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Résultats de stabilité en théorie des représentations par des méthodes géométriques

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Chapter 1

Introduction en français

1.1 Problème de branchement, exemples

Un groupe algébrique affine complexe est une variété algébrique affine définie sur le corps des nombres complexes $\mathbb{C}$ et telle que les opérations de multiplication et de passage à l'inverse sont données par des fonctions régulières sur la variété. Un tel groupe possède un radical, qui est la composante connexe de son sous-groupe fermé résoluble distingué maximal contenant l'élément neutre. Le sous-groupe des éléments unipotents de ce radical est appelé radical unipotent du groupe algébrique considéré. De manière équivalente, le radical unipotent d'un groupe algébrique complexe est son sous-groupe fermé distingué unipotent maximal.

Exemples : Si n est un entier strictement positif, les groupes $\mathrm{SL}_n(\mathbb{C})$ et $\mathrm{GL}_n(\mathbb{C})$, formés respectivement par les matrices carrées de déterminant 1 et et les matrices inversibles, sont des groupes algébriques affines complexes. Le radical de $\mathrm{SL}_n(\mathbb{C})$ est trivial, tandis que celui de $\mathrm{GL}_n(\mathbb{C})$ est composé des matrices d'homothétie $t\,\mathrm{I}_n$ $(t \in \mathbb{C}^*)$. Les radicaux unipotents de ces deux groupes sont triviaux.

Définition 1.1.1. Un groupe algébrique affine complexe dont le radical unipotent est trivial est dit réductif. Quand son radical est trivial et que le groupe est de plus connexe, on dit qu'il est semi-simple.

Exemples : D'après ce qui précède, $\mathrm{SL}_n(\mathbb{C})$ est semi-simple (et donc réductif) tandis que $\mathrm{GL}_n(\mathbb{C})$ est seulement réductif. On peut citer quelques autres exemples classiques de groupes réductifs complexes : $\mathrm{SO}_n(\mathbb{C})$ (groupe spécial orthogonal), $\mathrm{Sp}_{2n}(\mathbb{C})$ (groupe symplectique), les groupes finis...

Le terme « réductif » vient d'une propriété importante de ces groupes : on dit que leurs représentations sont complètement décomposables (« reducible » en anglais). Plus précisément, toute représentation complexe de dimension finie d'un groupe réductif complexe se décompose en somme directe de représentations irréductibles (i.e. qui ne contiennent pas de sous-représentation non triviale). Une remarque tout aussi intéressante est que, dans le cas d'un groupe connexe, on connaît ces représentations irréductibles.

Soit G un groupe réductif complexe connexe. On peut lui associer son algèbre de Lie $\mathfrak{g}$, qui est une algèbre de Lie réductive et possède donc une décomposition en sous-espaces radiciels, et ainsi un système de racines associé. Cette donnée combinatoire définit entre autres une notion de poids, dont certains sont appelés « poids entiers dominants ». Il se trouve que, à tout poids entier dominant λ de G, on peut associer une représentation irréductible de G, de dimension finie, appelée module de plus haut poids λ et notée $V_G(\lambda)$. De plus les $V_G(\lambda)$ sont exactement toutes les représentations complexes de G qui sont irréductibles, rationnelles, et de dimension finie.

Exemple : L'exemple le plus basique d'un groupe réductif complexe connexe est peut-être $\mathrm{GL}_n(\mathbb{C})$, pour lequel il est facile de décrire les poids entiers dominants : il s'agit exactement des suites finies décroissantes (au sens large) d'entiers, de longueur n. Une telle suite finie $\alpha = (\alpha_1, \dots, \alpha_n)$ donne un caractère du sous-groupe de $\mathrm{GL}_n(\mathbb{C})$ formé par les matrices diagonales – noté T – de la manière suivante :

$$e^\alpha : \begin{array}{ccc} T & \longrightarrow & \mathbb{C}^* \\ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} & \longmapsto & t_1^{\alpha_1} \dots t_n^{\alpha_n} \end{array}.$$

Le $\mathrm{GL}_n(\mathbb{C})$-module de plus haut poids α est alors noté $\mathbb{S}^\alpha(\mathbb{C}^n)$. Par isomorphisme, lorsque l'on s'intéresse au groupe $\mathrm{GL}(V)$ des automorphismes d'un $\mathbb{C}$-espace vectoriel V de dimension finie, toute suite finie décroissante α d'entiers, de longueur $\dim(V)$, donne une représentation irréductible $\mathbb{S}^\alpha V$ de $\mathrm{GL}(V)$.

On va uniquement s'intéresser à un type particulier de représentations de $\mathrm{GL}_n(\mathbb{C})$: celles qui sont appelées « polynomiales ». Il s'agit des représentations pour lesquelles l'action de tout élément $g \in \mathrm{GL}_n(\mathbb{C})$ est donnée par une famille fixée de polynômes en les entrées de g. Parmi les représentations irréductibles, celles qui sont polynomiales sont faciles à caractériser : si $\alpha = (\alpha_1, \dots, \alpha_n)$ est un poids entier dominant de $\mathrm{GL}_n(\mathbb{C})$, la représentation $\mathbb{S}^\alpha(\mathbb{C}^n)$ est polynomiale si et seulement si $\alpha_n \geq 0$. Les représentations complexes polynomiales irréductibles de dimension finie de $\mathrm{GL}_n(\mathbb{C})$ sont donc données par les partitions de longueur au plus n, qui sont des suites finies décroissantes $\alpha = (\alpha_1, \dots, \alpha_k)$ d'entiers strictement positifs, dont la longueur k est la longueur de la partition, notée $\ell(\alpha)$. On note de plus $|\alpha| = \sum_{i=1}^k \alpha_i$ la taille d'une telle partition, qui est alors qualifiée de « partition de l'entier $|\alpha|$ ».

Le problème de branchement : On considère à présent deux groupes réductifs complexes connexes, G et $\hat{G}$, et un morphisme $f : G \longrightarrow \hat{G}$. Alors, pour tout poids entier dominant $\hat{\lambda}$ de $\hat{G}$, le $\hat{G}$-module $V_{\hat{G}}(\hat{\lambda})$ est, via le morphisme f, une représentation (complexe, de dimension finie) de G, et donc se décompose en somme directe de représentations irréductibles de G :

$$V_{\hat{G}}(\hat{\lambda}) = \bigoplus_{\lambda \text{ poids entier dominant de } G} V_G(\lambda)^{\oplus c(\lambda,\hat{\lambda})}.$$

Définition 1.1.2. Les multiplicités $c(\lambda, \hat{\lambda})$ apparaissant dans la décomposition précédente sont des entiers positifs (ou nuls) appelés « coefficients de branchement ».

Le problème de branchement consiste à étudier ces coefficients, ce qui peut vouloir dire trouver une formule combinatoire pour les calculer (cela a été fait dans certains cas). Cela peut aussi vouloir dire étudier certains aspects plus qualitatifs de ceux-ci, comme on le fera dans la suite.

Exemples :

- Si $n \geq 2$, on peut introduire un morphisme de $\mathrm{GL}_{n-1}(\mathbb{C})$ dans $\mathrm{GL}_n(\mathbb{C})$ en envoyant toute matrice $A \in \mathrm{GL}_{n-1}(\mathbb{C})$ sur $\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$. Dans ce cas, les coefficients de branchements sont indexés par des couples de partitions : la première ayant une longueur d'au plus $n-1$, et la seconde de longueur au plus n. Pour une telle paire $\big(\underbrace{(\lambda_1, \ldots, \lambda_{n-1})}_{=\lambda}, \underbrace{(\mu_1, \ldots, \mu_n)}_{=\mu}\big)$, le coefficient de branchement correspondant est connu:

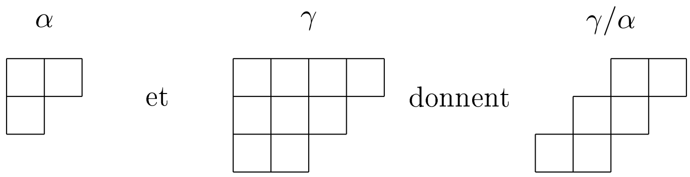
$$c(\lambda, \mu) = \begin{cases} 1 & \text{si } \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \cdots \geq \mu_{n-1} \geq \lambda_{n-1} \geq \mu_n \\ 0 & \text{sinon} \end{cases}.$$

- Si la situation de branchement est $T \subset \mathrm{GL}_n(\mathbb{C})$, où T est de nouveau le tore maximal de $\mathrm{GL}_n(\mathbb{C})$ composé des matrices diagonales (et le morphisme entre les deux est donc l'identité), alors on est en fait en train d'étudier la décomposition des modules de plus haut poids polynomiaux de $\mathrm{GL}_n(\mathbb{C})$ en somme directe de sous-espaces de poids (i.e. des sous-espaces sur lesquels T agit par un certain caractère λ). Les coefficients de branchement correspondants sont appelés « nombres de Kostka » et sont indexés par les couples formés d'une suite d'entiers positifs de longueur n (un poids entier dominant – disons λ – de T) et d'une partition de longueur au plus n (un poids entier dominant particulier – disons μ – de $\mathrm{GL}_n(\mathbb{C})$). Alors le nombre de Kostka $k_{\mu,\lambda}$ peut être calculé comme le nombre de tableaux de Young semi-standards de forme μ et de poids λ : c'est-à-dire le nombre de façons de remplir le diagramme de Young de la partition μ avec λ_1 fois le nombre 1, λ_2 fois le nombre 2, etc, de manière croissante (au sens large) en ligne et strictement croissante en colonne.

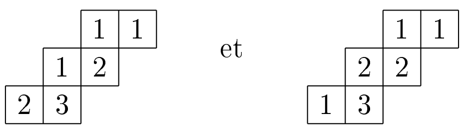
- Un exemple assez célèbre est le cas du produit tensoriel de deux représentations polynomiales irréductibles de $\mathrm{GL}_n(\mathbb{C})$: lorsque $G = \mathrm{GL}_n(\mathbb{C})$ est envoyé diagonalement dans $\hat{G} = G \times G$, on s'intéresse en fait à la manière dont le produit tensoriel de deux représentations polynomiales irréductibles de $\mathrm{GL}_n(\mathbb{C})$ se décompose en somme directe de telles représentations. Les coefficients de branchement correspondants sont appelés « coefficients de Littlewood-Richardson » :

$$\mathbb{S}^\alpha(\mathbb{C}^n) \otimes \mathbb{S}^\beta(\mathbb{C}^n) = \bigoplus_\gamma \mathbb{S}^\gamma(\mathbb{C}^n)^{\oplus c^\gamma_{\alpha,\beta}}.$$

Ils sont indexés par des triplets de partitions et il existe une règle combinatoire permettant de les calculer : la règle de Littlewood-Richardson. Elle exprime également les coefficients de Littlewood-Richardson en terme de tableaux de Young semi-standards particuliers : soient α, β, et γ des partitions vérifiant $|\alpha| + |\beta| = |\gamma|$ (il s'agit d'une condition nécessaire pour avoir $c^\gamma_{\alpha,\beta} \neq 0$). On considère alors le « diagramme de Young gauche » de forme γ/α : il s'agit simplement du diagramme obtenu en enlevant au diagramme de Young de γ celui de α (si ce n'est pas possible, c'est que le coefficient de Littlewood-Richardson est 0). Par exemple,



On appelle alors « tableau de Littlewood-Richardson » un tableau semi-standard gauche (i.e. la même chose qu'un tableau semi-standard, mais en partant d'un diagramme de Young gauche) tel que la suite obtenue en concaténant ses lignes inversées (i.e. lues de droite à gauche) est un mot de treillis : dans tout préfixe de cette suite, tout nombre i apparaît au moins autant de fois que le nombre $i+1$. La règle en question exprime alors que le coefficient de Littlewood-Richardson $c^\gamma_{\alpha,\beta}$ est égal au nombre de tableaux de Littlewood-Richardson de forme γ/α et de poids β. Voyons ce que cela donne sur un exemple : si $\alpha = (2,1)$, $\beta = (3,2,1)$, et $\gamma = (4,3,2)$, le coefficient est alors 2 car il y a exactement deux tableaux de Littlewood-Richardson de forme $(4,3,2)/(2,1)$ et de poids $(3,2,1)$:



Un autre problème très intéressant concernant ces coefficients était appelé la « Conjecture de Saturation » : a-t-on, pour tout triplet de partitions (α, β, γ),

$$\exists N \in \mathbb{N}^*,\ c^{N\gamma}_{N\alpha,N\beta} \neq 0 \quad \Longrightarrow \quad c^\gamma_{\alpha,\beta} \neq 0 \quad ?$$

(Le fait que $c_{\alpha,\beta}^{\gamma} \neq 0 \Rightarrow \forall N \in \mathbb{N}^*,\ c_{N\alpha,N\beta}^{N\gamma} \neq 0$ est bien plus facile à démontrer.) L'importance de cette question venait notamment de son lien avec la « Conjecture de Horn », provenant d'un problème assez ancien concernant les matrices hermitiennes : étant données deux matrices hermitiennes, que peut-on dire du spectre de leur somme ? Les personnes intéressées pourront par exemple se reporter à cet article introductif de W. Fulton : [Ful00]. La réponse finale à cette question vint de la démonstration de la Conjecture de Saturation par A. Knutson et T. Tao (voir [KT99] ou [Buc00]).

- L'exemple qui va le plus nous intéresser dans cette thèse est celui des coefficients de Kronecker. Comme pour l'exemple précédent, il s'agit de décomposer le produit tensoriel de deux représentations irréductibles d'un groupe réductif, qui se trouve cette fois être le groupe fini $\mathfrak{S}_k$ des permutations de l'ensemble $[\![1,k]\!]$ (où k est un entier strictement positif). Il est bien connu que les représentations irréductibles d'un groupe fini sont en bijection avec les classes de conjugaison de ce dernier. En particulier, les représentations irréductibles de $\mathfrak{S}_k$ sont même indexées par les partitions de l'entier k. Étant donnée une telle partition α, on notera M_α le $\mathfrak{S}_k$-module irréductible correspondant. Les coefficients de Kronecker sont alors les multiplicités apparaissant dans la décomposition :

$$M_\alpha \otimes M_\beta = \bigoplus_{\gamma \vdash k} M_\gamma^{\oplus g_{\alpha,\beta,\gamma}}$$

(où α et β sont des partitions de k). On utilise la notation $g_{\alpha,\beta,\gamma}$, sans distinction notable entre les trois partitions, car la valeur du coefficient ne dépend en fait pas de l'ordre de celles-ci. Cela est dû au fait que les $\mathfrak{S}_k$-modules irréductibles sont auto-duaux. Bien que l'on pourrait penser qu'étudier les coefficients de Kronecker serait plus simple qu'étudier ceux de Littlewood-Richardson (par exemple parce qu'ils proviennent de la théorie des représentations des groupes finis), on sait en fait que ces derniers sont des coefficients de Kronecker particuliers. Les coefficients de Kronecker forment alors une classe plus large de coefficients de branchement et il n'existe par exemple pour l'instant pas de règle combinatoire semblable à la règle de Littlewood-Richardson pour les calculer. Une autre illustration possible de la difficulté que leur étude peut présenter est que l'on sait que ces coefficients ne possèdent pas la propriété de saturation.

1.2 Organisation de la thèse et résultats principaux

La majeure partie de cette thèse concerne certaines notions de stabilité des coefficients de Kronecker. Remarquons tout d'abord que l'on peut étendre la définition de ces coefficients à des triplets de partitions n'ayant pas toutes la même taille en décrétant simplement que le coefficient de Kronecker est dans ce cas zéro.

Définition 1.2.1. Soit (α,β,γ) un triplet de partitions. On dit qu'il est :

- stable lorsque $g_{\alpha,\beta,\gamma} \neq 0$ et, pour tout triplet de partitions (λ, μ, ν), la suite
$$(g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma})_{d\in\mathbb{N}}$$
est stationnaire;
- faiblement stable lorsque, pour tout entier d strictement positif, $g_{d\alpha,d\beta,d\gamma} = 1$;
- presque stable lorsque, pour tout entier d strictement positif, $g_{d\alpha,d\beta,d\gamma} \leq 1$ et lorsqu'il existe un entier d_0 strictement positif tel que $g_{d_0\alpha,d_0\beta,d_0\gamma} \neq 0$.

Les notions de stabilité et de faible-stabilité proviennent des travaux de J. Stembridge dans [Ste14]. La première a été introduite dans le but de généraliser un comportement remarqué par F. Murnaghan en 1938 : avec la définition ci-dessus, Murnaghan s'est rendu compte que le triplet $\big((1),(1),(1)\big)$ était stable. La deuxième notion a alors été introduite par Stembridge dans le but de trouver une caractérisation de la stabilité qui serait plus facile à vérifier en pratique. En effet, Stembridge a démontré que tout triplet stable est faiblement stable et conjecturé que la réciproque est également vraie. S. Sam et A. Snowden ont ensuite prouvé cette conjecture, dans [SS16], par des méthodes algébriques. Notons aussi que P.-E. Paradan a également donné, dans [Par17], une autre preuve de ce fait dans un contexte plus large.

Le but du Chapitre 4[1] est de donner une nouvelle preuve du fait qu'un triplet faiblement stable est stable. Cette preuve est plus géométrique, et basée sur une autre expression classique des coefficients de Kronecker : pour tout triplet (α, β, γ) de partitions, il existe une variété projective X – un produit de variétés de drapeaux – sur laquelle agit un groupe réductif complexe connexe G (les deux ne dépendant que des longueurs des trois partitions), et un fibré en droites $\mathcal{L}_{\alpha,\beta,\gamma}$ G-linéarisé sur X tels que :
$$g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X, \mathcal{L}_{\alpha,\beta,\gamma})^G$$
(voir le Chapitre 3 pour les détails). Notre démonstration utilise de plus quelques notions de Théorie Géométrique des Invariants (qui sont également présentées dans le Chapitre 3), en particulier la notion de points semi-stables relativement à un fibré en droites G-linéarisé sur X (dont l'ensemble dans X est noté – si $\mathcal{L}$ est le fibré en droites – $X^{ss}(\mathcal{L})$). On se sert en outre de certaines conséquences des résultats de V. Guillemin et E. Sternberg dans [GS82], ou de C. Teleman dans [Tel00], sur ce que l'on appelle « quantisation commute à réduction ». On y utilise enfin un corollaire du théorème du Slice Étale de D. Luna (cf [Lun73]). On obtient un résultat en un certain sens un peu plus précis :

Théorème 1.2.2. *Soient (α, β, γ) et (λ, μ, ν) deux triplets de partitions, dont le premier est faiblement stable. Il existe alors un entier positif D tel que, pour tout entier $d \geq D$, $X^{ss}\left(\mathcal{L}_{\lambda,\mu,\nu} \otimes \mathcal{L}_{\alpha,\beta,\gamma}^{\otimes d}\right) \subset X^{ss}\left(\mathcal{L}_{\alpha,\beta,\gamma}\right)$. De plus, la suite de terme général $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ est constante si $d \geq D$. En particulier, (α, β, γ) est stable.*

[1] Précisons que ce chapitre forme avec les deux premiers paragraphes du Chapitre 5 un article soumis en janvier 2017.

Cette caractérisation d'une « borne de stabilisation » (l'entier D du théorème) nous donne la possibilité de calculer explicitement de telles bornes pour des exemples assez petits de triplets stables. C'est en effet ce que l'on fait dans la suite : en utilisant le critère de Hilbert-Mumford (voir dans le Chapitre 3), on peut calculer des bornes de stabilisation explicites pour les triplets stables $\big((1),(1),(1)\big)$ et $\big((1,1),(1,1),(2)\big)$.

Théorème 1.2.3. *Soit (λ,μ,ν) un triplet de partitions, tel que $n_1=\ell(\lambda)$ et $n_2=\ell(\mu)$. On pose* [2]

$$D_1=\left\lceil \frac{1}{2}\left(-\lambda_1+\lambda_2-\mu_1+\mu_2+2(\nu_2-\nu_{n_1n_2})+\sum_{k=1}^{n_1+n_2-4}(\nu_{k+2}-\nu_{n_1n_2-k})\right)\right\rceil .$$

Alors, pour tout $d\geq D_1$, $g_{\lambda+d(1),\mu+d(1),\nu+d(1)}=g_{\lambda+D_1(1),\mu+D_1(1),\nu+D_1(1)}$.

Théorème 1.2.4. *Dans le même contexte, on pose $m=\max(-\lambda_2-\mu_1,-\lambda_1-\mu_2)$, et*

$$D_2=\begin{cases}\left\lceil \frac{1}{2}\left(m+\lambda_3+\mu_3+2(\nu_2-\nu_{n_1n_2})+\sum_{k=1}^{n_1+n_2-4}(\nu_{k+2}-\nu_{n_1n_2-k})\right)\right\rceil & \text{si } n_1,n_2\geq 3\\ \left\lceil \frac{1}{2}\left(m+\mu_3+2\nu_2-\nu_{2n_2}+\sum_{k=1}^{n_2-1}\nu_{k+2}\right)\right\rceil & \text{si } n_1=2\\ \left\lceil \frac{1}{2}\left(m+\lambda_3+2\nu_2-\nu_{2n_1}+\sum_{k=1}^{n_1-1}\nu_{k+2}\right)\right\rceil & \text{si } n_2=2\end{cases}.$$

Alors, pour tout $d\geq D_2$, $g_{\lambda+d(1,1),\mu+d(1,1),\nu+d(2)}=g_{\lambda+D_2(1,1),\mu+D_2(1,1),\nu+D_2(2)}$.

Précisons que, pour raffiner légèrement les bornes obtenues (et ainsi obtenir celles écrites ci-dessus), on utilise un résultat classique de quasi-polynomialité qui concerne la dimension d'un sous-espace d'invariants dans une représentation irréductible d'un groupe réductif complexe. On écrit ainsi une démonstration de cette quasi-polynomialité dans un contexte suffisant pour l'utilisation que l'on en fait.

Dans le cas du triplet $\big((1),(1),(1)\big)$, il existait déjà certaines bornes, obtenues notamment par M. Brion (cf [Bri93]), E. Vallejo (cf [Val99]), et E. Briand, R. Orellana, et M. Rosas (cf [BOR11]). On observe que notre méthode permet de retrouver deux de ces bornes : celle due à Brion et une des deux dues à Briand-Orellana-Rosas. Pour le triplet $\big((1,1),(1,1),(2)\big)$ il n'existait à notre connaissance pas de telle borne.

Dans le Chapitre 5, on montre que les méthodes que l'on utilise s'applique de manière intéressante à d'autres types de coefficients de branchement (ce qui n'est pas surprenant, puisque l'on a expliqué que Paradan avait obtenu des résultats similaires à « stable $\Leftrightarrow$ faiblement stable » dans un cadre bien plus large). On s'intéresse donc d'abord, dans le paragraphe 5.1, à des coefficients appelés coefficients de pléthysme. Il s'agit

[2] La notation $\lceil x\rceil$ désigne la partie entière supérieure du réel x (i.e. l'entier vérifiant $\lceil x\rceil-1<x\leq\lceil x\rceil$).

des coefficients de branchement qui apparaissent lorsque l'on compose des foncteurs de Schur : si λ et μ sont deux partitions, si $\ell(\lambda)$ n'est « pas trop grande » par rapport à μ (voir la Définition 5.1.1 pour plus de précision), et si V est un $\mathbb{C}$-espace vectoriel de dimension finie au moins $\ell(\mu)$, alors $\mathbb{S}^\lambda(\mathbb{S}^\mu V)$ – qui par définition est un $\mathrm{GL}(\mathbb{S}^\mu V)$-module simple – est une représentation de $\mathrm{GL}(V)$. Les multiplicités dans sa décomposition en somme directe d'irréductibles sont les coefficients de pléthysme. On obtient sur ceux-ci un résultat de stabilité – que Sam et Snowden avaient déjà prouvé dans [SS16] –, que l'on applique ensuite pour redémontrer que deux exemples de suites de tels coefficients sont stationnaires. Ces deux exemples avaient déjà été obtenus par L. Colmenarejo dans [Col17].

Le deuxième autre exemple de coefficients de branchement considérés est celui du produit tensoriel de représentations irréductibles du groupe hyperoctaédral. Il s'agit d'un groupe fini qui est le groupe de Weyl W_n de type B_n (pour $n \geq 2$), et qui peut s'écrire sous la forme d'un produit semi-direct : $W_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ (cf Paragraphe 5.2 pour des précisions). On doit d'abord utiliser une sorte de dualité de Schur-Weyl pour ce groupe – due à M. Sakamoto et T. Shoji, dans [SS99] –, qui permet de ré-exprimer les coefficients de branchement considérés uniquement à l'aide de groupes connexes. On peut alors obtenir un analogue de l'équivalence « stable $\Leftrightarrow$ faiblement stable » et trouver une borne de stabilisation dans un cas similaire à la stabilité de Murnaghan pour les coefficients de Kronecker.

Le troisième et dernier exemple de ce chapitre concerne le produit de Heisenberg, introduit par M. Aguiar, W. Ferrer Santos, et W. Moreira dans [AFSM15]. Leur but était d'unifier différents produits ou co-produits définis dans différents contextes et, dans celui des représentations du groupe symétrique, ce produit mène à la définition par L. Ying (cf [Yin17]) des coefficients de Aguiar, qui en un certain sens généralisent les coefficients de Kronecker. Dans ce même article, Ying démontre un résultat de stabilité des coefficients de Aguiar similaire à la stabilité de Murnaghan. On parvient à redémontrer et généraliser ce résultat en prouvant que les coefficients de Aguiar sont également les coefficients de branchement pour la situation $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \to \hat{G} = \mathrm{GL}\big(V_1 \oplus (V_1 \otimes V_2) \oplus V_2\big)$ (pour V_1 et V_2 deux $\mathbb{C}$-espaces vectoriels de dimension finie). On s'intéresse également à quelques bornes de stabilisation.

Dans le Chapitre 6, on s'intéresse à présent à certaines faces de ce que l'on appelle le cône de Kronecker : pour n_1 et n_2 deux entiers strictement positifs fixés, on note $\mathcal{P}_{n_1,n_2}$ l'ensemble des triplets de partitions tels que $\ell(\alpha) \leq n_1$, $\ell(\beta) \leq n_2$, et $\ell(\gamma) \leq n_1 n_2$. L'expression précédente des coefficients de Kronecker induit par exemple facilement que $\big\{(\alpha,\beta,\gamma) \in \mathcal{P}_{n_1,n_2} \text{ t.q. } g_{\alpha,\beta,\gamma} \neq 0\big\}$ est un semi-groupe (i.e. est stable par addition). Ce qui nous intéresse alors est le cône engendré par ce semi-groupe :

$$\mathrm{PKron}_{n_1,n_2} = \big\{(\alpha,\beta,\gamma) \text{ t.q. } \exists N \in \mathbb{N}^*,\ g_{N\alpha,N\beta,N\gamma} \neq 0\big\}$$

(que l'on note comme dans [Man15a]). C'est un cône polyédral appelé le « cône de

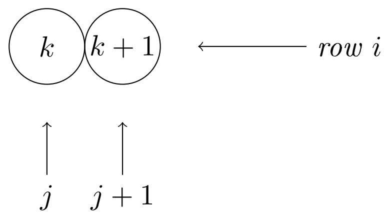
Kronecker » et un résultat connu (voir [Man15a], Paragraphe 2.4) qui illustre l'intérêt des triplets stables est le suivant : ils sont situés sur des faces de ce cône. On aimerait donc trouver un moyen de produire de telles faces qui contiennent uniquement des triplets stables, ou au moins presque stables.

Parmi les faces du cône de Kronecker, on s'intéressera tout particulièrement à certaines qui sont qualifiées de « régulières » : il s'agit de celles qui contiennent au moins un triplet (α, β, γ) tel que α, β, et γ sont régulières (c'est-à-dire possèdent respectivement n_1, n_2, et $n_1 n_2$ parts deux à deux distinctes). On se place alors de nouveau dans le contexte des coefficients de Kronecker (voir Chapitre 3) : autrement dit on pose $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ et on considère un sous-groupe à un paramètre du tore maximal T de G formé des matrices diagonales :

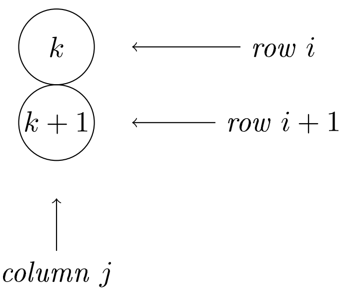
$$\begin{array}{rccl} \tau : & \mathbb{C}^* & \longrightarrow & T \\ & t & \longmapsto & \left(\begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_{n_1}} \end{pmatrix}, \begin{pmatrix} t^{b_1} & & \\ & \ddots & \\ & & t^{b_{n_2}} \end{pmatrix} \right) \end{array}$$

(avec $a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2} \in \mathbb{Z}$). Supposons de plus que τ est dominant et régulier (i.e. $a_1 > \cdots > a_{n_1} \geq 0$ et $b_1 > \cdots > b_{n_2} \geq 0$), et également $\hat{G}$-régulier (i.e. les entiers $a_i + b_j$ sont deux à deux distincts). On construit alors la matrice $M = (a_i + b_j)_{i,j}$, ainsi que ce que l'on appelle la « matrice d'ordre de τ » : il s'agit de la matrice dans laquelle chaque coefficient de M est remplacé par son rang dans la suite des coefficients $a_i + b_j$ ordonnés de manière décroissante. Il existe alors un résultat dû à L. Manivel (cf [Man15a]) et E. Vallejo (cf [Val14]) énonçant qu'une telle matrice d'ordre donne une face régulière explicite de PKron_{n_1,n_2}, de dimension minimale parmi ces faces (i.e. de dimension $n_1 n_2$), qui ne contient que des triplets stables. On étend ce résultat en prouvant qu'une matrice d'ordre donne en fait d'autres faces :

Théorème 1.2.5. *Pour tout sous-groupe à un paramètre τ de T dominant, régulier, et $\hat{G}$-régulier, toute configuration du type suivant dans la matrice d'ordre:*



donne une face régulière de dimension $n_1 n_2$ du cône de Kronecker PKron_{n_1,n_2}*, qui ne contient que des triplets stables, et donnée explicitement dans le Paragraphe 6.3.4. De même, toute configuration du type*



donne une telle face explicite de dimension $n_1 n_2$.

On définit aussi (cf Paragraphe 6.3.5) cinq autres sortes de configurations pouvant apparaître dans une matrice d'ordre (appelées Configurations Ⓐ à Ⓔ), qui mettent cette fois en jeu trois ou quatre de ses coefficients. On montre alors que :

Théorème 1.2.6. *Soit* τ *un sous-groupe à un paramètre dominant, régulier, et* $\hat{G}$*-régulier de* T. *Toute configuration d'un des types* Ⓐ *à* Ⓔ *apparaissant dans la matrice d'ordre de* τ *donne alors une face – pas nécessairement régulière et possiblement réduite à zéro – du cône de Kronecker* PKron_{n_1,n_2} *qui ne contient que des triplets presque stables.*

On conclut ce chapitre en regardant tous les exemples de matrices d'ordre possibles de taille 2×2, 3×2, et 3×3, afin de voir combien nos résultats précédents produisent de nouvelles (i.e. par rapport au résultat de Manivel et Vallejo) faces. Par exemple, dans le cas des matrices d'ordre de taille 3×2, on obtient 23 nouvelles faces régulières de $\mathrm{PKron}_{3,2}$ qui ne contiennent que des triplets stables, alors que 5 autres étaient déjà connues. On obtient également 2 autres nouvelles faces non régulières, qui elles ne contiennent que des triplets presque stables.

Dans le Chapitre 7, on s'intéresse à des « zéros » apparaissant dans le cône de Kronecker : par zéros, on entend des triplets $(\alpha, \beta, \gamma) \in \mathrm{PKron}_{n_1,n_2}$ tels que $g_{\alpha,\beta,\gamma} = 0$. L'existence de tels triplets correspond au fait que les coefficients de Kronecker ne possèdent pas la propriété de saturation, et les comprendre est un grand problème dans l'étude de ces coefficients. Quand on considère un tel zéro (α, β, γ), on va comme précédemment s'intéresser à la demi-droite $\mathbb{N}^*(\alpha, \beta, \gamma)$, et plus particulièrement à $\Lambda(\alpha, \beta, \gamma) = \{d \in \mathbb{N}^* \text{ t.q. } g_{d\alpha,d\beta,d\gamma} \neq 0\}$. On remarque que, dans quasiment tous les exemples connus, ce semi-groupe est de la forme $d_0\mathbb{N}^*$ pour un entier strictement positif d_0. On montre que c'est en fait toujours le cas lorsque le triplet de départ est presque stable (notons que ceci est également une conséquence immédiate des résultats de Paradan, en particulier du Théorème B de [Par17]) :

Théorème 1.2.7. *Soit* (α, β, γ) *un triplet de partitions presque stable. Il existe alors* $d_0 \in \mathbb{N}^*$ *tel que, pour tout* $d \in \mathbb{N}^*$,

$$d \in \Lambda(\alpha, \beta, \gamma) \quad \Longleftrightarrow \quad d_0 | d$$

Ce résultat n'est par contre pas vrai pour tous les triplets dans PKron_{n_1,n_2}. Il existe en effet une famille de contre-exemples, donnée par Briand, Orellana, et Rosas dans [BOR09] (Theorem 2.4), dont le plus petit est $\big((6,6),(7,5),(6,4,2)\big)$. Il s'agit à notre connaissance des seuls exemples connus où $\Lambda(\alpha,\beta,\gamma)$ n'est pas de la forme $d_0\mathbb{N}^*$. On étudie donc géométriquement et en détail cet exemple et on parvient à montrer que :

Proposition 1.2.8. *Notons Q le groupe des quaternions, vu comme un sous-groupe (de cardinal 8) de $\mathrm{SL}_2(\mathbb{C})$. Alors, pour tout entier d strictement positif,*

$$g_{d(6,6),d(7,5),d(6,4,2)} = \dim \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q = \dim\left(\mathbb{C}[x,y]_{2d}\right)^Q,$$

où $\mathbb{C}[x,y]_{2d}$ désigne le $\mathbb{C}$-espace vectoriel des polynômes homogènes en deux variables x et y, de degré $2d$, sur lequel $Q \subset \mathrm{SL}_2(\mathbb{C})$ agit par son action naturelle sur (x,y).

