of $\mathrm{SO}_{N+2}(\mathbb{C})$ as the special orthogonal group associated to the quadratic form on $\mathbb{C}^{N+2}$ defining Q_N inside $\mathbb{P}^{N+1}$. The mirror construction from [Rie08] applies in this setting and gives a regular function $\mathcal{F}_q$ on an N-dimensional affine subvariety $\mathcal{R}$ (generally larger than a torus) of the Langlands dual full flag variety. If N is odd then this Langlands dual full flag variety is $Sp_{N+1}(\mathbb{C})/B$. If N is even then it is $\mathrm{SO}_{N+2}(\mathbb{C})/B$.

One advantage of the mirrors $\mathcal{F}_q$ over the Laurent polynomials L_q is that the former have the expected number of critical points (at fixed generic value of q), namely $\dim(H^*(Q_N))$. This is not generally the case for Laurent polynomial mirrors, as was already observed in [EHX97]. In [EHX97] it was suggested to solve this problem using a partial compactification and this was carried out for the first time in the case of Q_4, albeit in an ad hoc fashion. Since then a partial compactification of the Hori-Vafa mirror in the case of all odd quadrics was obtained in [GS13], along with a proof of the isomorphism of D-modules. This partial compactification was then shown in [PR13a] to be isomorphic to the mirror $\mathcal{F}_q$.

We note that for type A flag varieties the mirrors $\mathcal{F}_q$ were shown to be partial compactifications of the Laurent polynomial mirrors of [BCFKvS00, BCFKvS98], see [Rie08, Rie06, MR13].

In this paper we will discuss and compare four different versions of the LG models for quadrics, and prove various identities predicted by mirror symmetry. Here is a summary of our results.

1.1. **A canonical mirror.** Suppose X is a homogeneous space for an adjoint simple complex algebraic group. For cominuscule X, such as Grassmannians, Lagrangian Grassmannians, and also quadrics, the Langlands dual group naturally acts on $H^*(X, \mathbb{C})$, by the geometric Satake correspondence [Lus83, MV07, Gin95]. We exploit this to give a very natural formulation of the mirror in the even quadrics case, compare [MR13, PR13b, PR13a]. Namely for even dimensional quadrics we prove an isomorphism between the domain $\mathcal{R}$ of $\mathcal{F}_q$ and the complement of an anti-canonical divisor in a 'mirror' quadric $\check{Q}_N$. This mirror quadric is obtained as a closed orbit of the Langlands dual group inside $\mathbb{P}(H^*(Q_N,\mathbb{C})^*)$. Therefore the cohomology classes of Q_N are naturally coordinate functions on the dual quadric $\check{Q}_N$. We then obtain an LG-model W_q on $\check{Q}_N$ by pulling back $\mathcal{F}_q$ and expressing it in the coordinates coming from the Schubert basis of $H^*(Q_N,\mathbb{C})$. We consider this to be the most canonical presentation of the LG-model for Q_N. For odd N we note that the analogous procedure gives an LG-model on $\mathbb{P}^N$, where $\mathbb{P}^N$ is viewed as a homogeneous space for $Sp_{N+1}(\mathbb{C})$, see [PR13a].

1.2. **An isomorphism of D-modules.** For even dimensional quadrics we construct an explicit isomorphism from the Dubrovin D-module M_A to a natural submodule of the Gauss-Manin D-module M_B. We conjecture that this submodule is in fact all of M_B so that M_A and M_B are isomorphic. Here we use the new version W_q of the mirror which takes place on a dual quadric. We note that there is a non-trivial cluster algebra structure on the coordinate ring of the mirror, which plays a role in our proof of the isomorphism.

1.3. **Laurent polynomial mirrors analogous to projective space.** By restricting to a natural choice of torus in $\mathcal{R}$ we obtain a further Laurent polynomial expression for the mirror. Combining this with results from [PR13b] we obtain a series of Laurent polynomial mirrors for all Q_N, which resemble the well-known Laurent polynomial mirrors for projective spaces (but differ from the Hori-Vafa mirrors).

1.4. **The hypergeometric series of the quadric.** We work out in two different ways a series expansion for the coefficient of the top class in Givental's J-function. On the one hand we obtain the series as a residue integral on the B-model side, using the Laurent polynomial formulation from 1.3. On the other hand the coefficients of the series can be interpreted as 1-point descendent Gromow-Witten invariants, and we determine these directly on the A-side, using Kontsevich-Manin reconstruction and the usual axioms. We identify this series as hypergeometric series and identify the differential equation which it satisfies, which is a 'quantum differential equation' of the quadric.

1.5. **A quiver version of the superpotential.** We interpret our Laurent polynomial version of the mirror from 1.3 in terms of a quiver, in the spirit of [BCFKvS98, BCFKvS00, Giv97]. The fundamental class coefficient of the J-function can be read off directly from the quiver. This is in analogy with the residue formula of [BCFKvS00, Section 5.1] for type A partial flag varieties, which was conjectured there to recover that coefficient of the J-function (now proved in [MR13] for Grassmannians, and a consequence of [Giv97] for the full flag variety).

1.6. **Comparison with the Hori-Vafa mirrors.** Finally we show that the Hori-Vafa mirrors arise out of W_q in the same way as the other Laurent polynomial mirrors, by restriction to a specific cluster torus.

2. Landau-Ginzburg models for odd quadrics

The quadrics are cominuscule homogeneous spaces (for the Spin groups). Therefore, in addition to the Hori-Vafa approach [HV00] for constructing LG models, there is another LG model for each quadric on an affine variety generally larger than a torus, which was defined by the second-named author using a Lie-theoretic construction [Rie08]. Namely for any projective homogeneous space $X = G/P$ of a simple complex algebraic group, [Rie08] constructed a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. It was shown in [Rie08] that this LG model recovers the Peterson variety presentation [Pet97] of the quantum cohomology of $X = G/P$. It therefore defines an LG model whose Jacobi ring has the correct dimension. In this section we will rewrite this LG model in terms of natural projective coordinates on $\mathbb{P}(H^*(Q_N, \mathbb{C})^*)$.

Note that for odd-dimensional quadrics Q_{2m-1} a recent paper [GS13] of Gorbounov and Smirnov constructed directly a partial compactification of the Hori-Vafa mirrors, without making use of [Rie08].

2.1. **The LG model for Q_{2m-1} on a Langlands dual projective space.** LG models for odd-dimensional quadrics with the expected number of critical points have been constructed in [Rie08] (where they appear as a special case), and [GS13], and finally [PR13a]. Here we recall the main results from the paper [PR13a], which contains the formulation for the LG model which we will adopt.

In this section our A-model variety $X = X_N = X_{2m-1}$ is the quadric $Q_N = Q_{2m-1}$. Recall that an odd-dimensional quadric has 1-dimensional cohomology groups in even degrees spanned by Schubert classes $\sigma_i \in H^{2i}(Q_{2m-1}, \mathbb{C})$ for $0 \leq i \leq 2m-1$, and no other cohomology. To construct its mirror first consider the projective space $\check{X} = \check{X}_{2m-1} = \mathbb{P}^{2m-1}$ with homogeneous coordinates $(p_0 : p_1 : \cdots : p_{2m-1})$ in one-to-one correspondence with these Schubert classes σ_i. Inside $\check{X}$ we have the open affine subvariety $\check{X}^\circ \subset \mathbb{P}^{2m-1}$ defined by:

$$\check{X}^\circ = \check{X}^\circ_{2m-1} := \check{X} \setminus D, \tag{2}$$

where $D := D_0 + D_1 + \ldots + D_{m-1} + D_m$, the divisors D_i being given by

$$\begin{aligned} D_0 &:= \{p_0 = 0\}, \\ D_\ell &:= \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-1-\ell+k} = 0 \right\} \text{ for } 1 \leq \ell \leq m-1, \\ D_m &:= \{p_{2m-1} = 0\}. \end{aligned}$$

The divisor D is an anti-canonical divisor. Indeed, the index of $\check{X} = \mathbb{P}^{2m-1}$ is $2m$. For simplicity, we will define

$$\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k}. \tag{3}$$

(For odd quadrics, N $= 2m-1$.) We have:

Theorem 2.1 ([PR13a, Theorem 1])**.** *The LG model $\mathcal{F}_q : \mathcal{R} \to \mathbb{C}$ from* [Rie08] *for $X = Q_{2m-1}$ is isomorphic to $W_q : \check{X}^\circ_{2m-1} \to \mathbb{C}$ defined by*

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1} p_{2m-1-\ell}}{\delta_\ell} + q \frac{p_1}{p_{2m-1}}. \tag{4}$$

We also have another expression for the superpotential:

Proposition 2.2 ([PR13a, Proposition 8])**.** *For $X = Q_{2m-1}$ and W_q as above, there is a torus $(\mathbb{C}^*)^{2m-1} \hookrightarrow \check{X}^\circ_{2m-1}$ to which W_q pulls back giving the Laurent* *polynomial expression*

$$W_q = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \dots a_{m-1} c b_{m-1} \dots b_1}. \tag{5}$$

2.2. **Comparison with the Hori-Vafa model for odd quadrics.** Here we check that once restricted to a certain torus, our LG model (4) is isomorphic to the Hori-Vafa LG model. Let us consider the change of coordinates:

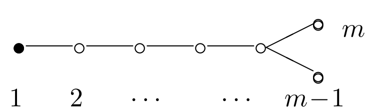
$$Y_i = \begin{cases} \frac{p_i}{p_{i-1}} & \text{for } 1 \leq i \leq m-1 \ ; \\ \frac{p_{2m-1-i}\delta_{2m-3-i}}{p_{2m-2-i}\delta_{2m-2-i}} & \text{for } m \leq i \leq 2m-3 \ ; \\ q\frac{p_1}{p_{2m-1}} & \text{for } i = 2m-2 \ ; \\ q\frac{\delta_{m-2}}{\delta_{m-1}} & \text{for } i = 2m-1. \end{cases}$$

This change of coordinates is well-defined on the torus $T = \{p_i \neq 0 \ \forall i\}$ inside $\check{X}^\circ$. Moreover, an easy calculation shows that it transforms our LG model (4) into the Hori-Vafa model (1) for odd quadrics. Note that this change of coordinates may also be obtained by combining the isomorphism between (4) and the Gorbounov-Smirnov mirror from [PR13a, Section 6], with the comparison between the Gourbounov-Smirnov mirror and Hori-Vafa's mirror in [GS13].

3. Landau-Ginzburg models for even quadrics

We view the quadric $X = X_{2m-2} := Q_{2m-2}$ of dimension $2m-2$ as a homogeneous space for the Spin group $\mathrm{Spin}_{2m}(\mathbb{C})$. In this section we will introduce a natural LG model for X_{2m-2} which will be defined on an open subvariety of a dual quadric $\check{X}_{2m-2} = P\backslash \mathrm{PSO}_{2m}(\mathbb{C})$, see Section 3.2. Note that the projective special orthogonal group PSO_{2m} is the Langlands dual group to Spin_{2m}, and both groups have the same Dynkin diagram, namely the Dynkin diagram of type D_m. The main result of this section, Proposition 3.6, shows that the new LG-model is isomorphic to one defined earlier [Rie08] on a Richardson variety $\mathcal{R}$ inside the full flag variety of $\mathrm{PSO}_{2m}(\mathbb{C})$.

