A comparison of Landau-Ginzburg models for odd dimensional Quadrics

C. Pech and K. Rietsch
King’s College London*

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Abstract

In [Rie08], the second author defined a Landau-Ginzburg model for homogeneous spaces \( G/P \), as a regular function on an affine subvariety of the Langlands dual group. In this paper, we reformulate this LG model in the case of the odd-dimensional quadric \( Q_{2m-1} \) as a regular function \( W_t \) on the complement \( \tilde{X}^\circ \) of a particular anticanonical divisor in the projective space \( \mathbb{P}^{2m} = \mathbb{P}(H^*(Q_{2m-1}, \mathbb{C}))^\ast \). In fact, we express \( W_t \) in terms of Plücker coordinates which are canonically identified with the Schubert basis of \( H^*(Q_{2m-1}, \mathbb{C}) \). Our construction of \( (\tilde{X}^\circ, W_t) \) compares with work by R. Marsh and the second author [MR12] for type A Grassmannians and by both authors [PR13] for Lagrangian Grassmannians.

We also obtain a change of coordinates relating \( (\tilde{X}^\circ, W_t) \) to the LG model obtained by Gorbounov and Smirnov [GS13] via an ad hoc compactification of the Hori-Vafa mirror [HV00]. We use this comparison of LG models and results of [GS13] to deduce part of [Rie08, Conj. 8.1] for odd-dimensional quadrics.

1 Introduction

In 2000 Hori and Vafa wrote down a conjectured LG model for any hypersurface in a (weighted) complex projective space [HV00], [Prz07, Rmk. 19]. This is a Laurent polynomial associated to the hypersurface which plays the part of the B-model to the hypersurface in mirror symmetry, meaning its singularities are meant to encode various structures to do with Gromov-Witten theory of the hypersurface. In the case of the smooth quadric \( Q_3 \) in \( \mathbb{P}^4 \) the LG model is

\[
Y_1 + Y_2 + \frac{(Y_3 + q)^2}{Y_1 Y_2 Y_3},
\]

and in this special case it was written down earlier by Eguchi, Hori, and Xiong [EHX97]. For a quadric \( Q_{2m-1} \) the formula of Hori and Vafa reads

\[
Y_1 + Y_2 + \ldots + Y_{2m-2} + \frac{(Y_{2m-1} + q)^2}{Y_1 Y_2 \ldots Y_{2m-1}}.
\]

One issue with these Laurent polynomial formulas is that they do not always have the expected number of critical points (at fixed generic value of \( q \)) which should be equal to \( \dim(H^*(Q_{2m-1})) = 2m \). This was already observed in [EHX97], where it was suggested to

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solve this problem using a partial compactification, and this was carried out for the first time albeit in an ad hoc fashion.

The quadratic hypersurfaces $Q_{2m-1}$ have a large symmetry group. Indeed $Q_{2m-1}$ is a cominuscule homogeneous space for the group $\text{Spin}_{2m+1}(\mathbb{C})$. Therefore there is already another LG model on an affine variety generally larger than a torus, which was defined by the second author using a Lie-theoretic construction [Rie08]. Namely for any projective homogeneous space $G/P$ of a simple complex algebraic group, [Rie08] constructs a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. It is shown in [Rie08] that this LG model recovers the Peterson variety presentation [Pet97] of the quantum cohomology of $G/P$. It therefore defines an LG model whose Jacobian ring has the correct dimension.

For odd-dimensional quadrics $Q_{2m-1}$ a recent paper [GS13] of Gorbounov and Smirnov constructed directly a partial compactification of the Hori-Vafa mirrors, without making use of [Rie08]. Moreover they proved a version of mirror symmetry, which identifies the initial data of the Frobenius manifold associated to the LG model with that constructed out of the quantum cohomology of $Q_{2m-1}$.

The goal of this note is twofold. We first express the LG model from [Rie08] in the case of $Q_{2m-1}$ in terms of natural coordinates on an open affine subvariety of a ‘mirror homogeneous space’ $\tilde{X}$ which should be thought of as $\mathbb{P}^{2m-1}$ viewed as a homogeneous space for $\text{PSp}_{2m}(\mathbb{C})$. For example in the case of $Q_3$ we obtain

$$W_q = p_1 + \frac{p_2^2}{p_1 p_2 - p_3} + q \frac{p_1}{p_3}$$

in terms of homogeneous coordinates on $\mathbb{P}^3$.

The first main result generalises this formula. Define

$$\tilde{X}^\circ := \tilde{X} \setminus D,$$

where $D := D_0 + D_1 + \ldots + D_{m-1} + D_m$, the divisors $D_i$ being given by

$$D_0 := \{ p_0 = 0 \},$$

$$D_l := \{ p_l p_{2m-l-1} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1} = 0 \} \text{ for } 1 \leq l \leq m-1,$$

$$D_m := \{ p_{2m-1} = 0 \}.$$

The divisor $D$ is an anticanonical divisor. Indeed, the index of $\tilde{X} = \mathbb{P}^{2m-1}$ is $2m$.