Notons que ce résultat est aussi valable pour les autres contre-exemples de la famille donnée dans [BOR09]. On aimerait alors utiliser ce genre de résultat pour produire de nouveaux exemples comme ceux-ci. Malheureusement il est déjà assez compliqué de trouver un autre groupe fini dont, comme Q, l'action sur des espaces de polynômes homogènes donne des dimensions intéressantes pour les espaces des invariants. On parvient seulement à trouver un exemple comme cela, mais on ne parvient pas vraiment à l'exploiter pour produire d'autres triplets comme $\big((6,6),(7,5),(6,4,2)\big)$.

Chapter 2

Introduction

2.1 Presentation of the branching problem

A complex affine algebraic group is a group that is an affine algebraic variety defined over the field $\mathbb{C}$ of complex numbers, such that the multiplication and inversion operations are given by regular maps on the variety. Such a group has a radical, which is the identity component of its maximal closed normal solvable subgroup. And the subgroup of this radical formed by unipotent elements is called the unipotent radical of the affine algebraic group. Equivalently, the unipotent radical of a complex affine algebraic group is its maximal closed unipotent normal subgroup.

Examples: For a positive integer n, the groups $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ (of matrices of determinant 1 and invertible matrices, respectively) are for instance complex affine algebraic groups. The radical of $\mathrm{SL}_n(\mathbb{C})$ is trivial, whereas the radical of $\mathrm{GL}_n(\mathbb{C})$ is the subgroup of the scalar matrices: $\{t\,\mathrm{I}_n\,;\,t \in \mathbb{C}^*\}$. Both unipotent radicals of these groups are trivial.

Definition 2.1.1. A complex affine algebraic group whose unipotent radical is trivial is said to be reductive. When the group is moreover connected and its radical is trivial, it is said to be semisimple.

Examples: According to what precedes, $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ are reductive, and the former is even semisimple. Other classical examples of complex reductive groups include $\mathrm{SO}_n(\mathbb{C})$ (group of orthogonal matrices of determinant 1), $\mathrm{Sp}_{2n}(\mathbb{C})$ (symplectic group), all the finite groups...

The name "reductive" comes from an important property of these groups: the complete reducibility of their representations. Indeed, every finite dimensional complex representation of a complex reductive group decomposes as a direct sum of simple (or irreducible) representations. What is moreover interesting is that, for connected groups, we know how to construct these irreducible representations.

To a complex connected reductive group G one can associate its Lie algebra $\mathfrak{g}$. It is a reductive Lie algebra and comes then with an associated root system. This combinatorial data brings in particular a notion of weights, and some of them are called "integral dominant". Then to any integral dominant weight λ of G one can associate an irreducible representation of G, called a highest weight module. We will denote the highest weight module of G of highest weight λ by $V_G(\lambda)$. Conversely, any finite dimensional rational complex irreducible representation of G is a $V_G(\lambda)$ for a certain integral dominant weight λ of G.

Example: Probably the most basic example of a connected reductive group is $\mathrm{GL}_n(\mathbb{C})$, for which the integral dominant weights are easy to describe: they are exactly the non-increasing finite sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ of integers, of length n. Such a finite sequence yields a character of T, which is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ formed by the diagonal matrices, in the following way:

$$\begin{array}{rccc} e^\alpha : & T & \longrightarrow & \mathbb{C}^* \\ & \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} & \longmapsto & t_1^{\alpha_1} \ldots t_n^{\alpha_n} \end{array}.$$

The irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ associated to such an integral dominant weight is denoted by $\mathbb{S}^\alpha(\mathbb{C}^n)$. Isomorphically, when one considers the group $\mathrm{GL}(V)$ – of automorphisms of a finite dimensional $\mathbb{C}$-vector space V –, every non-increasing finite sequence α of integers, of length $\dim(V)$, gives the irreducible representation $\mathbb{S}^\alpha V$ of $\mathrm{GL}(V)$.

We will only be interested in a particular sort of representations of $\mathrm{GL}_n(\mathbb{C})$: the ones which are said to be polynomial. They are the representations for which the action of each $g \in \mathrm{GL}_n(\mathbb{C})$ is given by a fixed family of polynomials in the entries of g. Among the irreducible ones, the polynomial representations are easy to characterise: for $\alpha = (\alpha_1, \ldots, \alpha_n)$ an integral dominant weight of $\mathrm{GL}_n(\mathbb{C})$, $\mathbb{S}^\alpha(\mathbb{C}^n)$ is polynomial if and only if $\alpha_n \geq 0$. Thus the finite dimensional irreducible polynomial complex representations of $\mathrm{GL}_n(\mathbb{C})$ are given by the partitions of length at most n, which are finite non-increasing sequences $\alpha = (\alpha_1, \ldots, \alpha_k)$ of positive integers, whose length k is the length of the partition, denoted $\ell(\alpha)$. We also denote by $|\alpha| = \sum_{i=1}^k \alpha_i$ the weight of such a partition α, which is then said to be a partition of the integer $|\alpha|$

The branching problem: Consider now two connected complex reductive groups, G and $\hat{G}$, and a morphism $f : G \to \hat{G}$. Then, for any dominant weight $\hat{\lambda}$ of $\hat{G}$, the highest weight module $V_{\hat{G}}(\hat{\lambda})$ is, via the morphism f, also a (finite dimensional complex) representation of G and, as such, it decomposes into a direct sum of irreducible representations of G:

$$V_{\hat{G}}(\hat{\lambda}) = \bigoplus_{\lambda \text{ dominant weight of } G} V_G(\lambda)^{\oplus c(\lambda, \hat{\lambda})}.$$

Definition 2.1.2. The multiplicities $c(\lambda, \hat{\lambda})$ appearing in the previous decomposition are non-negative integers which are called the branching coefficients.

The branching problem consists in studying these branching coefficients. Studying these can mean finding a combinatorial way of computing them, and in some cases it has been done. But it can also mean studying other more qualitative aspects of those, as we will see later.

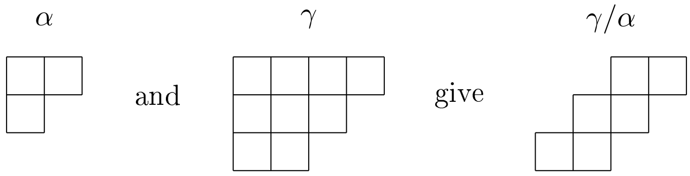
Examples:

- If $n \geq 2$, we can form a morphism from $\mathrm{GL}_{n-1}(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})$ by sending $A \in \mathrm{GL}_{n-1}(\mathbb{C})$ to $\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$. Then the branching coefficients for this situation are indexed by a pair of partitions, the first one with length at most $n-1$ and the second one with length at most n. For such a pair $\big(\underbrace{(\lambda_1, \dots, \lambda_{n-1})}_{=\lambda}, \underbrace{(\mu_1, \dots, \mu_n)}_{=\mu}\big)$, the corresponding branching coefficient is known:
$$c(\lambda, \mu) = \begin{cases} 1 & \text{if } \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \dots \geq \mu_{n-1} \geq \lambda_{n-1} \geq \mu_n \\ 0 & \text{otherwise} \end{cases}.$$

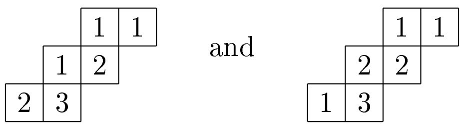
- If the branching situation is $T \subset \mathrm{GL}_n(\mathbb{C})$, where T is the maximal torus in $\mathrm{GL}_n(\mathbb{C})$ constituted of the diagonal matrices (and the morphism between the two is then the identity), then we are looking at how a polynomial highest weight module of $\mathrm{GL}_n(\mathbb{C})$ decomposes as a direct sum of weight spaces (spaces on which T acts by a certain character λ). The corresponding branching coefficients are called the Kostka numbers and are indexed by pairs composed of a sequence of non-negative integers of length n (a dominant weight – say λ – of T) and of a partition of length at most n (a particular dominant weight – say μ – of $\mathrm{GL}_n(\mathbb{C})$). Then the Kostka number $k_{\mu,\lambda}$ can be computed as the number of semistandard Young tableaux of shape μ and weight λ, i.e. the number of ways to fill the Young diagram of μ with λ_1 1's, λ_2 2's, etc, in a non-decreasing way along each row and an increasing way down each column.

- One famous example is the case of the tensor product of polynomial irreducible representations of $\mathrm{GL}_n(\mathbb{C})$: when $G = \mathrm{GL}_n(\mathbb{C})$ is embedded diagonally inside $\hat{G} = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, we are in fact looking at how the tensor product of two polynomial irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ decomposes into a direct sum of such representations. The corresponding branching coefficients are called the Littlewood-Richardson coefficients:
$$\mathbb{S}^\alpha(\mathbb{C}^n) \otimes \mathbb{S}^\beta(C^n) = \bigoplus_\gamma \mathbb{S}^\gamma(\mathbb{C}^n)^{\oplus c^\gamma_{\alpha,\beta}}.$$

They are indexed by triples of partitions, and there exists a combinatorial way of computing them: the Littlewood-Richardson rule. This also expresses the Littlewood-Richardson coefficients in terms of particular semistandard Young tableaux: let α, β, and γ be partitions such that $|\alpha| + |\beta| = |\gamma|$ (this is a simple necessary condition to have $c_{\alpha,\beta}^{\gamma} \neq 0$). Then we consider the "skew Young diagram" of shape γ/α: it is simply the diagram obtained by the set-theoretic difference of the Young diagrams of γ and α. For example,



Then a Littlewood-Richardson tableau is a skew semistandard tableau (i.e. the same as a semistandard tableau, but starting from a skew Young diagram) which has the additional property that the sequence obtained by concatenating its reversed rows is a lattice word: in every initial part of this sequence, any number i occurs at least as often as the number $i+1$. The rule thus states that the Littlewood-Richardson coefficient $c_{\alpha,\beta}^{\gamma}$ is the number of Littlewood-Richardson tableaux of shape γ/α and weight β. Let us return to our previous example: for $\alpha = (2,1)$, $\beta = (3,2,1)$, and $\gamma = (4,3,2)$, the coefficient is 2 because there are exactly two Littlewood-Richardson tableaux of shape $(4,3,2)/(2,1)$ and weight $(3,2,1)$:



Another really important problem concerning these coefficients was the Saturation Conjecture: is that true that, for all triple of partitions (α, β, γ),

$$\exists N \in \mathbb{N}^*,\ c_{N\alpha,N\beta}^{N\gamma} \neq 0 \quad \Longrightarrow \quad c_{\alpha,\beta}^{\gamma} \neq 0 \quad ?$$

(The fact that $c_{\alpha,\beta}^{\gamma} \neq 0 \Rightarrow \forall N \in \mathbb{N}^*,\ c_{N\alpha,N\beta}^{N\gamma} \neq 0$ is much easier to prove.) The importance of this question was highlighted by its connection with the so-called Horn conjecture, coming from an "old" problem concerning hermitian matrices: given two hermitian matrices, what can one say about the spectrum of their sum? Any interested reader can for instance read this survey by W. Fulton: [Ful00]. The final answer to this question came with the proof of the Saturation Conjecture by A. Knutson and T. Tao (see [KT99] or [Buc00]).

- The example in which we will be the most interested in this thesis is the example of the Kronecker coefficients. As in the previous example, it starts with the problem of decomposing a tensor product of two irreducible representations of a reductive group, which is this time the finite group $\mathfrak{S}_k$ of permutations of the set $[\![1,k]\!]$ (for some positive integer k). For a finite group, it is known that the complex irreducible representations are in bijection with the conjugacy classes of the group. Therefore, for $\mathfrak{S}_k$, the irreducible representations are even indexed by the partitions of the integer k. Given such a partition α, the corresponding irreducible $\mathfrak{S}_k$-module will be denoted by M_α. Then the Kronecker coefficients are the multiplicities appearing in the decomposition:

$$M_\alpha \otimes M_\beta = \bigoplus_{\gamma \vdash k} M_\gamma^{\oplus g_{\alpha,\beta,\gamma}}$$

(where α and β are two partitions of k). We use the notation $g_{\alpha,\beta,\gamma}$ (with no real difference between the three partitions) because the value of the coefficient does not depend on the order of the partitions indexing it. This is due to the fact that the irreducible $\mathfrak{S}_k$-modules are self-dual. Although one could think that studying Kronecker coefficients would be easier than studying Littlewood-Richardson coefficients (for instance because they come from the representation theory of finite groups), the latter are actually known to be special Kronecker coefficients. Then the Kronecker coefficients form a bigger class of branching coefficients and, for example, no combinatorial rule such as the Littlewood-Richardson rule is known for them. Another possible illustration of their added complexity is that it is known that they do not have the saturation property.

2.2 Organisation of the thesis and main results

A large part of this thesis will concern some notions of stability for Kronecker coefficients. Notice that we have only defined Kronecker coefficients associated to triples of partitions of the same size. For easier notations we can also define that the Kronecker coefficient associated to a triple of partitions in which (at least) two have different sizes is simply zero.

Definition 2.2.1. Let (α, β, γ) be a triple of partitions. It is said to be:

- stable if $g_{\alpha,\beta,\gamma} \neq 0$ and, for all triples of partitions (λ, μ, ν), the sequence

$$\left(g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}\right)_{d\in\mathbb{N}}$$

  eventually stabilises;

- weakly stable if, for all positive integers d, $g_{d\alpha,d\beta,d\gamma} = 1$;

- almost stable if, for all positive integers d, $g_{d\alpha,d\beta,d\gamma} \leq 1$ and there exists a positive integer d_0 such that $g_{d_0\alpha,d_0\beta,d_0\gamma} \neq 0$.

The notions of stability and weak-stability come from the work of J. Stembridge in [Ste14]. The notion of stability was introduced in order to generalise a behaviour noticed by F. Murnaghan in 1938: in the terms of Stembridge's definition, Murnaghan noticed that the triple $\big((1),(1),(1)\big)$ is stable. The notion of weak-stability was then introduced in order to find a characterisation of stability which would be much simpler to check. Indeed Stembridge proved that a stable triple is weakly stable[1], and conjectured that the converse is also true. S. Sam and A. Snowden then managed in [SS16] to prove this conjecture, by completely algebraic methods. Note that P.-E. Paradan also gave another proof – approximately at the same time as ours – of this fact in [Par17], in a much more general setting and with methods closer to ours.

The goal of Chapter 4[2] is to give another proof of the fact that a weakly stable triple is stable. The proof that we give is a geometric one, based on a well-known expression of Kronecker coefficients: for any triple (α,β,γ) of partitions, there exist a projective variety X – which is a product of flag varieties – on which acts a complex connected reductive group G (both depending only on the lengths of the partitions), and a G-linearised line bundle $\mathcal{L}_{\alpha,\beta,\gamma}$ on X such that

$$g_{\alpha,\beta,\gamma}=\dim \mathrm{H}^0(X,\mathcal{L}_{\alpha,\beta,\gamma})^G$$

(cf Chapter 3). Our proof moreover uses some notions of Geometric Invariant Theory (also presented in Chapter 3), and in particular the notion of semi-stable points relatively to some G-linearised line bundle on X (whose set in X is denoted – if $\mathcal{L}$ is the line bundle – by $X^{ss}(\mathcal{L})$), as well as some consequences of the work by V. Guillemin and E. Sternberg in [GS82], or later C. Teleman in [Tel00], on "Quantisation commutes with reduction". We also use a corollary of D. Luna's Etale Slice Theorem (cf [Lun73]). We obtain a result which is a little more precise:

Theorem 2.2.2. *Let (α,β,γ) and (λ,μ,ν) be two triples of partitions, the first one being weakly stable. Then there exists a non-negative integer D such that, for all integers $d\geq D$, $X^{ss}\left(\mathcal{L}_{\lambda,\mu,\nu}\otimes\mathcal{L}_{\alpha,\beta,\gamma}^{\otimes d}\right)\subset X^{ss}(\mathcal{L}_{\alpha,\beta,\gamma})$. Moreover the sequence of general term $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ is constant for $d\geq D$. In particular (α,β,γ) is stable.*

This characterisation of a "bound of stabilisation" (the D from the theorem) gives us hope to compute some explicit such bounds for not too difficult examples of stable triples. This is in fact what we do next: using the Hilbert-Mumford numerical criterion (cf Chapter 3), we are able to compute some bounds of stabilisation for two examples of stable triples, namely $\big((1),(1),(1)\big)$ and $\big((1,1),(1,1),(2)\big)$.

Theorem 2.2.3. *Let (λ,μ,ν) be a triple of partitions, with $n_1=\ell(\lambda)$ and $n_2=\ell(\mu)$. We set* [3]

$$D_1=\left\lceil\frac{1}{2}\left(-\lambda_1+\lambda_2-\mu_1+\mu_2+2(\nu_2-\nu_{n_1n_2})+\sum_{k=1}^{n_1+n_2-4}(\nu_{k+2}-\nu_{n_1n_2-k})\right)\right\rceil.$$

Then, for all $d\geq D_1$, $g_{\lambda+d(1),\mu+d(1),\nu+d(1)}=g_{\lambda+D_1(1),\mu+D_1(1),\nu+D_1(1)}$.

[1] We will therefore use this implication in – almost (see at the end of Section 7.1) – the whole thesis.

[2] Note that this chapter, together with the first two sections of Chapter 5, forms an article submitted in January 2017.

[3] The notation $\lceil x\rceil$ stands for the ceiling of the number x (i.e. the integer such that $\lceil x\rceil-1<x\leq\lceil x\rceil$).

Theorem 2.2.4. *In the same context, set* $m = \max(-\lambda_2 - \mu_1, -\lambda_1 - \mu_2)$*, and*

$$D_2 = \begin{cases} \left\lceil \frac{1}{2}\left(m + \lambda_3 + \mu_3 + 2(\nu_2 - \nu_{n_1 n_2}) + \sum_{k=1}^{n_1+n_2-4} (\nu_{k+2} - \nu_{n_1 n_2 - k}) \right) \right\rceil & \text{if } n_1, n_2 \geq 3 \\ \left\lceil \frac{1}{2}\left(m + \mu_3 + 2\nu_2 - \nu_{2n_2} + \sum_{k=1}^{n_2-1} \nu_{k+2} \right) \right\rceil & \text{if } n_1 = 2 \\ \left\lceil \frac{1}{2}\left(m + \lambda_3 + 2\nu_2 - \nu_{2n_1} + \sum_{k=1}^{n_1-1} \nu_{k+2} \right) \right\rceil & \text{if } n_2 = 2 \end{cases}.$$

Then, for all $d \geq D_2$, $g_{\lambda+d(1,1),\mu+d(1,1),\nu+d(2)} = g_{\lambda+D_2(1,1),\mu+D_2(1,1),\nu+D_2(2)}$.

Note that to refine slightly those bounds, we use a classical argument of quasipolynomiality concerning the behaviour of the dimension of invariants in an irreducible representation of a complex reductive group. Before using it we write a proof of this quasipolynomiality in a context – which is not meant to be optimal – sufficient for what we want.

In the case of the triple $\big((1),(1),(1)\big)$ there already existed some bounds, due notably to M. Brion (see [Bri93]), E. Vallejo (see [Val99]), and E. Briand, R. Orellana, and M. Rosas (see [BOR11]). We notice that our methods allow to re-obtain some of those bounds: the one by Brion and one of the two given by Briand-Orellana-Rosas. Moreover we test our bound and compare it on examples with the other ones. For the triple $\big((1,1),(1,1),(2)\big)$, there was not – as far as we know – any already existing such bounds.

In Chapter 5 we show that our methods apply interestingly to other branching coefficients. It comes as no surprise, since we explained quickly earlier that Paradan has proved – in [Par17] – a similar result to the "stable ⇔ weakly stable" one in a much more general setting. Actually the only thing necessary to make our techniques work is an expression of the branching coefficients as the one we gave for the Kronecker coefficients: of the type $\dim \mathrm{H}^0(X, \mathcal{L})^G$, where X, G, and $\mathcal{L}$ are the same kinds of objects as for the Kronecker coefficients (cf Chapter 3 for more precise statements and definitions).

At first we then use our techniques to obtain a stability result – already proven by Sam and Snowden in [SS16] – concerning plethysm coefficients, in Section 5.1. They are the branching coefficients that arise when one composes Schur functors: if one considers two partitions λ and μ, with $\ell(\lambda)$ "not too big" relatively to μ (see Definition 5.1.1 for precisions), and a complex vector space V of finite dimension at least $\ell(\mu)$, then $\mathbb{S}^\lambda(\mathbb{S}^\mu V)$ – which is an irreducible $\mathrm{GL}(\mathbb{S}^\mu V)$-module by definition – is a representation of $\mathrm{GL}(V)$, and thus decomposes as a direct sum of irreducible ones. The multiplicities in this decomposition are the plethysm coefficients. We see thereafter that we can re-obtain two stability properties for these coefficients given by L. Colmenarejo in [Col17].

The second other example of branching coefficients we look at is the example of the tensor product for irreducible representations of the hyperoctahedral group. It is a finite group, which is the Weyl group W_n of type B_n (for $n \geq 2$), and can be written as a semidirect product: $W_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ (see Section 5.2 for precisions). What we first have to do here is using a kind of Schur-Weyl duality for W_n – due to M. Sakamoto and T. Shoji in [SS99] – in order to rewrite the considered branching coefficients as branching coefficients for connected groups. We manage to do this for the following: if we consider complex finite dimensional vector spaces $V_1 = V_1^+ \oplus V_1^-$ and $V_2 = V_2^+ \oplus V_2^-$, then the branching situation considered is with $G = \mathrm{GL}(V_1^+) \times \mathrm{GL}(V_1^-) \times \mathrm{GL}(V_2^+) \times \mathrm{GL}(V_2^-)$ and $\hat{G} = \mathrm{GL}\left((V_1^+ \otimes V_2^+) \oplus (V_1^- \otimes V_2^-)\right) \times \mathrm{GL}\left((V_1^+ \otimes V_2^-) \oplus (V_1^- \otimes V_2^+)\right)$. This allows us to obtain an analogous of the equivalence “stable $\Leftrightarrow$ weakly stable” in that case and to compute an explicit bound of stabilisation in a case similar to the Murnaghan stability for Kronecker coefficients.

Our third and final example in that chapter concerns the Heisenberg product. It was introduced by M. Aguiar, W. Ferrer Santos, and W. Moreira in [AFSM15] in order to unify many related products and coproducts defined on various objects (species, representations of the symmetric groups, endomorphisms of graded connected Hopf algebras...). In the context of representations of the symmetric groups, it lead to the definition by L. Ying – in [Yin17] – of the Aguiar coefficients, which extend in a way the Kronecker ones. In that last article Ying proves also an analogous of Murnaghan's stability for these coefficients, as well as a bound of stabilisation in that case. The Aguiar coefficients are once again defined using finite groups, and thus the first part of our work is to express them as branching coefficients for connected reductive groups: the branching situation we have to consider here is $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ and $\hat{G} = \mathrm{GL}\left(V_1 \oplus (V_1 \otimes V_2) \oplus V_2\right)$, for V_1 and V_2 two finite dimensional complex vector spaces. Then we immediately obtain a general stability result for these coefficients, implying in particular the stability proved by Ying. We finally give some new examples of stable triples for Aguiar coefficients, and look at some bounds of stabilisation.

In Chapter 6 we are interested in some faces of what is called the Kronecker cone, which are related to the stable triples that we previously studied. Let n_1 and n_2 be two positive integers and denote by $\mathcal{P}_{n_1,n_2}$ the set of triples (α, β, γ) of partitions such that $\ell(\alpha) \leq n_1$, $\ell(\beta) \leq n_2$, and $\ell(\gamma) \leq n_1 n_2$. The expression of the Kronecker coefficients that we gave earlier for instance leads to the fact that $\left\{(\alpha, \beta, \gamma) \in \mathcal{P}_{n_1,n_2} \text{ s.t. } g_{\alpha,\beta,\gamma} \neq 0\right\}$ is a semigroup (i.e. is stable under addition). We then consider the cone spanned by this semigroup:

$$\mathrm{PKron}_{n_1,n_2} = \left\{(\alpha, \beta, \gamma) \text{ s.t. } \exists N \in \mathbb{N}^*,\ g_{N\alpha,N\beta,N\gamma} \neq 0\right\}$$

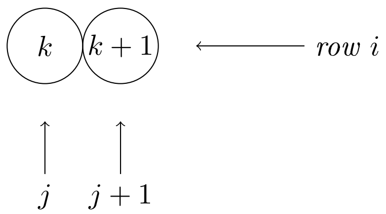
(we use the same notation as in [Man15a]). This is a rational polyhedral cone called the Kronecker cone. Then one of the classical results (see [Man15a], Paragraph 2.4) highlighting the interest for stable triples is that they are located on faces of the Kronecker cone. Therefore one would like to produce some faces of PKron_{n_1,n_2} which contain only stable triples, or at least almost stable ones (see Definition 2.2.1). Among the faces of the Kronecker cone, some particularly interesting ones are those that we will call “regular”: they are the ones which contain at least one triple (α, β, γ) such that α,

β, and γ are regular (meaning that they have respectively n_1, n_2, and n_1n_2 pairwise distinct parts). Then, in the settings of Kronecker coefficients (see Chapter 3 and set $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$), consider a one-parameter subgroup of the maximal torus T of G formed by diagonal matrices:

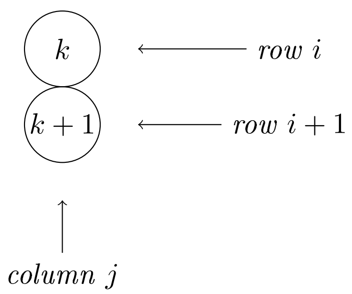
$$\begin{array}{rccl} \tau : & \mathbb{C}^* & \longrightarrow & T \\ & t & \longmapsto & \left(\begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_{n_1}} \end{pmatrix}, \begin{pmatrix} t^{b_1} & & \\ & \ddots & \\ & & t^{b_{n_2}} \end{pmatrix} \right) \end{array}$$

(with $a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2} \in \mathbb{Z}$). Assume moreover that τ is dominant and regular (i.e. $a_1 > \cdots > a_{n_1} \geq 0$ and $b_1 > \cdots > b_{n_2} \geq 0$), and also $\hat{G}$-regular (i.e. the $a_i + b_j$ are pairwise distinct). Then we build the matrix $M = (a_i + b_j)_{i,j}$ and what we call the "order matrix of τ": it is the matrix in which each coefficient of M is replaced by its rank (starting at 1) in the decreasingly ordered sequence of the $a_i + b_j$. Then one existing result, due to L. Manivel (see [Man15a]) and E. Vallejo (see [Val14]), is that each such order matrix gives one explicit regular face of PKron_{n_1,n_2}, of minimal dimension among the regular faces (i.e. of dimension n_1n_2), and which contains only stable triples. We extend this result by proving that an order matrix actually gives other such faces:

Theorem 2.2.5. *For any dominant, regular, $\hat{G}$-regular one-parameter subgroup τ of T, each configuration of the following type in the order matrix:*



gives a regular face of dimension n_1n_2 of the Kronecker cone PKron_{n_1,n_2}, containing only stable triples, and explicitly given in Section 6.3.4. Likewise, each configuration of the type



gives such an explicit face of dimension n_1n_2.

We also define (see Section 6.3.5) five other types of possible configurations in an order matrix (from Configuration Ⓐ to Ⓔ), involving this time three or four of its coefficients. We prove:

Theorem 2.2.6. *Let τ be a dominant, regular, $\hat{G}$-regular one-parameter subgroup of T. Each of the Configurations Ⓐ to Ⓔ appearing in the order matrix coming from τ then gives a face – not necessarily regular and even possibly reduced to zero – of the Kronecker cone PKron_{n_1,n_2} which only contains almost stable triples.*

We end this chapter by looking at the actual number of new (i.e. compared to the result of Manivel and Vallejo) faces that our two results produce in the cases of order matrices of size 2×2, 3×2, and 3×3. For instance in the 3×2 case, we obtain 23 new regular faces of $\mathrm{PKron}_{3,2}$ which contain only stable triples, whereas 5 others were already known. We also get 2 other new non-regular faces, containing only almost stable triples.

In Chapter 7 we are interested in "zeroes" appearing in the Kronecker cone. By such zeroes we mean triples $(\alpha, \beta, \gamma) \in \mathrm{PKron}_{n_1,n_2}$ such that the Kronecker coefficient $g_{\alpha,\beta,\gamma}$ is 0. The existence of such triples is equivalent to the fact that the Kronecker coefficients do not have the saturation property, and understanding them is a huge problem in the study of those coefficients. Once we have such a triple (α, β, γ), we will as before consider the half-line $\mathbb{N}^*(\alpha, \beta, \gamma)$. Then one can notice that, in most of the known examples, the set $\Lambda(\alpha, \beta, \gamma) = \{d \in \mathbb{N}^* \text{ s.t. } g_{d\alpha,d\beta,d\gamma} \neq 0\}$ – which is always a semigroup – has the form $d_0\mathbb{N}^*$, for some $d_0 \in \mathbb{N}^*$. We prove that, for almost stable triples, this is always true (note that this can also be seen as a direct consequence of the results by Paradan, and more specifically of Theorem B from [Par17]):

Theorem 2.2.7. *Let (α, β, γ) be an almost stable triple of partitions. Then there exists $d_0 \in \mathbb{N}^*$ such that, for all $d \in \mathbb{N}^*$,*

$$d \in \Lambda(\alpha, \beta, \gamma) \quad \Longleftrightarrow \quad d_0 | d$$

For triples in PKron_{n_1,n_2} that are not almost stable, this result does not hold. There is indeed a family of counter-examples due to Briand, Orellana, and Rosas (see [BOR09], Theorem 2.4), whose smallest example is the triple $\big((6,6),(7,5),(6,4,2)\big)$. But as far as we know, this family is the only known example where $\Lambda(\alpha, \beta, \gamma)$ does not have the form $d_0\mathbb{N}^*$. Therefore we study this example geometrically, in details, and manage (using the result of [BOR09]) to prove:

Proposition 2.2.8. *Denote by Q the quaternionic group, seen as a subgroup (of cardinal 8) of $\mathrm{SL}_2(\mathbb{C})$. Then, for all positive integers d,*

$$g_{d(6,6),d(7,5),d(6,4,2)} = \dim \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q = \dim\left(\mathbb{C}[x,y]_{2d}\right)^Q,$$

where $\mathbb{C}[x,y]_{2d}$ denotes the vector space of homogeneous polynomials in two variables x and y, of degree $2d$, on which $Q \subset \mathrm{SL}_2(\mathbb{C})$ acts by its natural action on (x,y).

Note that this result holds for the other triples in the family of counter-examples given in [BOR09]. We would then like to replicate this kind of result to produce other examples like these. But it is in fact already quite difficult to find finite groups like Q, whose action on spaces of homogeneous polynomials gives interesting dimensions for the spaces of invariants. We manage to find only one small example like this, but we cannot really use it to obtain other triples like $\big((6,6),(7,5),(6,4,2)\big)$.

Chapter 3

Some prerequisites

3.1 Branching coefficients expressed geometrically

When the reductive groups involved in the definition of the branching coefficients are connected, there is a nice geometric expression of these coefficients in terms of sections of line bundles on flag varieties. Let G be a complex connected reductive group. Consider moreover a Borel subgroup B of G, containing a maximal torus T of G, and P a parabolic subgroup of G containing B. Then G/B and G/P are projective varieties which are called flag varieties (the former is the "complete flag variety", whereas the latter is said to be a "partial flag variety" if P contains strictly B).

Example: When V is a complex vector space of dimension n and $G = \mathrm{GL}(V)$, the previous flag varieties can be easily described. Here we choose a basis of V and identify G with $\mathrm{GL}_n(\mathbb{C})$. Consider the Borel subgroup B of upper-triangular matrices, and T the maximal torus formed by the diagonal matrices. Then the complete flag variety associated to G is:

$$G/B \simeq \mathcal{F}\ell(V) = \Big\{ \big(\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V\big) \,|\, \forall i,\ V_i \text{ subspace of dimension } i \Big\}.$$

The isomorphism between the two is explicit:

$$\begin{array}{ccc} G/B & \longrightarrow & \mathcal{F}\ell(V) \\ \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \mathrm{mod} B & \longmapsto & \big(\{0\} \subset \mathrm{Vect}(v_1) \subset \mathrm{Vect}(v_1, v_2) \subset \cdots \subset \mathrm{Vect}(v_1, \dots, v_{n-1}) \subset V\big) \end{array}$$

If we consider a parabolic subgroup P containing strictly B, G/P will be a partial flag variety, meaning that it will contain flags which do not have subspaces of some particular dimensions. Let us give one example: for $V = \mathbb{C}^6$ and

$$P = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} \right\} \subset G,$$

the corresponding partial flag variety is

$$G/P \simeq \mathcal{F}\ell(\mathbb{C}^6; 2, 5) = \Big\{ \big(\{0\} \subset V_2 \subset V_5 \subset \mathbb{C}^6\big) \,|\, V_2 \text{ of dim } 2,\ V_5 \text{ of dim } 5 \Big\}$$

(and the isomorphism is given as before for G/B).

We can define interesting line bundles on these projective varieties: let λ be an integral dominant weight of G. Then $-\lambda$ is a weight of G and it gives a character $e^{-\lambda}$ of T, which can be extended uniquely to a character of B. We denote by $\mathbb{C}_{-\lambda}$ the one-dimensional representation of B associated to this character and set:

$$\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}.$$

This denotes a fibre product: B acts on G (on the right) and on $C_{-\lambda}$ (on the left), and we consider the quotient by this action. For a pair $(g, v) \in G \times \mathbb{C}_{-\lambda}$, the corresponding class modulo B is denoted by $[g : v]$. This defines a line bundle on the complete flag variety G/B, whose associated projection on G/B is simply $[g : v] \mapsto gB$. The group G acts on this line bundle by left multiplication, and $\mathcal{L}_\lambda$ is then a G-linearised line bundle.