Note that in the following we will denote the group $\mathrm{PSO}_{2m}(\mathbb{C})$ by G, since this is the group we will primarily be working with. Then the A-model symmetry group is $G^\vee = \mathrm{Spin}_{2m}(\mathbb{C})$, and we have $X_{2m-2} = G^\vee/P^\vee$, where $P^\vee$ is the parabolic subgroup associated to the first node of the Dynkin diagram of type D_m.



3.1. **Notations and definitions.** Let $V = \mathbb{C}^{2m}$ with fixed quadratic form

$$Q = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

In other words $Q(v_i, v_j) = (-1)^{\max(i,j)}\delta_{i+j,2m+1}$ where $\{v_i\}$ is the standard basis of $\mathbb{C}^{2m}$. For $G = \mathrm{PSO}(V, Q) = \mathrm{PSO}(V)$ we fix Chevalley generators $(e_i)_{1\leq i\leq m}$ and

$(f_i)_{1\le i\le m}$. To be explicit we embed $\mathfrak{so}(V,Q)$ into $\mathfrak{gl}(V)$ and set

$$e_i = \begin{cases} E_{i,i+1} + E_{2m-i,2m-i+1} & \text{if } 1 \le i \le m-1, \\ E_{m-1,m+1} + E_{m,m+2} & \text{if } i = m, \end{cases}$$

and $f_i := e_i^T$, the transpose matrix, for every $i = 1, \ldots, m$. Here $E_{i,j} = (\delta_{i,k}\delta_{l,j})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group $\mathrm{PSO}(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ consisting of upper-triangular and lower-triangular matrices in $\mathrm{PSO}(V)$, respectively. Here U_+ and U_- are the unipotent radicals of B_+ and B_-, respectively, and T is the maximal torus of $\mathrm{PSO}(V)$, consisting of diagonal matrices (d_{ij}) with non-zero entries $d_{i,i} = d^{-1}_{2m-i+1,2m-i+1}$. We let $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$, $R \subset X(T)$ the set of roots, and R^+ the positive roots. We denote the set of simple roots by $\Pi = \{\alpha_i \mid 1 \le i \le m\} \subset R^+ \subset R \subset X(T)$, and the set of fundamental weights (which is the dual basis in $X(T)$) by $\{\omega_i \mid 1 \le i \le m\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

The parabolic subgroup P of $\mathrm{PSO}(V)$ we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the e_i together with $f_2, \ldots, f_m$, leaving out f_1. Let $x_i(a) := \exp(ae_i)$ and $y_i(a) := \exp(af_i)$. The Weyl group W of $\mathrm{PSO}(V)$ is generated by simple reflections s_i for which we choose representatives

$$\dot{s}_i = y_i(-1)x_i(1)y_i(-1). \tag{6}$$

We let W_P denote the parabolic subgroup of the Weyl group W, namely $W_P = \langle s_2, \ldots, s_m\rangle$. The length of a Weyl group element w is denoted by $\ell(w)$. The longest element in W_P is denoted by w_P. We also let w_0 be the longest element in W. Next W^P is defined to be the set of minimal length coset representatives for W/W_P. The minimal length coset representative for w_0 is denoted by w^P.

We introduce the following notation for the elements of W^P. Namely, $W^P = \{e, w_1, \ldots, w_{m-1}, w'_{m-1}, w_m, w_{m+1}, \ldots w_{2m-2}\}$, where

$$w_k = \begin{cases} s_k s_{k-1} \ldots s_1 & \text{if } 1 \le k \le m-2, \\ s_m s_{m-2} \ldots s_1 & \text{if } k = m-1, \\ s_{m-1} s_m s_{m-2} \ldots s_1 & \text{if } k = m, \\ s_{2m-1-k} \ldots s_{m-2} s_{m-1} s_m s_{m-2} \ldots s_1 & \text{if } m+1 \le k \le 2m-2. \end{cases}$$

and $w'_{m-1} = s_{m-1}s_{m-2}\ldots s_1$.

For any $w \in W$ let $\dot{w}$ denote the representative of w in G obtained by setting $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m}$, where $w = s_{i_1} \cdots s_{i_m}$ is a reduced expression and $\dot{s}_i$ is as in (6). Moreover, each $\dot{w}_k \in \mathrm{PSO}(V)$ can be represented by a matrix $[w_k] \in \mathrm{SO}(V)$ such that

$$[w_k] \cdot v_{2m} = \begin{cases} v_{2m-k} & 1 \le k < m-1, \\ v_{2m-k-1} & m-1 < k \le 2m-2, \end{cases} \tag{7}$$

and $[w'_{m-1}] \cdot v_{2m} = v_{m+1}$ and $[w_{m-1}] \cdot v_{2m} = v_m$.

3.2. **The dual quadric and its Plücker coordinates.** Consider the homogeneous space $\check{X}_{2m-2} = P\backslash\mathrm{PSO}(V)$. It is canonically identified with the isotropic Grassmannian of lines in V^*, when this Grassmannian is viewed as a homogeneous space via the action of $\mathrm{PSO}(V)$ from the right. Moreover the isotropic Grassmannian of lines is also a $(2m-2)$-dimensional quadric $\check{X}_{2m-2} =: \check{Q}_{2m-2}$, now in $\mathbb{P}(V^*)$. So in this case, the varieties X and $\check{X}$ are (non-canonically) isomorphic. The reason for this isomorphism of varieties is that the group $G^\vee$ is of simply-laced type. However Lie-theoretically we still think of X_{2m-2} and $\check{X}_{2m-2}$ as being very different homogeneous spaces, with $X_{2m-2} = \mathrm{Spin}_{2m}(\mathbb{C})/P^\vee$ and $\check{X}_{2m-2} = P\backslash \mathrm{PSO}_{2m}(\mathbb{C})$.

Definition 3.1 (Plücker coordinates)**.** The Plücker coordinates for $\check{X} = P\backslash\mathrm{PSO}(V)$ are the homogeneous coordinates coming from the embedding of $\check{X}_{2m-2}$ into $\mathbb{P}(V^*)$ as the (right) G-orbit of the line $\mathbb{C}v_{2m}^*$:

$$\check{X}_{2m-2} = P\backslash\mathrm{PSO}(V) \to \mathbb{P}(V^*) : Pg \mapsto (\mathbb{C}v_{2m}^*)\cdot g.$$

We think of the Plücker coordinates as corresponding to the elements of W^P. Let $v_{\omega_i}^-$ (respectively $v_{\omega_i}^+$) denote lowest and highest weight vectors in the highest weight representation V_{ω_i}. Then the Plücker coordinates may be defined by:

$$\begin{aligned} p_0(g) &= \langle v_{\omega_1}^-\cdot[g], v_{\omega_1}^-\rangle \\ p_k(g) &= \langle v_{\omega_1}^-\cdot[g], [w_k]\cdot v_{\omega_1}^-\rangle \text{ for } 1\le k\le 2m-2, \text{ and} \\ p'_{m-1}(g) &= \langle v_{\omega_1}^-\cdot[g], [w'_{m-1}]\cdot v_{\omega_1}^-\rangle, \end{aligned}$$

where $[g]\in\mathrm{SO}(V)$ is any fixed matrix representing $g\in\mathrm{PSO}(V)$. The homogeneous coordinates of Pg are then given by

$$(p_0(g) : \ldots : p_{m-2}(g) : p'_{m-1}(g) : p_{m-1}(g) : p_m(g) : \ldots\ : p_{2m-2}(g)).$$

These are simply the bottom row entries of $[g]$ read from right to left, keeping in mind (7).

We note that as in the case of the odd quadric these Plücker coordinates are to be thought of as B-model incarnations of the Schubert classes of Q_{2m-2}. Namely recall that $H^*(Q_{2m-2},\mathbb{C})$ has a Schubert basis indexed by W^P. We will use the notation $\sigma_i = \sigma_{w_i}$ and $\sigma'_{m-1} = \sigma_{w'_{m-1}}$ and $\sigma_0 = \sigma_e$. As a special case of the geometric Satake correspondence [Lus83, Gin95, MV07] we have that the (defining) projective representation V of $PSO_{2m}(V)$ is identified with the cohomology of Q_{2m-2},

$$V = H^*(Q_{2m-2},\mathbb{C}),$$

and the standard basis v_i agrees with the Schubert basis via $v_{2m} = \sigma_0$ and

$$[w_i]\cdot v_{2m} = \sigma_i, \quad [w'_{m-1}]\cdot v_{2m} = \sigma'_{m-1}. \tag{8}$$

The Schubert classes σ_w are in this way naturally identified with the Plücker coordinates.

3.3. **The superpotential for Q_{2m-2} on a dual quadric.** In this section we state our theorem describing a superpotential for Q_{2m-2} in terms of Plücker coordinates on the dual quadric $\check{X}_{2m-2} = \check{Q}_{2m-2}$. Consider

$$\check{X}^\circ = \check{X}^\circ_{2m-2} := \check{X}\setminus D, \tag{9}$$

where $D := D_0 + D_1 + \ldots + D_{m-2} + D_{m-1} + D'_{m-1}$, the D_i's being given by

$$\begin{aligned} D_0 &:= \{p_0 = 0\}, \\ D_\ell &:= \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-2-\ell+k} = 0 \right\} \text{ for } 1 \le \ell \le m-3, \\ D_{m-2} &:= \{p_{2m-2} = 0\}, \\ D_{m-1} &:= \{p_{m-1} = 0\}, \\ D'_{m-1} &:= \{p'_{m-1} = 0\}. \end{aligned}$$

The divisor D is an anti-canonical divisor in $\check{X}$. For simplicity, we will define

$$\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k}. \tag{10}$$

(For even quadrics, $N = 2m-2$.) Our first result is the following theorem.

Theorem 3.1. *The LG model for $Q_{2m-2} = \mathrm{Spin}_{2m}/P^\vee$ from* [Rie08] *is isomorphic to $W_q : \check{X}^\circ_{2m-2} \to \mathbb{C}$ defined by*

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1} p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q \frac{p_1}{p_{2m-2}}. \tag{11}$$

Before we begin the proof we need to recall the definition of the LG-model from [Rie08].

3.4. **The superpotential for Q_{2m-2} on a Richardson variety.** Following [Rie08] consider the (open) Richardson variety $\mathcal{R} := R_{w_P, w_0} \subset G/B_-$, namely

$$\mathcal{R} := R_{w_P, w_0} = (B_+ \dot{w}_P B_- \cap B_- \dot{w}_0 B_-)/B_-.$$

This Richardson variety $\mathcal{R}$ is irreducible of dimension $2m-2$, and its closure is the Schubert variety $\overline{B_+ \dot{w}_P B_-/B_-}$. Let T^{W_P} be the W_P-fixed part of the maximal torus T. Note that since we are in the setting of Section 3.1 we have that $T^{W_P} \cong \mathbb{C}^*$ with isomorphism given by α_1. The inverse isomorphism is $\omega_1^\vee : \mathbb{C}^* \to T^{W_P}$. We fix a $d \in T^{W_P}$. Then one can define

$$Z_d := B_- \dot{w}_0 \cap U_+ d \dot{w}_P U_- \subset G, \tag{12}$$

and the map

$$\pi_R : Z_d \to \mathcal{R} : g \mapsto g B_-, \tag{13}$$

is an isomorphism from Z_d to the open Richardson variety [Rie08, Section 4.1].