**Theorem 1.** The LG model $\mathcal{F}_q : \mathcal{R} \rightarrow \mathbb{C}$ from [Rie08] is isomorphic to $W_q : \tilde{X}^\circ \rightarrow \mathbb{C}$ defined by

$$W_q = p_1 + \sum_{l=1}^{m-1} \frac{p_{l+1} p_{2m-l-1}}{p_l p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_{2m-1}} + q \frac{p_1}{p_{2m-1}}. \quad (2)$$

**Corollary 2.** There is an isomorphism

$$\mathbb{C}[\tilde{X}^\circ \times \mathbb{C}^*]/(\partial W_q) \rightarrow QH^*(X)[q^{-1}]$$

defined by sending $p_i$ to the Schubert class $\sigma_i \in H^{2i}(X)$.

This follows from Thm. 1 together with [Rie08]. Indeed the isomorphism in Cor. 2 fits in well with the geometric Satake correspondence (see [Lus83], [Gin95], [MV07]), by which

$$H^*(Q_{2m-1}, \mathbb{C}) = V_{\omega_1}^\text{PSp}_{2m}.$$
With this in mind it is natural to identify $\hat{X}$ with $\mathbb{P}(H^*(Q_{2m-1}, \mathbb{C})^*)$ and the coordinates $\{p_i\}$ with the Schubert basis $\{\sigma_i\}$ of $H^*(Q_{2m-1}, \mathbb{C})$.

It is interesting to note that under the isomorphism from Cor. 2, the denominators of $W_q$ actually map to something extremely simple inside the quantum cohomology of the quadric:

**Corollary 3.** For $1 \leq l \leq m-1$, the denominator $p_l p_{2m-1-l} - p_{l-1} p_{2m-1} + \cdots + (-1)^l p_{2m-1}$ represents an element in the Jacobi ring of $W_q$ which maps to

$$\sigma_l \sigma_{2m-1-l} - \sigma_{l-1} \sigma_{2m-1} + \cdots + (-1)^l \sigma_{2m-1} = q$$

inside $QH^*(X)$ under the isomorphism (3).

This is an easy consequence of quantum Schubert calculus on the quadric (which can be deduced from the quantum Chevalley formula of [FW04]).

Finally, in Sec. 6 we recall a partial compactification of the Hori-Vafa mirror defined by Gorbounov and Smirnov. We then show the following corollary.

**Corollary 4.** The partially compactified LG model defined in Gorbounov and Smirnov is related to the formula (2) by a change of coordinates. In particular the Gorbounov and Smirnov LG model is isomorphic to the LG model defined in [Rie08].

Together with Cor. 4, the work of Gorbounov and Smirnov implies a part of the mirror conjecture stated in [Rie08, Conjecture 8.1] for the groups Spin$_{2m+1}(\mathbb{C})$ with maximal parabolic $P = P_{\omega_1}$, see Sec. 7.

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2 Notations and Definitions

The LG model for $Q_{2m-1} = \text{Spin}_{2m+1}/P_{\omega_1}$ defined in [Rie08] takes place on an open Richardson variety inside the Langlands dual flag variety $\text{PSp}_{2m}/B_\pm$. We let $G = \text{PSp}_{2m}(\mathbb{C})$, since this is the group we will primarily be working with. Then $G' = \text{Spin}_{2m+1}(\mathbb{C})$ and $Q_{2m-1} = G'/P'$ for the parabolic subgroup $P'$ associated to the first node of the Dynkin diagram of type $B_m$:

$$\circ  \quad \circ  \quad \cdots  \quad \circ \Rightarrow \circ.$$

Let $V = \mathbb{C}^{2m}$ with fixed symplectic form

$$J = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -1
\end{pmatrix}.$$

For $G = \text{PSp}(V, J)$ we fix Chevalley generators $(e_i)_{1 \leq i \leq m}$ and $(f_i)_{1 \leq i \leq m}$. To be explicit we embed $\mathfrak{sp}(V, J)$ into $\mathfrak{gl}(V)$ and set

$$e_i = E_{i,i+1} + E_{2m-i,2m-i+1}, \quad \text{for } i = 1, \ldots, m-1, \quad \text{and } e_m = E_{m,m+1}.$$ 

and $f_i := e^T_i$, the transpose matrix, for every $i = 1, \ldots, m$. Here $E_{i,j} = (\delta_{i,k} \delta_{l,j})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group $\text{PSp}(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$. 

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consisting of upper-triangular and lower-triangular matrices in $\text{PSp}(V)$, respectively. $T$ is the maximal torus of $\text{PSp}(V)$, consisting of diagonal matrices $(d_{ij})$ with non-zero entries $d_{ii} = d_{2m-i-1,2m-i+1}^{-1}$.

The parabolic subgroup $P$ we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the $e_i$ together with $f_2, \ldots, f_m$, leaving out $f_1$. Let $x_i(a) := \exp(a e_i)$ and $y_i(a) = \exp(a f_i)$. The Weyl group $W$ of $\text{PSp}_{2m}$ is generated by simple reflections $s_i$ for which we choose representatives

$$s_i = y_i(-1)x_i(1)y_i(-1).$$

We let $W_P$ denote the parabolic subgroup of the Weyl group $W$, namely $W_P = \langle s_2, \ldots, s_m \rangle$. The length of a Weyl group element $w$ is denoted by $\ell(w)$. The longest element in $W_P$ is denoted by $w_P$. We also let $w_0$ be the longest element in $W$. Next $W^P$ is defined to be the set of minimal length coset representatives for $W/W_P$. The minimal length coset representative for $w_0$ is denoted by $w^P$. Let $\hat{w}$ denote the representative of $w \in W$ in $G$ obtained by setting $\hat{w} = \hat{s}_{i_1} \cdots \hat{s}_{i_m}$, where $w = s_{i_1} \cdots s_{i_m}$ is a reduced expression.