For line bundles on G/P one needs to consider particular weights: let λ be an integral dominant weight of G such that, for all simple roots α such that $-\alpha$ is a root of the Lie algebra $\mathfrak{p}$ of P, $\lambda(\alpha^\vee) = 0$ (where $\alpha^\vee$ is the simple coroot corresponding to α). The set of such λ's will be denoted by Λ_P^+. Then $e^{-\lambda}$ extends to a character of P, and we can set:

$$\mathcal{L}_\lambda = G \times_P \mathbb{C}_{-\lambda}.$$

As above, this is a G-linearised line bundle on the flag variety G/P. These line bundles being all G-linearised, there is a linear action on the vector space formed by their sections. The interesting fact about these spaces is the following:

Theorem 3.1.1 (Borel-Weil Theorem)**.** *For any integral dominant weight λ of G (resp. $\lambda \in \Lambda_P^+$), the space $\mathrm{H}^0(G/B, \mathcal{L}_\lambda)$ (resp. $\mathrm{H}^0(G/P, \mathcal{L}_\lambda)$) of sections of the line bundle $\mathcal{L}_\lambda$ defined on G/B (resp. G/P) is the dual of the irreducible representation $V_G(\lambda)$ of highest weight λ of G.*

This result can for instance be found in [CG10].

Example: Still for $G = \mathrm{GL}(V)$ and V of dimension n, an integral dominant weight is a partition λ of length at most n. The character $e^{-\lambda}$ of T is then

$$\begin{array}{cccc} e^{-\lambda}: & T & \longrightarrow & \mathbb{C}^* \\ & \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} & \longmapsto & t_1^{-\lambda_1}\dots t_n^{-\lambda_n} \end{array}$$

and its extension to B is simply:

$$\begin{array}{cccc} e^{-\lambda}: & B & \longrightarrow & \mathbb{C}^* \\ & \begin{pmatrix} b_1 & & (*) \\ & \ddots & \\ (0) & & b_n \end{pmatrix} & \longmapsto & b_1^{-\lambda_1}\dots b_n^{-\lambda_n} \end{array}.$$

The conclusion of the previous theorem is thus that $\mathrm{H}^0(\mathcal{F}\ell(V), \mathcal{L}_\lambda)$ is the dual of $\mathbb{S}^\lambda V$.

We can now give a well-known interesting expression for the branching coefficients. Assume that G and $\hat{G}$ are complex reductive groups, that $f : G \longrightarrow \hat{G}$ is a morphism, and that the groups are both connected. Then recall that Schur's Lemma states that every morphism of representations between two irreducible modules is either zero or an isomorphism. Furthermore, if these two irreducible representations are the same, then the vector space of endomorphisms of representations is of dimension 1. Therefore we can derive from the definition of the branching coefficients:

$$V_{\hat{G}}(\hat{\lambda}) = \bigoplus_{\lambda \text{ dominant weight of } G} V_G(\lambda)^{\oplus c(\lambda,\hat{\lambda})},$$

that

$$\dim\left(V_G(\lambda)^* \otimes V_{\hat{G}}(\hat{\lambda})\right)^G = c(\lambda, \hat{\lambda}).$$

Then an immediate consequence of Borel-Weil Theorem is the following:

Proposition 3.1.2. *Let G and $\hat{G}$ be two complex connected reductive groups, with respective Borel subgroups B and $\hat{B}$, and $f : G \longrightarrow \hat{G}$ be a morphism. Then, for all λ integral dominant weights of G and all $\hat{\lambda}$ integral dominant weights of $\hat{G}$,*

$$c(\lambda, \hat{\lambda}) = \dim \mathrm{H}^0(G/B \times \hat{G}/\hat{B}, \mathcal{L}_\lambda \otimes \mathcal{L}_{\hat{\lambda}}^*)^G.$$

Note that the dual $\mathcal{L}_{\hat{\lambda}}^*$ of $\mathcal{L}_{\hat{\lambda}}$ is simply the line bundle $\mathcal{L}_{-\hat{w}_0.\hat{\lambda}}$, where $\hat{w}_0$ is the longest element of the Weyl group $\hat{W}$ associated to $\hat{G}$. Moreover the external tensor product of the line bundles $\mathcal{L}_\lambda$ and $\mathcal{L}_{\hat{\lambda}}^*$ is simply the line bundle on $G/B \times \hat{G}/\hat{B}$ such that, for all $(x, y) \in G/B \times \hat{G}/\hat{B}$, the fibre over (x, y) is the tensor product of the fibre over x in $\mathcal{L}_\lambda$ and the fibre over y in $\mathcal{L}_{\hat{\lambda}}^*$.

We want to have this kind of expression for the Kronecker coefficients, but the definition that we gave used finite groups, which are obviously not connected. Fortunately they can be expressed as branching coefficients for another branching situation, this time involving connected groups. It is a consequence of Schur-Weyl duality (see [Wey39]): let V be a complex vector space of finite dimension n and k be a positive integer. Then consider the vector space $V^{\otimes k}$. It is a obviously a representation of $\mathrm{GL}(V)$ – since V is –, but it is also a representation of the symmetric group $\mathfrak{S}_k$, which acts – on the left – by permuting the factors in the tensor product. Moreover these two actions commute, and thus $V^{\otimes k}$ is a representation of the direct product $\mathrm{GL}(V) \times \mathfrak{S}_k$.

Theorem 3.1.3 (Schur-Weyl Duality). *As a representation of* $\mathrm{GL}(V) \times \mathfrak{S}_k$, $V^{\otimes k}$ *splits in a direct sum of irreducible modules in the following way:*

$$V^{\otimes k} \simeq \bigoplus_{\alpha \vdash k \text{ s.t. } \ell(\alpha) \leq n} \mathbb{S}^\alpha V \otimes M_\alpha.$$

Here is the classical consequence for Kronecker coefficients:

Proposition 3.1.4. *Let α, β, and γ be three partitions of the same positive integer k. Then, for any pair of finite dimensional vector spaces (V_1, V_2) such that* $\dim V_1 \geq \ell(\alpha)$ *and* $\dim V_2 \geq \ell(\beta)$, *the Kronecker coefficient $g_{\alpha,\beta,\gamma}$ is the multiplicity of the irreducible representation* $\mathbb{S}^\alpha V_1 \otimes \mathbb{S}^\beta V_2$ *of* $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ *inside the irreducible representation* $\mathbb{S}^\gamma(V_1 \otimes V_2)$ *of* $\hat{G} = \mathrm{GL}(V_1 \otimes V_2)$.

Note that the morphism from G to $\hat{G}$ is given in that case by: $(g_1, g_2) \in G$ defines an automorphism of the vector space $V_1 \otimes V_2$, denoted by ϕ_{g_1,g_2}, defined on elementary tensors $v_1 \otimes v_2$ by $\phi_{g_1,g_2}(v_1 \otimes v_2) = g_1(v_1) \otimes g_2(v_2)$.

Proof. Consider two finite dimensional vector spaces V_1 and V_2 as in the statement of the proposition. Then we look at the vector space

$$(V_1 \otimes V_2)^{\otimes k} \simeq V_1^{\otimes k} \otimes V_2^{\otimes k}.$$

At first we use Schur-Weyl duality on the left side of this identity:

$$\begin{aligned} (V_1 \otimes V_2)^{\otimes k} &\simeq \bigoplus_{\gamma \vdash k} \mathbb{S}^\gamma(V_1 \otimes V_2) \otimes M_\gamma \\ &\simeq \bigoplus_{\gamma \vdash k, \alpha, \beta} \left(\mathbb{S}^\alpha V_1 \otimes \mathbb{S}^\beta V_2 \otimes M_\gamma\right)^{\oplus n^\gamma_{\alpha,\beta}}, \end{aligned}$$

where the coefficients $n^\gamma_{\alpha,\beta}$ are (almost all zero) non-negative integers, which exist since $\mathbb{S}^\gamma(V_1 \otimes V_2)$ is a G-module and are the multiplicities mentioned in the statement of the proposition. If we now apply Schur-Weyl duality on the right side of the first identity:

$$\begin{aligned} V_1^{\otimes k} \otimes V_2^{\otimes k} &\simeq \bigoplus_{\alpha \vdash k} (\mathbb{S}^\alpha V_1 \otimes M_\alpha) \otimes \bigoplus_{\beta \vdash k} \left(\mathbb{S}^\beta V_2 \otimes M_\beta\right) \\ &\simeq \bigoplus_{\alpha, \beta, \gamma \vdash k} \left(\mathbb{S}^\alpha V_1 \otimes \mathbb{S}^\beta V_2 \otimes M_\gamma\right)^{\oplus g_{\alpha,\beta,\gamma}}. \end{aligned}$$

Then, for all triples (α, β, γ) of partitions of k, $g_{\alpha,\beta,\gamma} = n^\gamma_{\alpha,\beta}$. ☐

Proposition 3.1.5. *Let (α, β, γ) be a triple of partitions of the same integer. Then there exist a projective variety X and a complex connected reductive group G acting on X, both depending only on the lengths of the three partitions, together with a G-linearised line bundle $\mathcal{L}_{\alpha,\beta,\gamma}$ on X such that:*

$$g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X, \mathcal{L}_{\alpha,\beta,\gamma})^G.$$

Proof. This is a direct consequence of Propositions 3.1.2 and 3.1.4 with:

$$X = \mathcal{F}\ell(V_1) \times \mathcal{F}\ell(V_2) \times \mathcal{F}\ell(V_1 \otimes V_2),$$

$$G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2),$$

and

$$\mathcal{L}_{\alpha,\beta,\gamma} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{L}_\gamma^*.$$

□

3.2 Some notions from Geometric Invariant Theory

A notion from Geometric Invariant Theory, due D. Mumford, which will be central in this thesis, is the notion of semi-stable points. Let X be a complex projective variety on which a complex connected reductive group G acts. Consider furthermore a G-linearised line bundle $\mathcal{L}$ on X.

Definition 3.2.1. The line bundle $\mathcal{L}$ is said to be:

- base-point-free if, for any $x \in X$, there exists a section σ of $\mathcal{L}$ defined on X such that $\sigma(x) \neq 0$;
- very ample if there are a finite dimensional complex vector space V and an embedding $X \longrightarrow \mathbb{P}(V)$ such that the pull-back of $\mathcal{O}(1)$ is isomorphic to $\mathcal{L}$;
- ample if there exists a positive power of $\mathcal{L}$ which is very ample;
- semi-ample if there exists a positive power of $\mathcal{L}$ which is base-point-free.

Definition 3.2.2. A point $x \in X$ is said to be semi-stable (relatively to $\mathcal{L}$) if there exist $n \in \mathbb{N}^*$ and $\sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G$ such that $\sigma(x) \neq 0$. It is said to be unstable otherwise.

We denote respectively by $X^{ss}(\mathcal{L})$ and $X^{us}(\mathcal{L})$ the sets of semi-stable and unstable points in X relatively to $\mathcal{L}$. They are respectively open and closed subsets.

Remark 3.2.3. It is important to notice that this definition is not exactly the one that was given by Mumford (see e.g. [Dol03] for this one): one usually requires in addition that the set of points on which σ is not zero is affine. These two notions of semi-stability nevertheless coincide in the case of an ample line bundle, and the one that we use here is a little bit better adapted for the case of semi-ample line bundles, which is the case that we will always consider in this thesis.

The purpose of this notion was the construction of good quotients for projective varieties. For example the following result can be found in [Dol03] (Theorem 8.1):

Theorem 3.2.4 (Mumford)**.** *If $\mathcal{L}$ is ample, there exists a good categorical quotient*

$$\pi : X^{ss}(\mathcal{L}) \longrightarrow X^{ss}(\mathcal{L}) /\!\!/ G,$$

where $X^{ss}(\mathcal{L}) /\!\!/ G$ is a projective variety. Moreover the morphism π is affine.

Maybe now would be a good time to wonder what an ample or semi-ample line bundle looks like in our case, i.e. on a flag variety. So we temporarily set $X = \mathcal{F}\ell(V)$, with V a complex finite dimensional vector space whose dimension is denoted by n. Then we have seen that, for any partition λ of length at most n, we have a $\mathrm{GL}(V)$-linearised line bundle $\mathcal{L}_\lambda$ on X. Then this line bundle will be ample if and only if the partition is "regular", i.e. has n pairwise distinct parts (the last one can be zero). If one looks at a partial flag variety $Y = \mathcal{F}\ell(V; d_1, \dots, d_r)$, with $1 \leq d_1 < \cdots < d_r \leq n-1$, it is a little bit more complicated: from our construction, the line bundle $\mathcal{L}_\lambda$ will be well-defined if and only if $\{i \in [\![1, n-1]\!] \text{ s.t. } \lambda_i > \lambda_{i+1}\} \subset \{d_1, \dots, d_r\}$. Moreover it will be ample if and only if these two sets are equal.

Example: On $Y = \mathcal{F}\ell(\mathbb{C}^6; 2, 4, 5)$, for $\lambda = (5, 5, 3, 3, 2, 0)$, $\mu = (5, 5, 3, 3, 2, 2)$, and $\nu = (5, 4, 3, 3, 2, 0)$: $\mathcal{L}_\lambda$ is well-defined and ample, $\mathcal{L}_\mu$ is well-defined but not ample, and $\mathcal{L}_\nu$ is not well-defined on Y.

Notice now that one has a surjective map (which is a fibration)

$$p : \begin{array}{ccc} X & \longrightarrow & Y \\ (V_1 \subset \cdots \subset V_{n-1}) & \longmapsto & (V_{d_1} \subset \cdots \subset V_{d_r}) \end{array}.$$

If a partition λ is such that $\mathcal{L}_\lambda$ is well-defined on Y, we then have two line bundles defined by λ: one on X and one on Y, that we denote for now $\mathcal{L}_\lambda^{(B)}$ and $\mathcal{L}_\lambda^{(P)}$ respectively. Then $\mathcal{L}_\lambda^{(B)}$ is the pull-back of $\mathcal{L}_\lambda^{(P)}$ by p. Therefore if $\mathcal{L}_\lambda^{(P)}$ is ample, then $\mathcal{L}_\lambda^{(B)}$ is semi-ample. As a consequence, since for any non-zero partition one can find a partial flag variety on which $\mathcal{L}_\lambda^{(P)}$ is ample, every line bundle $\mathcal{L}_\lambda^{(B)}$ on the complete flag variety X is semi-ample. The same reasoning works to prove that every well-defined line bundle $\mathcal{L}_\lambda^{(P)}$ on a partial flag variety is semi-ample.

With the definition of semi-stability that we have chosen, there exists an important numerical criterion of semi-stability which remains true for semi-ample line bundles (it usually works for ample line bundles). Consider again any complex projective variety X on which a complex connected reductive group G acts. Let $\mathcal{L}$ be a G-linearised line bundle on X, τ be a one-parameter subgroup of G (we denote by $X_*(G)$ the set of such subgroups), and $x \in X$. Then, since X is complete, we can consider the limit

$$\tilde{x} = \lim_{t \to 0} \tau(t).x,$$

which is a point of X fixed by τ (i.e. by $\operatorname{Im}\tau$). Thus $\mathbb{C}^*$ acts via τ on the fibre $\mathcal{L}_{\tilde{x}}$ over $\tilde{x}$. This action is therefore given by an integer: there exists $\mu^{\mathcal{L}}(x,\tau) \in \mathbb{Z}$ such that

$$\forall t \in \mathbb{C}^*,\ \forall v \in \mathcal{L}_{\tilde{x}},\ \tau(t).v = t^{-\mu^{\mathcal{L}}(x,\tau)}v.$$

These integers have the following properties:

Lemma 3.2.5. *(i) For all $g \in G$, $\mu^{\mathcal{L}}(g.x, g\tau g^{-1}) = \mu^{\mathcal{L}}(x,\tau)$.*
(ii) The map $\mathcal{L} \mapsto \mu^{\mathcal{L}}(x,\tau)$ is a group homomorphism from $\operatorname{Pic}^G(X)$ *– the group of G-linearised line bundles on X – to $\mathbb{Z}$.*
(iii) For any G-variety and any G-equivariant morphism $f : Y \to X$, for all $y \in Y$, $\mu^{f^*(\mathcal{L})}(y,\tau) = \mu^{\mathcal{L}}(f(y),\tau)$.

Lemma 3.2.6. *Let v be a non-zero point in $\mathcal{L}_{\tilde{x}}$. Then, when t tends to 0:*

1. *if $\mu^{\mathcal{L}}(x,\tau) < 0$, then $\tau(t).v$ tends to zero in $\mathcal{L}_{\tilde{x}}$;*
2. *if $\mu^{\mathcal{L}}(x,\tau) = 0$, then $\tau(t).v$ tends to a non-zero point in $\mathcal{L}_{\tilde{x}}$;*
3. *if $\mu^{\mathcal{L}}(x,\tau) > 0$, then $\tau(t).v$ has no limit in $\mathcal{L}_{\tilde{x}}$.*

All these results can for instance be found in [Res10], as can be the adaptation of the following really important criterion to the case of semi-ample line bundles:

Theorem 3.2.7 (Hilbert-Mumford criterion)**.** *If $\mathcal{L}$ is semi-ample, then*

$$x \in X^{ss}(\mathcal{L}) \quad \Longleftrightarrow \quad \forall \tau \in X_*(G),\ \mu^{\mathcal{L}}(x,\tau) \leq 0.$$

Chapter 4

Characterisation of stability for Kronecker coefficients, bounds of stabilisation

4.1 Introduction

For a positive integer n, let $\mathfrak{S}_n$ be the symmetric group over n elements. The complex irreducible representations of this group are indexed by the partitions of n (i.e. non-increasing finite sequences of positive integers – called parts – whose sum is equal to n). For a partition α of n (for which the integer n is called the size, and denoted $|\alpha|$), we denote its length (i.e. the number of parts) by $\ell(\alpha)$, and write M_α for the associated complex irreducible representation of $\mathfrak{S}_n$. An important problem concerning the representation theory of this group is the understanding of the decomposition of the tensor product of two such irreducible representations:

$$M_\alpha \otimes M_\beta = \bigoplus_{\gamma \vdash n} M_\gamma^{\oplus g_{\alpha,\beta,\gamma}},$$

where the multiplicities $g_{\alpha,\beta,\gamma}$ are non-negative integers, which are called the Kronecker coefficients. These coefficients appear in various situations, and are quite difficult to study. Some of their properties are nevertheless known, one of which being that the order of the three partitions indexing a Kronecker coefficient does not matter.

There are several different ways of studying the Kronecker coefficients, and we will be interested in their asymptotic behaviour, in various senses. They hold indeed a remarkable asymptotic property, noticed by F. Murnaghan in 1938: let α, β, γ be partitions of the same integer; if one repetitively increases by 1 the first part of each of these partitions, the corresponding sequence of Kronecker coefficients ends up stabilising. J. Stembridge, in [Ste14], introduced two notions of stability of a triple of partitions in order to generalise this Murnaghan stability:

Definition 4.1.1. A triple (α, β, γ) of partitions such that $|\alpha| = |\beta| = |\gamma|$ is called:

- weakly stable if $g_{d\alpha,d\beta,d\gamma} = 1$ for all $d \in \mathbb{N}^*$;
- stable if $g_{\alpha,\beta,\gamma} > 0$ and, for any triple (λ, μ, ν) of partitions such that $|\lambda| = |\mu| = |\nu|$, the sequence of general term $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ is eventually constant.

The terminology "weakly stable" is in fact used by L. Manivel in [Man15a]. The notion of a stable triple is made to generalise the Murnaghan stability: the latter simply means that the triple $\big((1),(1),(1)\big)$ is stable. By introducing the notion of a weakly stable triple, Stembridge hoped to find a simpler criterion to determine whether a triple is stable. He proved in [Ste14] that a stable triple is weakly stable, and conjectured that the converse is true. S. Sam and A. Snowden proved shortly afterwards, in [SS16], that it is indeed verified. We also learned during the redaction of this article about a prepublication by P.-E. Paradan [Par17], who demonstrated this kind of result in a more general context which in particular contains the case of Kronecker coefficients (as well as the plethysm case). In the first part of this chapter, we give another new proof of this result:

Theorem 4.1.2. *If a triple (α, β, γ) of partitions is weakly stable, then it is stable.*

A question then arises: given a stable triple, can we determine when the associated sequences of Kronecker coefficients do stabilise? There have already been results on this, at least in the case of Murnaghan's stability: for instance, M. Brion – in 1993 – and E. Vallejo – in 1999 – calculated bounds from which these sequences are necessarily constant. In [BOR11], E. Briand, R. Orellana, and M. Rosas recall the two bounds from Brion and Vallejo, and determine two other ones, still in the case of the stable triple $\big((1),(1),(1)\big)$.

The interesting aspect of our proof of Theorem 4.1.2 is that it gives a nice "geometric bound" from which we can be certain that the sequence $(g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma})_d$ is constant, if the triple (α, β, γ) is stable. Indeed, the Kronecker coefficients can classically be related to the dimension of spaces of invariant sections from some line bundles: for all triples (α, β, γ) and (λ, μ, ν), there exist a reductive group G acting on a projective variety X, and two G-linearised line bundles $\mathcal{L}$ and $\mathcal{M}$ on X whose spaces of invariant sections respectively give -via their dimension- the coefficients $g_{\alpha,\beta,\gamma}$ and $g_{\lambda,\mu,\nu}$ (cf. Section 4.2.1). Then, for $d \in \mathbb{N}$, $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ is the dimension of $\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G$, the space of invariant sections of the line bundle $\mathcal{M} \otimes \mathcal{L}^{\otimes d}$ on X. Recall that, if $\mathcal{N}$ is a G-linearised line bundle on X, $X^{ss}(\mathcal{N})$ stands for the set of semi-stable points with respect to $\mathcal{N}$, i.e. the points x for which there exists a G-invariant section of a positive power of $\mathcal{N}$ whose value at x is not zero.

Proposition 4.1.3. *We suppose that the triple (α, β, γ) is weakly stable. Then:*

- *there exists an integer $D \in \mathbb{N}$ such that, for all $d \geq D$, $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$;*
- *for all $d \geq D$, the Kronecker coefficient $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ does not depend on d.*

What we prove precisely is in fact that $\mathrm{H}^0(X^{ss}(\mathcal{L}), \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G$ does not depend on d and that, if $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$, then the restriction morphism $\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \hookrightarrow \mathrm{H}^0(X^{ss}(\mathcal{L}), \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G$ is an isomorphism. A natural question could thus be: is the converse true? The answer here is "no". A counter-example can be found for the triples $(\alpha, \beta, \gamma) = \big((1),(1),(1)\big)$ and $(\lambda, \mu, \nu) = \big((4,1,1),(3,3),(2,2,2)\big)$: the Kronecker coefficient $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ does not depend on $d \geq 0$, but one can prove that $X^{ss}(\mathcal{M}) \not\subset X^{ss}(\mathcal{L})$.

We indeed manage, in Section 4.3.4, to improve slightly the previously-stated result by proving that the sequence $(g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma})_d$ can already stabilise for a d such that the inclusion $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$ is a priori not verified. The key points to obtain this extension are an argument of quasipolynomiality (which is a known result, of which we nevertheless write a proof in Section 4.3.4, inspired by [KP14]) and the structure of what is called the GIT-fan (see for instance [Res00] for this notion and the description of its structure).

In Section 4.3, we give a method allowing -at least for "small" weakly stable triples- to compute bounds from which the inclusion $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$ is realised. We perform the calculations for two examples of triples (namely $\big((1),(1),(1)\big)$ and $\big((1,1),(1,1),(2)\big)$). Taking into account the slight extension explained in the previous paragraph, it gives us:

Theorem 4.1.4. *If we denote $n_1 = \ell(\lambda)$, $n_2 = \ell(\mu)$, and set* [1]

$$D_1 = \left\lceil \frac{1}{2}\left(-\lambda_1 + \lambda_2 - \mu_1 + \mu_2 + 2(\nu_2 - \nu_{n_1 n_2}) + \sum_{k=1}^{n_1+n_2-4} (\nu_{k+2} - \nu_{n_1 n_2 - k})\right)\right\rceil,$$

we have, for all $d \geq D_1$, $g_{\lambda+d(1),\mu+d(1),\nu+d(1)} = g_{\lambda+D_1(1),\mu+D_1(1),\nu+D_1(1)}$.

(it is in this case legitimate to reorder the partitions λ, μ, and ν to get the lowest bound D_1 possible) and

Theorem 4.1.5. *If $m = \max(-\lambda_2 - \mu_1, -\lambda_1 - \mu_2)$, and*

$$D_2 = \begin{cases} \left\lceil \frac{1}{2}\left(m + \lambda_3 + \mu_3 + 2(\nu_2 - \nu_{n_1 n_2}) + \sum_{k=1}^{n_1+n_2-4} (\nu_{k+2} - \nu_{n_1 n_2 - k})\right)\right\rceil & \text{if } n_1, n_2 \geq 3 \\ \left\lceil \frac{1}{2}\left(m + \mu_3 + 2\nu_2 - \nu_{2n_2} + \sum_{k=1}^{n_2-1} \nu_{k+2}\right)\right\rceil & \text{if } n_1 = 2 \\ \left\lceil \frac{1}{2}\left(m + \lambda_3 + 2\nu_2 - \nu_{2n_1} + \sum_{k=1}^{n_1-1} \nu_{k+2}\right)\right\rceil & \text{if } n_2 = 2 \end{cases},$$

then for all $d \geq D_2$, $g_{\lambda+d(1,1),\mu+d(1,1),\nu+d(2)} = g_{\lambda+D_2(1,1),\mu+D_2(1,1),\nu+D_2(2)}$.

[1] The notation $\lceil x \rceil$ stands for the ceiling of the number x (i.e. the integer such that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$).

We then prove that our method allows to recover some of the bounds already existing in the case of Murnaghan's stability: we re-obtain Brion's bound, as well as the second one given by Briand, Orellana, and Rosas. Moreover, we get slight improvements for these in some cases. The bounds we obtained are in addition tested on some examples, in Section 4.3.6. We also make a comparison on these examples with the four already existing bounds that we cited.

4.2 Proof of the characterisation of stability

4.2.1 Link with invariant sections of line bundles

Thanks to Schur-Weyl duality, the Kronecker coefficients also appear in the decomposition of representations of the general linear group. If V_1 and V_2 are two (complex) vector spaces, γ is a partition, and if we denote by $\mathbb{S}$ the Schur functor[2],

$$\mathbb{S}^\gamma(V_1 \otimes V_2) \simeq \bigoplus_{\alpha,\beta} \left(\mathbb{S}^\alpha(V_1) \otimes \mathbb{S}^\beta(V_2)\right)^{\oplus g_{\alpha,\beta,\gamma}}$$

as representations of $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Then, by Schur's Lemma we have, for all triples (α, β, γ) of partitions (such that $|\alpha| = |\beta| = |\gamma|$) and all vector spaces V_1 and V_2 such that $\dim(V_1) \geq \ell(\alpha)$, $\dim(V_2) \geq \ell(\beta)$, and $\dim(V_1)\dim(V_2) \geq \ell(\gamma)$:

$$g_{\alpha,\beta,\gamma} = \dim\left((\mathbb{S}^\alpha V_1)^* \otimes (\mathbb{S}^\beta V_2)^* \otimes \mathbb{S}^\gamma(V_1 \otimes V_2)\right)^G .$$

Finally, we use Borel-Weil's Theorem: if V is a complex vector space of finite dimension, we denote by $\mathcal{F}\ell(V)$ the complete flag variety associated to V. We know that, if B is a Borel subgroup of $\mathrm{GL}(V)$, the variety $\mathcal{F}\ell(V)$ is isomorphic to $\mathrm{GL}(V)/B$. We can then define particular line bundles on $\mathrm{GL}(V)/B$: for any partition λ of length at most $\dim V$, the finite sequence of integers $-\lambda$ defines a character $e^{-\lambda}$ of B, and this allows us to define $\mathcal{L}_\lambda = \mathrm{GL}(V) \times_B \mathbb{C}_{-\lambda}$, where $\mathbb{C}_{-\lambda}$ is the one-dimensional complex representation of B given by the character $e^{-\lambda}$. The fibre product $\mathcal{L}_\lambda$ is a $\mathrm{GL}(V)$-linearised line bundle on $\mathrm{GL}(V)/B \simeq \mathcal{F}\ell(V)$. Then Borel-Weil's Theorem states that the representation $(\mathbb{S}^\alpha V_1)^*$ is isomorphic to $\mathrm{H}^0(\mathcal{F}\ell(V_1), \mathcal{L}_\alpha)$, the space of sections of the line bundle $\mathcal{L}_\alpha$ on $\mathcal{F}\ell(V_1)$. This is the same for $(\mathbb{S}^\beta V_2)^*$, and for $V_1 \otimes V_2$ this yields $\mathbb{S}^\gamma(V_1 \otimes V_2) \simeq \mathrm{H}^0(\mathcal{F}\ell(V_1 \otimes V_2), \mathcal{L}_\gamma^*)$. Hence, we have the following important proposition:

Proposition 4.2.1. *For any triple (α, β, γ) of partitions such that $|\alpha| = |\beta| = |\gamma|$, there exist a reductive group G, a projective variety X on which G acts, and a G-linearised line bundle $\mathcal{L}_{\alpha,\beta,\gamma}$ on X such that*

$$g_{\alpha,\beta,\gamma} = \dim\left(\mathrm{H}^0(X, \mathcal{L}_{\alpha,\beta,\gamma})^G\right) .$$

[2] In other words, if V is a complex vector space of dimension n, and λ a partition of length $\leq n$, then $\mathbb{S}^\lambda(V)$ is the corresponding irreducible representation of $\mathrm{GL}(V)$. Moreover, all complex irreducible polynomial representations of this group are obtained this way.

More precisely, the previous equality is true if V_1 and V_2 are finite-dimensional complex vector spaces such that $\ell(\alpha) \leq \dim(V_1)$, $\ell(\beta) \leq \dim(V_2)$, $\ell(\gamma) \leq \dim(V_1)\dim(V_2)$, and for:

$$G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2),$$

$$X = \mathcal{F}\ell(V_1) \times \mathcal{F}\ell(V_2) \times \mathcal{F}\ell(V_1 \otimes V_2),$$

$$\mathcal{L}_{\alpha,\beta,\gamma} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{L}_\gamma^*.$$

Proof. This follows directly from the Borel-Weil Theorem. □

Thus, from now on, we consider a weakly stable triple (α, β, γ) of partitions, and another triple (λ, μ, ν) of partitions (also satisfying $|\lambda| = |\mu| = |\nu|$). Then there exists a reductive group G, acting on a projective variety X, and two G-linearised line bundles $\mathcal{L}$ and $\mathcal{M}$ on X such that:

$$g_{\alpha,\beta,\gamma} = \dim\left(\mathrm{H}^0(X, \mathcal{L})^G\right) \qquad \text{and} \qquad g_{\lambda,\mu,\nu} = \dim\left(\mathrm{H}^0(X, \mathcal{M})^G\right)$$

(we denote by V_1 and V_2 the two vector spaces used to define those). We are interested in the behaviour of $\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G$, or rather its dimension, for $d \in \mathbb{N}$.

4.2.2 Semi-stable points

Definition and criterion of semi-stability

Definition 4.2.2. Given a G-linearised line bundle $\mathcal{N}$ on X, we define the semi-stable points in X (relatively to $\mathcal{N}$) as the elements of

$$X^{ss}(\mathcal{N}) = \{x \in X \text{ s.t. } \exists k \in \mathbb{N}^*,\ \exists \sigma \in \mathrm{H}^0(X, \mathcal{N}^{\otimes k})^G,\ \sigma(x) \neq 0\}.$$

The points which are not semi-stable are said to be unstable (relatively to $\mathcal{N}$), and we denote by $X^{us}(\mathcal{N})$ the set of unstable points.

Let us emphasise that this is not the standard definition of semi-stability (cf. for instance [Dol03], Chapter 8): most often there is an additional requirement to fulfil for a point to be semi-stable. The definition we gave coincides nevertheless with the usual one in the case of an ample line bundle. The following result is then due to V. Guillemin and S. Sternberg:

Proposition 4.2.3. *If $\mathcal{N}$ is a G-linearised semi-ample line bundle on X, then*

$$\mathrm{H}^0(X, \mathcal{N})^G \simeq \mathrm{H}^0(X^{ss}(\mathcal{N}), \mathcal{N})^G.$$

Proof. A demonstration of this result for ample line bundles can be found in [GS82], or for example in [Tel00], Theorem 2.11(a). It is given with the more usual definition of semi-stable points, which is not ours, but coincides with it in this case. Then, in the case of a semi-ample line bundle $\mathcal{N}$, there exists a G-equivariant projection $\pi : X \to \overline{X}$ (which is even a fibration with connected fibres) such that $\mathcal{N}$ is the pull-back by π of an

ample line bundle $\overline{\mathcal{N}}$ on a projective variety $\overline{X}$.
Indeed, X is a product of flag varieties and, on such a variety, a semi-ample line bundle is a $\mathcal{L}_\delta$ for δ a partition. Moreover this $\mathcal{L}_\delta$ is ample if and only if the type of the partition (i.e. the indices i such that $\delta_i > \delta_{i+1}$) coincides with the type of the flag variety. Henceforth, for every partition δ, there exists a projection as announced above, which consists simply in forgetting in the flag variety the dimensions which do not appear in the type of δ.