Let q be the non-vanishing coordinate on the 1-dimensional torus T^{W_P} given by $\alpha_1 : T^{W_P} \to \mathbb{C}^*$. The mirror LG model is a regular function on $\mathcal{R}$ depending also on q, and hence a regular function on $\mathcal{R} \times T^{W_P}$. It is defined as follows [Rie08]:

$$\mathcal{F} : (u_1 \dot{w}_P B_-, d) \ \mapsto \ g = u_1 d \dot{w}_P \bar{u}_2 \in Z_d \ \mapsto \ \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2), \tag{14}$$

where $u_1 \in U_+, \bar{u}_2 \in U_-$, and where $\bar{u}_2$ is determined by u_1 and the property that $u_1 d \dot{w}_P \bar{u}_2 \in Z_d$.

The corresponding map from $\mathcal{R}$, when the coordinate q is fixed, is denoted

$$\mathcal{F}_q : \mathcal{R} \to \mathbb{C} : u_1 \dot{w}_P B_- \mapsto \mathcal{F}(u_1 \dot{w}_P B_-, \omega_1^\vee(q)).$$

Remark 1. Note that if $g = u_1 d\dot{w}_P \bar{u}_2 \in Z_d$, then we have a simple identity concerning the Plücker coordinates:

$$(p_0(g) : \ldots : p_{2m-2}(g)) = (p_0(\bar{u}_2) : \ldots : p_{2m-2}(\bar{u}_2)).$$

The remainder of Section 3 will be devoted to proving Theorem 3.1, which now says that there is an isomorphism $\check{X}^\circ_{2m-2} \xrightarrow{\sim} \mathcal{R}$ under which W_q is identified with $\mathcal{F}_q$.

3.5. **Comparison of the superpotentials as rational functions.** $\mathcal{F}_q$ defines a rational function on the Schubert variety $\overline{\mathcal{R}} \in G/B_-$, and W_q defines a rational function on the quadric $\check{X}_{2m-2} = P\backslash G$. As a first step towards the proof of Theorem 3.1 we exhibit a birational isomorphism between these two projective varieties, under which the rational functions $\mathcal{F}_q$ and W_q are identified.

Recall the definition of the variety Z_d isomorphic to $\mathcal{R}$ from (12). We define another embedding of Z_d by

$$\pi_L : Z_d \to P\backslash \mathrm{PSO}(V) : \; g \mapsto Pg. \tag{15}$$

This embedding maps Z_d isomorphically to an open subvariety of a big cell in the homogeneous space $P\backslash \mathrm{PSO}(V)$.

We can now relate $\mathcal{F}_q$ to a rational function in the Plücker coordinates by using π_L from above and π_R from (13). To summarize, these maps are given by

$$\begin{array}{lcccl} (16) & \check{X} = P\backslash G & \xleftarrow{\pi_L} & B_-\dot{w}_0 \cap U_+ d\dot{w}_P U_- & \xrightarrow{\pi_R} \mathcal{R}, \\ (17) & Pg & \leftarrow\!\shortmid & g & \mapsto gB_-. \end{array}$$

Now let $\widetilde{W}_q$ be the rational function on $\check{X}_{2m-2}$ defined by

$$\widetilde{W}_q := (\pi_L)_* \pi_R^* \mathcal{F}_q. \tag{18}$$

In order to compare $\widetilde{W}_q$ with W_q we will express it as a rational function in the Plücker coordinates. We will then prove in Section 3.7 that the locus $\check{X}^\circ_{2m-2}$ is isomorphic to the open Richardson variety $\mathcal{R}$.

Proposition 3.2. *$\widetilde{W}_q$ equals*

$$\frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1} p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q\frac{p_1}{p_{2m-2}}$$

as a rational function on $\check{X}_{2m-2}$.

Along the way we will also prove the following useful proposition.

Proposition 3.3. *$\widetilde{W}_q$ restricted to a particular open torus chart inside $\check{X}_{2m-2}$ has the following Laurent polynomial expression*

$$a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_1}. \tag{19}$$

The torus chart used in Proposition 3.3 will be defined in Section 3.6.

3.6. **Proof of Propositions 3.2 and 3.3.** To prove the results of Section 3.5 we first recall that

$$\pi_R^*\mathcal{F}_q : g = u_1 d\dot{w}_P \bar{u}_2 \in Z_d \ \mapsto \ \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2).$$

Now $\bar{u}_2$ appearing in $u_1 d\dot{w}_P\bar{u}_2 \in Z_d$ can be assumed to lie in $U_- \cap B_+(\dot{w}^P)^{-1}B_+$. This is because we have two birational maps

$$\begin{array}{lllllll} \Psi_1 : & U_- \cap B_+(\dot{w}^P)^{-1}B_+ & \to & P\backslash G & \bar{u}_2 & \mapsto & P\bar{u}_2, \\ \pi_L : & B_-\dot{w}_0 \cap U_+ d\dot{w}^P U_- & \to & P\backslash G & b_-\dot{w}_0 = u_1 d\dot{w}_P\bar{u}_2 & \mapsto & Pb_-\dot{w}_0, \end{array}$$

which compose to give $\Psi_1^{-1}\circ\pi_L : b_-\dot{w}_0 \mapsto \bar{u}_2$. This gives a birational map

$$\Psi_1^{-1}\circ\pi_L : Z_d \to U_- \cap B_+(\dot{w}^P)^{-1}B_+.$$

Now a generic element $\bar{u}_2$ in $U_-\cap B_+(\dot{w}^P)^{-1}B_+$ can be assumed to have a particular factorisation. The smallest representative w^P in W of $[w_0] \in W/W_P$ has the following reduced expression:

$$w^P = s_1 \dots s_{m-2}s_{m-1}s_m s_{m-2}\dots s_1. \tag{20}$$

It follows by an application of Bruhat's lemma [Lus94] that a generic element $\bar{u}_2$ of $U_- \cap B_+(\dot{w}^P)^{-1}B_+$ can be written in the form

$$\bar{u}_2 = y_1(a_1)\dots y_{m-2}(a_{m-2})y_m(d)y_{m-1}(c)y_{m-2}(b_{m-2})\dots y_1(b_1), \tag{21}$$

where $a_i, c, d, b_j \neq 0$. We have the following standard expression for the p_k on factorized elements, which is a simple consequence of their definition.

Lemma 3.4. *Fix $0 \le k \le 2m-2$ an integer. Then if $\bar{u}_2$ is of the form (21) we have*

$$p_k(\bar{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1\dots a_{k-1}(a_k + b_k) & \text{if } 1 \le k \le m-2, \\ a_1\dots a_{m-2}c & \text{if } k = m-1, \\ a_1\dots a_{m-2}cd & \text{if } k = m, \\ a_1\dots a_{m-2}cdb_{m-2}\dots b_{2m-1-k} & \text{otherwise.} \end{cases}$$

and

$$p'_{m-1}(\bar{u}_2) = a_1\dots a_{m-2}d. \qquad \square$$

We will also need the following:

Lemma 3.5. *If $u_1 \in U_+$, $\bar{u}_2 \in U_-$, $u_1 d\dot{w}_P\bar{u}_2 \in Z_d$, and $\bar{u}_2$ can be written as in* (21)*, then we have the following identities:*

$$f_i^*(\bar{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \le i \le m-2, \\ c & \text{if } i = m-1, \\ d & \text{if } i = m. \end{cases} \tag{22}$$

$$e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \le i \le m, \\ q\frac{a_1+b_1}{a_1\dots a_{m-1}cdb_{m-1}\dots b_1} & \text{if } i = 1. \end{cases} \tag{23}$$

Proof. Equation (22) is obtained immediately from the definition of $\bar{u}_2$. For Equation (23), notice that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1}\cdot v_{\omega_i}^-, e_i\cdot v_{\omega_i}^-\rangle}{\langle u_1^{-1}\cdot v_{\omega_i}^-, v_{\omega_i}^-\rangle} \\ &= \frac{\langle d\dot{w}_P\bar{u}_2\cdot v_{\omega_i}^+, e_i\cdot v_{\omega_i}^-\rangle}{\langle d\dot{w}_P\bar{u}_2\cdot v_{\omega_i}^+, v_{\omega_i}^-\rangle}. \end{aligned}$$

Assume $2\le i\le m$. Then $e_i^*(u_1)=0$ if and only if $\langle \bar{u}_2\cdot v_{\omega_i}^+, \dot{w}_P^{-1}e_i\cdot v_{\omega_i}^-\rangle = 0$. Now the vector $w_P^{-1}e_i\cdot v_{\omega_i}^-$ is in the μ-weight space of the i-th fundamental representation, where $\mu = w_P^{-1}s_i(-\omega_i)$. Moreover, $\bar{u}_2\in B_+(\dot{w}^P)^{-1}B_+$, hence $\bar{u}_2\cdot v_{\omega_i}^+$ can have non-zero components only down to the weight space of weight $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. Since $l(w_P^{-1}s_i) > l(w_P^{-1})$ for $2\le i\le m$, this is higher than μ, which proves that $e_i^*(u_1)=0$.

Now assume $i=1$. We have

$$\begin{aligned} e_1^*(u_1) &= \frac{\langle d\dot{w}_P\bar{u}_2\cdot v_{\omega_1}^+, e_1\cdot v_{\omega_1}^-\rangle}{\langle d\dot{w}_P\bar{u}_2\cdot v_{\omega_1}^+, v_{\omega_1}^-\rangle} \\ &= (\omega_1+\alpha_1-\omega_1)(d)\frac{\langle \bar{u}_2\cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1\cdot v_{\omega_1}^-\rangle}{\langle \bar{u}_2\cdot v_{\omega_1}^+, \dot{w}_P v_{\omega_1}^-\rangle} \\ &= q\frac{\langle \bar{u}_2\cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1\cdot v_{\omega_1}^-\rangle}{\langle \bar{u}_2\cdot v_{\omega_1}^+, v_{\omega_1}^-\rangle}. \end{aligned}$$

First look at the denominator. The only way to go from the highest weight vector $v_{\omega_1}^+$ of the first fundamental representation to the lowest weight vector $v_{\omega_1}^-$ is to apply $g\in B_+wB_+$ for $w\ge (w^P)^{-1}$. Since $\bar{u}_2\in B_+(\dot{w}^P)^{-1}B_+$, it follows that we need to take all factors of $\bar{u}_2$, and normalising $v_{\omega_1}^-$ appropriately, we get

$$\langle \bar{u}_2\cdot v_{\omega_1}^+, v_{\omega_1}^-\rangle = a_1\dots a_{m-1}cdb_{m-1}\dots b_1.$$

Finally, we look at the numerator $\langle \bar{u}_2\cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1\cdot v_{\omega_1}^-\rangle$. The vector $\dot{w}_P^{-1}e_1\cdot v_{\omega_1}^-$ has weight

$$\mu' = \dot{w}_P^{-1}s_1(-\omega_1) = \dot{w}_P^{-1}(-\epsilon_2) = \epsilon_2.$$

Write $w_P^{-1}s_1$ as a prefix $w' = s_1s_2\dots s_{m-2}s_ms_{m-1}s_{m-2}\dots s_2$ of $(w^P)^{-1}$. We have $w's_1 = (w^P)^{-1}$, hence the way from $v_{\omega_1}^+$ to $w'\cdot v_{\omega_1}^-$ is through s_1. From the factorization of $\bar{u}_2$ in (21), it follows that $\langle \bar{u}_2\cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1\cdot v_{\omega_1}^-\rangle = a_1+b_1$. □

Using the expression (14) of the superpotential from [Rie08], we immediately deduce from Lemma 3.5 the intermediate expression for $\widetilde{W}_q$ as a Laurent polynomial in Proposition 3.3. Now with the help of Lemma 3.4 and Proposition 3.3, we prove the second expression of $\widetilde{W}_q$.