We consider the open Richardson variety $\mathcal{R} := R_{w_P,w_0} \subset G/B$, namely

$$\mathcal{R} := R_{w_P,w_0} = (B_+ \hat{w}_PB_+ \cap B_- \hat{w}_0B_-)/B_-.$$

Let $T^{W_P}$ be the $W_P$-fixed part of the maximal torus $T$, and fix $d \in T^{W_P}$. Then we also define

$$Z_d := B_- \hat{w}_0 \cap U_+ \hat{w}_P U_-.$$

The map

$$\pi_R : Z_d \rightarrow \mathcal{R} : z \mapsto zB_-,$$

is an isomorphism from $Z_d$ to the open Richardson variety.

Let $q$ be the coordinate $a_1$ on the 1-dimensional torus $T^{W_P}$. The mirror LG model is a regular function on $\mathcal{R}$ depending also on $q$, so a regular function on $\mathcal{R} \times T^{W_P}$. It is defined as follows [Rie08],

$$\mathcal{F} : (u_1 \hat{w}_P B_-, d) \mapsto z = u_1 \hat{w}_P d \hat{u}_2 \in Z_d \mapsto \sum e_i^*(u_1) + \sum f_i^*(\hat{u}_2). \quad (4)$$

The corresponding map from $\mathcal{R}$, when $d$ is fixed, is denoted

$$\mathcal{F}_d : \mathcal{R} \rightarrow \mathbb{C} : u_1 \hat{w}_P B_- \mapsto \mathcal{F}(u_1 \hat{w}_P B_-, d).$$

We also define another embedding

$$\pi_L : Z_d \rightarrow P' \text{PSp}(V) : z \mapsto Pz,$$

which maps $Z_d$ isomorphically to an open subvariety of a big cell in $P' \text{PSp}(V)$. Note that $P' \text{PSp}(V)$ is canonically the isotropic Grassmannian of lines in $V'$, when this Grassmannian is viewed as a homogeneous space via the action of $\text{PSp}(V)$ from the right. Moreover the isotropic Grassmannian of lines is just $\mathbb{P}(V')$, since any line is automatically isotropic. Therefore the second embedding $\pi_L$ has an advantage, that it is just an embedding into a projective space.

**Definition 2.1** (Plücker coordinates). First we introduce notation for the elements of $W^P$:

$$w_k = \begin{cases} s_k s_{k-1} \cdots s_1 & \text{if } k \leq m, \\ s_{2m-k} \cdots s_{m-1} s_m s_{m-1} \cdots s_1 & \text{if } m+1 \leq k \leq 2m-1. \end{cases}$$

The associated Plücker coordinates $p_k$ are defined by

$$p_k(g) = \langle v_{\omega_1}, w_k, v_{\omega_1} \rangle.$$
Note that the Plücker coordinates are just the homogeneous coordinates on the projective space \( \mathbb{P}(V^*) \). For a coset \( P q \) they are given by the bottom row entries of \( g \) read from right to left. If \( g = u_1 \bar{w} P d \bar{u}_2 \) then
\[
(p_0(g) : \ldots : p_{2m-1}(g)) = (p_0(\bar{u}_2) : \ldots : p_{2m-1}(\bar{u}_2)).
\]

Our goal is to express \( \mathcal{F} \) as a rational function in the Plücker coordinates and \( q = \alpha_1(d) \). We first illustrate our result in the smallest interesting example: that of the three-dimensional quadric \( Q_3 \).

3 The mirror to \( Q_3 \)

A generic element of \( Z_d := B_- \bar{w}_0 \cap U_+ d \bar{w} P U_- \) can be written as \( u_1 d \bar{w} P \bar{u}_2 \), where
\[
\bar{u}_2 = y_1(a_1)y_2(c)y_3(b_1)
\]
and \( a_1, c, b_1 \) are non-zero. Hence
\[
\bar{u}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
a_1 + b_1 & 1 & 0 & 0 \\
c b_1 & c & 1 & 0 \\
a_1 c b_1 & a_1 c & a_1 + b_1 & 1
\end{pmatrix}
\]

The map \( \pi_L : Z_d \to P \setminus \text{PSp}(V) \cong \mathbb{P}(V^*) \) takes \( z = u_1 d \bar{w} P \bar{u}_2 \) to \( P z = P \bar{u}_2 \). This may be interpreted as taking \( z \) to the span of the reverse row vector corresponding to the last row of \( \bar{u}_2 \) after the identification \( P \setminus \text{PSp}(V) \cong \mathbb{P}(V^*) \). The Plücker coordinates of \( \bar{u}_2 \) are given by \( p_0 = 1, p_1 = a_1 + b_1, p_2 = a_1 c, p_3 = a_1 c b_1 \).

If we are interested in the image of \( Z_d \) in \( \mathbb{P}(V^*) \) then first of all we can observe that it is independent of \( d \). So we may choose for \( d \) the identity element, and restrict our attention to \( B_- \bar{w}_0 \cap U_+ d \bar{w} P U_- \). It turns out that the image of \( Z_d \) in \( \mathbb{P}(V^*) \) is obtained from \( \mathbb{P}(V^*) \) in coordinates
\[
(p_0 : p_1 : p_2 : p_3) \in \mathbb{P}(V^*)
\]
by removing \( \{p_0 = 0\} \cup \{p_3 p_0 - p_2 p_1 = 0\} \cup \{p_3 = 0\} \). We call this variety \( \bar{X}^o \), and the isomorphism with \( Z_d \) in Prop. 9 shows that \( \bar{X}^o \) is also isomorphic to the open Richardson variety \( R \).