Then, with the properties of π,

$$\mathrm{H}^0(X,\mathcal{N})^G \simeq \mathrm{H}^0(\overline{X},\overline{\mathcal{N}})^G \simeq \mathrm{H}^0(\overline{X}^{ss}(\overline{\mathcal{N}}),\overline{\mathcal{N}})^G \simeq \mathrm{H}^0(X^{ss}(\mathcal{N}),\mathcal{N})^G,$$

since $\pi^{-1}(\overline{X}^{ss}(\overline{\mathcal{N}})) = X^{ss}(\mathcal{N})$. □

There is an extremely useful criterion of semi-stability which is called the Hilbert-Mumford criterion. It is generally stated for ample line bundles but, with the previously given definition of semi-stability, it holds for semi-ample line bundles (cf. [Res10], Lemma 2), which is the case for all the line bundles we consider. We are going to rephrase this criterion to get a more geometric one, in terms of polytopes. Let us begin with the case in which a torus T acts on X, and $\mathcal{N}$ is a T-linearised ample line bundle on X.

Then (see e.g. [Dol03], Section 9.4), as $\mathcal{N}$ is ample, we have a closed embedding of X in $\mathbb{P}(V)$, where V is a finite dimensional vector space, the action of T on X comes from a linear action on V, and some positive tensor power of $\mathcal{N}$ is the restriction of $\mathcal{O}(1)$ to X. Then, since T is a torus, V splits into a direct sum of eigensubspaces,

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi,$$

where $X^*(T)$ denotes the set of all characters of T and, for all $\chi \in X^*(T)$, $V_\chi = \{v \in V \text{ s.t. } \forall t \in T,\ t.v = \chi(t)v\}$ is the eigenspace associated to the character χ. Then, for $x \in X \subset \mathbb{P}(V)$ and a $v = \sum_\chi v_\chi \in V$ ($v_\chi \in V_\chi$) such that $x = \mathrm{Span}(v)$, we define the weight set of x as

$$\mathrm{Wt}(x) = \{\chi \in X^*(T) \text{ s.t. } v_\chi \neq 0\}.$$

Note that $\mathrm{Wt}(x)$ is a finite subset of $X^*(T) \simeq \mathbb{Z}^N \subset \mathbb{R}^N$ (N is the rank of T). We finally define the weight polytope of x as the convex hull $\mathrm{conv}(\mathrm{Wt}(x))$ of $\mathrm{Wt}(x)$ in $\mathbb{R}^N$. Then, Theorem 9.2 of [Dol03] states that the Hilbert-Mumford criterion means:

$$x \in X^{ss}(\mathcal{N}) \Longleftrightarrow 0 \in \mathrm{conv}(\mathrm{Wt}(x)).$$

We want to express this in a way which does not use an embedding in a $\mathbb{P}(V)$, and which involves explicitly $\mathcal{N}$. For this, one has to wonder which objects of X correspond to objects in $\mathbb{P}(V)$:

In $\mathbb{P}(V)$	In X
$\mathbb{P}(V_\chi)$ (for $V_\chi \neq \{0\}$)	fixed points of T
$\bigcup_\chi (\mathbb{P}(V_\chi) \cap X)$	$X^T = \{$fixed points of T in $X\}$
$\mathbb{P}(V_\chi) \cap X$	a union of some irreducible components $X_1, \dots, X_k$ of X^T
χ^{-1}	character giving the action of T on $\mathcal{N}\|_{X_i}$ for $i \in [\![1, k]\!]$

So we set, denoting by $X_1, \dots, X_s$ the irreducible components of X^T, for all $i \in [\![1, s]\!]$,

$$\begin{array}{rccl} \chi_i : & \mathrm{Pic}^T(X) & \longrightarrow & X^*(T) \\ & \mathcal{N} & \longmapsto & \text{the inverse of the character giving the action of } T \text{ on } \mathcal{N}|_{X_i} \end{array}.$$

Then, the Hilbert-Mumford criterion states:

$$x \in X^{ss}(\mathcal{N}) \Longleftrightarrow 0 \in \mathrm{conv}(\{\chi_i(\mathcal{N}) \ ; \ i \in [\![1, s]\!] \text{ s.t. } \chi_i(\mathcal{N}) \text{ is a vertex of } \mathrm{conv}(\mathrm{Wt}(x))\}).$$

And the only object left which uses an embedding of X in $\mathbb{P}(V)$ is $\mathrm{Wt}(x)$. But we can get rid of it thanks to the following lemma:

Lemma 4.2.4. *With the notations used above, if* $x = \mathrm{Span}\left(\sum_\chi v_\chi\right) \in X \subset \mathbb{P}(V)$,

$$\chi \textit{ is a vertex of } \mathrm{conv}(\mathrm{Wt}(x)) \Longleftrightarrow \mathbb{P}(V_\chi) \cap \overline{T.x} \neq \emptyset.$$

Proof. Let us recall that there is a duality pairing between $X^*(T)$ and the one-parameter subgroups of T, whose set is denoted by $X_*(T)$: for all $\chi \in X^*(T)$ and $\tau \in X_*(T)$, $\chi \circ \tau : \mathbb{C}^* \to \mathbb{C}^*$ is of the form $z \mapsto z^n$ with n integer. We set $\langle \chi, \tau \rangle = n$. Then, according to a classical property of convex polyhedra:

$$\chi \text{ is a vertex of } \mathrm{conv}(\mathrm{Wt}(x)) \Longleftrightarrow \exists \tau \in X_*(T) \text{ s.t. } \left\{ \begin{array}{l} \langle \chi, \tau \rangle = 0 \\ \forall \chi' \in \mathrm{conv}(\mathrm{Wt}(x)) \setminus \{\chi\},\ \langle \chi', \tau \rangle > 0 \end{array} \right. .$$

As a consequence, if χ is a vertex of $\mathrm{conv}(\mathrm{Wt}(x))$, we have such a $\tau \in X_*(T)$. Moreover,

$$\forall z \in \mathbb{C}^*,\ \tau(z).x = \mathrm{Span}\left(\sum_{\chi'} \chi' \circ \tau(z) v_{\chi'}\right) = \mathrm{Span}\left(\sum_{\chi'} z^{\langle \chi', \tau \rangle} v_{\chi'}\right).$$

And thus $\lim\limits_{z \to 0} (\tau(z).x) = \mathrm{Span}(v_\chi) \in \mathbb{P}(V_\chi) \cap \overline{T.x}$.

Conversely, if we suppose that χ is not a vertex of $\mathrm{conv}(\mathrm{Wt}(x))$, then for all $\tau \in X_*(T)$, there exists $\chi(\tau) \in \mathrm{conv}(\mathrm{Wt}(x)) \setminus \{\chi\}$ such that $\langle \chi(\tau), \tau \rangle = \langle \chi, \tau \rangle$. We want to prove that $\mathbb{P}(V_\chi) \cap \overline{T.x} = \emptyset$.

By contradiction, let us assume that $\mathbb{P}(V_\chi) \cap \overline{T.x} \neq \emptyset$. Then there exists $\tau \in X_*(T)$ such that $\lim_{z\to 0}(\tau(z).x) \in \mathbb{P}(V_\chi)$. On the other hand,

$$\forall z \in \mathbb{C}^*,\ \tau(z).x = \mathrm{Span}\left(\sum_{\chi'} z^{\langle \chi', \tau\rangle} v_{\chi'}\right).$$

So, for every $\chi' \in \mathrm{Wt}(x) \setminus \{\chi\}$, $\langle \chi', \tau\rangle > \langle \chi, \tau\rangle$. This contradicts the existence of $\chi(\tau)$, which is necessarily a convex combination involving at least one element of $\mathrm{Wt}(x) \setminus \{\chi\}$. □

Then,

$$x \in X^{ss}(\mathcal{N}) \Longleftrightarrow 0 \in \mathrm{conv}(\{\chi_i(\mathcal{N})\ ;\ i \in [\![1,s]\!] \text{ s.t. } X_i \cap \overline{T.x} \neq \emptyset\}),$$

which now does not involve anymore any embedding of X in $\mathbb{P}(V)$. So this is also true for line bundles which are semi-ample, and not necessarily ample (since Hilbert-Mumford criterion holds for such ones). We now extend this to the case when G is reductive. Then we take a maximal torus T in G and, using Theorem 9.3 of [Dol03], we finally get:

Proposition 4.2.5. *In our settings (a reductive group G acting on a flag variety X), if $\mathcal{N}$ is a G-linearised semi-ample line bundle on X, then*

$$x \in X^{ss}(\mathcal{N}) \Longleftrightarrow \forall g \in G,\ 0 \in \mathrm{conv}(\{\chi_i(\mathcal{N})\ ;\ i \in [\![1,s]\!] \text{ s.t. } X_i \cap \overline{T.(g.x)} \neq \emptyset\}),$$

where T is a maximal torus in G, and $X_1, \dots, X_s$ are the irreducible components of X^T.

Inclusions of sets of semi-stable points

The following proposition could be deduced from well-known results on the GIT-fan (see e.g. [DH98], Section 3.4, or [Res00], Section 5), but we give another demonstration specific to this case:

Proposition 4.2.6. *There exists $D \in \mathbb{N}$ such that, for all $d \geq D$, $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$.*

Proof. To all $x \in X$ and $g \in G$, we associate $E_{x,g} \in \mathcal{P}([\![1,s]\!])$ (i.e. a subset of $[\![1,s]\!]$) as follows:

$$E_{x,g} = \{i \in [\![1,s]\!] \text{ s.t. } X_i \cap \overline{T.(g.x)} \neq \emptyset\}.$$

With this notation, we know that:

$$x \in X^{ss}(\mathcal{L}) \Longleftrightarrow \forall g \in G,\ 0 \in \mathrm{conv}(\{\chi_i(\mathcal{L})\ ;\ i \in E_{x,g}\}).$$

So we set $A = \{E_{x,g} \text{ s.t. } 0 \notin \mathrm{conv}(\{\chi_i(\mathcal{L})\ ;\ i \in E_{x,g}\})\}$, which is finite since contained in $\mathcal{P}([\![1,s]\!])$. Then, for all $E \in A$, there exists $\varphi_E \in (\mathbb{R}^N)^*$ such that, for all $i \in E$, $\varphi_E(\chi_i(\mathcal{L})) > 0$. Moreover[3],

$$\forall E \in A,\ \forall i \in E,\ \varphi_E \circ \chi_i\left(\frac{\mathcal{M} \otimes \mathcal{L}^{\otimes d}}{d}\right) \xrightarrow[d\to+\infty]{} \varphi_E(\chi_i(\mathcal{L})) > 0,$$

[3] the additive maps $\varphi_E \circ \chi_i : \mathrm{Pic}^G(X) \to \mathbb{R}$ can be extended without problem to $\mathrm{Pic}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

so there exists $D_E \in \mathbb{N}^*$ such that, for all $d \geq D_E$, for all $i \in E$, $\varphi_E \circ \chi_i \left(\frac{\mathcal{M} \otimes \mathcal{L}^{\otimes d}}{d} \right) > 0$.

We then set $D = \max\{D_E \ ; \ E \in A\}$. Let $d \in \mathbb{N}$, $d \geq D$. Let $x \notin X^{ss}(\mathcal{L})$, which means that there exists $g \in G$ such that $0 \notin \mathrm{conv}(\{\chi_i(\mathcal{L}) \ ; \ i \in E_{x,g}\})$. In other words, $E_{x,g} \in A$. So, as $d \geq D \geq D_{E_{x,g}}$, $\varphi_{E_{x,g}}(\chi_i(\mathcal{M} \otimes \mathcal{L}^{\otimes d})) = d\varphi_{E_{x,g}} \circ \chi_i \left(\frac{\mathcal{M} \otimes \mathcal{L}^{\otimes d}}{d} \right) > 0$ for all $i \in E_{x,g}$. Hence

$$0 \notin \mathrm{conv}\left(\left\{ \chi_i \left(\mathcal{M} \otimes \mathcal{L}^{\otimes d} \right) \ ; \ i \in E_{x,g} \right\} \right), \text{ i.e. } x \notin X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}).$$

Thus, $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$. □

4.2.3 Use of Luna's Etale Slice Theorem

Let us recall that we considered a triple of partitions (α, β, γ) such that, for all $d \in \mathbb{N}^*$, $g_{d\alpha,d\beta,d\gamma} = 1$. This means that,

$$\forall d \in \mathbb{N}^*, \ \mathrm{H}^0(X, \mathcal{L}^{\otimes d})^G \simeq \mathbb{C}.$$

Then, using Proposition 8.1 of [Dol03], as X is projective,

$$X^{ss}(\mathcal{L}) /\!\!/ G \simeq \mathrm{Proj}(\mathbb{C}[t]).$$

So $X^{ss}(\mathcal{L}) /\!\!/ G$ is a point. Thus $X^{ss}(\mathcal{L})$ contains exactly one closed G-orbit, denoted by $G.x_0$. Moreover, $X^{ss}(\mathcal{L})$ is affine (since the canonical projection $X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L}) /\!\!/ G$ is affine). So we can use Corollary 2 to Luna's Slice Étale Theorem (cf. [Lun73]): there exist a reductive subgroup H – which is in fact the isotropy subgroup G_{x_0} – of G and an affine H-variety S such that

$$\begin{cases} S^H = \{x_0\} \\ \forall x \in S, \ x_0 \in \overline{H.x} \\ X^{ss}(\mathcal{L}) \simeq G \times_H S \end{cases}.$$

Furthermore, since X is smooth, S is isomorphic to $\mathrm{T}_{x_0}X/\mathrm{T}_{x_0}(G.x_0)$ as an H-variety. Thus S is a vector space of finite dimension on which H acts linearly.

4.2.4 Proof of the characterisation

We are now ready to prove Theorem 4.1.2. We still have our weakly stable triple (α, β, γ) and another triple of partitions (λ, μ, ν), which give rise to the two (semi-ample) line bundles $\mathcal{L}$ and $\mathcal{M}$.

Proposition 4.2.7. *If $D \in \mathbb{N}$ is such that, for all $d \geq D$, $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$, then*

$$\forall d \geq D, \ \mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(S, \mathcal{M})^H.$$

Proof. Let $D \in \mathbb{N}$ be as in the statement, and $d \in \mathbb{N}$, $d \geq D$. Then, thanks to Proposition 4.2.3,

$$\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}), \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G.$$

Consequently, since $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L}) \subset X$,

$$\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(X^{ss}(\mathcal{L}), \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G.$$

Now, using the consequence of Luna's Slice Étale Theorem:

$$\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(G \times_H S, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(S, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^H.$$

We are almost done; it only remains to prove that $\mathrm{H}^0(S, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^H$ does not depend on d. For this, we demonstrate that $\mathcal{L}$ is trivial on S, using the following lemma:

Lemma 4.2.8. *The map*

$$\begin{array}{rccc} \psi: & X^*(H) & \longrightarrow & \mathrm{Pic}^H(S) \\ & \chi & \longmapsto & \mathcal{L}_\chi \end{array},$$

where $\mathcal{L}_\chi$ is the trivial bundle $S \times \mathbb{C}$ whose H-linearisation is given by the character χ, is an isomorphism.

Proof. The only non trivial thing to prove is the surjectivity of ψ. Let $\mathcal{N} \in \mathrm{Pic}^H(S)$. We have seen that x_0 is a point of S fixed by H. So, H acts on the fibre $\mathcal{N}_{x_0}$. This action gives $\chi \in X^*(H)$. Moreover, $\mathcal{N}$ is trivial because S is a vector space. Necessarily, its linearisation is given by the character χ. □

We consider the character χ_0 given by the action of H on $\mathcal{L}_{x_0}$ and we want to prove that χ_0 is trivial. As $x_0 \in X^{ss}(\mathcal{L})$, there exist $k \in \mathbb{N}^*$ and $\sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes k})^G$ such that $\sigma(x_0) \neq 0$. Moreover, $\dim(\mathrm{H}^0(X, \mathcal{L})^G) = \dim(\mathrm{H}^0(X, \mathcal{L}^{\otimes k})^G) = 1$ so, if we take $\sigma_0 \in \mathrm{H}^0(X, \mathcal{L})^G \setminus \{0\}$, we have $\sigma_0^{\otimes k} = t\sigma$ with $t \in \mathbb{C}^*$. As a consequence, $\sigma_0^{\otimes k}(x_0) \neq 0$ and so $\sigma_0(x_0) \neq 0$.
Furthermore,

$$\forall h \in H,\ \sigma_0(x_0) = \sigma_0(h.x_0) = h.\sigma_0(x_0) = \chi_0(h)\sigma_0(x_0),$$

and then $\chi_0(h) = 1$ for all $h \in H$. Thus, χ_0 is trivial and so is $\mathcal{L}$ over S.

Finally,

$$\forall d \geq D,\ \mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(S, \mathcal{M})^H,$$

□

Proposition 4.2.6 and Proposition 4.2.7 together conclude the proof of Proposition 4.1.3, and as a consequence our proof of Theorem 4.1.2. Let us note that the formula we get for the limit coefficient (i.e. $\dim \mathrm{H}^0(S, \mathcal{M})^H$) corresponds to the one obtained by Paradan in [Par17], Theorem 5.12.

4.3 Explicit bounds of stabilisation in small cases

We saw in the previous section that the sequence $(g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma})_{d\in\mathbb{N}}$ stabilises as soon as $X^{ss}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})\subset X^{ss}(\mathcal{L})$. We now would like to see if one can compute the rank D from which this inclusion is realised. The computation of D from the proof of Proposition 4.2.6 appears to be too tricky, and so in the following we focus on two examples in which we can do explicit computations using another method.

4.3.1 Steps of the computation

The inclusion $X^{ss}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})\subset X^{ss}(\mathcal{L})$ we are interested in is equivalent to the following: $X^{us}(\mathcal{L})\subset X^{us}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})$. Here we are rather looking to prove this last one, principally because we find that the fact of being an unstable point has -thanks to the Hilbert-Mumford criterion- a more practical description. Here are the different steps we are then going to carry out on the two examples:

- The first step is to consider the projection $\pi : X\to\overline{X}$ onto the product of partial flag varieties such that $\mathcal{L}$ is the pull-back of an ample line bundle $\overline{\mathcal{L}}$ on $\overline{X}$.
- The second step is to study the set $\overline{X}^{us}(\overline{\mathcal{L}})$ of unstable points in $\overline{X}$. More precisely, we want to express this set as the union of some orbit closures: $\mathrm{cl}(G.\overline{x_1}),\dots,\mathrm{cl}(G.\overline{x_p})$.
- Then one can prove that, thanks to good properties of the projection π, $X^{us}(\mathcal{L})$ is the union of the closures of $\pi^{-1}(G.\overline{x_1}),\dots,\pi^{-1}(G.\overline{x_p})$. As a consequence, since $X^{us}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})$ is closed and π is G-equivariant, to prove that $X^{us}(\mathcal{L})\subset X^{us}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})$ we only need to show for all $i\in[\![1,p]\!]$ that $\pi^{-1}(\overline{x_i})\subset X^{us}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})$.
- In the fourth step we want to use the Hilbert-Mumford criterion. Let us write it in a way different from before:

  Definition 4.3.1. Let Y be a projective variety on which a reductive group H acts, and $\mathcal{N}$ a H-linearised line bundle on Y. Let $y\in Y$ and τ be a one-parameter subgroup of H (denoted $\tau\in X_*(H)$). Since Y is projective, $\lim\limits_{t\to 0}\tau(t).y$ exists. We denote it by z. This point is fixed by the image of τ, and so $\mathbb{C}^*$ acts via τ on the fibre $\mathcal{N}_z$. Then there exists an integer $\mu^{\mathcal{N}}(y,\tau)$ such that, for all $t\in\mathbb{C}^*$ and $\tilde{z}\in\mathcal{N}_z$,
  $$\tau(t).\tilde{z}=t^{-\mu^{\mathcal{N}}(y,\tau)}\tilde{z}.$$

  The Hilbert-Mumford criterion can then be stated as (see e.g. [Res10], Lemma 2):

  Proposition 4.3.2. *In the settings of the previous definition, if in addition $\mathcal{N}$ is semi-ample, then:*
  $$y\in Y^{ss}(\mathcal{N})\quad\Longleftrightarrow\quad\forall\tau\in X_*(H),\ \mu^{\mathcal{N}}(y,\tau)\le 0.$$

Set $i \in [\![1, p]\!]$. Since $\overline{x_i} \in \overline{X}^{us}(\overline{\mathcal{L}})$, we can find a destabilising one-parameter subgroup for $\overline{x_i}$: τ_i such that $\mu^{\overline{\mathcal{L}}}(\overline{x_i}, \tau_i) > 0$.

- Let us keep in mind that we want to get $\pi^{-1}(\overline{x_i}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$. By Hilbert-Mumford criterion, this will be true when, for all $x \in \pi^{-1}(\overline{x_i})$, $\mu^{\mathcal{M}\otimes\mathcal{L}^{\otimes d}}(x, \tau_i) > 0$. But, for such an x, we have:

  $$\mu^{\mathcal{M}\otimes\mathcal{L}^{\otimes d}}(x, \tau_i) = \mu^{\mathcal{M}}(x, \tau_i) + d\mu^{\overline{\mathcal{L}}}(\overline{x_i}, \tau_i).$$

  So we only need to calculate $\mu^{\mathcal{M}}(x, \tau_i)$ for all $x \in \pi^{-1}(\overline{x_i})$:

  - From the definition of the integers $\mu^{\mathcal{M}}(., \tau_i)$, we see that we can restrict to the case when $x \in \pi^{-1}(\overline{x_i})$ is a fixed point of τ_i. Then at first we determine the form of such a fixed point.
  - Finally we calculate explicitly the action of τ_i on the fibre of $\mathcal{M}$ over such a point.

- As a conclusion, as soon as

  $$d > -\frac{\mu^{\mathcal{M}}(x, \tau_i)}{\mu^{\overline{\mathcal{L}}}(\overline{x_i}, \tau_i)}$$

  for all $i \in [\![1, p]\!]$ and $x \in \pi^{-1}(\overline{x_i})^{\tau_i}$, we have the inclusion we were looking for.

4.3.2 Case of Murnaghan's stability

Reduction to ample line bundles

In this case, the stable triple we are interested in is simply $\big((1), (1), (1)\big)$. It has been known for a long time that it is a stable triple. Consider

$$\begin{array}{cccc} \pi : & X & \longrightarrow & \overbrace{\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}((V_1 \otimes V_2)^*)}^{\text{denoted } \overline{X}} \\ & ((W_{1,i})_i, (W_{2,i})_i, (W'_i)_i) & \longmapsto & (W_{1,1}, W_{2,1}, \{\varphi \in (V_1 \otimes V_2)^* \text{ s.t. } \ker\varphi = W'_{n_1 n_2 - 1}\}) \end{array}.$$

Since $\alpha = \beta = \gamma = (1)$, we have that $\mathcal{L} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{L}_\gamma^*$ is the pull-back of $\mathcal{O}(1) \otimes \mathcal{O}(1) \otimes \mathcal{O}(1)$ (denoted $\overline{\mathcal{L}}$ from now on) by π. Moreover,

$$\mathrm{H}^0(\overline{X}, \overline{\mathcal{L}})^G \simeq (V_1^* \otimes V_2^* \otimes V_1 \otimes V_2)^G \simeq \mathbb{C}.$$

So $\overline{X}^{ss}(\overline{\mathcal{L}}) = \{x \in \overline{X} \text{ s.t. } \overline{\sigma}_0(x) \neq 0\}$ for any $\overline{\sigma}_0 \in \mathrm{H}^0(\overline{X}, \overline{\mathcal{L}})^G \setminus \{0\}$. A simple non-zero section on $\overline{X}$ is

$$\mathbb{C}v_1 \otimes \mathbb{C}v_2 \otimes \mathbb{C}\varphi \longmapsto \varphi(v_1 \otimes v_2).$$

And

$$\overline{X}^{ss}(\overline{\mathcal{L}}) = \{(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}\varphi) \in \overline{X} \text{ s.t. } v_1 \otimes v_2 \notin \ker\varphi\}.$$

Determination of $\overline{X}^{us}(\overline{\mathcal{L}})$

Let us take $(e_1, \dots, e_{n_1})$ a basis in V_1 (with $n_1 \geq 2$), and $(f_1, \dots, f_{n_2})$ a basis in V_2 ($n_2 \geq 2$). Their dual bases are denoted with upper stars. Moreover, we set $n = \min(n_1, n_2)$.

Proposition 4.3.3. *The set $\overline{X}^{us}(\overline{\mathcal{L}})$ consists in the closure of the G-orbit of the element $\overline{x} = (\mathbb{C}e_1, \mathbb{C}f_2, \mathbb{C}\varphi_n)$, where $\varphi_n = \sum_{i=1}^{n} e_i^* \otimes f_i^* \in V_1^* \otimes V_2^* \simeq (V_1 \otimes V_2)^*$.*

Proof. At first, since $\overline{X}^{us}(\overline{\mathcal{L}}) = \{(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}\varphi) \in \overline{X} \text{ s.t. } \varphi(v_1 \otimes v_2) = 0\}$, $\overline{X}^{us}(\overline{\mathcal{L}})$ is pure of codimension 1.

Then $\mathbb{P}((V_1 \otimes V_2)^*) \simeq \mathbb{P}(V_1^* \otimes V_2^*) \simeq \mathbb{P}(\mathrm{Hom}(V_1, V_2^*))$. So we consider $(l_1, l_2, \mathbb{C}\psi) \in \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(\mathrm{Hom}(V_1, V_2^*))$. The action of G is then:

$$\forall (g_1, g_2) \in G,\ (g_1, g_2).(l_1, l_2, \mathbb{C}\psi) = (g_1(l_1), g_2(l_2), \mathbb{C}{}^t g_2^{-1} \circ \psi \circ g_1^{-1}).$$

So we know that the orbits of the action on the third part ($\mathbb{C}\psi$) are classified by the rank of ψ. Moreover, this triple $(l_1, l_2, \mathbb{C}\psi)$ defines several subspaces:

in V_1	in V_2	in V_1^*	in V_2^*
l_1	l_2	$H_1 = l_1^\perp$	$H_2 = l_2^\perp$
$\ker\psi$	$\ker {}^t\psi$	$\mathrm{Im}\, {}^t\psi = (\ker\psi)^\perp$	$\mathrm{Im}\,\psi = (\ker {}^t\psi) \perp$
$\psi^{-1}(H_2)$	${}^t\psi^{-1}(H_1)$	${}^t\psi(l_2) = \psi^{-1}(H_2)^\perp$	$\psi(l_1) = {}^t\psi^{-1}(H_1)^\perp$

and the different possible positions of l_1 and l_2 with respect to $\ker\psi$, $\psi^{-1}(H_2)$, and respectively $\ker {}^t\psi$, ${}^t\psi^{-1}(H_1)$, shall help us to describe the orbits. Furthermore, $\ker\psi \subset \psi^{-1}(H_2)$, $\ker {}^t\psi \subset {}^t\psi^{-1}(H_1)$, and $l_1 \subset \psi^{-1}(H_2) \Leftrightarrow l_2 \subset {}^t\psi^{-1}(H_1)$.

First case: $n_1 = n_2$ (so $n = n_1 = n_2$).
Let us first assume that $\mathrm{rk}\,\psi = n$. Then, $\ker\psi = \{0\}$ and $\ker {}^t\psi = \{0\}$. So this leaves two possibilities for the positions of l_1 and l_2:

- $l_1 \subset \psi^{-1}(H_2)$ and $l_2 \subset {}^t\psi^{-1}(H_1)$. One can check that such $(l_1, l_2, \mathbb{C}\psi)$ form one orbit, $\mathcal{O}_1$.
- $l_1 \not\subset \psi^{-1}(H_2)$ and $l_2 \not\subset {}^t\psi^{-1}(H_1)$. One can also check that such triples form a second orbit, $\mathcal{O}_2$.

We can see that $\mathcal{O}_1$ is unstable, whereas $\mathcal{O}_2$ is semi-stable.
What if $\mathrm{rk}\,\psi \leq n-1$? The closed subset $Y = \{(l_1, l_2, \mathbb{C}\psi) \text{ s.t. } \mathrm{rk}\,\psi \leq n-1\}$ satisfies $\mathrm{codim}(Y \cap \overline{X}^{us}(\overline{\mathcal{L}})) \geq 2$ because, for all l_1 and l_2, $\{\mathbb{C}\psi\,;\, \mathrm{rk}\,\psi \leq n-1 \text{ and } \psi(l_1)(l_2) = \{0\}\}$ has codimension 2 in $\mathbb{P}(\mathrm{Hom}(V_1, V_2^*))$. So the complement of $Y \cap \overline{X}^{us}(\overline{\mathcal{L}})$ intersects every irreducible components of $\overline{X}^{us}(\overline{\mathcal{L}})$. Thus, $Y^c = \{(l_1, l_2, \mathbb{C}\psi) \text{ s.t. } \mathrm{rk}\,\psi = n\}$ intersects every irreducible components of $\overline{X}^{us}(\overline{\mathcal{L}})$
Conclusion for this case: $\overline{X}^{us}(\overline{\mathcal{L}}) = \mathrm{cl}(\mathcal{O}_1)$, the closure of orbit $\mathcal{O}_1$. Furthermore, a representative of $\mathcal{O}_1$ is $\overline{x} = (\mathbb{C}e_1, \mathbb{C}f_2, \mathbb{C}\varphi_n)$.

Second case: $n_1 < n_2$ (and then $n = n_1$).
In this case, $\{\mathbb{C}\psi \text{ s.t. } \operatorname{rk}\psi \leq n-1\}$ has codimension at least 2 (because the minors of rank n must be zero, and there are at least 2). So, as in the previous case, it suffices to consider the case where $\operatorname{rk}\psi = n$, for which $\ker\psi = \{0\}$ and $\ker {}^t\psi \neq \{0\}$. This leads to three possibilities for l_1 and l_2:

- $l_1 \subset \psi^{-1}(H_2)$ and $l_2 \subset \ker {}^t\psi \subset {}^t\psi^{-1}(H_1)$. One can check that such $(l_1, l_2, \mathbb{C}\psi)$ form one orbit, $\mathcal{O}_1$.

- $l_1 \subset \psi^{-1}(H_2)$ and $l_2 \not\subset \ker {}^t\psi$, but $l_2 \subset {}^t\psi^{-1}(H_1)$. Once again, one can check that this gives only one orbit, $\mathcal{O}_2$.

- $l_1 \not\subset \psi^{-1}(H_2)$ and $l_2 \not\subset {}^t\psi^{-1}(H_1)$. One can still check that these triples form one orbit, $\mathcal{O}_3$.

The orbit $\mathcal{O}_3$ is semi-stable, whereas $\mathcal{O}_1$ and $\mathcal{O}_2$ are unstable. In addition, $\mathcal{O}_1 \subset \operatorname{cl}(\mathcal{O}_2)$ because, if $\operatorname{rk}\psi = n$ and $(l_1, l_2, \mathbb{C}\psi)$ is unstable, $(l_1, l_2, \mathbb{C}\psi) \in \mathcal{O}_1 \Leftrightarrow {}^t\psi(l_2) = \{0\}$ and $(l_1, l_2, \mathbb{C}\psi) \in \mathcal{O}_2 \Leftrightarrow {}^t\psi(l_2) \neq \{0\}$.
Conclusion for that case: Here, $\overline{X}^{us}(\overline{\mathcal{L}}) = \operatorname{cl}(\mathcal{O}_2)$ and a representative of $\mathcal{O}_2$ is the same $\overline{x}$ as before: $\overline{x} = (\mathbb{C}e_1, \mathbb{C}f_2, \mathbb{C}\varphi_n)$.

Third and last case: $n_1 > n_2$.
Everything happens similarly to the previous case, if we exchange the roles of V_1 and V_2. So we have also the orbit of $\overline{x} = (\mathbb{C}e_1, \mathbb{C}f_2, \mathbb{C}\varphi_n)$ which is dense in $\overline{X}^{us}(\overline{\mathcal{L}})$. □

Restriction to $\pi^{-1}(\overline{x})$

The projection π we use is of the form

$$\pi : \tilde{G}/\tilde{B} \longrightarrow \tilde{G}/\tilde{P},$$

with $\tilde{G}$ a complex reductive group, $\tilde{B}$ a Borel subgroup, and $\tilde{P}$ a parabolic subgroup containing $\tilde{B}$. So the fibres are all isomorphic to $\tilde{P}/\tilde{B}$ (π is even a fibration). This is also true for its restriction to $X^{us}(\mathcal{L}) = \pi^{-1}(\overline{X}^{us}(\overline{\mathcal{L}}))$. Thus, since $G.\overline{x}$ is dense in $\overline{X}^{us}(\overline{\mathcal{L}})$, $\pi^{-1}(G.\overline{x})$ is dense in $X^{us}(\mathcal{L})$. As a consequence, $X^{us}(\mathcal{L}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$ if $\pi^{-1}(G.\overline{x}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$ (because $X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$ is closed). And finally, if $\pi^{-1}(\overline{x}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$, then $\pi^{-1}(G.\overline{x}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$ since π is G-equivariant. Hence the following lemma:

Lemma 4.3.4. *If $d_0 \in \mathbb{N}$ is such that, for all $d \geq d_0$, $\pi^{-1}(\overline{x}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$, then*

$$\forall d \geq d_0,\ g_{\lambda+d(1),\mu+d(1),\nu+d(1)} = g_{\lambda+d_0(1),\mu+d_0(1),\nu+d_0(1)}.$$

Computation of the bound

We identify $\mathrm{GL}(V_1)$, $\mathrm{GL}(V_2)$, and $\mathrm{GL}(V_1 \otimes V_2)$ respectively with $\mathrm{GL}_{n_1}(\mathbb{C})$, $\mathrm{GL}_{n_2}(\mathbb{C})$, and $\mathrm{GL}_{n_1 n_2}(\mathbb{C})$ thanks to the bases $(e_1, \dots, e_{n_1})$ and $(f_1, \dots, f_{n_2})$ of V_1 and V_2 respectively that we considered. The basis in $V_1 \otimes V_2$ is then $(e_i \otimes f_j)_{i,j}$, ordered lexicographically. Moreover we use the following notation for one-parameter subgroups of some $\mathrm{GL}_{m_1}(\mathbb{C}) \times \dots \times \mathrm{GL}_{m_p}(\mathbb{C})$:

$$\begin{array}{cccc} \tau: & \mathbb{C}^* & \longrightarrow & \mathrm{GL}_{m_1}(\mathbb{C}) \times \dots \times \mathrm{GL}_{m_p}(\mathbb{C}) \\ & t & \longmapsto & \left(\begin{pmatrix} t^{a_1^{(1)}} & & & \\ & t^{a_2^{(1)}} & & \\ & & \ddots & \\ & & & t^{a_{m_1}^{(1)}} \end{pmatrix}, \dots, \begin{pmatrix} t^{a_1^{(p)}} & & & \\ & t^{a_2^{(p)}} & & \\ & & \ddots & \\ & & & t^{a_{m_p}^{(p)}} \end{pmatrix} \right) \end{array}$$

is denoted by $\tau = \left(a_1^{(1)}, a_2^{(1)}, \dots, a_{m_1}^{(1)} | \dots | a_1^{(p)}, \dots, a_{m_p}^{(p)}\right)$.