Proof of Proposition 3.2. From Lemma 3.4, it follows that for $\bar{u}_2$ as in (21)

$$p_{\ell+1}(\bar{u}_2)p_{2m-2-\ell}(\bar{u}_2) = (a_{\ell+1}+b_{\ell+1})(a_1\dots a_\ell)^2a_{\ell+1}\dots a_{m-2}cdb_{m-2}\dots b_{\ell+1}$$

for $0\le\ell\le m-3$. We also get

(24)

$$p_k(\bar{u}_2)p_{2m-2-k}(\bar{u}_2) = \begin{cases} a_1\dots a_{m-2}cdb_{m-2}\dots b_1 & \text{if } k=0\\ (a_1+b_1)a_1\dots a_{m-2}cdb_{m-2}\dots b_2 & \text{if } k=1\\ (a_k+b_k)(a_1\dots a_{k-1})^2a_k\dots a_{m-2}cdb_{m-2}\dots b_{k+1} & \text{if } 2\le k\le m-3.\end{cases}$$

Using (24), we find that most terms in $\delta_\ell(\bar{u}_2) = \sum_{k=0}^{\ell}(-1)^k p_{\ell-k}(\bar{u}_2)p_{2m-2+k-\ell}(\bar{u}_2)$ cancel, and

$$\delta_\ell(\bar{u}_2) = (a_1 \dots a_\ell)^2 a_{\ell+1} \dots a_{m-2} c d b_{m-2} \dots b_{\ell+1}.$$

This proves that

$$\frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell}(\bar{u}_2) = a_{\ell+1} + b_{\ell+1}$$

for $0 \le \ell \le m-3$. Moreover:

$$\frac{p_m}{p_{m-1}}(\bar{u}_2) = \frac{a_1 \dots a_{m-2}cd}{a_1 \dots a_{m-2}c} = d$$

and

$$\frac{p_m}{p'_{m-1}}(\bar{u}_2) = \frac{a_1 \dots a_{m-2}cd}{a_1 \dots a_{m-2}d} = c.$$

For the first and last terms, we obtain

$$\frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1$$

and

$$\frac{p_1}{p_{2m-2}}(\bar{u}_2) = \frac{a_1 + b_1}{a_1 \dots a_{m-1} c d b_{m-1} \dots b_1}$$

as easy consequences of Lemma 3.4. □

3.7. **Isomorphism with the Richardson variety.** To prove Theorem 3.1, it now only remains to prove that $\check{X}^\circ_{2m-2}$ is isomorphic to the open Richardson variety $\mathcal{R}$. Indeed, we have proved that $\mathcal{F}_q$ pulls back to W_q as a rational map on $\check{X}$, where $\alpha_1(d) = q$. Recall from (16) that for fixed $d \in T^{W_P}$, or equivalently, fixed value of the parameter q, we have the following maps

$$\begin{array}{ccccc} \check{X} = P\backslash G & \xleftarrow{\pi_L} & B_-\dot{w}_0 \cap U_+ d\dot{w}_P U_- & \xrightarrow{\pi_R} & \mathcal{R}, \\ Pg & \leftarrow\!\shortmid & g & \mapsto & gB_-. \end{array}$$

given by taking left and right cosets, respectively. Note that $g = b_-\dot{w}_0$ in our previous notation and factorizes as

$$g = u_1 d \dot{w}_P \bar{u}_2.$$

Moreover Ψ_R is an isomorphism, so we have $\Psi := \Psi_L \circ \Psi_R^{-1} : \mathcal{R} \to \check{X}_{2m-2}$. We now prove:

Proposition 3.6. *Ψ defines an isomorphism from $\mathcal{R}$ to $\check{X}^\circ_{2m-2}$.*

The proof uses a presentation of the coordinate ring of the open Richardson variety $\mathcal{R}$ due to [GLS11]. More precisely, the result describes the coordinate ring of the unipotent cell $U_-^P := U_- \cap B_+(\dot{w}^P)^{-1}B_+$, which is isomorphic to $\mathcal{R}$ by the standard map $g \mapsto gB_-$. In the particular case of $\check{X}^\circ_{2m-2}$, it can be stated as follows.

Let us define the generalized minors involved in this presentation. Let G^{sc} be the simply-connected covering group of $G = \mathrm{PSO}(V)$ and with Borel subgroup B_-^{sc} and unipotent radical U_-^{sc} projecting to B_- and U_- in G. Here $G^{sc} = \mathrm{Spin}(V)$. Since $U_-^{sc} \cong U_-$ via this projection, we may use representations of G^{sc} to define minors of elements of U_-. For $u \in U_-$ we denote by u^{sc} its lift to U_-^{sc}.

Definition 3.2. Let $w \in W$ and ω_j be a fundamental weight of G^{sc}. Let V_{ω_j} be the irreducible representation of G^{sc} with highest weight ω_j and $v^+_{\omega_j}$ be a fixed highest weight vector. Define for any $u \in U_-$:

$$\Delta_{\omega_j, w\cdot\omega_j}(u) = \langle u^{sc}\dot{w}\cdot v^+_{\omega_j}, v^+_{\omega_j}\rangle.$$

Theorem 3.7 ([GLS11, Section 8])**.** *Let $s_{i_1}\dots s_{i_{2m-2}} = s_1\dots s_{m-2}s_m s_{m-1}s_{m-2}\dots s_1$ be the reduced expression for $(\dot{w}^P)^{-1}$ coming from (20). The coordinate ring of the unipotent cell $U^P_- := U_-\cap B_+(\dot{w}^P)^{-1}B_+$ inside PSO_{2m} is*

$$\mathbb{C}\left[U^P_-\right] = \mathbb{C}\left[\Delta_{\omega_{i_r},(\dot{w}^P)^{-1}_{\leq r}\cdot\omega_{i_r}}, \Delta^{-1}_{\omega_{2m-2-s},(\dot{w}^P)^{-1}_{\leq s}\cdot\omega_{2m-2-s}}\right]$$

where

- $1 \leq r \leq 2m-2$ *;* $m-1 \leq s \leq 2m-2$ *;*
- $(\dot{w}^P)^{-1}_{\leq r} := s_{i_1}\dots s_{i_r}$.

If $j < m$ then $\Delta_{\omega_j, w\cdot\omega_j}(u)$ is a regular minor of the matrix $u^{sc}\in \mathrm{SO}_{2m}$. We denote the minor of u^{sc} with row set $\{i_1,\dots,i_p\}$ and column set $\{j_1,\dots,j_p\}$ by $D^{i_1,\dots,i_p}_{j_1,\dots,j_p}(u)$. We now reformulate Theorem 3.7 as follows.

Corollary 3.8. *The coordinate ring $\mathbb{C}\left[U^P_-\right]$ is generated by the minors*

$$D^{2,\dots,r,r+1}_{1,2,\dots,r}, \quad 1\leq r\leq m-2;$$

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}, \quad m+1\leq s\leq 2m-3, \text{ and } D^{2m}_1;$$

the functions

$$\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m]} \text{ and } \Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m]},$$

which are Pfaffians; the inverses of minors

$$\left(D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}\right)^{-1}, \quad m+1\leq s\leq 2m-3, \text{ and } \left(D^{2m}_1\right)^{-1};$$

and the inverses of Pfaffians

$$\Delta^{-1}_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m]} \text{ and } \Delta^{-1}_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m]}.$$

A consequence of Corollary 3.8 is that the minors $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}$ and D^{2m}_1 and the Pfaffians $\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m)]}$ and $\Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m)]}$ do not vanish for any $\bar{u}_2\in U^P_-$. We will use that fact to prove that the map Ψ lands in fact in $\check{X}^\circ$. We need two lemmas.

Lemma 3.9. *We have the following equalities of generalised minors and Plücker coordinates:*

$$\begin{aligned} p_{2m-2} &= D^{2m}_1 \\ p_{m-1} &= \Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m)]} \\ p'_{m-1} &= \Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m)]} \end{aligned}$$

Proof. The lemma follows immediately from the definitions of Plücker coordinates and of generalised minors. □

Recall the definition of the elements $\bar{u}_2$, which have a factorisation given by (21).

Lemma 3.10. *The minor* $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)$ *is equal to*

$$\delta_{s-m}(\bar u_2)=\sum_{k=s}^{m}(-1)^{s-k}p_{k-m}(\bar u_2)p_{3m-2-k}(\bar u_2).$$

for $m+1\le s\le 2m-3$.

Proof. Developing $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)$ with respect to the $(2m-1-s)$-th column, we see that it is equal to

$$D^{m+1}_{2m-1-s}(\bar u_2)\times D^{2,\dots,2m-1-s}_{1,2,\dots,2m-2-s}(\bar u_2)-D^{2,\dots,2m-2-s,m+1}_{1,\dots,2m-2-s}(\bar u_2).$$

Since $\bar u_2$ is orthogonal for Q, we have

$$D^{2,\dots,2m-1-s}_{1,2,\dots,2m-2-s}(\bar u_2)=D^{1,\dots,s+1,2m}_{1,\dots,s+2}(\bar u_2),$$

and since $\bar u_2$ is in U_-,

$$D^{1,\dots,s+1,2m}_{1,\dots,s+2}(\bar u_2)=D^{2m}_{s+2}(\bar u_2)=p_{2m-2-s}(\bar u_2).$$

Finally

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)=D^{m+1}_{2m-1-s}(\bar u_2)p_{2m-2-s}(\bar u_2)-D^{2,\dots,2m-2-s,m+1}_{1,\dots,2m-2-s}(\bar u_2),$$

hence

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)=\sum_{k=s}^{2m-2}(-1)^{k-s}D^{m+1}_{2m-1-s}(\bar u_2)p_{2m-2-s}(\bar u_2).$$

We also have $D^{m+1}_{2m-1-s}(\bar u_2)=db_{2m-2}\dots b_{2m-1-s}$ for $m+1\le s\le 2m-2$. Indeed, by definition

$$D^{m+1}_{2m-1-s}(\bar u_2)=\langle v^*_{m+1}\cdot\bar u_2,v_{2m-1-s}\rangle=db_{2m-2}\dots b_{2m-1-s}.$$

Hence

$$\begin{aligned}D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)&=\sum_{k=s}^{2m-2}(-1)^{k-s}db_{2m-2}\dots b_{2m-1-s}p_{2m-2-s}\\&=\sum_{k=s}^{m}(-1)^{s-k}p_{k-m}(\bar u_2)p_{3m-2-k}(\bar u_2).\end{aligned}$$ □

We can now prove that the image of Ψ is contained in $\check X^\circ_{2m-2}$. Indeed, if $\bar u_2\in U^P_-$, then the minors $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar u_2)$ and $D^{2m}_1(\bar u_2)$ and the Pfaffians $\Delta_{\omega_m,\frac12[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m)]}(\bar u_2)$ and $\Delta_{\omega_{m-1},\frac12[-\epsilon_1+\epsilon_2+\dots+\epsilon_m)]}(\bar u_2)$ do not vanish. Since we have proved in Lemmas 3.9 and 3.10 that those correspond precisely the divisors involved in defining $\check X^\circ_{2m-2}$, it follows that $P\bar{u_2}\in\check X^\circ_{2m-2}$. We may now prove the isomorphism between $\mathcal R$ and $\check X^\circ_{2m-2}$.