Let us denote by \( W : \bar{X}^o \times \mathbb{C}^* \to \mathbb{C} \) the map obtained from \( \mathcal{F} \), see (4), after the identifications \( R \cong \bar{X}^o \) and \( (T)^{WP} \cong \mathbb{C}^* \) via \( d \mapsto \alpha_1(d) = q \). In this way we can compute the superpotential \( \mathcal{F} \) from [Rie08] in the coordinates on \( \mathbb{P}(V^*) \):
\[
W = \frac{p_1}{p_0} + \frac{p_2}{p_0 p_2 - p_0 p_3} + q \frac{p_1}{p_3}.
\]

This is equivalent to the following Landau-Ginzburg model of [GS13]:
\[
g = y + yz + q \frac{x^2}{(xy - 1) z}
\]
via the change of coordinates:
\[
x = \frac{p_0 p_2}{p_1 p_2 - p_0 p_3}; y = \frac{p_1}{p_0}; z = \frac{q p_0}{p_3}.
\]

Note that in [GS13] the superpotential denoted \( \tilde{f} \) is \( g \) where \( z \) is replaced by \( z + 1 \).
4 The mirror to $Q_{2m-1}$

We now write down $W_q = (\pi_L)_* \pi_R^* F_d$ as a rational function on $\bar{X}$, where $d \in (T)^W$ is such that $\alpha_1(d) = q$. We will then prove in the next section that the locus $\bar{X}$ where it is defined is isomorphic to the open Richardson variety $\mathcal{R}$.

**Proposition 5.** As a rational function on $\bar{X}$

$$W_q = \frac{p_1}{p_0} + \sum_{i=1}^{m-1} \frac{p_i p_{2m-1-i}}{p_0 p_{2m-1-i} - p_{i-1} p_{2m-1-i} + \cdots + (-1)^{i} p_0 p_{2m-1}} + q \frac{p_1}{p_2}.$$

To prove the result, we first recall that

$$\pi_R^* F_d : z = u_1 \hat{w} p \hat{u}_2 \in Z_d \mapsto \sum e_i^*(u_1) + \sum f_i^*(\hat{u}_2).$$

Now $\hat{u}_2$ appearing in $u_1 \hat{w} p \hat{u}_2 \in Z_d \hat{w}_0$ can be assumed to lie in $U_\pm B_+ (\hat{w}^P)^{-1} B_+$. This is because we have two birational maps

$$\Psi_1 : U_\pm B_+ (\hat{w}^P)^{-1} B_+ \to P\backslash G : \hat{u}_2 \mapsto P \hat{u}_2,$$

$$\Psi_2 : B_+ \cap U^\pm \hat{w}^P U_\pm \to P \backslash G : b_\pm = u_1 \hat{w} p \hat{u}_2 \mapsto P b_\pm,$$

which compose to give $\Psi_1^{-1} \circ \Psi_2 : b_\pm \mapsto \hat{u}_2$. This gives a birational map

$$\Psi_1^{-1} \circ \Psi_2 : \hat{Z} \hat{w}_0 \to U_\pm \cap B_+ (\hat{w}^P)^{-1} B_+.$$

Now a generic element $\hat{u}_2$ in $U_\pm \cap B_+ (\hat{w}^P)^{-1} B_+$ can be assumed to have a particular factorisation. The smallest representative $\hat{w}^P$ in $W$ of $[w_0] \in W/W_P$ has the following reduced expression:

$$\hat{w}^P = s_1 \cdots s_{m-1} s_m s_{m-1} \cdots s_m.$$  

It follows that as a generic element of $U_\pm \cap B_+ (\hat{w}^P)^{-1} B_+$, the element $\hat{u}_2$ can be assumed to be written as

$$\hat{u}_2 = y_1(1) \cdots y_{m-1}(a_{m-1}) y_m(c) y_{m-1}(b_{m-1}) \cdots y_1(b_1),$$  

where $a_i, c, b_j \neq 0$. We have the following standard expression for the $p_k$ on factorized elements, which is a simple consequence of their definition.

**Lemma 6.** Fix $0 \leq k \leq 2m-1$ an integer. Then if $\hat{u}_2$ is of the form (5) we have

$$p_k(\hat{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1 \cdots a_{k-1}(a_k + b_k) & \text{if } 1 \leq k \leq m - 1, \\ a_1 \cdots a_{m-1} c b_{m-1} \cdots b_{2m-k} & \text{otherwise}. \end{cases}$$

We will also need the following:

**Lemma 7.** If $u_1$ and $\hat{u}_2$ are as above then we have the following identities

$$f_i^*(\hat{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \leq i \leq m, \\ c & \text{otherwise}. \end{cases}$$  

$$e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \leq i \leq m, \\ e_i & \text{if } i = 1. \end{cases}$$
Proof. Equation (6) is obtained immediately from the definition of $\bar{u}_2$. For Equation (7), notice that

$$e_i^+(u_1) = \frac{\langle u_1^{-1} \cdot v_{\omega_i}, e_i \cdot v_{\omega_i} \rangle}{\langle u_1^{-1} \cdot v_{\omega_i}, v_{\omega_i} \rangle} = \frac{\langle e^h w p \bar{u}_2 \cdot v_{\omega_i}^+, e_i \cdot v_{\omega_i}^- \rangle}{\langle e^h w p \bar{u}_2 \cdot v_{\omega_i}^+, v_{\omega_i}^- \rangle}.$$ 