Destabilising one-parameter subgroup for $\overline{x}$: We set the following one-parameter subgroup of G:

$$\tau_0 = \big(1, -1, 0, \dots, 0 \,\big|\, -1, 1, 0, \dots, 0\big).$$

Then, since the action of $\tau_0(t)$ on the lines $\mathbb{C}e_1$, $\mathbb{C}f_2$, and $\mathbb{C}\varphi_n$ is the multiplication by t, t, and 1 respectively, we have

$$\mu^{\overline{\mathcal{L}}}(\overline{x}, \tau_0) = 2.$$

Let now $x \in \pi^{-1}(\overline{x})$. We want to calculate $\mu^{\mathcal{M}}(x, \tau_0)$. Thanks to the way μ is defined (first, one has to take the limit when $t \to 0$ from $\tau_0(t).x$ and gets a fixed point of τ_0), and since $\overline{x}$ is fixed by τ_0, it suffices to calculate $\mu^{\mathcal{M}}(x, \tau_0)$ for $x \in \pi^{-1}(\overline{x})^{\tau_0}$. So we take $x \in \pi^{-1}(\overline{x})^{\tau_0}$.

Form of an element $x \in \pi^{-1}(\overline{x})^{\tau_0}$: First of all, the action of τ_0 on V_1 has three different weights: 1,-1, and 0, whose corresponding subspaces are

$$W_1 = \mathbb{C}e_1, \ W_{-1} = \mathbb{C}e_2, \text{ and } W_0 = \mathbb{C}e_3 + \dots + \mathbb{C}e_{n_1}.$$

Thus, the component of x in $\mathcal{F}\ell(V_1)$ is a flag given by a basis of V_1 composed of: e_1 at first, e_2 in a position i between 2 and n_1, and $n_1 - 2$ vectors forming a basis of W_0. For the same reasons, there exists an integer j between 2 and n_2 such that the second component of x (in $\mathcal{F}\ell(V_2)$) is a flag given by a basis of V_2 composed of f_2 at first, f_1 in position j, and $n_2 - 2$ vectors forming a basis of $\mathbb{C}f_3 + \dots + \mathbb{C}f_{n_2}$.

For the third component (in $\mathcal{F}\ell(V_1 \otimes V_2)$) of x: the action of τ_0 on $V_1 \otimes V_2$ has now five different weights, 2, -2, 1, -1, and 0, whose respective corresponding subspaces are

$$W_2 = \mathbb{C}e_1 \otimes f_2, \ W_{-2} = \mathbb{C}e_2 \otimes f_1, \ W_1 = \mathbb{C}e_1 \otimes f_3 + \dots + \mathbb{C}e_1 \otimes f_{n_2} + \mathbb{C}e_3 \otimes f_2 + \dots + \mathbb{C}e_{n_1} \otimes f_2,$$

$$W_{-1} = \mathbb{C}e_2 \otimes f_3 + \cdots + \mathbb{C}e_2 \otimes f_{n_2} + \mathbb{C}e_3 \otimes f_1 + \cdots + \mathbb{C}e_{n_1} \otimes f_1,$$

$$W_0 \text{ spanned by the rest of the } e_i \otimes f_j.$$

Thus, the component of x in $\mathcal{F}\ell(V_1 \otimes V_2)$ is a flag given by a basis of $V_1 \otimes V_2$ of the form:

- $e_1 \otimes f_2$ at a position k_2 between 1 and $n_1n_2 - 1$,
- $e_2 \otimes f_1$ at a position k_{-2} between 1 and $n_1n_2 - 1$,
- n_1+n_2-4 vectors forming a basis of W_1 at positions $m_1^{(1)}, \dots, m_{n_1+n_2-4}^{(1)}$ (between 1 and $n_1n_2 - 1$),
- $n_1 + n_2 - 4$ vectors forming a basis of W_{-1} at positions $m_1^{(-1)}, \dots, m_{n_1+n_2-4}^{(-1)}$ (between 1 and $n_1n_2 - 1$),
- the other vectors forming a basis of W_0.

Calculation of the action of τ_0 on the fibre of $\mathcal{M}$ over x: (We denote this fibre by $\mathcal{M}_x$). Let us recall another description, for δ a partition, of the line bundle $\mathcal{L}_\delta$ on a flag variety $\mathcal{F}\ell(V)$ (with $\dim V = n \geq \ell(\delta)$). We have the embedding

$$\begin{array}{lccc} \iota: & \mathcal{F}\ell(V) & \longrightarrow & \prod_{k=1}^n \mathbb{P}(\bigwedge^k V) \\ & (\mathbb{C}v_1, \mathbb{C}v_1 \oplus \mathbb{C}v_2, \dots, \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n) & \longmapsto & (\mathbb{C}v_1, \mathbb{C}(v_1 \wedge v_2), \dots, \mathbb{C}(v_1 \wedge \cdots \wedge v_n) \end{array}.$$

Then $\mathcal{L}_\delta$ is the pull-back of the line bundle $\mathcal{O}(\delta_1 - \delta_2) \otimes \cdots \otimes \mathcal{O}(\delta_{n-1} - \delta_n) \otimes \mathcal{O}(\delta_n)$ by ι (for all the partitions that we use, we take the convention that, if $i > \ell(\delta)$, δ_i is simply 0). Using this description and the form of an element $x \in \pi^{-1}(\overline{x})^{\tau_0}$, we can easily get the following:

Lemma 4.3.5. *For $x \in \pi^{-1}(\overline{x})^{\tau_0}$, there exist $i \in [\![2, n_1]\!]$, $j \in [\![2, n_2]\!]$, and $2(n_1 + n_2 - 3)$ distinct integers $k_2, k_{-2}, m_1^{(1)}, \dots, m_{n_1+n_2-4}^{(1)}, m_1^{(-1)}, \dots, m_{n_1+n_2-4}^{(-1)} \in [\![1, n_1n_2 - 1]\!]$ such that*

$$\mu^{\mathcal{M}}(x, \tau_0) = \lambda_1 - \lambda_i + \mu_1 - \mu_j + 2(\nu'_{k_{-2}} - \nu'_{k_2}) + \sum_{k=1}^{n_1+n_2-4} (\nu'_{m_k^{(-1)}} - \nu'_{m_k^{(1)}}),$$

with $(\nu'_1, \dots, \nu'_{n_1n_2}) = (\nu_{n_1n_2}, \dots, \nu_1)$. Moreover, all the possibilities for i, j, k_2, k_{-2}, the $m_k^{(1)}$'s, and the $m_k^{(-1)}$'s arise when x varies in $\pi^{-1}(\overline{x})^{\tau_0}$.
As a consequence,

$$\max_{x \in \pi^{-1}(\overline{x})} \big(-\mu^{\mathcal{M}}(x, \tau_0)\big) = -\lambda_1 + \lambda_2 - \mu_1 + \mu_2 + 2(\nu_2 - \nu_{n_1n_2}) + \sum_{k=1}^{n_1+n_2-4} (\nu_{k+2} - \nu_{n_1n_2-k}).$$

Finally, Lemma 4.3.4 leads to the following result:

Proposition 4.3.6. *If we set*

$$d_0 = \frac{1}{2}\left(-\lambda_1 + \lambda_2 - \mu_1 + \mu_2 + 2(\nu_2 - \nu_{n_1n_2}) + \sum_{k=1}^{n_1+n_2-4} (\nu_{k+2} - \nu_{n_1n_2-k})\right),$$

we have for all $d \in \mathbb{N}$ such that $d > d_0$,
$g_{\lambda+d(1),\mu+d(1),\nu+d(1)} = g_{\lambda+\lfloor d_0+1\rfloor(1),\mu+\lfloor d_0+1\rfloor(1),\nu+\lfloor d_0+1\rfloor(1)}$.

Proof. For all $x \in \pi^{-1}(\overline{x})$ and all $d > d_0$,

$$\mu^{\mathcal{M}\otimes\mathcal{L}^{\otimes d}}(x,\tau_0) = \mu^{\mathcal{M}}(x,\tau_0) + d\mu^{\overline{\mathcal{L}}}(\overline{x},\tau_0) = \mu^{\mathcal{M}}(x,\tau_0) + 2d > 0$$

because $d > d_0 \geq -\frac{1}{2}\mu^{\mathcal{M}}(x,\tau_0)$. Thus, by Hilbert-Mumford criterion, $x \in X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$, and we conclude using Lemma 4.3.4. □

Remark 4.3.7. We even have the inclusion $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$ which is true for all $d > d_0$, $d \in \mathbb{Q}$. Indeed, the definition of $X^{ss}(\mathcal{N})$ (and the one from $\mu^{\mathcal{N}}(.,\tau_0)$) can be extended to $\mathcal{N} \in \mathrm{Pic}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$X^{ss}(\mathcal{N}) = \{x \in X \mid \exists k \in \mathbb{N}^* \text{ s.t. } \mathcal{N}^{\otimes k} \in \mathrm{Pic}^G(X) \text{ and } \exists \sigma \in \mathrm{H}^0(X, \mathcal{N}^{\otimes k})^G,\ \sigma(x) \neq 0\}.$$

4.3.3 Case of the triple $\big((1,1),(1,1),(2)\big)$

We now have a look at the triple $\big((1,1),(1,1),(2)\big)$ which is also stable (cf. for instance [Ste14]). We consider

$$\begin{array}{rccc} \pi: & X & \longrightarrow & \overbrace{\mathcal{F}\ell(V_1;1,2) \times \mathcal{F}\ell(V_2;1,2) \times \mathbb{P}((V_1 \otimes V_2)^*)}^{\text{denoted } \overline{X}} \\ & ((W_i)_i, (W_i')_i, (W_i'')_i) & \longmapsto & ((W_1,W_2),(W_1',W_2'),\{\varphi \in (V_1\otimes V_2)^* \ / \ \ker\varphi = W''_{n_1n_2-1}\}) \end{array}.$$

Similarly as before, the line bundle $\mathcal{L}$ is the pull-back by π of $\overline{\mathcal{L}} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{O}(2)$. The same arguments that we have used throughout the previous section are also going to work here. The only changes will be the orbits of G in $\overline{X}$ which are unstable:

Proposition 4.3.8. *If $n_1 \geq 3$ or $n_2 \geq 3$, then the set $\overline{X}^{us}(\overline{\mathcal{L}})$ of unstable points consists in the union of the closures of two G-orbits: that of $\overline{x}_1 = ((\mathbb{C}e_1, \mathbb{C}e_1 + \mathbb{C}e_2), (\mathbb{C}f_3, \mathbb{C}f_3 + \mathbb{C}f_1), \mathbb{C}\varphi_n)$ and that of $\overline{x}_2 = ((\mathbb{C}e_1, \mathbb{C}e_1 + \mathbb{C}e_2), (\mathbb{C}f_2, \mathbb{C}f_2 + \mathbb{C}f_3), \mathbb{C}\varphi_n)$.*

Proof. It is completely similar to the proof of Proposition 4.3.3. □

We then set two destabilising one-parameter subgroups of G for the two elements $\overline{x}_1$ and $\overline{x}_2$ (we still consider the case when $n_1, n_2 \geq 3$):

$$\tau_1 = \big(0,1,-1,0,\dots,0 \ \big|\ 0,-1,1,0,\dots,0\big)$$

and

$$\tau_2 = \big(1,0,-1,0,\dots,0 \,\big|\, -1,0,1,0,\dots,0\big),$$

which give

$$\mu^{\overline{\mathcal{L}}}(\overline{x}_1,\tau_1)=2=\mu^{\overline{\mathcal{L}}}(\overline{x}_2,\tau_2).$$

As before, we only have to get a bound from which $\pi^{-1}(\overline{x}_1)\subset X^{us}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})$ and $\pi^{-1}(\overline{x}_2)\subset X^{us}(\mathcal{M}\otimes\mathcal{L}^{\otimes d})$. We have already seen the form of elements of $\pi^{-1}(\overline{x}_1)^{\tau_1}$ and $\pi^{-1}(\overline{x}_2)^{\tau_2}$, and as a consequence we get:

Lemma 4.3.9. *If $n_1\geq 3$ and $n_2\geq 3$,*

$$\max_{x_1\in\pi^{-1}(\overline{x}_1)}(-\mu^{\mathcal{M}}(x_1,\tau_1))=-\lambda_2+\lambda_3-\mu_1+\mu_3+2(\nu_2-\nu_{n_1n_2})+\sum_{k=1}^{n_1+n_2-4}(\nu_{k+2}-\nu_{n_1n_2-k})$$

and

$$\max_{x_2\in\pi^{-1}(\overline{x}_2)}(-\mu^{\mathcal{M}}(x_2,\tau_2))=-\lambda_1+\lambda_3-\mu_2+\mu_3+2(\nu_2-\nu_{n_1n_2})+\sum_{k=1}^{n_1+n_2-4}(\nu_{k+2}-\nu_{n_1n_2-k}).$$

What remains to be seen is what happens when $n_1=2$ or $n_2=2$. Let us focus on the case where $n_1=2$ and $n_2\geq 3$. Then, τ_1 and τ_2 become:

$$\tau_1=\big(0,1\;\big|\;0,-1,1,0,\dots,0\big),\qquad \tau_2=\big(1,0\;\big|\;-1,0,1,0,\dots,0\big).$$

We still have $\mu^{\overline{\mathcal{L}}}(\overline{x}_1,\tau_1)=2=\mu^{\overline{\mathcal{L}}}(\overline{x}_2,\tau_2)$, but this time

$$\max_{x_1\in\pi^{-1}(\overline{x}_1)}(-\mu^{\mathcal{M}}(x_1,\tau_1))=-\lambda_2-\mu_1+\mu_3+2\nu_2-\nu_{2n_2}+\sum_{k=1}^{n_2-1}\nu_{k+2},$$

and

$$\max_{x_2\in\pi^{-1}(\overline{x}_2)}(-\mu^{\mathcal{M}}(x_2,\tau_2))=-\lambda_1-\mu_2+\mu_3+2\nu_2-\nu_{2n_2}+\sum_{k=1}^{n_2-1}\nu_{k+2}.$$

By exchanging the roles of V_1 and V_2 (that is to say λ and μ), we easily get the result for the case $n_1\geq 3$, $n_2=2$. Only the case $n_1=2=n_2$ remains, and we could do exactly the same. But the result we would get would be exactly the formula for $n_1=2$, $n_2\geq 3$ in which we take μ_3 to be zero. Finally we have:

Proposition 4.3.10. *If we set $m=\max(-\lambda_2-\mu_1,-\lambda_1-\mu_2)$ and*

$$d_0=\begin{cases}\dfrac{1}{2}\left(m+\lambda_3+\mu_3+2(\nu_2-\nu_{n_1n_2})+\displaystyle\sum_{k=1}^{n_1+n_2-4}(\nu_{k+2}-\nu_{n_1n_2-k})\right) & \text{if } n_1,n_2\geq 3\\ \dfrac{1}{2}\left(m+\mu_3+2\nu_2-\nu_{2n_2}+\displaystyle\sum_{k=1}^{n_2-1}\nu_{k+2}\right) & \text{if } n_1=2\\ \dfrac{1}{2}\left(m+\lambda_3+2\nu_2-\nu_{2n_1}+\displaystyle\sum_{k=1}^{n_1-1}\nu_{k+2}\right) & \text{if } n_2=2\end{cases},$$

then we have, for all $d \in \mathbb{N}$ such that $d > d_0$,

$$g_{\lambda+d(1,1),\mu+d(1,1),\nu+d(2)} = g_{\lambda+\lfloor d_0+1\rfloor(1,1),\mu+\lfloor d_0+1\rfloor(1,1),\nu+\lfloor d_0+1\rfloor(2)}.$$

Proof. It is exactly in the same way as the proof of Proposition 4.3.6. □

Remark 4.3.11. We can notice that, in the cases where $n_1 = 2$ or $n_2 = 2$, we have two possible bounds: the one which concerns only these cases, or the general one, which we can use by considering λ (respectively μ) of length 3 by setting $\lambda_3 = 0$ (respectively $\mu_3 = 0$). We will come back to this in Remark 4.3.16.

Remark 4.3.12. As after Proposition 4.3.6, we have also here that the inclusion $X^{ss}(\mathcal{M}\otimes\mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$ is true for all $d > d_0$, $d \in \mathbb{Q}$.

4.3.4 Slight improvement of the previous bounds

In Propositions 4.3.6 and 4.3.10, we got an integer or half-integer d_0 such that the sequence $(g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma})_d$ is constant for all integers strictly greater than d_0. We now want to prove that, if this d_0 is an integer, this sequence of Kronecker coefficients already stabilises for our bound d_0. We need at first, in the following subsection, to expose a well-known result of quasipolynomiality.

Piecewise quasipolynomial behaviour of the dimension of invariants in an irreducible representation

This part of the chapter is quite disconnected with the others. The inspiration for the proofs given here is the article [KP14], in which the case of T-invariants is studied. Note also that the quasipolynomial behaviour of this kind of multiplicities can be seen as a consequence of the work of E. Meinrenken and R. Sjamaar on $[Q, R] = 0$ (it is explained in Section 13 of [PV16]). The following settings concern this subsection and only this one.

Let G be a connected complex reductive group, and H be a subgroup of G, also reductive. We consider a maximal torus T, a Borel subgroup B of G such that $T \subset B$, and the corresponding flag variety $X = G/B$. We denote by $X^*(T)$ the (multiplicative) group of characters of T, and by Q and Λ respectively the root lattice and the weight lattice. Λ^+ (resp. Λ^{++}) denotes the dominant (resp. dominant regular) weights.

Let us recall that $X^*(T)$ can be embedded as a sublattice of Λ (let set $\iota : X^*(T) \hookrightarrow \Lambda$) and that

$$Q \subset \iota(X^*(T)) \subset \Lambda.$$

Set $X^*(T)^+ = \iota(X^*(T)) \cap \Lambda^+$, and

$$\begin{array}{rccc} m: & X^*(T)^+ & \longrightarrow & \mathbb{N} \\ & \lambda & \longmapsto & \dim V(\lambda)^H \end{array},$$

where $V(\lambda)$ is the irreducible G-module with highest weight λ. The result we want to show is that m is piecewise quasipolynomial. For a more precise statement, let us consider $X^*(\mathbb{R}) = \iota(X^*(T)) \otimes_{\mathbb{Z}} \mathbb{R}$, $X^*(\mathbb{R})^+$ the cone spanned by $X^*(T)^+$, and $X^*(\mathbb{R})^{++}$ the relative interior of this cone.
Here we use the more standard definition of semi-stability: if $\mathcal{L}$ is a H-linearised line bundle on X, a point $x \in X$ is said semi-stable (with respect to $\mathcal{L}$) if there exist $n \in \mathbb{N}^*$ and $\sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^H$ such that $\{y \in X \text{ s.t. } \sigma(y) \neq 0\}$ is affine and contains x. To avoid confusion with the notion of semi-stability that we use everywhere but in this subsection, we denote by $X^{ss}_{\mathrm{st}}(\mathcal{L})$ the set of these semi-stable points with respect to $\mathcal{L}$. Finally let us denote by $C_1, \dots, C_N$ the chambers in $X^*(\mathbb{R})^{++}$, i.e. the GIT-classes of maximal dimension. Let us recall that the chambers are the relative interiors of convex rational polyhedral cones in $X^*(\mathbb{R})^{++}$ (see [Res00]). For all k, denote by $X^{ss}_{\mathrm{st}}(C_k)$ the set of semi-stable points common to all $\mathcal{L}_\lambda$ for $\lambda \in C_k$.

Lemma 4.3.13. *There exists a sublattice Γ of $\iota(X^*(T))$ of finite index such that, for all $k \in [\![1, N]\!]$, for all $\lambda \in \Gamma$, the H-linearised line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$ descends to a line bundle on $X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H$ (i.e. the restriction of $\mathcal{L}_\lambda$ to $X^{ss}_{\mathrm{st}}(C_k)$ is H-isomorphic to the pull-back of a line bundle on $X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H$).*

Proof. For better readability we have divided this demonstration into four steps.
First step: we want to prove that, for all $k \in [\![1, N]\!]$, there exists a sublattice Γ_k of $\iota(X^*(T))$ of finite index such that, for all $\lambda \in \Gamma_k \cap C_k^\ell$ (where $C_k^\ell = C_k \cap \iota(X^*(T))$), $\mathcal{L}_\lambda$ descends to $X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H$.
Let $k \in [\![1, N]\!]$. We set

$$A_k = \{\lambda \in C_k^\ell \text{ s.t. } \mathcal{L}_\lambda \text{ descends to } X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H\}.$$

Then it is clear that A_k is stable by addition. Thus consider Γ_k the lattice generated by A_k. It satisfies $A_k = \Gamma_k \cap C_k^\ell$ and so, for all $\lambda \in \Gamma_k \cap C_k^\ell$, $\mathcal{L}_\lambda$ descends to $X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H$.
Let us now check that Γ_k is of finite index in $\iota(X^*(T))$. It suffices to prove that there exists $n \in \mathbb{N}^*$ such that, for all $\lambda \in C_k^\ell$, $n\lambda \in \Gamma_k$, i.e. $\mathcal{L}_{n\lambda} \simeq \mathcal{L}_\lambda^{\otimes n}$ descends to $X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H$.
For all $\lambda \in \overline{C_k^\ell} = \overline{C_k} \cap \iota(X^*(T))$ and $x \in X^{ss}_{\mathrm{st}}(C_k) \subset X^{ss}_{\mathrm{st}}(\mathcal{L}_\lambda)$, by definition we know that there exist $n_{x,\lambda} \in \mathbb{N}^*$ and $\sigma_{x,\lambda} \in \mathrm{H}^0(X, \mathcal{L}_\lambda^{\otimes n_{x,\lambda}})^H$ such that $\sigma_{x,\lambda}(x) \neq 0$. Let $\lambda \in C_k^\ell$. Then the algebra

$$R = \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}_\lambda^{\otimes n})^H$$

is of finite type. Let us set $\sigma_1, \dots, \sigma_r$ a system of generators of R (we can choose $\sigma_i \in \mathrm{H}^0(X, \mathcal{L}_\lambda^{\otimes n_i})^H$ for some $n_i \in \mathbb{N}^*$). Write $n_\lambda = \prod_{i=1}^r n_i \in \mathbb{N}^*$. Then, for $x \in X^{ss}_{\mathrm{st}}(C_k)$, there exists

$$\sigma_{x,\lambda} = \sigma_1^{\otimes a_1} \otimes \cdots \otimes \sigma_r^{\otimes a_r}$$

with $a_1, \dots, a_r \in \mathbb{N}$ not all zero such that $\sigma_{x,\lambda}(x) \neq 0$. So there exists $i \in [\![1, r]\!]$ such that $\sigma_i(x) \neq 0$. Hence $\sigma_i^{\otimes n_1 \dots n_{i-1} n_{i+1} \dots n_r}(x) \neq 0$ with $\sigma_i^{\otimes n_1 \dots n_{i-1} n_{i+1} \dots n_r} = \sigma_0 \in$

$\mathrm{H}^0(X, \mathcal{L}_\lambda^{\otimes n_\lambda})^H$.
Thus, if we denote by χ the character by which H_x acts on the fiber $\left(\mathcal{L}_\lambda^{\otimes n_\lambda}\right)_x$, we have

$$\forall h \in H_x,\ \chi(h)\sigma_0(x) = h.\sigma_0(x) = \sigma_0(h.x) = \sigma_0(x),$$

and so χ is trivial. We have just proven that, for all $x \in X_{\mathrm{st}}^{ss}(C_k)$, H_x acts trivially on $\left(\mathcal{L}_\lambda^{\otimes n_\lambda}\right)_x$. In other words, by Kempf's Descent Lemma (see e.g. Lemma 3.8 in [Kum08]), $\mathcal{L}_{n_\lambda \lambda}$ descends to $X_{\mathrm{st}}^{ss}(C_k)$, i.e. $n_\lambda \lambda \in \Gamma_k$.
Now, $\overline{C_k^\ell}$ is finitely generated, since it is the intersection of a lattice and a closed convex rational polyhedral cone (see e.g. Section 5.18 from [Sch03] on Hilbert bases). So if we take $\lambda_1, \dots, \lambda_p$ generators, by setting $n = \prod_{i=1}^p n_{\lambda_i} \in \mathbb{N}^*$ we get:

$$\forall \lambda \in C_k^\ell,\ n\lambda \in \Gamma_k.$$

Thus Γ_k is of finite index in $\iota(X^*(T))$.

Second step: Now we set

$$\Gamma = \bigcap_{k=1}^N \Gamma_k.$$

It is a sublattice of $\iota(X^*(T))$ of finite index, since $\Gamma_1, \dots, \Gamma_N$ are. Moreover, for all $k \in [\![1, N]\!]$, for all $\lambda \in \Gamma \cap C_k \subset \Gamma_k \cap C_k$, $\mathcal{L}_\lambda$ descends to a line bundle denoted $\hat{\mathcal{L}}_\lambda^{(k)}$ on $X_{\mathrm{st}}^{ss}(C_k)$.

Third step: Let $k \in [\![1, N]\!]$. We can notice that $\Gamma \cap C_k$ is a semigroup: it is the intersection between a lattice and the interior of a convex rational polyhedral cone. Let us consider Z_k the subgroup of Γ generated by $\Gamma \cap C_k$. Let $\lambda \in Z_k$. It can be written as $\lambda_1 - \lambda_2$, with $\lambda_1, \lambda_2 \in \Gamma \cap C_k$. Then we define

$$\hat{\mathcal{L}}_\lambda^{(k)} = \hat{\mathcal{L}}_{\lambda_1}^{(k)} \otimes \left(\hat{\mathcal{L}}_{\lambda_2}^{(k)}\right)^*,$$

which is a line bundle on $X_{\mathrm{st}}^{ss}(C_k) /\!\!/ H$. If $\lambda = \lambda_1' - \lambda_2'$ also ($\lambda_1', \lambda_2' \in \Gamma \cap C_k$), then $\lambda_1 + \lambda_2' = \lambda_1' + \lambda_2 \in \Gamma \cap C_k$ and so $\hat{\mathcal{L}}_{\lambda_1+\lambda_2'}^{(k)} \simeq \hat{\mathcal{L}}_{\lambda_1'+\lambda_2}^{(k)}$. Moreover, by uniqueness of the line bundle to which a line bundle can descend (cf. [Tel00], §3), $\hat{\mathcal{L}}_{\lambda_1+\lambda_2'}^{(k)} \simeq \hat{\mathcal{L}}_{\lambda_1}^{(k)} \otimes \hat{\mathcal{L}}_{\lambda_2'}^{(k)}$, and similarly for $\hat{\mathcal{L}}_{\lambda_1'+\lambda_2}^{(k)}$. Thus,

$$\hat{\mathcal{L}}_{\lambda_1}^{(k)} \otimes \left(\hat{\mathcal{L}}_{\lambda_2}^{(k)}\right)^* \simeq \hat{\mathcal{L}}_{\lambda_1'}^{(k)} \otimes \left(\hat{\mathcal{L}}_{\lambda_2'}^{(k)}\right)^*,$$

and our $\hat{\mathcal{L}}_\lambda^{(k)}$ is well defined. As a consequence $\mathcal{L}_\lambda$ descends to a line bundle on $X_{\mathrm{st}}^{ss}(C_k) /\!\!/ H$ for all $\lambda \in Z_k$.

Fourth step: To conclude, let us prove that $Z_k = \Gamma$. We consider $\gamma_1, \dots, \gamma_r$ a system of generators of Γ, and a norm $\|.\|$ on $X^*(\mathbb{R})$. Set $d = \max\{\|\gamma_i\|\,;\, i \in [\![1, r]\!]\}$. Then there

exists $\lambda \in \Gamma \cap C_k$ such that $B(\lambda, d)$, the closed ball of center λ and radius d, is contained in C_k. Hence $\lambda + \gamma_i \in B(\lambda, d) \subset C_k$ for all $i \in [\![1, r]\!]$.
So, for all i, $\lambda + \gamma_i \in \Gamma \cap C_k$, and thus $\gamma_i \in Z_k$. Hence $Z_k = \Gamma$, which proves the lemma. □

The following result is then a classical one. The proof we write here is an adaptation (but with less quantitative results) from the one by Kumar and Prasad in [KP14], which was in the case of T-invariants.

Theorem 4.3.14. *Let $\bar{\mu} = \mu + \Gamma$ be a coset of Γ in $\iota(X^*(T))$ and $k \in [\![1, N]\!]$. Then there exists a polynomial $f_{\bar{\mu},k} : X^*(\mathbb{R}) \to \mathbb{R}$ with rational coefficients such that,*

$$\forall \lambda \in \overline{C_k} \cap \bar{\mu},\ m(\lambda) = f_{\bar{\mu},k}(\lambda).$$

Proof. Let $\bar{\mu}$ and k be as in the above statement. Applying the Borel-Weil-Bott Theorem we get that, for all $\lambda \in X^*(T)^+$, $\mathrm{H}^0(X, \mathcal{L}_\lambda) \simeq V(\lambda)^*$ and, for all $p > 0$, $\mathrm{H}^p(X, \mathcal{L}_\lambda) = \{0\}$. As a consequence, since $\dim \left(V(\lambda)^H\right) = \dim \left(\left(V(\lambda)^*\right)^H\right)$,

$$m(\lambda) = \dim \left(\mathrm{H}^0(X, \mathcal{L}_\lambda)^H\right).$$

Let us begin by considering $\lambda \in C_k \cap \bar{\mu}$. Denote by π the standard quotient map $X^{ss}_{\mathrm{st}}(C_k) \to X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H$ and, for any H-equivariant sheaf $\mathcal{S}$ on $X^{ss}_{\mathrm{st}}(C_k)$, by $\pi_*(\mathcal{S})^H$ the H-invariant direct image sheaf of $\mathcal{S}$ by π (it is then a sheaf on the GIT-quotient).
Then, by [Tel00], Remark 3.3(i),

$$\mathrm{H}^p(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H, \pi_*(\mathcal{L}_\lambda)^H) \simeq \begin{cases} \{0\} & \text{if } p > 0 \\ \mathrm{H}^0(X, \mathcal{L}_\lambda)^H & \text{if } p = 0 \end{cases}.$$

And thus, if χ is the Euler-Poincaré characteristic,

$$\chi(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H, \pi_*(\mathcal{L}_\lambda)^H) = \sum_{p \geq 0} (-1)^p \dim \left(\mathrm{H}^p(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H, \pi_*(\mathcal{L}_\lambda)^H)\right) = m(\lambda).$$

Take now $\lambda \in \overline{C_k} \cap \bar{\mu}$. We consider P (containing B) the unique parabolic subgroup of G such that $\mathcal{L}_\lambda$ descends as an ample line bundle $\mathcal{L}^P_\lambda$ on G/P via the standard projection $q : X = G/B \to G/P$. Let $\nu \in C_k \cap \iota(X^*(T))$.
Then, by [Tel00], §1.2, for any small enough rational $\varepsilon > 0$, the pull-back $q^*(\mathcal{L}^P_\lambda)$ is adapted to the stratification on X coming from $q^*(\mathcal{L}^P_\lambda) \otimes \mathcal{L}^{\otimes \varepsilon}_\nu$. So, by [Tel00], Remark 3.3(ii),

$$\forall p \in \mathbb{N}, \quad \mathrm{H}^p\left(X^{ss}_{\mathrm{st}}(q^*(\mathcal{L}^P_\lambda) \otimes \mathcal{L}^{\otimes \varepsilon}_\nu) /\!\!/ H, \pi_*(q^*(\mathcal{L}^P_\lambda)^H)\right) \simeq \mathrm{H}^p(X, q^*(\mathcal{L}^P_\lambda))^H.$$

Moreover, $q^*(\mathcal{L}^P_\lambda) = \mathcal{L}_\lambda$ and $X^{ss}_{\mathrm{st}}(q^*(\mathcal{L}^P_\lambda) \otimes \mathcal{L}^{\otimes \varepsilon}_\nu) = X^{ss}_{\mathrm{st}}(\mathcal{L}_{\lambda + \varepsilon\nu}) = X^{ss}_{\mathrm{st}}(C_k)$ (because $\lambda + \varepsilon\nu \in C_k$ if ε is small enough), and thus

$$\forall p \in \mathbb{N}, \quad \mathrm{H}^p(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H, \pi_*(\mathcal{L}_\lambda)^H) \simeq \mathrm{H}^p(X, \mathcal{L}_\lambda)^H.$$

Consequently we have once again

$$m(\lambda) = \chi(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H, \pi_*(\mathcal{L}_\lambda)^H).$$

We now introduce a $\mathbb{Z}$-basis $(\gamma_1, \dots, \gamma_r)$ of the lattice Γ. For any $\lambda = \mu + \sum_{i=1}^r a_i\gamma_i \in \overline{C_k} \cap \bar{\mu}$ (i.e. with $a_1, \dots, a_r \in \mathbb{Z}$),

$$\pi_*(\mathcal{L}_\lambda)^H \simeq \pi_*(\mathcal{L}_\mu)^H \otimes \hat{\mathcal{L}}^{(k)}_{a_1\gamma_1+\cdots+a_r\gamma_r}$$

by definition of the lattice Γ and the projection formula for π_*, and with the notation $\hat{\mathcal{L}}$ defined in the proof of Lemma 4.3.13. Finally, for any such λ we apply the Riemann-Roch Theorem for singular varieties (see e.g. [Ful84], Theorem 18.3), to the sheaf $\pi_*(\mathcal{L}_\lambda)^H$ and get

$$\begin{aligned} m(\lambda) &= \chi(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H, \pi_*(\mathcal{L}_\lambda)^H) \\ &= \sum_{n\geq 0} \int_{X^{ss}_{\mathrm{st}}(C_k)/\!\!/ H} \frac{(a_1c_1(\gamma_1) + \cdots + a_rc_1(\gamma_r))^n}{n!} \cap \tau(\pi_*(\mathcal{L}_\mu)^H), \end{aligned}$$

where, for all i, $c_1(\gamma_i)$ is the first Chern class of the line bundle $\hat{\mathcal{L}}^{(k)}_{\gamma_i}$, and $\tau(\pi_*(\mathcal{L}_\mu)^H)$ is a certain class in the Chow group $A_*(X^{ss}_{\mathrm{st}}(C_k) /\!\!/ H) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence $m(\lambda)$ is a polynomial with rational coefficients in the variables a_i. □

Improvement of the bounds of Sections 4.3.2 and 4.3.3

We now come back to the notations of Section 4.2.