Proof of Proposition 3.6. The map $\Psi:U^P_-\to\check X^\circ_{2m-2}$ is an algebraic map between affine varieties, which induces a pullback map $\mathbb C[\check X^\circ_{2m-2}]\to\mathbb C[U^P_-]$ between their coordinate rings. Injectivity of the pullback map is a simple consequence of the fact that the map $U^P_-\to\check X_{2m-2}$ is dominant.

We now prove that $\mathbb C[\check X^\circ_{2m-2}]\to\mathbb C[U^P_-]$ is surjective. To do this, it is enough to find a pre-image for each of the functions (minors, Pfaffians, inverses of minors, inverses of Pfaffians) generating $\mathbb C[U^P_-]$.

We have already seen that the inverses of minors and Pfaffians correspond to the inverses of denominators of W, which are by definition well-defined on $\check X^\circ_{2m-2}$.

Let us now consider the minors $D^{2,\dots,r,r+1}_{1,2,\dots,r}(\bar u_2)$ for $1\le r\le m-2$ and

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s} \text{ for } m+1\le s\le 2m-3.$$

In Lemma 3.10, we proved

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}=\delta_{s-m}.$$

As in Lemma 3.10, we have:

$$D^{2,\dots,r+1}_{1,\dots,r}=D^{1,\dots,2m-1-r}_{1,\dots,2m-r}=D^{2m}_{2m-r}=p_r.$$

Finally, $D^{2m}_1=p_{2m}$. So these minors are all well-defined functions on $\check X^\circ_{2m-2}$.

Let us finally consider the Pfaffians

$$\Delta_{\omega_m,\frac12[-\epsilon_1+\epsilon_2+\cdots+\epsilon_{m-1}-\epsilon_m)]} \text{ and } \Delta_{\omega_{m-1},\frac12[-\epsilon_1+\epsilon_2+\cdots+\epsilon_m)]}.$$

We have seen in Lemma 3.9 that they are in fact the Plücker coordinates p'_{m-1} and p_{m-1}. Those being well-defined functions on $\check X^\circ_{2m-2}$, this concludes the proof. $\square$

3.8. **Comparison with the Hori-Vafa model for even quadrics.** Here we check that just like for odd quadrics, once restricted to the subset $T_1:=\{x\in \check X^\circ \mid p_i(x)\neq 0 \text{ for all } 0\le i\le m-3\}$, our LG model is isomorphic to the Hori-Vafa model. Let us consider the change of coordinates:

$$Y_i=\begin{cases}\frac{p_i}{p_{i-1}} & \text{for } 1\le i\le m-2\ ;\\ \frac{p_{2m-3-i}\delta_{2m-5-i}}{p_{2m-4-i}\delta_{2m-4-i}} & \text{for } m-1\le i\le 2m-5\ ;\\ \frac{p_m}{p_{m-1}} & \text{for } i=2m-4\ ;\\ \frac{p_m}{p'_{m-1}} & \text{for } i=2m-3\ ;\\ q\frac{\delta_{m-3}}{\delta_{m-2}} & \text{for } i=2m-2.\end{cases}$$

Again, an easy calculation shows that it transforms our LG model (11) into the Hori-Vafa model (1) for even quadrics.

4. The A-model connection

Our expression for the LG-model W in terms of homogeneous coordinates coming from $\check X^\circ\subset\mathbb{P}(H^*(X,\mathbb{C})^*)$ makes it possible to compare the (small) Dubrovin connection on the A side and the Gauss-Manin connection on the B side. We recall the relevant definitions on the A-side.

Let $X=Q_N$. Consider $H^*(X,\mathbb{C}[\hbar,q])$ as a space of sections on a trivial bundle with fiber $H^*(X,\mathbb{C})$. Let Gr be the operator on sections defined on the fibres as the 'grading operator' $H^*(X,\mathbb{C})\to H^*(X,\mathbb{C})$ which multiplies $\sigma\in H^{2k}(X,\mathbb{C})$ by k. We define the Dubrovin connection by

$$ {}^A\nabla_{q\partial_q}S \;:=\; q\frac{\partial S}{\partial q}+\frac{1}{\hbar}\sigma_1\star_q S \tag{25}$$

$$ {}^A\nabla_{\hbar\partial_\hbar}S \;:=\; \hbar\frac{\partial S}{\partial \hbar}-\frac{1}{\hbar}c_1(TX)\star_q S+\mathrm{Gr}(S), \tag{26}$$

following the conventions of Iritani [Iri09], where $\star_q$ denotes the quantum cup product in the quantum cohomology. This defines a meromorphic flat connection, see also [Dub96, Giv96, CK99]. Moreover it therefore turns $H^*(X,\mathbb{C}[\hbar^{\pm1},q^{\pm1}])$ into a D-module sometimes called the *quantum cohomology D-module* for $\mathbb{C}[\hbar^{\pm1},q^{\pm1}]\langle\partial_\hbar,\partial_q\rangle$. This is the connection or D-module we consider on the A-model side.

4.1. **The dual Dubrovin connection and the J-function.** In this section we define Givental's J-function and the quantum differential operators. Consider the dual connection to ${}^A\nabla$ with respect to the pairing

$$\langle \sigma, \tau \rangle = \frac{1}{(2\pi i\hbar)^N} \int_X \sigma \cup \tau.$$

Here $\sigma \cup \tau$ is the usual cup product of σ and τ, which we will subsequently also denote by $\sigma\tau$. Explicitly, the dual connection is given by the formulas:

$$(27) \qquad {}^A\nabla^\vee_{q\partial_q} S := q\frac{\partial S}{\partial q} - \frac{1}{\hbar}\sigma_1 \star_q S$$

$$(28) \qquad {}^A\nabla^\vee_{\hbar\partial_\hbar} S := \hbar\frac{\partial S}{\partial \hbar} + \frac{1}{\hbar}c_1(TX) \star_q S + \mathrm{Gr}(S).$$

For the purposes of the J-function we ignore the ${}^A\nabla^\vee_{\hbar\partial_\hbar}$ part of the covariant derivative and consider ${}^A\nabla^\vee_{q\partial_q}$ as a family of connections (in the parameter $\hbar$). Formal flat sections indexed by the cohomology basis were written down by Givental [Giv96] in terms of descendent Gromov-Witten invariants. We denote these sections by $S_0, \dots, S_{2m-1}$ in the case of Q_{2m-1}, and by $S_0, \dots, S_{m-1}, S'_{m-1}, S_m, \cdots, S_{2m-2}$ for Q_{2m-2}, in keeping with the notation from (8) for Schubert classes. See [CK99, (10.14)] for a precise definition of the sections S_i.

We also consider the *quantum differential operators*, see for example [CK99, Definition 10.3.2], as the differential operators P which are formal power series in

$$\hbar q\partial_q, q, \hbar$$

and which annihilate the top coefficients of Givental's flat sections, for example, $P \cdot \langle S_j, \sigma_0 \rangle = 0$ for the flat section S_j.

Definition 4.1. We define Givental's J-function in our setting as

$$J = (2\pi i\hbar)^N \sum \langle S_j, \sigma_0 \rangle \sigma_{PD(j)}$$

where the sum is over all the Schubert classes, including σ'_{m-1} in the even case, and where $\sigma_{PD(j)}$ stands for the Poincaré dual cohomology class to σ_j.

4.2. **The hypergeometric series of Q_N.** A special role is played by the term $\langle S_N, \sigma_0 \rangle$, appearing as the coefficient of the fundamental class in the definition of J-function. This term is special in that it is a power series in $\mathbf{q} = \hbar^{-N} q$. We define it as in [BCFKvS98]:

Definition 4.2. The *hypergeometric series* A_X of X is the unique power series of the form $A_X = 1 + \sum_{k=1}^\infty a_k q^k$, for which $P(q\partial_q, q, 1)A_X = 0$ for all quantum differential operators $P(\hbar q\partial_q, q, \hbar)$ specialized to $\hbar = 1$.

The hypergeometric series of the quadric may be obtained by setting $\hbar$ to 1 in $\langle S_N, \sigma_0 \rangle$. Or in our example $\langle S_N, 1 \rangle = A_X(\hbar^{-N} q)$.

The hypergeometric series of the quadric counts certain 1-pointed Gromov-Witten invariants. Let

$$(29) \qquad I_d(\psi_1^{a_1}\gamma_1, \dots, \psi_r^{a_r}\gamma_r)$$

denote the degree d descendant Gromov-Witten invariant associated to the cohomology classes $\gamma_1, \dots, \gamma_r$, where the ψ-class ψ_i denotes the first Chern class of the ith cotangent bundle of the moduli space of degree d genus 0 stable maps with r marked points. [CK99,]. Let ψ denote ψ_1.

Indeed, if we write

$$J^{Q_N} = \sum J_i^{Q_N} \sigma_{\mathrm{PD}(i)},$$

we have

$$\begin{aligned} J_N^{Q_N} &= 1 + \sum_{d=1}^{\infty} q^d I_d \left(\frac{\sigma_N e^{\frac{\ln(q)\sigma_1}{\hbar}}}{\hbar - \psi}, \sigma_0 \right) \\ &= 1 + \sum_{d=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^d}{\hbar} I_d \left(\sigma_N \left(\frac{\ln(q)\sigma_1}{\hbar} \right)^j \frac{1}{j!} \left(\frac{\psi}{\hbar} \right)^k, \sigma_0 \right) \end{aligned}$$

The cup-product $\sigma_N \left(\frac{\ln(q)\sigma_1}{\hbar} \right)^j$ is nonzero if and only if $j = 0$. Therefore we have

$$J_N^{Q_N} = 1 + \sum_{d=1}^{\infty} \sum_{k=0}^{\infty} \frac{q^d}{\hbar} I_d \left(\sigma_N \left(\frac{\psi}{\hbar} \right)^k, \sigma_0 \right).$$

Now we use the fact that the dimension of the moduli space of stable maps $\overline{\mathcal{M}}_{0,2}(Q_N, d)$ is $(d+1)N - 1$, which gives

$$J_N^{Q_N} = 1 + \sum_{d=1}^{\infty} \frac{q^d}{\hbar} I_d \left(\sigma_N \left(\frac{\psi}{\hbar} \right)^{dN-1}, \sigma_0 \right).$$

Next we use the fundamental class axiom to get

$$J_N^{Q_N} = 1 + \sum_{d=1}^{\infty} \left(\frac{q}{\hbar^N} \right)^d I_d \left(\sigma_N \psi^{dN-2} \right).$$

When we set $\hbar = 1$, this is exactly the hypergeometric series of the quadric, so we obtain the following geometric interpretation of $A_X(q)$:

$$a_d = I_d \left(\sigma_N \psi^{dN-2} \right). \tag{30}$$

5. The B-model connection

For the B-model, recall that $\check{X}^\circ$ is the complement of an anti-canonical divisor in $\check{X}$. Therefore there is an up to scalar unique non-vanishing holomorphic n-form on $\check{X}^\circ$ which we will fix and call ω. Let $\Omega^k(\check{X}^\circ)$ denote the space of all holomorphic k-forms.