Assume $2 \leq i \leq m$. Then $e_i^+(u_1) = 0$ if and only if $\langle \bar{u}_2 \cdot v_{\omega_i}^+, \bar{w}_p^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$. Now the vector $w_p^{-1} e_i \cdot v_{\omega_i}^-$ is in the $\mu$-weight space of the $i$-th fundamental representation, where $\mu = w_p^{-1} s_i (-\omega_i)$. Moreover, $\bar{u}_2 \in B_+(wP)^{-1} B_+$, hence $\bar{u}_2 \cdot v_{\omega_i}^+$ can have non-zero components only down to the weight space of weight $(wP)^{-1} (\omega_i) = w_p^{-1} (-\omega_i)$. Since $l(w_p^{-1} s_i) > l(w_p^{-1})$ for $2 \leq i \leq m$, this is higher than $\mu$, which proves that $e_i^+(u_1) = 0$.

Now assume $i = 1$. We have

$$e_1^+(u_1) = \frac{\langle e^h w p \bar{u}_2 \cdot v_{\omega_1}^+, e_1 \cdot v_{\omega_1}^- \rangle}{\langle e^h w p \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle} = \frac{(\omega_1 + \alpha_1 - \omega_1)(e^h) \langle \bar{u}_2 \cdot v_{\omega_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{\omega_1}^- \rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \bar{w}_p v_{\omega_1}^- \rangle} = e_1 (\bar{u}_2 \cdot v_{\omega_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{\omega_1}^-).$$

First look at the denominator. The only way to go from the highest weight vector $v_{\omega_1}^+$ of the first fundamental representation to the lowest $v_{\omega_1}^-$ is to apply $g \in B_+ w B_+$ for $w \geq (wP)^{-1}$. Since $\bar{u}_2 \in B_+(\bar{w}P)^{-1} B_+$, it follows that we need to take all factors of $\bar{u}_2$, and normalising $v_{\omega_1}^-$ appropriately, we get

$$\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^- \rangle = a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1.$$

Finally, we look at the numerator $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{\omega_1}^- \rangle$. The vector $\bar{w}_p^{-1} e_1 \cdot v_{\omega_1}^-$ has weight $\mu' = \bar{w}_p^{-1} s_1 (-\omega_1) = \bar{w}_p^{-1} (-e_2) = e_2$.

Write $w_p^{-1} s_1$ as a prefix $w' = s_1 s_2 \ldots s_m s_{m-1} \ldots s_2$ of $(wP)^{-1}$. We have $w' s_1 = (wP)^{-1}$, hence the way from $v_{\omega_1}^+$ to $w' \cdot v_{\omega_1}^-$ is through $s_1$. From the shape of $\bar{u}_2$, it follows that $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{\omega_1}^- \rangle = a_1 + b_1$. 

Using the expression (4) of the superpotential from [Rie08], we immediately deduce from Lem. 7 a intermediate expression for the Landau-Ginzburg model $W_q$ of the odd-dimensional quadric as a Laurent polynomial:

**Proposition 8.**

$$W_q = a_1 + \ldots + a_{m-1} + c + b_{m-1} + \ldots + b_1 + q \frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1}$$

(8)

Now with the help of Lem. 6 and Prop. 8, we prove the second expression of $W_q$:

**Proof of Prop. 5.** From Lem. 6, it follows that for $\bar{u}_2$ as in (5)

$$p_{l+1} P_{2m-1-l}(\bar{u}_2) = \begin{cases} (a_{l+1} + b_{l+1})(a_1 \ldots a_l)^2 a_{l+1} \ldots a_{m-1} c b_{m-1} \ldots b_{l+1} & \text{if } l \leq m - 2, \\ (a_1 \ldots a_{m-1} c)^2 & \text{if } l = m - 1. \end{cases}$$

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and
\[ p_k p_{2m-1-k}(\bar{u}_2) = (a_k + b_k)(a_1 \ldots a_{k-1})^2 a_k \ldots a_{m-1} b_{m-1} \ldots b_{k+1}. \]
Hence most terms in \( \sum_{k=0}^{l} (-1)^k p_{l-k} p_{2m-1+k-l}(\bar{u}_2) \) cancel, and
\[ \sum_{k=0}^{l} (-1)^k p_{l-k} p_{2m-1+k-l}(\bar{u}_2) = (a_1 \ldots a_l)^2 a_{l+1} \ldots a_{m-1} b_{m-1} \ldots b_{l+1}. \]
This proves that
\[ \frac{p_{l+1} p_{2m-1-l}}{p_{l} p_{2m-1-l} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1}}(\bar{u}_2) = \begin{cases} a_{l+1} + b_{l+1} & \text{if } l \leq m - 2, \\ c & \text{if } l = m - 1. \end{cases} \]
For the first and last terms, we obtain
\[ \frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1 \]
and
\[ \frac{p_1}{p_{2m-1}}(\bar{u}_2) = \frac{a_1 + b_1}{a_1 \ldots a_{m-1} b_{m-1} \ldots b_1} \]
as easy consequences of Lem. 6.