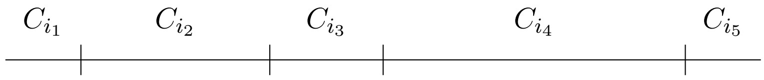
Proposition 4.3.15. *If $d_0 \in \mathbb{N}$ is such that, for all $d \in \mathbb{Q}$ such that $d > d_0$, $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$, then*

$$\dim\left(\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d_0})^G\right) = \dim\left(\mathrm{H}^0(S, \mathcal{M})^H\right).$$

Proof. Let us write $\ell = \dim\left(\mathrm{H}^0(S, \mathcal{M})^H\right)$, consider a $d_0 \in \mathbb{N}$ as in the statement above, and denote by $C_1, \dots, C_N$ the chambers (i.e. GIT-classes of maximal dimension) in $\mathbb{Q}\mathcal{L} \oplus \mathbb{Q}\mathcal{M} = \{\mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}\,;\, a, b \in \mathbb{Q}\}$ for the action of G on X. Since, for all $d > d_0$ ($d \in \mathbb{Q}$), $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}) \subset X^{ss}(\mathcal{L})$, and thanks to the results by N. Ressayre (cf. [Res00]) concerning the GIT-fan, the situation is necessarily the following:

- the $\mathcal{M} \otimes \mathcal{L}^{\otimes d}$ for $d > d_0$ are in a chamber, say for instance C_1;
- $\mathcal{L}$ belongs to $\overline{C_1}$, the closure of this chamber;
- $\mathcal{M} \otimes \mathcal{L}^{\otimes d_0}$ belongs also to $\overline{C_1}$.

We can draw a picture of this situation: in $\mathbb{Q}\mathcal{L} \oplus \mathbb{Q}\mathcal{M}$, the set of semi-ample line bundles is a closed convex cone. As a consequence, up to multiplication by a positive rational number, this set can be represented by a line or a segment. The two cases can here be treated in the same way, so we assume for instance to be in the case of a line. Then the situation of the chambers is typically:



If $\mathcal{M} \otimes \mathcal{L}^{\otimes d_0} \in C_1$, then $X^{ss}(\mathcal{M} \otimes \mathcal{L}^{\otimes d_0}) \subset X^{ss}(\mathcal{L})$ and Proposition 4.2.7 gives immediately that $\dim \mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d_0})^G = \ell$. So we assume from now on that $\mathcal{M} \otimes \mathcal{L}^{\otimes d_0} \in \overline{C_1} \setminus C_1$:

(if $\mathcal{L}$ belongs to the boundary of C_1, this does not change what follows).

Applying Lemma 4.3.13 and Theorem 4.3.14, we get that there exists a sublattice Γ of finite index of the lattice $\mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{M} = \{\mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}\,;\, a, b \in \mathbb{Z}\}$ such that, for all $\bar{\gamma} = \gamma \mathrm{mod}\Gamma \in (\mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{M})/\Gamma$, there is a polynomial $P_{\bar{\gamma}}$ with rational coefficients such that

$$\forall \mathcal{N} \in \overline{C_1} \cap \bar{\gamma},\ \dim \left(\mathrm{H}^0(X, \mathcal{N})^G\right) = P_{\bar{\gamma}}(\mathcal{N}).$$

In particular, if we denote $\bar{\gamma}_0 = (\mathcal{M} \otimes \mathcal{L}^{\otimes d_0})\mathrm{mod}\Gamma$,

$$\dim \left(\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d_0})^G\right) = P_{\bar{\gamma}_0}(\mathcal{M} \otimes \mathcal{L}^{\otimes d_0}).$$

We then consider the polynomial function in one variable

$$\tilde{P} : d \longmapsto P_{\bar{\gamma}_0}(\mathcal{M} \otimes \mathcal{L}^{\otimes d}).$$

We want to prove that $\tilde{P}$ is constant and we know that, for all integers $d > d_0$ such that $(\mathcal{M} \otimes \mathcal{L}^{\otimes d})\mathrm{mod}\Gamma = \bar{\gamma}_0$, $\tilde{P}(d) = \dim \left(\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G\right) = \ell$. It is consequently sufficient to notice that there exist infinitely many such d's: if we denote by $\mathcal{N}_1 = \mathcal{M}^{\otimes a_1} \otimes \mathcal{L}^{\otimes b_1}$ and $\mathcal{N}_2 = \mathcal{M}^{\otimes a_2} \otimes \mathcal{L}^{\otimes b_2}$ the elements of a $\mathbb{Z}$-basis of Γ, each $d \in \mathbb{N}(b_1 a_2 - a_1 b_2) + d_0$ does the trick. Finally, $\tilde{P}$ is constant and $\dim \left(\mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d_0})^G\right) = \tilde{P}(d_0) = \ell$. □

Thanks to this result we can improve slightly Propositions 4.3.6 and 4.3.10, and get Theorems 4.1.4 and 4.1.5.

Remark 4.3.16. We had previously noticed that, in the case of the triple $\big((1,1),(1,1),(2)\big)$, if $n_1 = 2$ (or $n_2 = 2$), there were two ways to compute a bound:

- by using the formula in the previous theorem which is special to this case,
- by using the formula valid for $n_1, n_2 \geq 3$, setting $\lambda_3 = 0$ (or $\mu_3 = 0$) and considering λ (or μ) as a partition of length 3.

Let us compare the two bounds we can obtain. For instance for three partitions of the form $(\lambda_1, \dots, \lambda_{n_1})$, (μ_1, μ_2), and $(\nu_1, \dots, \nu_{2n_1})$ (with $n_1 \geq 3$), we obtain by the first method:

$$D_2 = \left\lceil \frac{1}{2}(m + \lambda_3 + 2\nu_2 - \nu_{2n_1} + \nu_3 + \nu_4 + \cdots + \nu_{n_1+1}) \right\rceil.$$

And by the second method we get:

$$D_2' = \left\lceil \frac{1}{2}(m + \lambda_3 + 2\nu_2 + \nu_3 + \nu_4 + \cdots + \nu_{n_1+1}) \right\rceil.$$

So we have $D_2 \leq D_2'$ and $D_2' - D_2 = \lfloor \frac{\nu_{2n_1}}{2} \rfloor$. Similarly, for (λ_1, λ_2), (μ_1, μ_2), and $(\nu_1, \ldots, \nu_4)$,

$$D_2 = \left\lceil \frac{1}{2}(m + 2\nu_2 - \nu_4 + \nu_3) \right\rceil, \text{ whereas } D_2' = \left\lceil \frac{1}{2}(m + 2\nu_2 + \nu_3 + \nu_4) \right\rceil.$$

Once again, $D_2 \leq D_2'$. And, this time, $D_2' - D_2 = \nu_4$.
As a conclusion, it is better to use the first way of computing the bound, and that is what we do later on the examples.

4.3.5 Possibility of recovering already existing bounds by our method

In the case of Murnaghan's stability, there are some already existing bounds for the stabilisation of the sequence (see the Introduction). An interesting fact is that we can recover (and sometimes improve) some of them by our method, if we choose one-parameter subgroups different from the one that we had chosen. We focus only on two of the four bounds we cited: Brion's one (denoted by D_B), and the second one from Briand, Orellana, and Rosas, which we denote by D_{BOR2} (these authors introduced two bounds, and this one is the second). They are the ones who have a form similar to our bound; the two other ones seem far too different to be obtained this way.

Conversion to our settings

In the article [BOR11], the settings are different from ours. So, if we want to recover the bounds given here, the first thing is to convert them into our settings. For the authors, the bound given (for a triple of partitions (α, β, γ)) is the first integer n for which $\overline{\alpha}[n] = (n - |\alpha|, \alpha_1, \ldots, \alpha_{\ell(\alpha)})$, $\overline{\beta}[n]$, $\overline{\gamma}[n]$ are partitions and the sequence

$$(g_{\overline{\alpha}[n], \overline{\beta}[n], \overline{\gamma}[n]})_n$$

reaches its limit value (we know that it is a stationary sequence). Whereas for us, our bound for a triple (λ, μ, ν) of partitions (such that $|\lambda| = |\mu| = |\nu|$) is the first integer d such that the sequence

$$(g_{\lambda+(d), \mu+(d), \nu+(d)})_d$$

reaches its limit value.
The correspondence between the two points of view is then (we adopt the following useful notation: for a partition δ, $\delta_{\geq 2}$ denotes the partition obtained by removing the first -i.e. biggest- part of δ):

$$\begin{cases} \alpha = \lambda_{\geq 2} \\ \beta = \mu_{\geq 2} \\ \gamma = \nu_{\geq 2} \\ n = d + |\lambda| = d + |\mu| = d + |\nu| \end{cases}.$$

M. Brion's bound, which in [BOR11] notations is $M_B(\alpha,\beta;\gamma) = |\alpha| + |\beta| + \gamma_1$, then becomes

$$D_B(\lambda,\mu,\nu) = |\mu| - \lambda_1 - \mu_1 + \nu_2.$$

Similarly the bound D_{BOR2}, which in their notations is

$$N_2(\alpha,\beta,\gamma) = \left\lfloor \frac{|\alpha| + |\beta| + |\gamma| + \alpha_1 + \beta_1 + \gamma_1}{2} \right\rfloor,$$

becomes

$$D_{BOR2}(\lambda,\mu,\nu) = \left\lfloor \frac{-\lambda_1 + |\mu_{\geq 2}| - \nu_1 + \lambda_2 + \mu_2 + \nu_2}{2} \right\rfloor.$$

One parameter subgroups corresponding to D_B and D_{BOR2}

Case of D_B: We define the following one parameter subgroup of G:

$$\tau_B = \big(1,0,\dots,0 \;\big|\; -1,0,-1,\dots,-1\big).$$

Thus τ_B satisfies $\mu^{\mathcal{L}}(\overline{x},\tau_B) = 1$ and, for all $x \in \pi^{-1}(\overline{x})$,

$$\begin{aligned}\mu^{\mathcal{M}\otimes\mathcal{L}^{\otimes d}}(x,\tau_B) > 0 \Longleftrightarrow d &> \max_{x\in\pi^{-1}(\overline{x})}(-\mu^{\mathcal{M}}(x,\tau_B)) \\ &= -\lambda_1 + \mu_2 + \mu_3 + \dots + \mu_{n_2} + \nu_2 - \nu_{n_1+n_2} - \dots - \nu_{n_1 n_2}.\end{aligned}$$

Until now, we did not make any particular assumption on the flag varieties we considered. We had always taken complete ones, but we could also consider partial ones. Here, let us consider the partial flag variety $\mathcal{F}\ell(V_1 \otimes V_2; 1, 2, \dots, n_1+n_2-1)$ for the third factor of X. This corresponds to forgetting the terms $-\nu_{n_1+n_2}\cdots - \nu_{n_1 n_2}$ in the right-hand side of the inequality above. This way, this right-hand side is just $D_B(\lambda,\mu,\nu)$. Hence the bound D_B can be recovered by our method, with the one-parameter subgroup τ_B.

Remark 4.3.17. We can thus have an improvement of D_B in the case of a "long" partition ν: if we keep on with complete flag varieties, we keep the terms $-\nu_{n_1+n_2}\cdots - \nu_{n_1 n_2}$ at the end of the bound, and so it gives a lower value (and then better one) for partitions ν of length at least $n_1 + n_2$.

Case of B-O-R 2: We define the following one parameter subgroup of G:

$$\tau_{BOR2} = \big(1,-1,0,\dots,0 \;\big|\; -2,0,-1,\dots,-1\big).$$

This τ_{BOR2} satisfies $\mu^{\mathcal{L}}(\overline{x},\tau_{BOR2}) = 2$ and, for all $x \in \pi^{-1}(\overline{x})$,

$$\begin{aligned}\mu^{\mathcal{M}\otimes\mathcal{L}^{\otimes d}}(x,\tau_{BOR2}) > 0 \Longleftrightarrow d &> \frac{1}{2}\max_{x\in\pi^{-1}(\overline{x})}(-\mu^{\mathcal{M}}(x,\tau_{BOR2})) \\ &= \frac{1}{2}(-\lambda_1 + \lambda_2 + 2\mu_2 + |\mu_{\geq 3}| - \nu_1 + \nu_2 - \nu_{n_1+n_2-1} \\ &\quad - \dots - \nu_{n_1 n_2 - n_1 - n_2 + 2} - 2(\nu_{n_1 n_2 - n_1 - n_2 + 3} + \dots \\ &\quad + \nu_{n_1 n_2 - 1}) - 3\nu_{n_1 n_2}).\end{aligned}$$

Once again, considering the partial flag variety $\mathcal{F}\ell(V_1 \otimes V_2; 1, 2, \ldots, n_1 + n_2 - 2)$ (slightly different from the previous case), we can "forget" the terms concerning the last parts of partition ν (i.e. $-\nu_{n_1+n_2-1} - \cdots - \nu_{n_1 n_2 - n_1 - n_2 + 2} - 2(\nu_{n_1 n_2 - n_1 - n_2 + 3} + \cdots + \nu_{n_1 n_2 - 1}) - 3\nu_{n_1 n_2}$) and thus recognise $D_{BOR2}(\lambda, \mu, \nu)$ in the right-hand side of the previous inequality. Hence the bound D_{BOR2} can be recovered by our method, with the one-parameter subgroup τ_{BOR2}.

Remark 4.3.18. As for D_B, we can also have an improvement of D_{BOR2} by keeping the complete flag variety $\mathcal{F}\ell(V_1 \otimes V_2)$: if $\ell(\nu) \geq n_1 + n_2 - 1$, our method gives a lower bound.

4.3.6 Tests of our bounds and comparison with existing results

Tests and comparison for $\big((1), (1), (1)\big)$

We are now going to test the bound D_1 from Theorem 4.1.4 on a dozen examples. We also compare it to the four other bounds exposed in [BOR11] (Vallejo's bound is denoted by D_V, and the first one from Briand, Orellana, and Rosas by D_{BOR1}).

The following array presents the results of these bounds on chosen examples. We also added a column giving the minimal integer coming from all the bounds obtainable by our method: ours, D_B (a little improved, by Remark 4.3.17), and D_{BOR2} (likewise, cf. Remark 4.3.18). We denote this by D_m. Finally, we calculated with Sage[4] the first integer – denoted D_{real} – from which the sequence $(g_{\lambda+d(1),\mu+d(1),\nu+d(1)})_{d\in\mathbb{N}}$ actually stabilises.

triple λ, μ, ν	D_{real}	D_1	D_m	D_B	D_{BOR2}	D_V	D_{BOR1}
$(8,5,2), (6,5,2,2), (4,4,3,3,1)$	5	6	5	5	6	5	5
$(4,3,3), (3,2^3,1), (2^3,1^4)$	3	4	4	5	4	5	4
$(5,5,4,4), (6^3), (3,3,2^4,1^4)$	5	5	5	10	9	11	6
$(6,5,5), (8,8), (4,4,3,3,2)$	4	4	4	6	7	7	4
$(5^4), (4^5), (2^4,1^{12})$	4	5	4	13	10	14	6
$(6^3), (3^6), (2^6,1^6)$	6	7	6	11	9	11	7
$(5,5,4,4), (6^3), (3,2^6,1^3)$	4	4	4	9	8	11	5
$(7,6), (6,5,2), (7,3,2,1)$	3	3	3	3	3	4	3
$(8,4,3,3,1), (7,3^4), (14,3,2)$	0	0	0	0	0	0	0
$(8,5,3,1), (2,1^{15}), (4,3,3,2,2,1^3)$	1	3	1	6	6	7	2
$(6,6,4), (8,8), (5,5,4,1,1)$	6	7	6	7	8	7	7
$(8,6,6,2,1), (14,5,4), (5^4,3)$	5	6	6	6	6	8	5

We can notice (see e.g. the third row in the array) that there exist cases in which our bound is optimal whereas the other known bounds compared here are not. Ours is of course not always better: see e.g. the last row.

[4] http://www.sagemath.org/

Tests of the bound for $\big((1,1),(1,1),(2)\big)$

Here we compute the bound D_2 from Theorem 4.1.5 for a dozen examples and compare it, in the following array, to the first integer D_{real} from which the sequence actually stabilises. This last integer was once again computed with Sage.

λ	μ	ν	D_2	D_{real}
$(5,5,4,4)$	(6^3)	$(3,3,2^4,1^4)$	5	4
(5^4)	(4^5)	$(2^4,1^{12})$	5	4
$(6,5,5)$	$(6,5,5)$	$(3,3,2^4,1,1)$	4	4
$(8,5,2)$	$(6,5,2,2)$	$(4,4,3,2,2)$	4	4
$(4,3,3)$	$(4,3,3)$	$(2^3,1^4)$	3	3
$(5,4,4)$	$(5,4,4)$	$(3,2^3,1^4)$	3	3
$(6,5,5)$	$(8,8)$	$(4,4,3,3,2)$	3	2
$(6,6,6)$	$(9,9)$	$(6,4,3,3,2)$	3	1
$(10,8,6)$	$(12,12)$	$(6,5,4,4,3,2)$	1	1
$(8,2)$	$(6,4)$	$(5,4,1)$	1	1
$(6,6)$	$(8,4)$	$(6,4,2)$	0	0
$(20,5)$	$(13,12)$	$(11,10,3,1)$	2	1

Chapter 5

Application to other branching coefficients

In literature one can find results of stability similar to those observed for the Kronecker coefficients in various contexts (see [Wil14, SS16, Col17, Yin17]). In this chapter we use our previous techniques to re-obtain and improve quite a few of these results. Actually, as soon as – similarly to Chapter 4 – we are looking at sequences ($d \in \mathbb{N}$ being the variable) of coefficients given by $\dim \mathrm{H}^0(X, \mathcal{M} \otimes \mathcal{L}^{\otimes d})^G$, with G a connected complex reductive group acting on a projective variety X (both independent of d) and $\mathcal{L}$ and $\mathcal{M}$ two G-linearised line bundles on X, our previous methods can be applied.

This allows us to look at three other examples of branching coefficients, and we begin in Section 5.1 with a look at plethysm coefficients. They are the branching coefficients arising when one looks at the composition of Schur functors, and are considered to form an extremely difficult problem. A result of stabilisation about them was proven in [SS16], and we manage to reprove it. As an application it allows to re-obtain some examples of sequences of plethysm coefficients that were proven – by L. Colmenarejo in [Col17] – to be eventually constant.

The second example that we consider, in Section 5.2, concerns the multiplicities in the tensor product of two irreducible representations of the hyperoctahedral group. This (finite) group is the Weyl group of type B_n, which we denote by W_n, and can be expressed as a semidirect product: $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$. Then we explain that the irreducible W_n-modules are indexed by double partitions of n, i.e. ordered pairs of partitions whose sizes sum up to n. This allows to define the branching coefficients corresponding to the tensor product of two such W_n-modules. A key point is then that there exists a kind of Schur-Weyl duality in that case, which allows to interpret these coefficients as branching coefficients for connected reductive groups (see Section 5.2.2). Therefore we obtain in Section 5.2.3 a stability result similar to the one concerning Kronecker coefficients and apply it for instance to obtain and study an analogue of Murnaghan's stability.

Finally we study a third example in Section 5.3: M.Aguiar, W. Ferrer Santos, and W. Moreira introduced in [AFSM15] a product called the Heisenberg product, that they defined on various objects (their goal was to unify many existing related products and coproducts). We are interested in their definition of it in the context of representations of symmetric groups, particularly because it defines coefficients – called "Aguiar coefficients" in [Yin17] – which extend in a way the Kronecker coefficients. Moreover L. Ying proved a kind of Murnaghan's stability for these coefficients. In Section 5.3.2 we manage to express the Aguiar coefficients as branching coefficients for connected reductive groups, which allows us to prove the usual general stability result (i.e. the equivalent of "stable $\Leftrightarrow$ weakly stable") and to get some new examples of stability such as the one proven by Ying. We also discuss bounds of stabilisation in Section 5.3.3, because Ying in addition computed in [Yin17] such a bound in the case of this analogue of Murnaghan's stability. We compute therefore a bound of stabilisation for this example too, as well as for two other examples.

5.1 Application to plethysm coefficients

5.1.1 Definition and some known stability properties

The plethysm coefficients were introduced by J. Littlewood in 1950. To define them we still denote by $\mathbb{S}$ the Schur functor. For any partition λ, we also denote by n_λ the dimension of the representation $\mathbb{S}^\lambda(V)$. By Weyl's Dimension Formula, if $\ell(\lambda) \leq \dim(V)$,

$$n_\lambda = \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

(see e.g. [GW09]). The difficult problem of the composition of Schur functors gives rise to the following definition:

Definition 5.1.1. Let λ and μ be partitions such that $\ell(\lambda) \leq n_\mu$ and V a complex vector space such that $n = \dim V \geq \ell(\mu)$. Then $\mathbb{S}^\lambda(\mathbb{S}^\mu(V))$ is a representation of $\mathrm{GL}(V)$ and thus splits as a direct sum of irreducible ones:

$$\mathbb{S}^\lambda(\mathbb{S}^\mu(V)) = \bigoplus_{\nu \text{ s.t. } \ell(\nu) \leq n} a^\nu_{\lambda,\mu} \mathbb{S}^\nu(V).$$

The coefficients $a^\nu_{\lambda,\mu}$ are called the plethysm coefficients.

Remark 5.1.2. There is a necessary condition (known since the work of Littlewood) on the sizes of the partitions for these coefficients to be non zero: if $|\lambda|.|\mu| \neq |\nu|$, then $a^\nu_{\lambda,\mu} = 0$.

There exist for those coefficients some stability properties similar to the ones we studied concerning Kronecker coefficients. The following four are for example given in [Col17]:

Proposition 5.1.3. *For any partitions λ, μ, and ν, such that $|\lambda|.|\mu| = |\nu|$, the following four sequences of plethysm coefficients are constant for n sufficiently large:*

1. $(a^{\nu+(|\mu|n)}_{\lambda+(n),\mu})_n$,
2. $(a^{\nu+n\mu}_{\lambda+(n),\mu})_n$,
3. $(a^{\nu+(|\lambda|n)}_{\lambda,\mu+(n)})_n$,
4. $(a^{\nu+n|\lambda|\pi)}_{\lambda,\mu+n\pi})_n$ *for any partition π.*

Furthermore, the first one has limit zero when $\ell(\mu) > 1$, and the second and fourth are non-decreasing.

5.1.2 Relation with invariant sections of line bundles

Starting from Definition 5.1.1 we get, thanks to Schur's Lemma:

$$a^{\nu}_{\lambda,\mu} = \dim(\mathbb{S}^{\lambda}(\mathbb{S}^{\mu}(V)) \otimes (\mathbb{S}^{\nu}V)^*)^G$$

(denoting $\mathrm{GL}(V)$ by G). Then, Borel-Weil's Theorem gives

$$(\mathbb{S}^{\nu}V)^* \simeq \mathrm{H}^0(\mathcal{F}\ell(V), \mathcal{L}_{\nu})$$

and

$$\mathbb{S}^{\lambda}(\mathbb{S}^{\mu}(V)) \simeq \mathrm{H}^0(\mathcal{F}\ell(\mathbb{S}_{\mu}(V)), \mathcal{L}^*_{\lambda}).$$

Let us keep in mind that, as a vector space, $\mathbb{S}^{\mu}(V)$ is simply $\mathbb{C}^{n_{\mu}}$. So we obtain the following proposition:

Proposition 5.1.4. *If V is a complex vector space of dimension n and λ, μ, ν are three partitions such that $\ell(\lambda) \leq n_{\mu}$, $\ell(\mu) \leq n$, and $\ell(\nu) \leq n$, then*

$$a^{\nu}_{\lambda,\mu} = \dim\left(\mathrm{H}^0(X_{\mu}, \mathcal{L}_{\lambda,\nu})^G\right),$$

where $G = \mathrm{GL}(V)$, $X_{\mu} = \mathcal{F}\ell(V) \times \mathcal{F}\ell(\mathbb{C}^{n_{\mu}})$, and $\mathcal{L}_{\lambda,\nu} = \mathcal{L}_{\nu} \otimes \mathcal{L}^*_{\lambda}$.

For instance, it gives interesting things for two of the sequences cited earlier:

$$a^{\nu+d\mu}_{\lambda+(d),\mu} = \dim\left(\mathrm{H}^0(X_{\mu}, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}^{\otimes d}_{(1),\mu})^G\right)$$

and

$$a^{\nu+(d|\mu|)}_{\lambda+(d),\mu} = \dim\left(\mathrm{H}^0(X_{\mu}, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}^{\otimes d}_{(1),(|\mu|)})^G\right).$$

As, in these cases, the projective variety X_{μ} does not depend on d, we can apply our techniques. For comparison, for the two other sequences cited, it would give a variety depending on d and so it would be a lot different.

More generally, we are going to consider sequences of general term

$$a^{\nu+d\gamma}_{\lambda+d\alpha,\mu} = \dim\left(\mathrm{H}^0(X_{\mu}, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}^{\otimes d}_{\alpha,\gamma})^G\right),$$

where α and γ are partitions such that $|\alpha|.|\mu| = |\gamma|$.

5.1.3 Application of the previous techniques

Using exactly the same method as for Kronecker coefficients, we get the following result (Sam and Snowden obtained the same in [SS16] by completely different methods, whereas Paradan reproved it in [Par17]):

Theorem 5.1.5. *Let λ, μ, ν and α, γ be partitions such that $|\lambda|.|\mu| = |\nu|$ and, for all $d \in \mathbb{N}^*$, $a_{d\alpha,\mu}^{d\gamma} = 1$. Then the sequence $\left(a_{\lambda+d\alpha,\mu}^{\nu+d\gamma}\right)_{d\in\mathbb{N}}$ is non-decreasing and stabilises for d large enough.*

Proof. The fact that this sequence is non-decreasing is, as in the case of Kronecker coefficients, quite easy: let $\sigma_0 \in \mathrm{H}^0(X_\mu, \mathcal{L}_{\alpha,\gamma})^G \setminus \{0\}$ (such a section exists because $a_{\alpha,\mu}^{\gamma} = 1$). Then, for all $d \in \mathbb{N}$, we have the following injection:

$$\begin{array}{rccc} \iota_d : & \mathrm{H}^0(X_\mu, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G & \longrightarrow & \mathrm{H}^0(X_\mu, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes(d+1)})^G \\ & \sigma & \longmapsto & \sigma \otimes \sigma_0 \end{array},$$

and thus $a_{\lambda+d\alpha,\mu}^{\nu+d\gamma} \leq a_{\lambda+(d+1)\alpha,\mu}^{\nu+(d+1)\gamma}$.

For the fact that it stabilises, since it is exactly the same method as for Kronecker coefficients, we are not going to write every detail. But here are the principal steps of the proof. First of all,

$$\begin{aligned} \mathrm{H}^0(X_\mu, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G &\simeq \mathrm{H}^0(X_\mu^{ss}(\mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d}), \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G \\ &\simeq \mathrm{H}^0(X_\mu^{ss}(\mathcal{L}_{\alpha,\gamma}), \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G \qquad \text{for } d \gg 0 \end{aligned}$$

(because $X_\mu^{ss}(\mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d}) \subset X_\mu^{ss}(\mathcal{L}_{\alpha,\gamma})$ for $d \gg 0$). Then, since $\mathrm{H}^0(X_\mu, \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G \simeq \mathbb{C}$ for all $d \in \mathbb{N}^*$ and using Luna's Slice Étale Theorem,

$$\begin{aligned} \mathrm{H}^0(X_\mu, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G &\simeq \mathrm{H}^0(G \times_H S, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G \\ &\simeq \mathrm{H}^0(S, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^H \end{aligned}$$

(notations are the same as in the Kronecker coefficients' case). Finally, we have also here that the line bundle $\mathcal{L}_{\alpha,\gamma}$ is trivial on S. Thus

$$\mathrm{H}^0(X_\mu, \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G \simeq \mathrm{H}^0(S, \mathcal{L}_{\lambda,\nu})^H \qquad \text{for } d \gg 0.$$

□

This theorem applies to one of the examples given above: the sequence $(a_{\lambda+(d),\mu}^{\nu+d\mu})_{d\in\mathbb{N}}$. To see that, one just has to check that, for all $d \in \mathbb{N}^*$, $a_{(d),\mu}^{d\mu} = 1$. Let us set $d \in \mathbb{N}^*$. The coefficient $a_{(d),\mu}^{d\mu}$ is by definition the multiplicity of the irreducible representation $\mathbb{S}^{d\mu}(V)$ in the decomposition of $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$ (Sym denotes the symmetric power).

- First, if $v \in \mathbb{S}^\mu(V)$ is of weight μ (denoted $v \in \mathbb{S}^\mu(V)_\mu$), then $v^d \in \mathrm{Sym}^d(\mathbb{S}^\mu(V))$ is of weight $d\mu$. So $\dim\left(\mathrm{Sym}^d(\mathbb{S}^\mu(V))_{d\mu}\right) \geq 1$.

- Moreover, $\dim \mathbb{S}^\mu(V)_\mu = 1$ and the set of weights in $\mathbb{S}^\mu(V)$ is $\mathrm{Wt}(\mathbb{S}^\mu(V)) = \{\mu\} \sqcup \{\text{weights} < \mu\}$. So, since a well-known (and easy to understand) fact is that the weights of $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$ are among $\{\chi_1 + \cdots + \chi_d \ ; \ \chi_1, \ldots, \chi_d \in \mathrm{Wt}(\mathbb{S}^\mu(V))\}$, $\dim\left(\mathrm{Sym}^d(\mathbb{S}^\mu(V))_{d\mu}\right) = 1$.

- Finally, $\mathrm{Wt}(\mathrm{Sym}^d(\mathbb{S}^\mu(V))) \subset \{\chi_1 + \cdots + \chi_d \ ; \ \chi_1, \ldots, \chi_d \in \mathrm{Wt}(\mathbb{S}^\mu(V))\}$ also gives us that $\mathrm{Wt}(\mathrm{Sym}^d(\mathbb{S}^\mu(V))) = \{d\mu\} \sqcup \{\text{weigths} < d\mu\}$.

Thus we have $a_{(d),\mu}^{d\mu} = 1$.

5.1.4 Another example with different proof

Now what about the other sequence cited as example: $(a_{\lambda+(d),\mu}^{\nu+(d|\mu|)})_{d\in\mathbb{N}}$? When $\ell(\mu) = 1$, it is the same as before. So assume $\ell(\mu) > 1$.

Let us set $d \in \mathbb{N}^*$ and compute $a_{(d),\mu}^{(d|\mu|)}$. This coefficient is the multiplicity of $\mathrm{Sym}^{d|\mu|}(V)$ inside $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$. If $\mathrm{Sym}^{d|\mu|}(V)$ appears in $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$, then there exist vectors of weight $(d|\mu|)$ in $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$. But we already explained what weights of $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$ look like. So, if $\mathrm{Sym}^{d|\mu|}(V)$ appears in $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$, then $(d|\mu|) = \chi_1 + \cdots + \chi_d$ with $\chi_1, \ldots, \chi_d \in \mathrm{Wt}(\mathbb{S}^\mu(V))$. Then, for all $i \in [\![1, d]\!]$, $\chi_i = (|\mu|)$. But $(|\mu|)$ is not a weight of $\mathbb{S}^\mu(V)$ (because $\ell(\mu) > 1$ and the weights of $\mathbb{S}^\mu(V)$ are in the convex hull of $W.\mu$, where W denotes the Weyl group of G). Thus $\mathrm{Sym}^{d|\mu|}(V)$ does not appear in $\mathrm{Sym}^d(\mathbb{S}^\mu(V))$, which means that $a_{(d),\mu}^{(d|\mu|)} = 0$.

As a consequence, $X_\mu^{ss}(\mathcal{L}_{(1),(|\mu|)}) = \emptyset$ and there exists $D \in \mathbb{N}$ such that, for all $d \geq D$, $X_\mu^{ss}(\mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{(1),(|\mu|)}^{\otimes d}) = \emptyset$. Thus, for all $d \geq D$,

$$a_{\lambda+(d),\mu}^{\nu+(d|\mu|)} = \dim\left(\mathrm{H}^0(X_\mu^{ss}(\mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d}), \mathcal{L}_{\lambda,\nu} \otimes \mathcal{L}_{\alpha,\gamma}^{\otimes d})^G\right) = 0.$$

We recover the result from Proposition 5.1.3.