Definition 5.1. Define the $\mathbb{C}[\hbar, q]$-module

$$G_0^{W_q} := \Omega^n(\check{X}^\circ)[\hbar, q] / (\hbar d + dW_q \wedge -) \Omega^{n-1}(\check{X}^\circ)[\hbar, q].$$

It has a meromorphic (Gauss-Manin) connection given by

$$^B\nabla_{q\partial_q}[\alpha] = q \frac{\partial}{\partial q}[\alpha] - \frac{1}{\hbar} \left[\frac{\partial W_q}{\partial q} \alpha \right], \tag{31}$$

$$^B\nabla_{\hbar\partial_\hbar}[\alpha] = \hbar \frac{\partial}{\partial \hbar}[\alpha] + \frac{1}{\hbar}[W_q \alpha]. \tag{32}$$

Let $G^{W_q} = G_0^{W_q} \otimes_{\mathbb{C}[\hbar,q]} \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$. We view G^{W_q} as a $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$-module with $q\partial_q$ acting by $^B\nabla_{q\partial_q}$ and $\hbar\partial_\hbar$ acting by $^B\nabla_{\hbar\partial_\hbar}$.

5.1. **The case of odd-dimensional quadrics.** For odd-dimensional quadrics, an isomorphism between the connections (or D-modules) on the two sides has been proved by Gorbounov and Smirnov in [GS13], for their LG model constructed there. And the two first-named authors have established in [PR13a] that the Gorbounov-Smirnov LG model is isomorphic to the one obtained from the general construction of [Rie08] for homogeneous spaces. Hence we obtain the following result.

Theorem 5.1. *The map*

$$\begin{array}{ccc} H^*(Q_{2m-1},\mathbb{C}) & \to & H^{2m-1}_{\mathrm{dR}}(\check{X}^\circ_{2m-1}, d+dW_q\wedge -) \\ \sigma_i & \mapsto & [p_i\omega] \end{array}$$

defines an isomorphism of bundles with connection between the A-model and the B-model for $X = Q_{2m-1}$.

5.2. **The case of even-dimensional quadrics.** We need to prove a similar result to Theorem 5.1 for even quadrics Q_{2m-2}. To do this we will use the cluster algebra structure on our mirror $\check{X}^\circ_{2m-2}$ introduced in Section 3.7. We want to prove the following theorem.

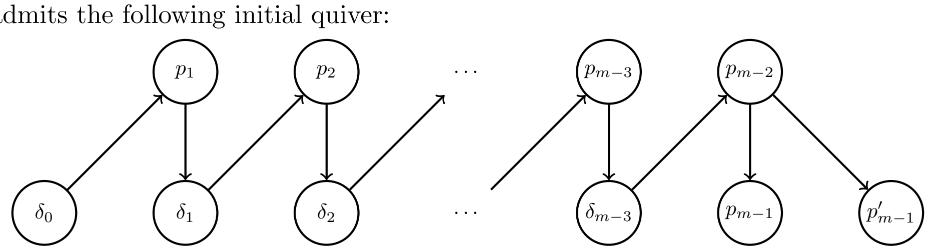
Theorem 5.2. *For $X = Q_{2m-2}$ with its mirror LG-model $(\check{X}^\circ_{2m-2}, W)$ from Theorem 3.1, the map*

$$\begin{array}{ccc} H^*(Q_{2m-2},\mathbb{C}[\hbar^{\pm1},q^{\pm1}]) & \to & G^{W_q} \\ \sigma_i & \mapsto & [p_i\omega] \end{array}$$

defines an injective homomorphism of D-modules. Here the D-module on the left hand side is the one defined in terms of the (small) Dubrovin connection in the A-model, and the D-module on the right hand side is the one defined via the B-model Gauss-Manin connection.

It would be interesting to see if the proof of cohomological tameness of the superpotential in the odd quadrics case given in [GS13] with Nemethi and Sabbah could be adapted to give a proof of the same property in the even case. This would imply that the injective homomorphism in Theorem 5.2 is an isomorphism.

We recall from Section 3.7 and [GLS11] that the cluster structure on $\mathbb{C}[\check{X}^\circ_{2m-2}]$ admits the following initial quiver:



Here the initial cluster variables correspond to the vertices in the top row of the quiver, while the frozen variables (or coefficients) correspond to the vertices in the bottom row. Recall that the p_i's are Plücker coordinates, and the δ_i's are defined as in (10). Hence we see that the coordinate ring of $\check{X}^0_{2m-2}$ has a cluster structure of type A_1^{m-2}. In particular, it is of finite type, and there are 2^{m-2} different clusters, consisting of

- the cluster variables $q_1,\ldots,q_{m-2}$, where $q_i \in \{p_i, p_{2m-2-i}\}$;
- the frozen variables (or coefficients) $\delta_0,\ldots,\delta_{m-3}$, p_{m-1} and p'_{m-1}.

The exchange relations are

$$p_i p_{2m-2-i} = \begin{cases} \delta_{i-1} + \delta_i & \text{for } 1 \leq i \leq m-3 \ ; \\ \delta_{m-3} + p_{m-1}p'_{m-1} & \text{for } i = m-2. \end{cases} \tag{33}$$

To prove Theorem 5.2, consider the following identities in $QH^*(Q_{2m-2}, \mathbb{C})$, which are a special case of results in [FW04]:

$$\sigma_1 \star_q \sigma_i = \begin{cases} \sigma_{i+1} & \text{for } 0 \leq i \leq m-3 \text{ or } m-1 \leq i \leq 2m-4 \ ; \\ \sigma_{m-1} + \sigma'_{m-1} & \text{for } i = m-2 \ ; \\ \sigma_{2m-2} + q\sigma_0 & \text{for } i = 2m-3 \ ; \\ q\sigma_1 & \text{for } i = 2m-2, \end{cases} \tag{34}$$

and

$$\sigma_1 \star_q \sigma'_{m-1} = \sigma_m. \tag{35}$$

We need to prove that there are similar identities on the B side:

(36)

$$\left[q \frac{\partial W_q}{\partial q} p_i \omega \right] = \begin{cases} [p_{i+1}\omega] & \text{for } 0 \leq i \leq m-3 \text{ or } m-1 \leq i \leq 2m-4 \ ; \\ [(p_{m-1} + p'_{m-1})\omega] & \text{for } i = m-2 \ ; \\ [(p_{2m-2} + q)\omega] & \text{for } i = 2m-3 \ ; \\ [qp_1\omega] & \text{for } i = 2m-2, \end{cases}$$

and

$$\left[q \frac{\partial W_q}{\partial q} p'_{m-1} \omega \right] = [p_m \omega]. \tag{37}$$

The proof of these identities in G^{W_q} proceeds by constructing closed $(2m-3)$-forms ν_i and ν'_{m-1} such that the relation corresponding to p_i will follow from

$$[dW_q \wedge \nu_i] = [(\hbar d + dW_q \wedge -)\nu_i] = 0$$

and similarly for p'_{m-1}.

Concretely, we will pick a cluster $\mathcal{C}$ containing a particular Plücker coordinate, say p_i, and use the following Ansatz for constructing ν_i. We define a vector field,

$$\xi_i = p_i \left(\sum_{c \in \mathcal{C} \setminus \{p_i\}} m_c c \partial_c \right) \tag{38}$$

and define an associated $(2m-3)$-form by insertion $\nu_i = \iota_{\xi_i}\omega$, and analogously for $\nu'_{m-1} = \iota_{\xi'_{m-1}}\omega$.

To prove those identities, we will work in two cluster charts:

- the chart $\mathcal{C}_1$ corresponding to the initial cluster
$$\{p_1, \dots, p_{m-2}, \delta_1, \dots, \delta_{m-3}, p_{m-1}, p'_{m-1}\};$$
- the chart $\mathcal{C}_2$ corresponding to the cluster
$$\{p_{2m-3}, \dots, p_m, \delta_1, \dots, \delta_{m-3}, p_{m-1}, p'_{m-1}\}.$$

Let us first start with $\mathcal{C}_1$ and express W_q in this chart using the exchange relations (33), having set $p_0 = 1$:

$$W_q = p_1 + \sum_{\ell=1}^{m-3} \left(\frac{p_{\ell+1}\delta_{\ell-1}}{p_\ell \delta_\ell} + \frac{p_{\ell+1}}{p_\ell} \right) + \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p_{m-1}}{p_{m-2}} + \frac{p'_{m-1}}{p_{m-2}} + q\frac{p_1}{\delta_0}. \tag{39}$$

The partial derivatives of W_q are:

$$p_1 \frac{\partial W_q}{\partial p_1} = p_1 - \frac{p_2\delta_0}{p_1\delta_1} - \frac{p_2}{p_1} + q\frac{p_1}{\delta_0} \tag{40}$$

$$p_i \frac{\partial W_q}{\partial p_i} = \frac{p_i\delta_{i-2}}{p_{i-1}\delta_{i-1}} + \frac{p_i}{p_{i-1}} - \frac{p_{i+1}\delta_{i-1}}{p_i\delta_i} - \frac{p_{i+1}}{p_i} \text{ for } 2 \le i \le m-3 \tag{41}$$

$$p_{m-2} \frac{\partial W_q}{\partial p_{m-2}} = \frac{p_{m-2}\delta_{m-4}}{p_{m-3}\delta_{m-3}} + \frac{p_{m-2}}{p_{m-3}} - \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} - \frac{p_{m-1}}{p_{m-2}} - \frac{p'_{m-1}}{p_{m-2}} \tag{42}$$

$$\delta_0 \frac{\partial W_q}{\partial \delta_0} = \frac{p_2\delta_0}{p_1\delta_1} - q\frac{p_1}{\delta_0} \tag{43}$$

$$\delta_i \frac{\partial W_q}{\partial \delta_i} = -\frac{p_{i+1}\delta_{i-1}}{p_i\delta_i} + \frac{p_{i+2}\delta_i}{p_{i+1}\delta_{i+1}} \text{ for } 1 \le i \le m-4 \tag{44}$$

$$\delta_{m-3} \frac{\partial W_q}{\partial \delta_{m-3}} = -\frac{p_{m-2}\delta_{m-4}}{p_{m-3}\delta_{m-3}} + \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} \tag{45}$$

$$p_{m-1} \frac{\partial W_q}{\partial p_{m-1}} = -\frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p_{m-1}}{p_{m-2}} \tag{46}$$

$$p'_{m-1} \frac{\partial W_q}{\partial p'_{m-1}} = -\frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p'_{m-1}}{p_{m-2}}. \tag{47}$$

Hence

$$\sum_{j=1}^{m-1} p_j \frac{\partial W_q}{\partial p_j} + p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + 2\sum_{j=0}^{m-3} \delta_j \frac{\partial W_q}{\partial \delta_j} = p_1 - q\frac{p_1}{\delta_0} \tag{48}$$

$$\sum_{j=i}^{m-1} p_j \frac{\partial W_q}{\partial p_j} + p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + \sum_{j=0}^{m-3} \delta_j \frac{\partial W_q}{\partial \delta_j} + \sum_{j=i-1}^{m-3} \delta_j \frac{\partial W_q}{\partial \delta_j} = \frac{p_i}{p_{i-1}} - q\frac{p_1}{\delta_0} \quad \text{for } 2 \le i \le m-2 \tag{49}$$

$$p_{m-1}\frac{\partial W_q}{\partial p_{m-1}} + p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + \sum_{j=0}^{m-3} \delta_j \frac{\partial W_q}{\partial \delta_j} = -q\frac{p_1}{\delta_0} + \frac{p_{m-1} + p'_{m-1}}{p_{m-2}}, \tag{50}$$

which is equivalent to the identity (37) for $0 \le i \le m-2$.