5 The open Richardson variety

We now prove that the affine subvariety \( \tilde{X}^o \) defined in Equation (1) is isomorphic to the open Richardson variety \( \mathcal{R} \).

Recall that \( \tilde{X}^o = X \setminus D \), where
\[ D := D_0 + D_1 + \ldots + D_{m-1} + D_m \]
and
\[ D_0 := \{ p_0 = 0 \}, \]
\[ D_l := \{ p_{p_{2m-1-l}} - p_{l-1} p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1} = 0 \} \text{ for } 1 \leq l \leq m - 1, \]
\[ D_m := \{ p_{2m-1} = 0 \}. \]
By definition, \( \tilde{X}^o \) is the locus where \( W_q \) is regular. Since \( p_0 \) is non-zero on \( \tilde{X}^o \), we may assume that \( p_0 = 1 \). Hence we have affine coordinates \( (p_1, \ldots, p_{2m-1}) \) on \( \tilde{X}^o \). We also set, for \( 1 \leq j \leq 2m - 1 \):
\[ r_j := \sum_{k=0}^{j} (-1)^k p_{j-k} p_{2m-1+k-j}. \]

**Proposition 9.** The map \( \pi_L \circ \pi_R^{-1} : \mathcal{R} \to \tilde{X}^o \) is an isomorphism.

We will prove the result by constructing the inverse map. But first, let us check that the image of this map is indeed inside \( \tilde{X}^o \) (and not just inside \( \tilde{X} \)). Clearly, \( \mathcal{F}_d \) equals \( W_q \circ \pi_L \circ \pi_R^{-1} \) as a rational map. Since \( \mathcal{F}_d \) is regular on \( \mathcal{R} \), it means that \( W_q \circ \pi_L \circ \pi_R^{-1} \) also is, hence that \( W_q \) is regular on the image of \( \pi_L \circ \pi_R^{-1} \). This proves that this image is contained in \( \tilde{X}^o \).
We now define a map $\Phi : \tilde{X}^0 \to B_{\mathbb{PSL}}^P \tilde{w}_0$, where $B_{\mathbb{PSL}}^P$ is the Borel of lower triangular matrices in $\mathbb{PSL}_{2m}$, so that $\Phi(p_1, \ldots, p_{2m-1}) \cdot v_j$ is equal to

$$
\begin{cases}
  p_{2m-1}v_{2m} & \text{if } j = 1, \\
  (-1)^j\frac{p_{2m-j}}{r_{2m-j}} + p_{2m-j} \left( \sum_{k=1}^{j-2} (-1)^{t+1} \frac{p_t}{r_t} v_{2m-t} + v_{2m} \right) & \text{if } 2 \leq j \leq m, \\
  (-1)^j\frac{p_{2m-1-j}}{r_{2m-1-j}} p_{2m-j} \left( \sum_{k=m+1}^{j-1} (-1)^k \frac{p_k}{r_k} v_{2m-k} + \sum_{k=1}^{m-1} (-1)^k (-1)^{j-k} \frac{p_k}{r_k} v_{2m-k} \right) & \text{if } m+1 \leq j \leq 2m-1, \\
  \frac{-1}{r_0} v_1 + \sum_{k=1}^{m-1} (-1)^k \frac{p_{2m-1-k}}{r_k} v_{k+1} + \sum_{k=1}^{m-1} (-1)^k \frac{p_k}{r_k} v_{2m-k} & \text{if } j = 2m.
\end{cases}
$$

Let $\Omega$ be the open dense subset of $\tilde{X}^0$ where the coordinates $p_m, p_{m-1}, \ldots, p_{2m-2}$ do not vanish and define coordinates on $\Omega$ (as follows from Lem. 10) by

$$
a_i = \frac{p_{2m-1-i} r_i}{p_{2m-1-i} r_{2m-1-i}} \text{ for all } 1 \leq i \leq m-1; \\
b_i = \frac{p_{2m-1-i}}{p_{2m-1-i}} \text{ for all } 1 \leq i \leq m-1; \\
c = \frac{p_m^2}{r_m}.
$$

**Lemma 10.** For all $(p_1, \ldots, p_{2m-1})$, $\Phi(p_1, \ldots, p_{2m-1})$ factorizes as $u_1 \tilde{w} p \tilde{u}_2$, where

$$
\tilde{u}_2 = y_1(a_1) \ldots y_{m-1}(a_{m-1}) y_m(c) y_{m-1}(b_{m-1}) \ldots y_1(b_1)
$$

and $u_1$ equals

$$
\begin{pmatrix}
  1 & \frac{a_1+b_1}{a_1 \cdots a_{m-1} c b_{m-1} \cdots b_1} & \frac{a_{m-1}+b_{m-1}}{a_1 \cdots a_{m-1} c b_{m-1} \cdots b_1} & \frac{1}{a_1} & \frac{-1}{a_1} \\
  & \ddots & 1 & \frac{-1}{a_1} & \frac{(-1)^m}{a_1 \cdots a_{m-1}} \\
  & & \ddots & \ddots & 1 \\
  & & & \frac{a_1+b_1}{a_1 \cdots a_{m-1} c b_{m-1} \cdots b_1} & 1
\end{pmatrix}
$$