5.2 Application for the hyperoctahedral group

5.2.1 Notations and coefficients studied

For $n \geq 2$, we consider the group $W_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$, which is the Weyl group in type B_n (if we see the root system of type B_n in $\mathbb{R}^n$ with basis $(\varepsilon_1, \ldots, \varepsilon_n)$, $\mathfrak{S}_n$ acts by permuting the ε_i, whereas $1_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in (\mathbb{Z}/2\mathbb{Z})^n$ acts just by $\varepsilon_i \mapsto -\varepsilon_i$). It is called the hyperoctahedral group, and it is known (cf. [Wil14] or [GP00]) that its rational irreducible complex representations can be built up from the ones of $\mathfrak{S}_n$ and are classified by double partitions of n. These are ordered pairs of partitions (α^+, α^-) such that $|\alpha^+| + |\alpha^-| = n$.

When (α^+, α^-) is a double partition, we choose to denote by $M_{\alpha^\pm}$ the associated irreducible representation of $W_{|\alpha^\pm|}$ (where $|\alpha^\pm|$ stands for $|\alpha^+|+|\alpha^-|$). Given two double partitions (α^+, α^-) and (β^+, β^-) of the same integer, consider the non-negative integers $n^{\gamma^\pm}_{\alpha^\pm,\beta^\pm}$ such that

$$M_{\alpha^\pm} \otimes M_{\beta^\pm} = \bigoplus_{(\gamma^+,\gamma^-)} M_{\gamma^\pm}^{\oplus n^{\gamma^\pm}_{\alpha^\pm,\beta^\pm}},$$

where the direct sum runs over all double partitions of $|\alpha^\pm|$.

5.2.2 Schur-Weyl duality in that case

Let V^+ and V^- be two complex vector spaces and set $V = V^+ \oplus V^-$. Then the groups $\mathrm{GL}(V^\pm) = \mathrm{GL}(V^+) \times \mathrm{GL}(V^-)$ and W_n act on $V^{\otimes n}$. (For W_n, $\mathfrak{S}_n$ acts simply by permuting the factors in $V^{\otimes n}$, and $1_i \in (\mathbb{Z}/2\mathbb{Z})^n$ acts by multiplying by -1 the i-th factor in $V^{\otimes n}$.) Furthermore, these two actions commute and thus $\mathrm{GL}(V^\pm) \times W_n$ acts on $V^{\otimes n}$.

Proposition 5.2.1. *As a representation of* $\mathrm{GL}(V^\pm) \times W_n$*,* $V^{\otimes n}$ *decomposes as a direct sum of irreducible representations in the following way:*

$$V^{\otimes n} = \bigoplus_{(\alpha^+,\alpha^-)} V_{\alpha^\pm}(\mathrm{GL}(V^\pm)) \otimes M_{\alpha^\pm},$$

where the direct sum runs over all double partitions of n *such that* $\ell(\alpha^+) \leq \dim(V^+)$ *and* $\ell(\alpha^-) \leq \dim(V^-)$*. Moreover,* $V_{\alpha^\pm}(\mathrm{GL}(V^\pm))$ *denotes the irreducible representation* $\mathbb{S}^{\alpha^+}(V^+) \otimes \mathbb{S}^{\alpha^-}(V^-)$ *of* $\mathrm{GL}(V^\pm)$*.*

Proof. This result comes from [SS99]. □

We now consider complex vector spaces $V_1 = V_1^+ \oplus V_1^-$ and $V_2 = V_2^+ \oplus V_2^-$ and we set $\mathrm{GL}(V_1^\pm) = \mathrm{GL}(V_1^+) \times \mathrm{GL}(V_1^-)$, $\mathrm{GL}(V_2^\pm) = \mathrm{GL}(V_2^+) \times \mathrm{GL}(V_2^-)$. Then, on the one hand,

$$\begin{aligned} V_1^{\otimes n} \otimes V_2^{\otimes n} &= \left(\bigoplus_{(\alpha^+,\alpha^-)} V_{\alpha^\pm}(\mathrm{GL}(V_1^\pm)) \otimes M_{\alpha^\pm} \right) \otimes \left(\bigoplus_{(\beta^+,\beta^-)} V_{\beta^\pm}(\mathrm{GL}(V_2^\pm)) \otimes M_{\beta^\pm} \right) \\ &= \bigoplus_{\alpha^\pm,\beta^\pm,\gamma^\pm} \left(\mathbb{S}^{\alpha^+}(V_1^+) \otimes \mathbb{S}^{\alpha^-}(V_1^-) \otimes \mathbb{S}^{\beta^+}(V_2^+) \otimes \mathbb{S}^{\beta^-}(V_2^-) \otimes M_{\gamma^\pm} \right)^{\oplus n^{\gamma^\pm}_{\alpha^\pm,\beta^\pm}}, \end{aligned}$$

with the direct sum over all triples $(\alpha^+,\alpha^-),(\beta^+,\beta^-),(\gamma^+,\gamma^-)$ of double partitions of n.
On the other hand,

$$V_1^{\otimes n}\otimes V_2^{\otimes n}=(V_1\otimes V_2)^{\otimes n}=\bigoplus_{(\gamma^+,\gamma^-)} V_{\gamma^\pm}(\mathrm{GL}(V^\pm))\otimes M_{\gamma^\pm},$$

where $\mathrm{GL}(V^\pm)=\mathrm{GL}(V^+)\times\mathrm{GL}(V^-)$ and $V^+=(V_1^+\otimes V_2^+)\oplus(V_1^-\otimes V_2^-)$, $V^-=(V_1^+\otimes V_2^-)\oplus(V_1^-\otimes V_2^+)$ (then $V_1\otimes V_2=V^+\oplus V^-$).
Moreover, one has a branching

$$\underbrace{\mathrm{GL}(V_1^+)\times\mathrm{GL}(V_1^-)\times\mathrm{GL}(V_2^+)\times\mathrm{GL}(V_2^-)}_{\text{denoted by } G}\longrightarrow\underbrace{\mathrm{GL}(V^+)\times\mathrm{GL}(V^-)}_{\text{denoted by } \hat{G}}$$

and then a decomposition

$$V_{\gamma^\pm}(\mathrm{GL}(V^\pm))=\bigoplus_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}\left(\mathbb{S}^{\alpha^+}(V_1^+)\otimes\mathbb{S}^{\alpha^-}(V_1^-)\otimes\mathbb{S}^{\beta^+}(V_2^+)\otimes\mathbb{S}^{\beta^-}(V_2^-)\right)^{\oplus\cdots}.$$

Thus, by identification:

Proposition 5.2.2. *The coefficients $n_{\alpha^\pm,\beta^\pm}^{\gamma^\pm}$ are also the coefficients in the branching situation $G\to\hat{G}$, i.e. for all double partitions (γ^+,γ^-),*

$$\mathbb{S}^{\gamma^+}(V^+)\otimes\mathbb{S}^{\gamma^-}(V^-)=\bigoplus_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}\left(\mathbb{S}^{\alpha^+}(V_1^+)\otimes\mathbb{S}^{\alpha^-}(V_1^-)\otimes\mathbb{S}^{\beta^+}(V_2^+)\otimes\mathbb{S}^{\beta^-}(V_2^-)\right)^{\oplus n_{\alpha^\pm,\beta^\pm}^{\gamma^\pm}}.$$

This new expression yields, by Schur's Lemma,

$$n_{\alpha^\pm,\beta^\pm}^{\gamma^\pm}=\dim\left(\mathbb{S}^{\gamma^+}(V^+)\otimes\mathbb{S}^{\gamma^-}(V^-)\otimes\mathbb{S}^{\alpha^+}(V_1^+)^*\otimes\mathbb{S}^{\alpha^-}(V_1^-)^*\otimes\mathbb{S}^{\beta^+}(V_2^+)^*\otimes\mathbb{S}^{\beta^-}(V_2^-)^*\right)^G.$$

And finally we prove the following proposition:

Proposition 5.2.3. *Let (α^+,α^-), (β^+,β^-), and (γ^+,γ^-) be three double partitions of the same integer. Then there exist a complex reductive group G acting on a projective variety X, and a G-linearised line bundle $\mathcal{L}_{\alpha^\pm,\beta^\pm,\gamma^\pm}$ on X such that*

$$n_{\alpha^\pm,\beta^\pm}^{\gamma^\pm}=\dim\left(\mathrm{H}^0(X,\mathcal{L}_{\alpha^\pm,\beta^\pm,\gamma^\pm})^G\right).$$

Proof. According to what precedes and thanks to Borel-Weil's Theorem, it is sufficient to consider complex vector spaces V_1^+, V_1^-, V_2^+, and V_2^- such that $\dim(V_1^+)\geq\ell(\alpha^+)$, $\dim(V_1^-)\geq\ell(\alpha^-)$, $\dim(V_2^+)\geq\ell(\beta^+)$, and $\dim(V_2^-)\geq\ell(\beta^-)$. Then one sets

$$X=\mathcal{F}\ell(V_1^+)\times\mathcal{F}\ell(V_1^-)\times\mathcal{F}\ell(V_2^+)\times\mathcal{F}\ell(V_2^-)\times\mathcal{F}\ell(\underbrace{V_1^+\otimes V_2^+\oplus V_1^-\otimes V_2^-}_{V^+})\times\mathcal{F}\ell(\underbrace{V_1^+\otimes V_2^-\oplus V_1^-\otimes V_2^+}_{V^-}),$$

$$G=\mathrm{GL}(V_1^+)\times\mathrm{GL}(V_1^-)\times\mathrm{GL}(V_2^+)\times\mathrm{GL}(V_2^-),$$

and

$$\mathcal{L}_{\alpha^\pm,\beta^\pm,\gamma^\pm}=\mathcal{L}_{\alpha^+}\otimes\mathcal{L}_{\alpha^-}\otimes\mathcal{L}_{\beta^+}\otimes\mathcal{L}_{\beta^-}\otimes\mathcal{L}_{\gamma^+}^*\otimes\mathcal{L}_{\gamma^-}^*.$$

□

5.2.3 Stability results and analogue of Murnaghan's stability

General result and examples

According to the previous section, we find ourselves in the same situation as for Kronecker coefficients. As a consequence, the same demonstration as in Section 4.2 can be applied here.

Theorem 5.2.4. *If $\alpha^{\pm} = (\alpha^+, \alpha^-)$, $\beta^{\pm} = (\beta^+, \beta^-)$, and $\gamma^{\pm} = (\gamma^+, \gamma^-)$ are three double partitions such that*

$$\forall d \in \mathbb{N}^*,\ n_{d\alpha^{\pm}, d\beta^{\pm}}^{d\gamma^{\pm}} = 1,$$

then the triple they form is stable in the sense that, for every double partition $\lambda^{\pm} = (\lambda^+, \lambda^-)$, $\mu^{\pm} = (\mu^+, \mu^-)$, and $\nu^{\pm} = (\nu^+, \nu^-)$, the sequence

$$\left(n_{\lambda^{\pm}+d\alpha^{\pm}, \mu^{\pm}+d\beta^{\pm}}^{\nu^{\pm}+d\gamma^{\pm}} \right)_{d \in \mathbb{N}}$$

stabilises for d large enough.

Example 1: There is in this situation an analogue of Murnaghan's stability. It has already been observed and proven in [Wil14], and we retrieve it here: according for instance to Proposition 5.2.3, we notice that

$$n_{\left((1),\emptyset\right),\left((1),\emptyset\right)}^{\left((1),\emptyset\right)} = g_{(1),(1),(1)}$$

($\emptyset$ here stands for the empty partition, of size and length zero). Then we can apply the previous theorem to conclude that, for all double partitions (λ^+, λ^-), (μ^+, μ^-), and (ν^+, ν^-) of the same integer, if we increase repetitively by one the first part of the partitions λ^+, μ^+, and ν^+, the associated sequence of coefficients c eventually stabilises.

Example 2: Let us consider the following triple of double partitions:

$$\Big(\big((2),(2)\big), \big((2),(2)\big), \big((2),(2)\big) \Big)$$

Lemma 5.2.5. *For all $d \in \mathbb{N}^*$,*

$$n_{d\left((2),(2)\right),d\left((2),(2)\right)}^{d\left((2),(2)\right)} = 1.$$

Proof. Let us set $d \in \mathbb{N}^*$. We proved that the coefficient $n_{d\left((2),(2)\right),d\left((2),(2)\right)}^{d\left((2),(2)\right)}$ is the multiplicity of

$$\mathrm{Sym}^{2d}(V_1^+) \otimes \mathrm{Sym}^{2d}(V_2^+) \otimes \mathrm{Sym}^{2d}(V_1^-) \otimes \mathrm{Sym}^{2d}(V_2^-)$$

in

$$\mathrm{Sym}^{2d}(V_1^+ \otimes V_2^+ \oplus V_1^- \otimes V_2^-) \otimes \mathrm{Sym}^{2d}(V_1^+ \otimes V_2^- \oplus V_1^- \otimes V_2^+)$$

(if V_1^+, V_1^-, V_2^+, and V_2^- are large enough vector spaces). But we have (cf. for example [FH91], Exercise 6.11)

$$\begin{aligned}\mathrm{Sym}^{2d}(V_1^+ \otimes V_2^+ \oplus V_1^- \otimes V_2^-) &= \bigoplus_{m+n=d} \mathrm{Sym}^m(V_1^+ \otimes V_2^+) \otimes \mathrm{Sym}^n(V_1^- \otimes V_2^-) \\ &= \bigoplus_{m+n=d,\ \lambda^+ \vdash m,\ \lambda^- \vdash n} \mathbb{S}^{\lambda^+}(V_1^+) \otimes \mathbb{S}^{\lambda^+}(V_2^+) \\ &\quad \otimes \mathbb{S}^{\lambda^-}(V_1^-) \otimes \mathbb{S}^{\lambda^-}(V_2^-).\end{aligned}$$

And the same kind of formula exists for $\mathrm{Sym}^{2d}(V_1^+ \otimes V_2^- \oplus V_1^- \otimes V_2^+)$. Hence,

$$\begin{aligned}&\mathrm{Sym}^{2d}(V_1^+ \otimes V_2^+ \oplus V_1^- \otimes V_2^-) \otimes \mathrm{Sym}^{2d}(V_1^+ \otimes V_2^- \oplus V_1^- \otimes V_2^+) \\ =& \bigoplus_{(\lambda^+,\lambda^-),(\mu^+,\mu^-) \text{ s.t. } |\lambda^\pm|=|\mu^\pm|=2d} \mathbb{S}^{\lambda^+}(V_1^+) \otimes \mathbb{S}^{\mu^+}(V_1^+) \otimes \mathbb{S}^{\lambda^+}(V_2^+) \otimes \mathbb{S}^{\mu^-}(V_2^+) \\ & \otimes \mathbb{S}^{\lambda^-}(V_1^-) \otimes \mathbb{S}^{\mu^-}(V_1^-) \otimes \mathbb{S}^{\lambda^-}(V_2^-) \otimes \mathbb{S}^{\mu^+}(V_2^-) \\ =& \bigoplus_{\lambda^+,\lambda^-,\mu^+,\mu^-,\nu_1,\nu_2,\nu_3,\nu_4} \left(\mathbb{S}^{\nu_1}(V_1^+) \otimes \mathbb{S}^{\nu_2}(V_2^+) \otimes \mathbb{S}^{\nu_3}(V_1^-) \otimes \mathbb{S}^{\nu_4}(V_2^-)\right)^{\oplus m_{\lambda^\pm,\mu^\pm}^{\nu_1,\nu_2,\nu_3,\nu_4}},\end{aligned}$$

where this last sum runs over partitions verifying $|\lambda^\pm| = |\mu^\pm| = 2d$, and

$$m_{\lambda^\pm,\mu^\pm}^{\nu_1,\nu_2,\nu_3,\nu_4} = c_{\lambda^+,\mu^+}^{\nu_1} c_{\lambda^+,\mu^-}^{\nu_2} c_{\lambda^-,\mu^-}^{\nu_3} c_{\lambda^-,\mu^+}^{\nu_4}$$

is a product of four Littlewood-Richardson coefficients. Henceforth, the multiplicity of $\mathrm{Sym}^{2d}(V_1^+) \otimes \mathrm{Sym}^{2d}(V_2^+) \otimes \mathrm{Sym}^{2d}(V_1^-) \otimes \mathrm{Sym}^{2d}(V_2^-)$ is

$$\sum_{\lambda^+,\lambda^-,\mu^+,\mu^-} c_{\lambda^+,\mu^+}^{(2d)} c_{\lambda^+,\mu^-}^{(2d)} c_{\lambda^-,\mu^-}^{(2d)} c_{\lambda^-,\mu^+}^{(2d)},$$

where we take the sum over partitions such that $|\lambda^+|+|\lambda^-| = |\mu^+|+|\mu^-| = |\lambda^+|+|\mu^+| = |\lambda^+| + |\mu^-| = |\lambda^-| + |\mu^-| = |\lambda^-| + |\mu^+| = 2d$, i.e. $|\lambda^+| = |\lambda^-| = |\mu^+| = |\mu^-| = d$. Then

$$n_{d\left((2),(2)\right),d\left((2),(2)\right)}^{d\left((2),(2)\right)} = \sum_{\lambda^+,\lambda^-,\mu^+,\mu^- \vdash d} c_{\lambda^+,\mu^+}^{(2d)} c_{\lambda^+,\mu^-}^{(2d)} c_{\lambda^-,\mu^-}^{(2d)} c_{\lambda^-,\mu^+}^{(2d)}.$$

Finally, the Littlewood-Richardson rule shows that $c_{\lambda,\mu}^{(2d)} = 0$ unless $\lambda = \mu = (d)$. And in that last case, the coefficient is 1. This concludes the proof of the lemma. □

Proposition 5.2.6. *For every triple* $\left((\lambda^+,\lambda^-),(\mu^+,\mu^-),(\nu^+,\nu^-)\right)$ *of double partitions, the sequence*

$$\left(n_{\lambda^\pm+d\left((2),(2)\right),\mu^\pm+d\left((2),(2)\right)}^{\nu^\pm+d\left((2),(2)\right)}\right)_{d\in\mathbb{N}}$$

stabilises for d *large enough.*

Proof. This is a direct consequence of the previous lemma and of Theorem 5.2.4. □

An example of an explicit bound

We can also here compute in some special and not too difficult cases a bound for the stabilisation of the sequence of coefficients. We do this in the case analogous to Murnaghan's stability (Example 1). As before we set $\mathcal{L} = \mathcal{L}_{((1),\emptyset),((1),\emptyset),((1),\emptyset)}$ and $\mathcal{M} = \mathcal{L}_{\lambda^\pm,\mu^\pm,\nu^\pm}$. If we consider the usual projection (cf. Section 4.3.1)

$$\begin{array}{ccc} \mathcal{L} & \dashrightarrow & \overline{\mathcal{L}} \\ \downarrow & & \downarrow \\ \pi : X & \longrightarrow & \overline{X} \end{array}$$

such that $\mathcal{L}$ is the pull-back of an ample line bundle $\overline{\mathcal{L}}$, we notice that $\overline{X}$ and $\overline{\mathcal{L}}$ are exactly the same as in Section 4.3.2. Then we know that it is sufficient to determine when $\pi^{-1}(\overline{x}) \subset X^{us}(\mathcal{M} \otimes \mathcal{L}^{\otimes d})$ (same notation as in 4.3.2 for $\overline{x}$). As a consequence, if we consider for instance the one-parameter subgroup

$$\tau_0 = \big(1, -1, 0, \dots, 0 \;\big|\; 0, \dots, 0 \;\big|\; -1, 1, 0, \dots, 0 \;\big|\; 0, \dots, 0\big)$$

of G, we have as before $\mu^{\overline{\mathcal{L}}}(\overline{x}, \tau_0) = 2$. And

$$\begin{aligned} \max_{x\in\pi^{-1}(\overline{x})}(-\mu^{\mathcal{M}}(x,\tau_0)) \;=\; & -\lambda_1^+ + \lambda_2^+ - \mu_1^+ + \mu_2^+ + 2\left(\nu_2^+ - \nu^+_{\ell(\lambda^+)\ell(\mu^+)+\ell(\lambda^-)\ell(\mu^-)}\right) \\ & + \sum_{k=1}^{\ell(\lambda^+)+\ell(\mu^+)-4} \left(\nu^+_{k+2} - \nu^+_{\ell(\lambda^+)\ell(\mu^+)+\ell(\lambda^-)\ell(\mu^-)-k}\right) \\ & + \sum_{k=1}^{\ell(\lambda^-)+\ell(\mu^-)} \left(\nu^-_{k} - \nu^-_{\ell(\lambda^+)\ell(\mu^-)+\ell(\lambda^-)\ell(\mu^+)-k+1}\right). \end{aligned}$$

Theorem 5.2.7. *Let (λ^+, λ^-), (μ^+, μ^-), and (ν^+, ν^-) be double partitions of the same integer. We set $m = \ell(\lambda^+)\ell(\mu^+) + \ell(\lambda^-)\ell(\mu^-)$, $n = \ell(\lambda^+)\ell(\mu^-) + \ell(\lambda^-)\ell(\mu^+)$, and*

$$D = \left\lceil \frac{1}{2}\left(-\lambda_1^+ + \lambda_2^+ - \mu_1^+ + \mu_2^+ + 2\left(\nu_2^+ - \nu_m^+\right) + \sum_{k=1}^{\ell(\lambda^+)+\ell(\mu^+)-4} \left(\nu^+_{k+2} - \nu^+_{m-k}\right) + \sum_{k=1}^{\ell(\lambda^-)+\ell(\mu^-)} \left(\nu^-_k - \nu^-_{n-k+1}\right) \right) \right\rceil.$$

Then, for all $d \geq D$ ($d \in \mathbb{N}$),

$$n^{(\nu^+ +(d),\nu^-)}_{(\lambda^+ +(d),\lambda^-),(\mu^+ +(d),\mu^-)} = n^{(\nu^+ +(D),\nu^-)}_{(\lambda^+ +(D),\lambda^-),(\mu^+ +(D),\mu^-)}.$$

5.3 The Heisenberg product and the Aguiar coefficients

5.3.1 Definition and first properties

Construction

The Heisenberg product was first defined by Marcelo Aguiar, Walter Ferrer Santos, and Walter Moreira in [AFSM15]. They defined it in different contexts, but we will only be interested in one of them, related to what we have done before: the representations of the symmetric group. In this context, this product extends in particular what is sometimes called the "Kronecker product" (meaning the tensor product of $\mathfrak{S}_k$-modules).

Remark 5.3.1. Let us recall that, for all nonnegative integers a and b, $\mathfrak{S}_a \times \mathfrak{S}_b$ can naturally be seen as a subgroup of $\mathfrak{S}_{a+b}$. We denote the corresponding injective group morphism by $\iota_{a,b} : \mathfrak{S}_a \times \mathfrak{S}_b \hookrightarrow \mathfrak{S}_{a+b}$.
On a different side, for any nonnegative integer a, $\mathfrak{S}_a$ can be considered as a subgroup of $\mathfrak{S}_a \times \mathfrak{S}_a$ through the diagonal embedding $\Delta_a : \mathfrak{S}_a \hookrightarrow \mathfrak{S}_a \times \mathfrak{S}_a$.

Consider from now on two symmetric groups: $\mathfrak{S}_k$ and $\mathfrak{S}_l$. Here is the definition of the Heisenberg product:

Definition 5.3.2. Let V and W be two (complex) representations of $\mathfrak{S}_k$ and $\mathfrak{S}_l$ respectively. Let $i \in [\![\max(k,l), k+l]\!]$. One has the following inclusions:

$$\begin{array}{ccc}
\mathfrak{S}_{i-l} \times \mathfrak{S}_{k+l-i} \times \mathfrak{S}_{k+l-i} \times \mathfrak{S}_{i-k} & \xhookrightarrow{\iota_{i-l,k+l-i} \times \iota_{k+l-i,i-k}} & \mathfrak{S}_k \times \mathfrak{S}_l \\
\Big\uparrow {\scriptstyle \mathrm{Id}_{\mathfrak{S}_{i-l}} \times \Delta_{k+l-i} \times \mathrm{Id}_{\mathfrak{S}_{i-k}}} & \nearrow & \\
\mathfrak{S}_{i-l} \times \mathfrak{S}_{k+l-i} \times \mathfrak{S}_{i-k} & \xhookrightarrow{\iota_{i-l,l} \circ \left(\mathrm{Id}_{\mathfrak{S}_{i-l}} \times \iota_{k+l-i,i-k}\right)} & \mathfrak{S}_i
\end{array}$$

We then set

$$(V \sharp W)_i = \mathrm{Ind}_{\mathfrak{S}_{i-l} \times \mathfrak{S}_{k+l-i} \times \mathfrak{S}_{i-k}}^{\mathfrak{S}_i} \mathrm{Res}_{\mathfrak{S}_{i-l} \times \mathfrak{S}_{k+l-i} \times \mathfrak{S}_{i-k}}^{\mathfrak{S}_k \times \mathfrak{S}_l} (V \otimes W)$$

(which is an $\mathfrak{S}_i$-module), and the Heisenberg product of V and W is

$$V \sharp W = \bigoplus_{i=\max(k,l)}^{k+l} (V \sharp W)_i.$$

A remarkable result proven in [AFSM15] is that this product is associative.

Definition 5.3.3. Let $\lambda \vdash k$ and $\mu \vdash l$. The Heisenberg product between the associated irreducible representations of the symmetric group decomposes as:

$$M_\lambda \sharp M_\mu = \bigoplus_{i=\max(k,l)}^{k+l} \bigoplus_{\nu \vdash i} M_\nu^{\oplus a_{\lambda,\mu}^{\nu}}.$$

The coefficients $a_{\lambda,\mu}^{\nu}$ are called the Aguiar coefficients.

We will adopt the convention that, if the weights of the partitions λ, μ, and ν are not compatible to define an Aguiar coefficient (i.e. $|\nu| \notin [\![\max(|\lambda|,|\mu|), |\lambda| + |\mu|]\!]$), then $a_{\lambda,\mu}^{\nu} = 0$.

Remark 5.3.4. As written earlier, the Heisenberg product extends the Kronecker one: when $k = l$, the lower term $(V \sharp W)_k$ of $V \sharp W$ is just $\mathrm{Res}_{\mathfrak{S}_k}^{\mathfrak{S}_k \times \mathfrak{S}_k}(V \otimes W)$. As a consequence, when the three partitions λ, μ, and ν have the same size, the Aguiar coefficient $a_{\lambda,\mu}^{\nu}$ coincides with the Kronecker coefficient $g_{\lambda,\mu,\nu}$.

First stability results by Li Ying

In this paragraph we recall some results from [Yin17]. If λ is a partition and n a positive integer, $\lambda[n]$ is the sequence $(n-|\lambda|, \lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$. The main result of [Yin17] is then:

Theorem 5.3.5. *Let λ and μ be two partitions, and d and h be two nonnegative integers. Then the decomposition of the $\mathfrak{S}_{n+h}$-module $(M_{\lambda[n]}\sharp M_{\mu[n-d]})_{n+h}$ stabilises when $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3h + 2d$. Moreover, the stabilisation begins exactly at this particular integer.*

The first part of the theorem can be expressed in terms of Aguiar coefficients as follows:

Proposition 5.3.6. *For all partitions λ and μ, nonnegative integers d and h, integer n such that $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3h + 2d$, and $\nu \vdash n+h$,*

$$a^{\nu}_{\lambda[n],\mu[n-d]} = a^{\nu+(1)}_{\lambda[n+1],\mu[n-d+1]}.$$

The proof of the previous proposition is strongly based on a remarkable expression of the Aguiar coefficients in terms of Littlewood-Richardson and Kronecker coefficients:

Proposition 5.3.7. *For all partitions λ, μ, and ν,*

$$a^{\nu}_{\lambda,\mu} = \sum_{\alpha,\beta,\delta,\eta,\rho,\tau} c^{\lambda}_{\alpha,\beta}\ c^{\mu}_{\eta,\rho}\ g_{\beta,\eta,\delta}\ c^{\tau}_{\alpha,\delta}\ c^{\nu}_{\tau,\rho}.$$

Li Ying deduces also from Proposition 5.3.6 a bound for the stabilisation of a sequence of Aguiar coefficients once λ, μ, and ν are fixed:

Corollary 5.3.8. *For all partitions λ, μ, and ν, and nonnegative integers d and h, the sequence of general term $a^{\nu[n+h]}_{\lambda[n],\mu[n-d]}$ stabilises when $n \geq \frac{1}{2}(|\lambda| + |\mu| + |\nu| + \lambda_1 + \mu_1 + \nu_1 - 1) + h + d$.*

5.3.2 Stability results by the previous methods

The Aguiar coefficients as branching coefficients

In order to use on the Aguiar coefficients the same methods that we used on Kronecker coefficients, we express these as branching coefficients for connected complex reductive groups.

Theorem 5.3.9. *The Aguiar coefficients are the branching coefficients for the groups $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \hookrightarrow \mathrm{GL}\big(V_1 \oplus (V_1 \otimes V_2) \oplus V_2\big)$. More precisely, if λ, μ, and ν are three partitions then, for all (complex) finite dimensional vector spaces V_1 and V_2 such that $\ell(\lambda) \leq \dim(V_1)$, $\ell(\mu) \leq \dim(V_2)$, and $\ell(\nu) \leq \dim\big(V_1 \oplus (V_1 \otimes V_2) \oplus V_2\big)$,*

$$\mathbb{S}^{\nu}\big(V_1 \oplus (V_1 \otimes V_2) \oplus V_2\big) = \bigoplus_{\lambda,\mu} \Big(\mathbb{S}^{\lambda} V_1 \otimes \mathbb{S}^{\mu} V_2\Big)^{\oplus a^{\nu}_{\lambda,\mu}}$$

(as representations of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$).

Proof. Let λ, μ, ν, V_1, and V_2 be as above. Then, using well-known properties of the Littlewood-Richardson and the Kronecker coefficients (recalled for instance in Part 1 of [SS12]: (3.11.1), (3.12.1), and (3.9.1)):

$$\begin{aligned}
\mathbb{S}^\nu(V_1 \oplus V_1 \otimes V_2 \oplus V_2) &= \bigoplus_{\tau,\rho} c^\nu_{\tau,\rho}\ \mathbb{S}^\tau(V_1 \oplus V_1 \otimes V_2) \otimes \mathbb{S}^\rho V_2 \\
&= \bigoplus_{\tau,\rho,\alpha,\delta} c^\nu_{\tau,\rho}\ c^\tau_{\alpha,\delta}\ \mathbb{S}^\alpha V_1 \otimes \mathbb{S}^\delta(V_1 \otimes V_2) \otimes \mathbb{S}^\rho V_2 \\
&= \bigoplus_{\tau,\rho,\alpha,\delta,\beta,\eta} c^\nu_{\tau,\rho}\ c^\tau_{\alpha,\delta}\ g_{\beta,\eta,\delta}\ \mathbb{S}^\alpha V_1 \otimes \mathbb{S}^\beta V_1 \otimes \mathbb{S}^\eta V_2 \otimes \mathbb{S}^\rho V_2 \\
&= \bigoplus_{\tau,\rho,\alpha,\delta,\beta,\eta,\lambda} c^\nu_{\tau,\rho}\ c^\tau_{\alpha,\delta}\ g_{\beta,\eta,\delta}\ c^\lambda_{\alpha,\beta}\ \mathbb{S}^\lambda V_1 \otimes \mathbb{S}^\eta V_2 \otimes \mathbb{S}^\rho V_2 \\
&= \bigoplus_{\tau,\rho,\alpha,\delta,\beta,\eta,\lambda,\mu} c^\nu_{\tau,\rho}\ c^\tau_{\alpha,\delta}\ g_{\beta,\eta,\delta}\ c^\lambda_{\alpha,\beta}\ c^\mu_{\eta,\rho}\ \mathbb{S}^\lambda V_1 \otimes \mathbb{S}^\mu V_2 \\
&= \bigoplus_{\lambda,\mu} a^\nu_{\lambda,\mu}\ \mathbb{S}^\lambda V_1 \otimes \mathbb{S}^\mu V_2,
\end{aligned}$$

using Proposition 5.3.7. □

Corollary 5.3.10. *Let λ, μ, ν be three partitions. Taking V_1 and V_2 as in the previous theorem, we set:*

$$G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2),$$

$$X = \mathcal{F}\ell(V_1) \times \mathcal{F}\ell(V_2) \times \mathcal{F}\ell\big(V_1 \oplus (V_1 \otimes V_2) \oplus V_2\big),$$

and

$$\mathcal{L} = \mathcal{L}_\lambda \otimes \mathcal{L}_\mu \otimes \mathcal{L}^*_\nu$$

(G-linearised line bundle on X). Then

$$a^\nu_{\lambda,\mu} = \dim \mathrm{H}^0(X, \mathcal{L})^G.$$

Proof. It works exactly as in the case of Kronecker coefficients (i.e. one uses Schur Lemma and Borel-Weil Theorem). □

Consequences and new examples of stable triples

Since the Aguiar coefficients can be expressed as $\dim \mathrm{H}^0(X, \mathcal{L})^G$, for well-chosen G, X, and $\mathcal{L}$ (cf previous paragraph), the same techniques as for Kronecker coefficients apply. This allows to obtain the following:

Theorem 5.3.11. *Let α, β, and γ be three partitions such that, for all $d \in \mathbb{N}^*$, $a^{d\gamma}_{d\alpha,d\beta} = 1$. Then, for all triple $(\lambda, \mu; \nu)$ of partitions, the sequence $(a^{\nu+d\gamma}_{\lambda+d\alpha,\mu+d\beta})_{d\in\mathbb{N}}$ stabilises.*

Definition 5.3.12. A triple $(\alpha, \beta; \gamma)$ of partitions such that $a^\gamma_{\alpha,\beta} \neq 0$ and that, for all triple $(\lambda, \mu; \nu)$ of partitions, $(a^{\nu+d\gamma}_{\lambda+d\alpha,\mu+d\beta})_{d\in\mathbb{N}}$ stabilises is said to be Aguiar-stable.