To prove the remaining identities, we use the cluster chart $\mathcal{C}_2$. In this chart, W_q takes the following form:

$$W_q = \frac{\delta_0}{p_{2m-3}} + \frac{\delta_1}{p_{2m-3}} + \sum_{\ell=1}^{m-4}\left(\frac{p_{2m-2-\ell}}{p_{2m-3-\ell}} + \frac{p_{2m-2-\ell}\delta_{\ell+1}}{p_{2m-3-\ell}\delta_\ell}\right) + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + \frac{p_{m+1}}{p_m} + \frac{p_{m-1}p'_{m-1}p_{m+1}}{p_m\delta_{m-3}} + \frac{q}{p_{2m-3}} + q\frac{\delta_1}{p_{2m-3}\delta_0}. \tag{51}$$

Working out the partial derivatives of W_q as before, we get

$$p'_{m-1}\frac{\partial W_q}{\partial p'_{m-1}} + \sum_{j=m}^{2m-3} p_j\frac{\partial W_q}{\partial p_j} + \sum_{j=0}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j} = -\frac{p_m}{p_{m-1}} + \frac{q}{p_{2m-3}} + q\frac{\delta_1}{p_{2m-3}\delta_0} \tag{52}$$

$$p_{m-1}\frac{\partial W_q}{\partial p_{m-1}} + \sum_{j=m}^{2m-3} p_j\frac{\partial W_q}{\partial p_j} + \sum_{j=0}^{m-3}\delta_j\frac{\partial W_q}{\partial \delta_j} = -\frac{p_m}{p'_{m-1}} + \frac{q}{p_{2m-3}} + q\frac{\delta_1}{p_{2m-3}\delta_0} \tag{53}$$

$$\sum_{j=i}^{2m-3} p_j\frac{\partial W_q}{\partial p_j} + \sum_{j=0}^{2m-2-i}\delta_j\frac{\partial W_q}{\partial \delta_j} = \frac{p_i}{p_{i-1}} - \frac{q}{p_{2m-3}} - q\frac{\delta_1}{p_{2m-3}\delta_0} \quad \text{for } m+1 \le i \le 2m-4 \tag{54}$$

$$\delta_0\frac{\partial W_q}{\partial \delta_0} = \frac{\delta_0}{p_{2m-3}} - q\frac{\delta_1}{p_{2m-3}\delta_0} \tag{55}$$

$$0 = q\frac{\delta_0+\delta_1}{p_{2m-3}\delta_0}\delta_0 - q\frac{\delta_0+\delta_1}{p_{2m-3}} \tag{56}$$

This gives us the identities (36) for $m-1 \le i \le 2m-2$, as well as the identity (37).

6. The hypergeometric series of Q_N

Recall from Section 4 the definition of the quantum differential operators and the hypergeometric series of Q_N. We will denote by $A_N(q)$ the hypergeometric series of the quadric Q_N.

The main result of this section is the following.

Theorem 6.1. *The hypergeometric series of the quadric Q_N is*

$$A_N(q) = 1 + \sum_{k\ge 1}\frac{1}{(k!)^N}\binom{2k}{k}q^k.$$

The theorem allows us to deduce a formula for some 1-pointed Gromov-Witten invariants, using Theorem 6.1 and Equation (30).

Corollary 6.2. *The Gromov-Witten invariant $I_d(\sigma_N\psi^{Nd-2})$ satisfies*

$$I_d(\sigma_N\psi^{Nd-2}) = \frac{1}{(d!)^N}\binom{2d}{d}.$$

We give an A-model and a B-model proof of Theorem 6.1.

B-model proof. Our B-model proof works by calculating the constant term of the exponential $\exp(\frac{1}{\hbar}W_q)$ of the superpotential, and showing that it equals

$$A_N(q,\hbar) = \sum_{k\ge 0}\frac{1}{\hbar^{kN}}\frac{1}{(k!)^N}\binom{2k}{k}q^k.$$

Let us consider the case that $N = 2m+1$. In this case recall from (5) that the superpotential is

$$W_q = a_1 + \cdots + a_m + c + b_m + \cdots + b_1 + \frac{q}{a_2 \dots a_m c b_m \dots b_1} + \frac{q}{a_1 \dots a_m c b_m \dots b_2}.$$

To compute the constant term of $\exp(\frac{1}{\hbar} W_q)$, we consider $1 + \frac{1}{\hbar} W_q + \frac{1}{\hbar^2}\frac{W_q^2}{2!} + \frac{1}{\hbar^3}\frac{W_q^3}{3!} + \dots$, and we pick out from each $\frac{W_q^i}{\hbar^i i!}$ any term which is a monomial in q alone, i.e. any term λq^j where $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$. Here we just need to look at each $\frac{W_q^{kN}}{\hbar^{kN}(kN)!}$ for $k = 0, 1, 2, \dots$, because the expansion of $\frac{W_q^i}{\hbar^i i!}$ for i not a multiple of N will contain no terms of the form λq^j for $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$.

Now let us analyze $\frac{W_q^{kN}}{\hbar^{kN}(kN)!}$ for $N = 2m+1$. A (Laurent) monomial in the expansion of $W_q^{k(2m+1)}$ is obtained by choosing one term in each of the $k(2m+1)$ factors. Some of the monomials in the expansion will be pure in the variable q alone – in which case they will equal q^k. We need to show that the number of such monomials divided by $(k(2m+1))!$ equals $\binom{2k}{k}/(k!)^{k(2m+1)}$. To count the number of such monomials, we need to pick one term in each of the $k(2m+1)$ factors so that we:

- choose i terms which are $\frac{q}{a_2 \dots a_m c b_1 \dots b_m}$ for some $0 \le i \le k$;
- choose $k-i$ terms which are $\frac{q}{a_1 \dots a_m c b_2 \dots b_m}$;
- choose k terms which are c;
- choose i terms which are b_1;
- choose $k-i$ terms which are a_1;
- for each j such that $2 \le j \le m$, choose k terms which are a_j;
- for each j such that $2 \le j \le m$, choose k terms which are b_j.

The number of ways to do this is the sum of multinomial coefficients

$$\sum_{i=0}^{k} \binom{k(2m+1)}{i, k-i, k, i, k-i, k \dots k},$$

where the number of k's in the string $k \dots k$ above is $2m-2$. Recall here that if $q_1 + q_2 + \cdots + q_r = p$ are positive integers, then the corresponding multinomial coefficient is defined by

$$\binom{p}{q_1, q_2, \dots, q_r} = \frac{p!}{q_1! q_2! \dots q_r!}.$$

So the coefficient of q^k in

$$\frac{W_q^{k(2m+1)}}{(k(2m+1))!}$$

equals

$$\begin{aligned}
\frac{1}{(k(2m+1))!} \sum_{i=0}^{k} \binom{k(2m+1)}{k, \dots, k, i, k-i, i, k-i} &= \frac{1}{(k!)^{2m-1}} \sum_{i=0}^{k} \frac{1}{i!(k-i)!i!(k-1)!} \\
&= \frac{1}{(k!)^{2m+1}} \sum_{i=0}^{k} \binom{k}{i}^2 \\
&= \frac{1}{(k!)^{2m+1}} \binom{2k}{k},
\end{aligned}$$

and therefore the coefficient of q^k in

$$\frac{W_q^{k(2m+1)}}{\hbar^{k(2m+1)}(k(2m+1))!}$$

equals $\frac{1}{\hbar^{k(2m+1)}}\frac{1}{(k!)^{2m+1}}\binom{2k}{k}$.

This completes the proof when $N = 2m+1$. The proof when $N = 2m$ is completely analogous, using the formula from Proposition 3.3 for the superpotential. □

A-model proof. Our A-model proof works by recovering Corollary 6.2 from Kontsevich-Manin's recurrence relations for Gromov-Witten invariants [KM98]. Define

$$\beta_{k,d} = I_d(\psi^{Nd-1-k}\sigma_N, \sigma_k),$$

so that $I_d(\sigma_N\psi^{dN-2}) = \frac{1}{d}\beta_{1,d}$ by the divisor axiom.

Let us first assume that $N = 2m-1$ is odd. Using the divisor axiom and topological recursion, we get:

$$d\beta_{k,d} = I_d(\psi^{Nd-1-k}\sigma_N, \sigma_k, \sigma_1) = \begin{cases} \beta_{k+1,d} & \text{if } k \notin \{m-1, N-1, N\} \\ 2\beta_{m,d} & \text{if } k = m-1 \\ \beta_{N,d} + \beta_{0,d-1} & \text{if } k = N-1 \\ \beta_{1,d-1} & \text{if } k = N. \end{cases}$$

An easy computation then gives $\beta_{1,d+1} = \beta_{1,d}\frac{2(2d+1)}{d(d+1)^N}$, and $\beta_{1,1} = 2$, which yields Corollary 6.2. □

Similarly, in the case where $N = 2m-2$ is even:

$$d\beta_{k,d} = I_d(\psi^{Nd-1-k}\sigma_N, \sigma_k, \sigma_1) = \begin{cases} \beta_{k+1,d} & \text{if } k \notin \{m-2, N-1, N\} \\ \beta_{m-1,d} + \beta'_{m-1,d} & \text{if } k = m-2 \\ \beta_{N,d} + \beta_{0,d-1} & \text{if } k = N-1 \\ \beta_{1,d-1} & \text{if } k = N, \end{cases}$$

and

$$d\beta'_{m-1,d} = \beta_{m,d}.$$

Corollary 6.2 is then easily checked. □

We also compute the constant term of $p_\ell \exp(\frac{1}{\hbar}W_q)$ for each Plücker coordinate p_ℓ. This is a series in q which also has an interpretation in terms of descendant Gromov-Witten invariants.

Theorem 6.3. *Let Q_N be an even or odd quadric. Then the constant term coefficient of $p_\ell \exp(\frac{1}{\hbar}W_q)$ is given by:*

$$\sum_{k\geq 0} \frac{1}{\hbar^{kN-\ell}} \cdot \frac{1}{(k!)^{N+1}} \cdot \binom{2k}{k} k \cdot q^k \qquad \text{if } \ell = 1,$$

$$\sum_{k\geq 0} \frac{1}{\hbar^{kN-\ell}} \cdot \frac{1}{(k!)^{N}} \cdot \frac{1}{2}\binom{2k}{k} k^{\ell-1}(k-1) \cdot q^k \qquad \text{if } 2 \leq \ell \leq \left\lfloor \frac{N-1}{2} \right\rfloor,$$

$$\sum_{k\geq 0} \frac{1}{\hbar^{kN-\ell}} \cdot \frac{1}{(k!)^{N}} \cdot \frac{1}{2}\binom{2k}{k} k^{\ell} \cdot q^k \qquad \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor \leq \ell \leq N-1,$$

$$\sum_{k\geq 0} \frac{1}{\hbar^{(k+1)N-\ell}} \cdot \frac{1}{(k!)^{N}} \cdot \frac{k}{k+1} \cdot \binom{2k}{k} \cdot q^{k+1} \qquad \text{if } \ell = N,$$

Proof. The proof is entirely analogous to the B-model proof of Theorem 6.1. We will give the proof in one representative case, but omit the other cases, which are extremely similar.