**Proof.** Using the definition of the $y_i's$, it is easy to check that $\tilde{u}_2 \cdot v_j$ is equal to

$$
v_j + \sum_{k=0}^{m-1-j} (a_{j+k} + b_{j+k}) b_{j+k-1} \ldots b_{j+1} b_j v_{j+k+1} + \sum_{k=0}^{m-1} a_{m-k} \cdots a_{m-1} c b_{m-1} \cdots b_j v_{m+1+k}
$$

if $1 \leq j \leq m-1$, 

$$
v_m + \sum_{k=0}^{m-1} a_{m-k} \cdots a_{m-1} c \text{ if } j = m,
$$

$$
v_j + (a_{2m-j} + b_{2m-j}) \sum_{k=0}^{2m-1-j} a_{2m-k-j} \cdots a_{2m-2-j} a_{2m-1-j} v_{j+k+1} \text{ if } m+1 \leq j \leq 2m,
$$

Now a straightforward, if slightly tedious, computation shows that $\Phi(p_1, \ldots, p_{2m-1}) = u_1 \tilde{w} p \tilde{u}_2$. \square
We now need to prove that the entire image of Φ is in fact contained in $B_-\check{w}_0 \cap U_+\check{w}U_-$ inside PSp$_{2m}$:

**Lemma 11.**

$$\Phi(\check{X}^o) \subset B_-\check{w}_0 \cap U_+\check{w}U_-.$$ 

**Proof.** We first prove that $\Phi(\Omega) \subset B_-\check{w}_0 \cap U_+\check{w}U_-$ inside PSp$_{2m}$. Indeed, from Lem. 10, we know that for all $(p_1,\ldots,p_{2m-1}) \in \Omega$, $\Phi(p_1,\ldots,p_{2m-1})$ factorises as $u_1\check{w}_2$, where $u_1$ and $\check{w}_2$ are defined in the statement of the lemma. The factorisation of $\check{w}_2$ means that $\check{w}_2$ is in $U_-$ (hence in particular in PSp$_{2m}$). Now we prove that $u_1$ is also in PSp$_{2m}$, by showing directly that $u_1J u_1 = J$ using the formula from Lem. 10. This is the result of a straightforward computation. It follows that $u_1 \in U_+$, hence $\Phi(p_1,\ldots,p_{2m-1}) \in U_+\check{w}U_-$ in this case. Now also $\Phi(p_1,\ldots,p_{2m-1}) \in B^-_{\text{PSL}}\check{w}_0 \cap \text{PSp}_{2m} = B_-\check{w}_0$. Therefore $\Phi(\Omega) \subset B_-\check{w}_0 \cap U_+\check{w}U_-.$

Since $\Omega$ is open dense in $\check{X}^o$ we now have that $\Phi(\check{X}^o) \subset B_-\check{w}_0 \cap \overline{U_+\check{w}U_-}$. Suppose there exists $(p_1,\ldots,p_{2m-1})$ in $\check{X}^o$ such that $\Phi(p_1,\ldots,p_{2m-1}) \notin \overline{U_+\check{w}U_-}$. Then from Bruhat decomposition, we get $\Phi(p_1,\ldots,p_{2m-1})\check{w}_0^{-1} \notin U_+\check{w}U_-$ with $w < \check{w}w_0$. It follows that we must have

$$(\Phi(p_1,\ldots,p_{2m-1})\check{w}_0^{-1}v_0,\check{w}_0^{-1}v_0) = (\Phi(p_1,\ldots,p_{2m-1})v_0,\check{w}_0v_0) = 0,$$

hence the lower-right corner of the matrix $\Phi(p_1,\ldots,p_{2m-1})$ has to be zero. But this coefficient is always 1, hence the result. \hfill \Box

We can now prove Prop. 9:

**Proof of Prop. 9.** We have showed that the image of $\pi_L$ is contained inside $\check{X}^o$. Moreover, we have defined a map $\Phi : \check{X}^o \to Z_1$, and a straightforward computation shows that it is the inverse of $\pi_L$. Hence $\pi_L$ is an isomorphism. Since we saw in Sec. 2 that $\pi_R$ is also an isomorphism, the proposition follows. \hfill \Box

The proof of Thm. 1 then follows from Prop. 5 and 9.

## 6 Comparison with the LG model of [GS13]

We now want to prove that our Landau-Ginzburg model (2) is isomorphic to the one stated in [GS13], which goes as follows

$$g = \sum_{i=1}^{m-1} y_i (1 + z_i) + q \frac{x^2}{(xy_1y_2 \cdots y_{m-1} - 1)z_1z_2 \cdots z_{m-1}}. \quad (9)$$

Note that as for $Q_3$, in [GS13] the superpotential denoted $\tilde{f}$ is $g$ where the $z_i$ are replaced by $z_i + 1$.

Assume $p_0 = 1$ and consider the change of variables:

$$y_1 = p_1; \quad y_i = \frac{p_i}{p_{i-1}} \quad \forall \ 2 \leq i \leq m - 1;$$
$$z_1 = \frac{q}{p_5}; \quad z_i = \frac{\sum_{k=0}^{i-2} (-1)^k p_{i-2-k}p_{2m+1+k-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k}p_{2m+k-i}} \quad \forall \ 2 \leq i \leq m - 1;$$
$$x = \frac{p_m}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k}p_{m+k}}.$$
Proposition 12. The above change of coordinates \( \{x, y, z_1\} \mapsto \{p_i\} \) defines an isomorphism between the Landau-Ginzburg model (9) and ours (2).

Proof. We have \( y_1(1 + z_1) = p_1 + \frac{q}{p_{2m-1}}, \) and

\[
y_i(1 + z_i) = \frac{p_i p_{2m-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1 - k} p_{2m+k-i}}.
\]

Moreover

\[
x y_1 \cdots y_{m-1} = \frac{\sum_{k=0}^{m-2} (-1)^k p_{m-2 - k} p_{m+1+k}}{\sum_{k=0}^{m-1} (-1)^k p_{m-1 - k} p_{m+k}},
\]

and

\[
z_1 \cdots z_{m-1} = \frac{q}{\sum_{k=0}^{m-2} (-1)^k p_{m-2 - k} p_{m+1+k}},
\]

hence

\[
x^2 = \frac{p^2_m}{\left(\sum_{k=0}^{m-1} (-1)^k p_{m-1 - k} p_{m+k}\right)^2},
\]

and

\[
q(xy_1y_2 \cdots y_{m-1} - 1) z_1 z_2 \cdots z_{m-1} = \frac{p^2_m}{\sum_{k=0}^{m-1} (-1)^k p_{m-1 - k} p_{m+k}}.
\]

Hence the change of variables maps (9) to (2). Finally, it is clear that both domains of definition are the same. \(\square\)

This proves Cor. 4.

7 Consequences

Let \( \mathcal{H}_A \) be the sheaf of regular functions of the trivial vector bundle with fiber \( H^*(X, \mathbb{C}) \) over \( \mathbb{C}_h^* \times \mathbb{C}_q^* \) the two-dimensional complex torus with coordinates \( h \) and \( q \). The A-model connection is defined on \( \mathcal{H}_A \) by

\[
A \nabla_q = q \frac{\partial}{\partial q} + \frac{1}{h} p_1 * q \bullet
\]

\[
A \nabla_h = h \frac{\partial}{\partial h} + \text{gr} - \frac{1}{h} c_1(TX) * q \bullet,
\]

where gr is a diagonal operator on \( H^*(X) \) given by gr(\( \alpha \)) = \( k \) for \( \alpha \in H^{2k}(X) \). Here we are using the conventions of [Iri09]. Let \( \mathcal{H}_A^\vee \) be the vector bundle on \( \mathbb{C}_h^* \times \mathbb{C}_q^* \) defined by \( \mathcal{H}_A^\vee = j^* \mathcal{H}_A \) for \( j : (h, q) \mapsto (-h, q) \). This vector bundle with the pulled back connection \( A \nabla^\vee = j^*(A \nabla) \) is dual to \( (\mathcal{H}_A, A \nabla) \) via the flat non-degenerate pairing,

\[
\langle \sigma_i, \sigma_j \rangle = (2\pi i h)^N \int_{[X]} \sigma_i \cup \sigma_j = (2\pi i h)^N \delta_{i+1,j,N},
\]

where \( N = 2m - 1 \) is the dimension of \( \hat{X}^\circ \). The dual A-model connection \( A \nabla^\vee \) defines a system of differential equations called the (small) quantum differential equations

\[
A \nabla_q^\vee S = 0.
\]

Define the \( \mathbb{C}[\hat{h}^{±1}, q^{±1}] \)-module

\[
G = \Omega^N(\hat{X}^\circ)[\hat{h}^{±1}, q^{±1}] / (d - \frac{1}{h} dW_q \wedge \bullet) \Omega^{N-1}(\hat{X}^\circ)[\hat{h}^{±1}, q^{±1}],
\]

11
where $\Omega^k(\hat{X}^\circ)$ is the space of holomorphic $k$-forms on $\hat{X}^\circ$. We denote by $\mathcal{H}_B$ the sheaf with global sections $G$. Because $W_q$ is cohomologically tame [GS13], $G$ is a free $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$-module of rank $2m$ (cf. [Sab99]), and $\mathcal{H}_B$ a trivial vector bundle of that dimension. It has a (Gauss-Manin) connection given by

\[
B_\nabla_{\eta \partial_{\eta}}[\eta] = q \frac{\partial}{\partial q}[\eta] + \frac{1}{\hbar} \left[ q \frac{\partial W_q}{\partial \eta} \eta \right]
\]

\[
B_\nabla_{\partial_{\hbar}}[\eta] = \hbar \frac{\partial}{\partial \eta}[\eta] - \frac{1}{\hbar} [W_q \eta].
\]

Let $\omega$ be the canonical $N$-form on $\hat{X}^\circ$.

**Corollary 13.** The two bundles with connection $(\mathcal{H}_A, A_\nabla)$ and $(\mathcal{H}_B, B_\nabla)$ are isomorphic via $\sigma_i \mapsto [p_i \omega]$.

**Proof.** The corollary is a consequence of the isomorphism of our LG model $W_q$ and the one of [GS13] (see Cor. 4) together with the results of Gorbounov and Smirnov. \hfill $\square$

Let $\Gamma_0$ be a compact oriented real $N$-dimensional submanifold of $\hat{X}^\circ$ representing a cycle in $H^N(\hat{X}^\circ, \mathbb{Z})$ dual to $\omega$, in the sense that $\frac{1}{(2i\pi)^N} \int_{\Gamma_0} \omega = 1$. Then :

**Corollary 14.** The integral

\[
S_0(z, q) = \frac{1}{(2i\pi)^N} \int_{\Gamma_0} e^{\frac{W_q}{\hbar}} \omega
\]

is a solution to the quantum differential equation (10).

This implies part of [Rie08, Conj. 8.1] for odd-dimensional quadrics.

**References**