Let us consider the case that $N = 2m$, and $m + 2 \leq \ell \leq 2m - 1$. In this case recall that $p_\ell = a_1 \dots a_{m-1} c d b_{m-1} \dots b_{2m+1-\ell}$, and recall from (19) that the superpotential W_q equals

$$a_1 + \dots + a_{m-1} + c + d + b_{m-1} + \dots + b_1 + \frac{q}{a_2 \dots a_{m-1} c d b_{m-1} \dots b_1} + \frac{q}{a_1 \dots a_{m-1} c d b_{m-1} \dots b_2}.$$

To compute the constant term of $p_\ell \exp(\frac{1}{\hbar}W_q)$, we consider $p_\ell(1 + \frac{1}{\hbar}W_q + \frac{1}{\hbar^2}\frac{W_q^2}{2!} + \frac{1}{\hbar^3}\frac{W_q^3}{3!} + \dots)$, and we pick out from each $p_\ell \frac{W_q^i}{\hbar^i i!}$ every term which has the form λq^j where $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$. Here we just need to look at each $\frac{W_q^{kN-\ell}}{\hbar^{kN-\ell}(kN-\ell)!}$ for $k = 1, 2, \dots$, because the expansion of $p_\ell \frac{W_q^i}{\hbar^i i!}$ for i not of the form $kN - \ell$ will contain no terms of the form λq^j for $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$.

Now let us analyze $p_\ell \frac{W_q^{kN-\ell}}{\hbar^{kN-\ell}(kN-\ell)!}$ for $N = 2m$. A (Laurent) monomial in the expansion of $p_\ell W_q^{k(2m)-\ell}$ is obtained by choosing one term in each of the $k(2m) - \ell$ factors. Some of the monomials in the expansion will be pure in the variable q alone – in which case they will equal q^k. We need to show that the number of such monomials divided by $(k(2m) - \ell)!$ equals $\frac{1}{2}\binom{2k}{k} k^\ell/(k!)^{k(2m)}$. To count the number of such monomials, we need to pick one term in each of the $k(2m) - \ell$ factors so that we:

- choose i terms which are $\frac{q}{a_2 \dots a_{m-1} c d b_{m-1} \dots b_1}$ for some $0 \leq i \leq k$;
- choose $k - i$ terms which are $\frac{q}{a_1 \dots a_{m-1} c d b_{m-1} \dots b_2}$;
- choose $k - 1$ terms which are c;
- choose $k - 1$ terms which are d;
- choose i terms which are b_1;
- choose $k - i - 1$ terms which are a_1;
- for each j such that $2 \leq j \leq m - 1$, choose $k - 1$ terms which are a_j;
- for each j such that $2 \leq j \leq 2m - \ell$, choose k terms which are b_j.
- for each j such that $2m - \ell < j \leq m - 1$, choose $k - 1$ terms which are b_j.

The number of ways to do this is the sum of multinomial coefficients

$$\sum_{i=0}^{k} \binom{k(2m)-\ell}{i,i,k-i,k-i-1,k\dots k,k-1\dots k-1}, \tag{57}$$

where the number of k's in the string $k\dots k$ above is $2m-\ell-1$, and the number of $k-1$'s in the string $k-1\dots k-1$ above is $\ell-1$. When we simplify (57) and divide by $(k(2m)-\ell)!$, we obtain $\frac{1}{2}\binom{2k}{k}k^{\ell}/(k!)^{k(2m)}$, as desired. □

7. A quiver description of the Laurent polynomial mirrors

As $Gr_2(4)$ is defined by a single (quadratic) Plücker relation, the hypergeometric series for $Gr_2(4)$ must agree with the one for Q_4. This hypergeometric series was obtained earlier in [BCFKvS98], and it was shown to agree with a residue integral for a (conjectural) Laurent polynomial superpotential. Indeed [BCFKvS98] described conjectural Laurent polynomial mirrors for all Grassmannians using quivers, along the lines of Givental's mirrors for SL_n/B from [Giv97], and worked out the residue integrals which give rise to the hypergeometric series of the Grassmannians. These are also described in terms of the quivers.

For $Gr_2(4)$ the quiver from [BCFKvS98] is shown in Figure 1. The superpotential can be read off easily. There are two versions. In the left hand picture the coordinates t_{ij} of $(\mathbb{C}^*)^4$ are in bijection vertices of the quiver. To each arrow we associate a Laurent monomial by taking the coordinate at the head of the arrow divided by the coordinate at the tail. The Laurent polynomial corresponding to the quiver is the sum of all of the Laurent monomials associated to the arrows.

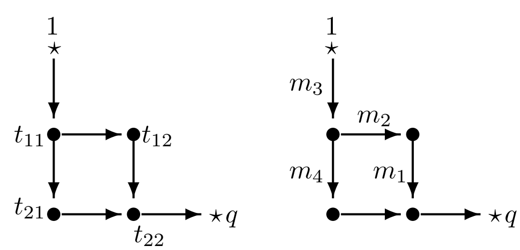


Figure 1. The quiver for $Gr_2(4)$.

The labels m_i of the arrows in the right hand version are another natural choice of coordinates on the torus. Indeed these are coordinates coming from factorizations into one-parameter subgroups of Lie theoretic mirrors, compare [MR13]. We suppose the remaining arrows are labelled in such a way that the square commutes and a/any path leading from 1 to q has labels whose product equals q. These are Laurent monomials in the variables m_i. Then the Laurent polynomial superpotential is obtained in [BCFKvS98] as the sum of the labels of all of the arrows of the quiver. In the case of $Gr_2(4)$ it is

$$m_1+m_2+m_3+m_4+\frac{m_1m_2}{m_3}+q\frac{1}{m_1m_2m_3}.$$

This is equivalent to the superpotential for Q_4,

$$a_1+c+d+b_1+q\frac{a_1+b_1}{a_1b_1cd}.$$

This superpotential comes from the quiver

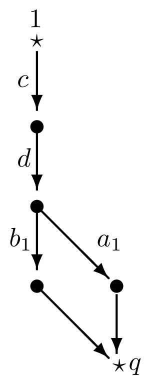


FIGURE 2. The quiver for Q_4.

More generally, our Laurent polynomial mirrors for Q_N can be described using quivers in a completely analogous way, see Figure 3. Here the top $N-2$ vertical arrows are labeled from top to bottom by $a_2, a_3, \dots, a_{m-1}, c, b_{m-1}, \dots, b_2$ for odd quadrics Q_{2m-1}, and by $a_2, a_3, \dots, a_{m-2}, c, d, b_{m-2}, \dots, b_2$ for even quadrics Q_{2m-2}. Note the relation with the factorization (21).

Remark 2. It is interesting to note that our quivers (restricted to the vertices which are not labeled by q) are orientations of type D Dynkin diagrams. So we have three ways to associate a Dynkin diagram to a quadric: the type of its symmetry group, the type of the cluster algebra associated to its coordinate ring, and the type of the quiver defining its superpotential. See Table 1 below.

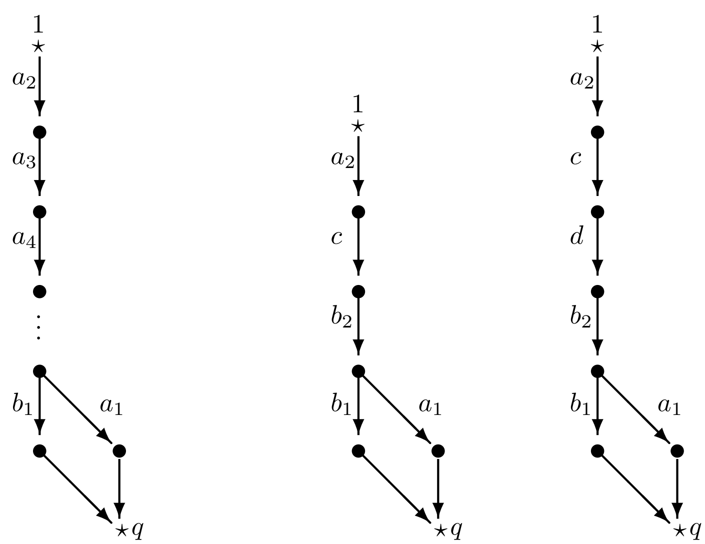


FIGURE 3. The quiver for Q_N, plus the labeled quivers for Q_5 and Q_6.

Quadric	Type of symmetry group	Cluster type	Superpotential Quiver
Q_3	B_2	A_1	D_4
Q_4	D_3	A_1	D_5
Q_5	B_3	A_1^2	D_6
Q_6	D_4	A_1^2	D_7
Q_7	B_4	A_1^3	D_8
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 1. Dynkin diagrams associated to quadrics

8. The hypergeometric equation of a quadric

Justifying its name, the hypergeometric series of the quadric computed in Theorem 6.1 is a generalised hypergeometric series; indeed, the general k-th coefficient of the series is a rational function of k. Following standard notation we will denote by

$$ {}_pF_r\left(a_1,\ldots,a_p;b_1,\ldots,b_r;z\right) $$

the series whose general term β_k is such that

$$ \frac{\beta_{k+1}}{\beta_k}=\frac{(a_1+k)\ldots(a_p+k)}{(b_1+k)\ldots(b_r+k)(1+k)} $$

and $\beta_0=1$. We immediately get that

$$ A_N(q,\hbar)={}_1F_N\left(\frac{1}{2};1,\ldots,1;\frac{4}{\hbar^N}q\right). $$

It is well-known that the hypergeometric series $w={}_pF_r\left(a_1,\ldots,a_p;b_1,\ldots,b_r;z\right)$ satisfies the differential equation

$$ z\prod_{n=1}^{p}(z\frac{\partial}{\partial z}+a_n)w=z\frac{\partial}{\partial z}\prod_{n=1}^{r}(z\frac{\partial}{\partial z}+b_n-1)w, $$

see for example [AOD10, (16.8.3)]. As a consequence, we obtain a differential equation satisfied by the hypergeometric series of the quadric.

Proposition 8.1. *The hypergeometric series of the N-dimensional quadric Q_N satisfies the following differential equation :*

$$ \left[\left(\hbar q\frac{\partial}{\partial q}\right)^{N+1}-q\left(4\hbar q\frac{\partial}{\partial q}+2\hbar\right)\right]A_N(q,\hbar)=0. $$

Let us check that this quantum differential equation gives rise to a relation in quantum cohomology. Indeed (see for instance [CK99]), if $P(\hbar q\frac{\partial}{\partial q},q,\hbar)A_N=0$, then $P(\sigma_1,q,0)=0$ in $QH^*(Q_N,\mathbb{C})$. Here we should have:

$$ \sigma_1^{N+1}-4q\sigma_1=0, $$

which indeed holds in $QH^*(Q_N,\mathbb{C})$, for example by an application of the quantum Chevalley formula, see [FW04].

Note that the differential system for the flat sections of the Dubrovin connection on Q_N can also be rewritten as a generalised hypergeometric differential equation, along the lines of [Dub99, Example 4.4] for projective spaces. We expect in this way to obtain the hypergeometric equation appearing in Proposition 8.1 directly from the A-model side.

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