With Theorem 5.3.11 we re-obtain immediately Li Ying's result on the stabilisation of the Aguiar coefficients (minus the bound of stabilisation), which can be reformulated as follows:

Corollary 5.3.13. *The triple* $\big((1),(1);(1)\big)$ *is Aguiar-stable.*

Proof. For all $d \in \mathbb{N}^*$, according to Remark 5.3.4, $a_{(d),(d)}^{(d)} = g_{(d),(d),(d)} = 1$. □

Remark 5.3.14. On a more general note, the same reasoning shows that every stable triple (i.e. in the sense of Kronecker coefficients) is Aguiar-stable. For results producing stable triples, see [Ste14], [Man15a], [Val14], and Chapter 6.

We can also give some other explicit examples of "small" Aguiar-stable triples:

Proposition 5.3.15. *The triples*

$$\big((2),(1);(2)\big),\ \ \big((2),(1);(1,1)\big),\ \ \big((2),(1);(3)\big),\ \ \textit{and}\ \big((2),(1);(2,1)\big)$$

are all Aguiar-stable triples.

Proof. Let us write the proof in detail for $\big((2),(1);(2)\big)$, for instance. The three other ones work similarly. Let $d \in \mathbb{N}^*$. Then

$$a_{d(2),d(1)}^{d(2)} = \sum_{\alpha,\rho,\tau,\beta,\eta,\delta} c_{\alpha,\beta}^{(2d)}\ c_{\eta,\rho}^{(d)}\ g_{\beta,\eta,\delta}\ c_{\alpha,\delta}^{\tau}\ c_{\tau,\rho}^{(2d)}.$$

But the Littlewood-Richardson rule shows that the coefficient $c_{\alpha,\beta}^{(2d)}$ is zero unless α and β have only one part, and $|\alpha| + |\beta| = 2d$ (and then this coefficient is 1). As a consequence,

$$a_{d(2),d(1)}^{d(2)} = \sum_{\substack{\rho,\tau,\eta,\delta,\\ n\in[\![0,2d]\!]}} c_{\eta,\rho}^{(d)}\ g_{(n),\eta,\delta}\ c_{(2d-n),\delta}^{\tau}\ c_{\tau,\rho}^{(2d)}.$$

The same is true for the coefficient $c_{\eta,\rho}^{(d)}$ and the partitions η and ρ. So

$$a_{d(2),d(1)}^{d(2)} = \sum_{\substack{\tau,\delta,\\ n\in[\![0,2d]\!]\\ m\in[\![0,d]\!]}} g_{(n),(d-m),\delta}\ c_{(2d-n),\delta}^{\tau}\ c_{\tau,(m)}^{(2d)}.$$

And then the Kronecker coefficient $g_{(n),(d-m),\delta}$ is zero unless $n = d - m$. Moreover, if this is verified, $g_{(n),(n),\delta}$ is zero unless $\delta = (n)$ (and then this coefficient is 1). Hence

$$a_{d(2),d(1)}^{d(2)} = \sum_{\substack{\tau,\\ n\in[\![0,d]\!]}} c_{(2d-n),(n)}^{\tau}\ c_{\tau,(d-n)}^{(2d)}.$$

The coefficient $c^{\tau}_{(2d-n),(n)}$ is then zero unless $|\tau| = 2d$. Furthermore, the other coefficient $c^{(2d)}_{\tau,(d-n)}$ is zero unless $|\tau| = 2d - d + n = d + n$. So

$$a^{d(2)}_{d(2),d(1)} = \sum_{\tau \vdash 2d} c^{\tau}_{(2d-d),(d)} \; c^{(2d)}_{\tau,(d-d)} = \sum_{\tau \vdash 2d} c^{\tau}_{(d),(d)} \; c^{(2d)}_{\tau,(0)}.$$

Finally this product is zero unless $\tau = (2d)$ (by the Littlewood-Richardson rule, for instance). Thus

$$a^{d(2)}_{d(2),d(1)} = c^{(2d)}_{(d),(d)} \; c^{(2d)}_{(2d),(0)} = 1,$$

and $\big((2),(1);(2)\big)$ is Aguiar-stable by Theorem 5.3.11. □

5.3.3 Some explicit bounds of stabilisation

The same method as for Kronecker coefficients can be used to get explicit bounds of stabilisation. Let us fix from now on an Aguiar-stable triple $(\alpha, \beta; \gamma)$ and a triple $(\lambda, \mu; \nu)$ of partitions. We also consider vector spaces V_1 and V_2 as before (of dimension at least 2), and denote $V = V_1 \oplus V_2 \oplus V_1 \otimes V_2$, such that

$$a^{\gamma}_{\alpha,\beta}(=1) = \dim \mathrm{H}^0(X, \mathcal{L})^G \qquad \text{and} \qquad a^{\nu}_{\lambda,\mu} = \dim \mathrm{H}^0(X, \mathcal{M})^G,$$

with $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$, $X = \mathcal{F}\ell(V_1) \times \mathcal{F}\ell(V_2) \times \mathcal{F}\ell(V)$, $\mathcal{L} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{L}^*_\gamma$, and $\mathcal{M} = \mathcal{L}_\lambda \otimes \mathcal{L}\mu \otimes \mathcal{L}_\nu$. We fix finally a basis $(e_1, \ldots, e_{n_1})$ of V_1 and a basis $(f_1, \ldots, f_{n_2})$ of V_2.

For the three examples of Aguiar-stable triples that we are going to study in this section (namely $\big((1),(1);(1)\big)$, $\big((2),(1);(2)\big)$, and $\big((2),(1);(3)\big)$), we must begin by considering the projection:

$$\begin{array}{cccc} \pi : & X & \longrightarrow & \overline{X} = \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V^*) \\ & \big((W_{1,i})_i, (W_{2,i})_i, (W'_i)_i\big) & \longmapsto & \big(W_{1,1}, W_{2,1}, \{\varphi \in V^* \text{ s.t. } \ker\varphi = W'_{n_1n_2+n_1+n_2-1}\}\big) \end{array}.$$

We also denote by $\overline{\mathcal{L}}$ the ample line bundle on $\overline{X}$ whose pull-back by π is $\mathcal{L}$.

Proposition 5.3.16. *The G-orbit $\mathcal{O}_0$ of $\overline{x}_0 = \big(\mathbb{C}e_1, \mathbb{C}f_1, \mathbb{C}(e^*_1 + e^*_{n_1} + f^*_1 + f^*_{n_2} + \varphi_n)\big) \in \overline{X}$ (where $n = \min(n_1, n_2)$ and $\varphi_n = \sum_{i=1}^n e^*_i \otimes f^*_i$) is open in $\overline{X}$.*
Moreover, if we denote respectively by $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ the G-orbits of

$$\overline{x}_1 = \big(\mathbb{C}e_1, \mathbb{C}f_2, \mathbb{C}(e^*_1 + e^*_{n_1} + f^*_2 + f^*_{n_2} + \varphi_n)\big),$$
$$\overline{x}_2 = \big(\mathbb{C}e_1, \mathbb{C}f_1, \mathbb{C}(e^*_{n_1} + f^*_1 + f^*_{n_2} + \varphi_n)\big),$$

and

$$\overline{x}_3 = \big(\mathbb{C}e_1, \mathbb{C}f_1, \mathbb{C}(e^*_1 + e^*_{n_1} + f^*_{n_2} + \varphi_n)\big)$$

in $\overline{X}$, then $\overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2} \cup \overline{\mathcal{O}_3} = \{\big(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}(\underbrace{\varphi_1}_{\in V_1^*} + \underbrace{\varphi_2}_{\in V_2^*} + \underbrace{\varphi}_{\in (V_1 \otimes V_2)^*})\big) \in \overline{X}$ s.t. $\varphi_1(v_1)\varphi_2(v_2)\varphi(v_1 \otimes v_2) = 0\}$. In addition, among $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$, no orbit is contained in the closure of another one.

Proof. We consider an element $\big(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}(\underbrace{\varphi_1}_{\in V_1^*} + \underbrace{\varphi_2}_{\in V_2^*} + \underbrace{\varphi}_{\in (V_1 \otimes V_2)^*})\big) \in \overline{X}$. Similarly to what we did in the case of Kronecker coefficients (see the proof of Proposition 4.3.3), we are only interested in the orbits which will contain all others in their closures. Then, considering the usual isomorphism $(V_1 \otimes V_2)^* \simeq \mathrm{Hom}(V_1, V_2^*)$, we say that φ corresponds to a linear map $\varphi' : V_1 \to V_2^*$, on which G acts by conjugation. As a consequence we only need to consider the case when φ' is of maximal rank (n, that is), since all the orbits with φ' of lower rank will be contained in the closure of such an orbit.

Thus we rather consider an element $\overline{x} = \big(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}(\underbrace{\varphi_1}_{\in V_1^*} + \underbrace{\varphi_2}_{\in V_2^*} + \varphi_n)\big) \in \overline{X}$, with $\varphi_n = \sum_{i=1}^{n} e_i^* \otimes f_i^* \in V_1^* \otimes V_2^*$, corresponding to a linear map $\varphi_n' : V_1 \to V_2^*$. Then the linear maps $\varphi_1, \varphi_2, \varphi_n', \varphi_n$, together with the vectors v_1 and v_2, give some vector subspaces of V_1, V_2, and $V_1 \otimes V_2$, whose relative positions will give us descriptions of the orbits we are interested in:

- in V_1 : $\mathbb{C}v_1$, $\ker \varphi_n' \subset (\varphi_n')^{-1}\big((\mathbb{C}v_2)^\perp\big)$, and $\ker \varphi_1$;
- in V_2 : $\mathbb{C}v_2$, $\ker {}^t\varphi_n' \subset ({}^t\varphi_n')^{-1}\big((\mathbb{C}v_1)^\perp\big)$, and $\ker \varphi_2$;
- in $V_1 \otimes V_2$: $Cv_1 \otimes v_2$ and $\ker \varphi_n$.

Then we see that there is an open orbit, $\mathcal{O}_0$, characterised by:

- $\varphi_1(v_1) \neq 0$, $\varphi_n'(v_1) \neq 0$, $\ker \varphi_n' \not\subset \ker \varphi_1$ (or rather, if $n = n_1$, $(\varphi_n')^{-1}\big((\mathbb{C}v_2)^\perp\big) \not\subset \ker \varphi_1$), $\ker \varphi_1 \not\subset (\varphi_n')^{-1}\big((\mathbb{C}v_2)^\perp\big)$,
- $\varphi_2(v_2) \neq 0$, ${}^t\varphi_n'(v_2) \neq 0$, $\ker {}^t\varphi_n' \not\subset \ker \varphi_2$ (or rather, if $n = n_2$, $({}^t\varphi_n')^{-1}\big((\mathbb{C}v_1)^\perp\big) \not\subset \ker \varphi_2$), $\ker \varphi_2 \not\subset ({}^t\varphi_n')^{-1}\big((\mathbb{C}v_1)^\perp\big)$,
- $\varphi_n(v_1 \otimes v_2) \neq 0$.

And the point $\overline{x}_0$ given above verifies all these conditions.

Finally the subset $\{\big(\mathbb{C}v_1, \mathbb{C}v_2, \mathbb{C}(\varphi_1 + \varphi_2 + \varphi)\big) \in \overline{X}$ s.t. $\varphi_1(v_1)\varphi_2(v_2)\varphi(v_1 \otimes v_2) = 0\}$ can be written as $\overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2} \cup \overline{\mathcal{O}_3}$ for three orbits $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$, characterised by the same equations as $\mathcal{O}_0$ except for:

- $\varphi_n(v_1 \otimes v_2) = 0$ for $\mathcal{O}_1$,
- $\varphi_1(v_1) = 0$ for $\mathcal{O}_2$,
- $\varphi_2(v_2) = 0$ for $\mathcal{O}_3$.

Then it is easy to check that $\overline{x}_1 \in \mathcal{O}_1$, $\overline{x}_2 \in \mathcal{O}_2$, and $\overline{x}_3 \in \mathcal{O}_3$. □

Murnaghan case and comparison with the results by Li Ying

In the case when $(\alpha,\beta;\gamma) = \big((1),(1);(1)\big)$, $\overline{\mathcal{L}} = \mathcal{O}(1)\otimes\mathcal{O}(1)\otimes\mathcal{O}(1)$. Moreover, since $\dim \mathrm{H}^0(\overline{X},\overline{\mathcal{L}})^G = 1$ and since $\big(\mathbb{C}v_1,\mathbb{C}v_2,\mathbb{C}(\varphi_1+\varphi_2+\varphi)\big)\in\overline{X}\mapsto\varphi(v_1\otimes v_2)$ gives a non-zero G-invariant section of $\overline{\mathcal{L}}$ over $\overline{X}$,

$$\overline{X}^{us}(\overline{\mathcal{L}}) = \{\big(\mathbb{C}v_1,\mathbb{C}v_2,\mathbb{C}(\varphi_1+\varphi_2+\varphi)\big)\in\overline{X} \text{ s.t. } \varphi(v_1\otimes v_2)=0\}.$$

Thus, according to Proposition 5.3.16 (and its proof),

$$\overline{X}^{us}(\overline{\mathcal{L}}) = \overline{\mathcal{O}_1}.$$

Then the one-parameter subgroup $\tau_1 = \big(1,0,\dots,0\,\big|\,0,1,0,\dots,0\big)$ is destabilising for $\overline{x}_1$:
$\mu^{\overline{\mathcal{L}}}(\overline{x}_1,\tau_1)=1$.
Moreover a calculation similar to what we did (several times) for Kronecker coefficients yields:

$$\max_{x\in\pi^{-1}(\overline{x}_1)}\big(-\mu^{\mathcal{M}}(x,\tau_1)\big) = -\lambda_1-\mu_1+\nu_1+2\nu_2+\sum_{k=1}^{n_1+n_2-1}\nu_{k+2}.$$

As usual, it follows that:

Theorem 5.3.17. *The sequence of general term $a^{\nu+(d)}_{\lambda+(d),\mu+(d)}$ is constant when $d\geq-\lambda_1-\mu_1+\nu_1+2\nu_2+\displaystyle\sum_{k=1}^{n_1+n_2-1}\nu_{k+2}$.*

Retrieving Li Ying's bound with our method: This is possible by choosing a different one-parameter subgroup destabilising $\overline{x}_1$. First we express this bound in our settings: if we rewrite the sequence $\left(a^{\nu[n+h]}_{\lambda[n],\mu[n-m]}\right)_n$ (settings from [Yin17]) as $\left(a^{\gamma+(d)}_{\alpha+(d),\beta+(d)}\right)_d$, then

$$\begin{aligned} & n\geq\frac{|\lambda|+|\mu|+|\nu|+\lambda_1+\mu_1+\nu_1-1}{2}+h+m \\ \iff & d\geq\frac{-|\alpha|-\alpha_1+\alpha_2-|\beta|-\beta_1+\beta_2+3|\gamma|-\gamma_1+\gamma_2-1}{2}.\end{aligned}$$

As a consequence, if we consider the one-parameter subgroup

$$\tau_1' = \big(2,0,1,\dots,1\,\big|\,0,2,1,\dots,1\big),$$

it destabilises $\overline{x}_1$: $\mu^{\overline{\mathcal{L}}}(\overline{x}_1,\tau_1')=2$. Moreover,

$$\begin{aligned}\max_{x\in\pi^{-1}(\overline{x}_1)}\big(-\mu^{\mathcal{M}}(x,\tau_1')\big) &= -2\lambda_1-\lambda_3-\dots-\lambda_{n_1}-2\mu_1-\mu_3-\dots-\mu_{n_2}+2\nu_1+4\nu_2 \\ &\quad +3(\nu_3+\dots+\nu_{n_1+n_2-2})+2(\nu_{n_1+n_2-1}+\dots+\nu_{n_1n_2-n_1-n_2+5}) \\ &\quad +\nu_{n_1n_2-n_1-n_2+6}+\dots+\nu_{n_1n_2+n_1+n_2-3},\end{aligned}$$

which gives even a slight improvement of Li Ying's bound for "long" partitions ν (i.e. of length $>n_1+n_2-2$), according to the previous expression of this bound.

Examples: The bound of Theorem 5.3.17 is for instance 15 for the triple $\big((7,3),(5,4,2);(6,6,5,4)\big)$, whereas Li Ying's (cf Corollary 5.3.8) is 18 (note that the improved bound obtained right above is 17). But the contrary is also possible: for the triple $\big((3,3,3),(4,3,2,1);(5,4,1)\big)$, Li Ying's bound is 4 whereas ours is 7.

Two other cases

For $(\alpha,\beta;\gamma)=\big((2),(1);(2)\big)$: Then $\overline{\mathcal{L}}=\mathcal{O}(2)\otimes\mathcal{O}(1)\otimes\mathcal{O}(2)$ and a non-zero G-invariant section of $\overline{\mathcal{L}}$ over $\overline{X}$ is given by $\mathbb{C}(\varphi_1+\varphi_2+\varphi)\big)\in\overline{X}\mapsto\varphi_1(v_1)\varphi(v_1\otimes v_2)$. As a consequence,

$$\overline{X}^{us}(\overline{\mathcal{L}})=\{\big(\mathbb{C}v_1,\mathbb{C}v_2,\mathbb{C}(\varphi_1+\varphi_2+\varphi)\big)\in\overline{X}\text{ s.t. }\varphi_1(v_1)\varphi(v_1\otimes v_2)=0\}=\overline{\mathcal{O}_1}\cup\overline{\mathcal{O}_2},$$

thanks to Proposition 5.3.16 and its proof.
Then we take the same τ_1 as before to destabilise $\overline{x}_1$ (still $\mu^{\overline{\mathcal{L}}}(\overline{x}_1,\tau_1)=1$), and $\tau_2=\big(1,0,\dots,0\,\big|-1,0,\dots,0\big)$ which destabilises $\overline{x}_2$: $\mu^{\overline{\mathcal{L}}}(\overline{x}_2,\tau_2)=1$. Finally,

$$\max_{x\in\pi^{-1}(\overline{x}_2)}\big(-\mu^{\mathcal{M}}(x,\tau_2)\big)=-\lambda_1+\mu_1+\sum_{k=1}^{n_2}\nu_{k+1}-\sum_{k=1}^{n_1}\nu_{n_1n_2+n_1+n_2+1-k}.$$

Theorem 5.3.18. *The sequence of general term* $a_{\lambda+(2d),\mu+(d)}^{\nu+(2d)}$ *is constant when*

$$d\geq-\lambda_1+\max\left(-\mu_1+\nu_1+2\nu_2+\sum_{k=1}^{n_1+n_2-1}\nu_{k+2},\mu_1+\sum_{k=1}^{n_2}\nu_{k+1}-\sum_{k=1}^{n_1}\nu_{n_1n_2+n_1+n_2+1-k}\right).$$

For $(\alpha,\beta;\gamma)=\big((2),(1);(3)\big)$: Then $\overline{\mathcal{L}}=\mathcal{O}(2)\otimes\mathcal{O}(1)\otimes\mathcal{O}(3)$ and a non-zero G-invariant section of $\overline{\mathcal{L}}$ over $\overline{X}$ is given by $\mathbb{C}(\varphi_1+\varphi_2+\varphi)\big)\in\overline{X}\mapsto\varphi_1(v_1)\varphi_1(v_1)\varphi_2(v_2)$. As a consequence,

$$\overline{X}^{us}(\overline{\mathcal{L}})=\{\big(\mathbb{C}v_1,\mathbb{C}v_2,\mathbb{C}(\varphi_1+\varphi_2+\varphi)\big)\in\overline{X}\text{ s.t. }\varphi_1(v_1)\varphi_2(v_2)=0\}=\overline{\mathcal{O}_2}\cup\overline{\mathcal{O}_3}.$$

Then we take the same τ_2 as before to destabilise $\overline{x}_2$ (still $\mu^{\overline{\mathcal{L}}}(\overline{x}_2,\tau_2)=1$), and $\tau_3=\big(-3,-2,\dots,-2\,\big|\,1,0,\dots,0,-2\big)$ which destabilises $\overline{x}_3$: $\mu^{\overline{\mathcal{L}}}(\overline{x}_3,\tau_3)=1$. Finally,

$$\begin{aligned}\max_{x\in\pi^{-1}(\overline{x}_3)}\big(-\mu^{\mathcal{M}}(x,\tau_3)\big)&=3\lambda_1+2\sum_{k=1}^{n_1-1}\lambda_{k+1}-\mu_1+2\mu_2-2\nu_1+\nu_2-\sum_{k=1}^{n_1-1}\nu_{n_2+k}\\&\quad-2\sum_{k=1}^{n_1n_2-n_1-n_2+2}\nu_{n_1+n_2-1+k}-3\sum_{k=1}^{n_2-1}\nu_{n_1n_2+1+k}\\&\quad-4\sum_{k=1}^{n_1-1}\nu_{n_1n_2+n_2+k}-5\nu_{n_1n_2}.\end{aligned}$$

Theorem 5.3.19. *The sequence of general term* $a_{\lambda+(2d),\mu+(d)}^{\nu+(3d)}$ *is constant when*

$$d\geq\max\left(-\lambda_1+\mu_1+\sum_{k=1}^{n_2}\nu_{k+1}-\sum_{k=1}^{n_1}\nu_{n_1n_2+n_1+n_2+1-k},\max_{x\in\pi^{-1}(\overline{x}_3)}\big(-\mu^{\mathcal{M}}(x,\tau_3)\big)\right).$$

Chapter 6

Production of stable triples

6.1 Introduction

In this chapter we come back to the Kronecker coefficients, and more precisely to the notion of a stable triple of partitions related to them. Let us recall the definition of such a triple (due to J. Stembridge in [Ste14]), as well as the characterisation conjectured by Stembridge and proven by S. Sam and A. Snowden in [SS16] (that we re-proved in Chapter 4):

Definition 6.1.1. A triple (α, β, γ) of partitions is said to be stable if $g_{\alpha,\beta,\gamma} \neq 0$ and, for any triple (λ, μ, ν) of partitions, the sequence of general term $g_{\lambda+d\alpha,\mu+d\beta,\nu+d\gamma}$ is constant when $d \gg 0$.

Proposition 6.1.2. *A triple (α, β, γ) of partitions is stable if and only if, for any positive integer d, $g_{d\alpha,d\beta,d\gamma} = 1$.*

This characterisation in particular leads us to define another notion close to this one:

Definition 6.1.3. A triple (α, β, γ) of partitions is said to be almost stable if, for any positive integer d, $g_{d\alpha,d\beta,d\gamma} \leq 1$ and there exists $d_0 \in \mathbb{N}^*$ such that $g_{d_0\alpha,d_0\beta,d_0\gamma} \neq 0$.

Stable triples are of particular interest because of a well-known result that we give now: let us consider the following set, for n_1 and n_2 positive integers:

$$\mathrm{Kron}_{n_1,n_2} = \{(\alpha, \beta, \gamma) \text{ s.t. } \ell(\alpha) \leq n_1,\ \ell(\beta) \leq n_2,\ \ell(\gamma) \leq n_1 n_2 \text{ and } g_{\alpha,\beta,\gamma} \neq 0\},$$

where the notations are the same as in [Man15a]. A classical result is that Kron_{n_1,n_2} is a finitely generated semigroup, and we consider the cone generated by this semigroup:

$$\mathrm{PKron}_{n_1,n_2} = \{(\alpha, \beta, \gamma) \text{ s.t. } \ell(\alpha) \leq n_1,\ \ell(\beta) \leq n_2,\ \ell(\gamma) \leq n_1 n_2 \text{ and } \exists N \in \mathbb{N}^*,\ g_{N\alpha,N\beta,N\gamma} \neq 0\}.$$

It is a rational polyhedral cone called the Kronecker cone, or the Kronecker polyhedron. Then a result highlighting an important aspect of the stable triples is the following, which can for instance be found in [Man15b] (Proposition 2):

Proposition 6.1.4. *The set of stable triples in* Kron_{n_1,n_2} *is the intersection of* Kron_{n_1,n_2} *with a union of faces of the Kronecker cone* PKron_{n_1,n_2}.

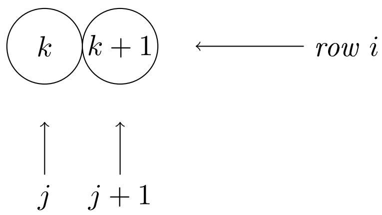
As a consequence we want to find ways to produce such faces of PKron_{n_1,n_2}, which contain only stable triples. There already exists one result in this direction, proven independently by L. Manivel (in [Man15a]) and E. Vallejo (in [Val14]), expressed in terms of additive matrices: a matrix $A = (a_{i,j})_{i,j} \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ having entries which are non-negative integers is said to be additive if there exist integers $x_1 > \cdots > x_{n_1}$ and $y_1 > \cdots > y_{n_2}$ such that, for all $(i,j),(k,l) \in [\![1,n_1]\!] \times [\![1,n_2]\!]$,

$$a_{i,j} > a_{k,l} \Longrightarrow x_i + y_j > x_k + y_l.$$

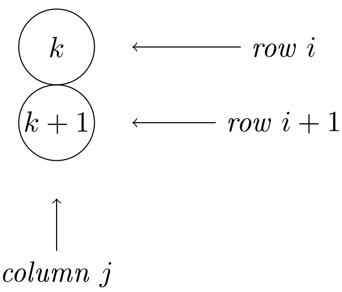
The result of Manivel and Vallejo is then that any such additive matrix gives an explicit face of PKron_{n_1,n_2} which contains only stable triples. This face is moreover regular, which means that it contains some triple (α,β,γ) of regular partitions (i.e. α, β, and γ have respectively n_1, n_2, and n_1n_2 pairwise distinct parts, with the last one being possibly 0), and it has the minimal dimension possible for a regular face: n_1n_2.

In this chapter we obtain results producing, from an additive matrix, more faces of this kind. Actually, rather than looking precisely at an additive matrix, we look instead at what we call the order matrix, which sort of "encodes the type" of the additive matrix: considering an additive matrix $A = (a_{i,j})_{i,j}$ whose coefficients are pairwise distinct[1], instead of the coefficients of A we write their rank in the decreasingly-ordered sequence of the $a_{i,j}$'s. Then our first result (see Section 6.3.4) is:

Theorem 6.1.5. *Any configuration of the following type in the order matrix:*



gives an explicit regular face of the Kronecker cone PKron_{n_1,n_2}*, of dimension* n_1n_2*, containing only stable triples. The same result is true for each configuration of the type*



[1] This assumption in fact does not make us produce less faces at the end.

Then we also obtain another result, concerning other types of configurations in the order matrix (called Configurations Ⓐ to Ⓔ, and involving now three or four coefficients of the order matrix, see Section 6.3.5), and which is more about almost stable triples:

Theorem 6.1.6. *Each one of the Configurations Ⓐ to Ⓔ in the order matrix gives a face – not necessarily regular and possibly reduced to zero – of the Kronecker cone* PKron_{n_1,n_2} *which contains only almost stable triples.*

Obtaining these results is based on the notions of dominant and well-covering pairs, coming from the work of N. Ressayre and that we present in Section 6.2, as well as the usual interpretation that we use for the Kronecker coefficients in terms of sections of line bundles on flag varieties (see Proposition 3.1.5). At the end of this chapter (in Section 6.4), we apply our results as well as Manivel and Vallejo's to all possible order matrices of small size (namely 2×2, 3×2, and 3×3) in order to have a look at the number of new interesting faces of PKron_{n_1,n_2} that we can produce.

6.2 Definitions and a few general results

6.2.1 Definitions in the general context

For now G is a connected complex reductive group acting on a smooth projective variety X. Let us consider a maximal torus T in G, τ a one-parameter subgroup of T (denoted by $\tau \in X_*(T)$), and C an irreducible component of X^τ, the set of points in X fixed by τ. We denote by G^τ the centraliser of τ (i.e. of $\operatorname{Im}\tau$) in G and set

$$P(\tau) = \{g \in G \text{ s.t. } \lim_{t\to 0} \tau(t) g \tau(t^{-1}) \text{ exists}\}.$$

Notice that $P(\tau)$ is a parabolic subgroup of G and that G^τ is the Levi subgroup of $P(\tau)$ containing T. Consider then

$$C^+ = \{x \in X \text{ s.t. } \lim_{t\to 0} \tau(t).x \in C\},$$

which is a $P(\tau)$-stable locally closed subvariety. For any $x \in C$, we define the following subspaces of $\mathrm{T}_x X$, the Zariski tangent space of X at x:

$$\begin{aligned}
&\mathrm{T}_x X_{>0} = \{\xi \in \mathrm{T}_x X \text{ s.t. } \lim_{t\to 0} \tau(t).\xi = 0\},\\
&\mathrm{T}_x X_{<0} = \{\xi \in \mathrm{T}_x X \text{ s.t. } \lim_{t\to 0} \tau(t^{-1}).\xi = 0\},\\
&\mathrm{T}_x X_0 = (\mathrm{T}_x X)^\tau,\\
&\mathrm{T}_x X_{\geq 0} = \mathrm{T}_x X_{>0} \oplus \mathrm{T}_x X_0,\\
&\mathrm{T}_x X_{\leq 0} = \mathrm{T}_x X_{<0} \oplus \mathrm{T}_x X_0.
\end{aligned}$$

Theorem 6.2.1 (Białynicki-Birula)**.**

(i) C *is smooth and, for any* $x \in C$, $\mathrm{T}_x C = \mathrm{T}_x X_0$.

(ii) C^+ *is smooth and irreducible and, for any* $x \in C$, $\mathrm{T}_x C^+ = \mathrm{T}_x X_{\geq 0}$.

We can now consider
$$\begin{array}{rccc} \eta: & G \times_{P(\tau)} C^+ & \longrightarrow & X \\ & [g:x] & \longmapsto & g.x \end{array}.$$
The following definition comes from [Res10].

Definition 6.2.2. The pair (C, τ) is said to be dominant if η is, and covering if η is birational. It is said to be well-covering when η induces an isomorphism onto an open subset of X intersecting C.

6.2.2 In the context of Kronecker coefficients

We consider from now on V_1 and V_2 two complex vector spaces of dimension respectively n_1 and n_2. We then set $G_1 = \mathrm{GL}(V_1)$, $G_2 = \mathrm{GL}(V_2)$, $G = G_1 \times G_2$, $\hat{G} = \mathrm{GL}(V_1 \otimes V_2)$. We also choose T_1, T_2, $T = T_1 \times T_2$, and $\hat{T} \supset T$ respective maximal tori, and B_1, B_2, $B = B_1 \times B_2$, $\hat{B}$ respective Borel subgroups containing the corresponding tori. Recall then our usual interpretation of the Kronecker coefficients (see for instance Proposition 3.1.5): if α, β, and γ are partitions of lengths at most n_1, n_2 and n_1n_2 respectively, then there exist explicit line bundles $\mathcal{L}_\alpha$, $\mathcal{L}_\beta$, and $\mathcal{L}_\gamma$, on G_1/B_1, G_2/B_2, and $\hat{G}/\hat{B}$ respectively, giving a G-linearised line bundle $\mathcal{L}_{\alpha,\beta,\gamma} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{L}_\gamma^*$ on $G/B \times \hat{G}/\hat{B}$ such that:
$$g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(G/B \times \hat{G}/\hat{B}, \mathcal{L}_{\alpha,\beta,\gamma})^G.$$
We consider in addition parabolic subgroups P of G and $\hat{P}$ of $\hat{G}$ containing the Borel subgroups. All corresponding Lie algebras will be denoted with lower case gothic letters. We consider
$$X = G/P \times \hat{G}/\hat{P},$$
on which G acts diagonally. We finally denote by W and $\hat{W}$ the Weyl groups associated to G and $\hat{G}$ respectively, and by W_P (resp.$\hat{W}_{\hat{P}}$) the Weyl group of the Levi subgroup of P (resp. $\hat{P}$) containing T (resp. $\hat{T}$). The latter is canonically a subgroup of W (resp. $\hat{W}$).

We also give notations concerning the root systems: let us denote by Φ (resp. $\hat{\Phi}$) the set of roots of G (resp. $\hat{G}$), with Φ^+ and Φ^- (resp. $\hat{\Phi}^+$ and $\hat{\Phi}^-$) the subsets of positive and negative ones with respect to the choice of B (resp. $\hat{B}$). Finally, the set of roots of $\hat{\mathfrak{p}}$ is denoted by $\hat{\Phi}_{\hat{\mathfrak{p}}}$.

Let us consider $\tau \in X_*(T)$. It is known that the irreducible components of X^τ are the $G^\tau v^{-1}P/P \times \hat{G}^\tau \hat{v}^{-1}\hat{P}/\hat{P}$, for $v \in W_P \backslash W / W_{P(\tau)}$ and $\hat{v} \in \hat{W}_{\hat{P}} \backslash \hat{W} / \hat{W}_{\hat{P}(\tau)}$. We then fix two such v and $\hat{v}$, and denote by C the corresponding irreducible component of X^τ. Therefore, if (α, β, γ) is a triple of partitions such that $\mathcal{L}_{\alpha,\beta,\gamma}$ descends to a line bundle on X – that we will also denote by $\mathcal{L}_{\alpha,\beta,\gamma}$ – then, for any $x \in C$, $\mathbb{C}^*$ acts via τ on the fibre $(\mathcal{L}_{\alpha,\beta,\gamma})_x$ over x. This action is given by an integer n which, since C is an irreducible component, does not depend on $x \in C$. We then set $\mu^{\mathcal{L}_{\alpha,\beta,\gamma}}(C, \tau) = -n$.

Lemma 6.2.3. *For any dominant pair (C,τ) we consider the set of all triples $(\alpha,\beta,\gamma) \in \mathrm{PKron}_{n_1,n_2}$ such that $\mu^{\mathcal{L}_{\alpha,\beta,\gamma}}(C,\tau)=0$. Then it is a face of PKron_{n_1,n_2} (possibly reduced to zero). Moreover it can also be described as:*

$$\{(\alpha,\beta,\gamma) \text{ s.t. } X^{ss}(\mathcal{L}_{\alpha,\beta,\gamma}) \cap C \neq \emptyset\}.$$

We denote this face by $\mathcal{F}(C)$.

Note that this result is actually valid in the general context of Section 6.2.1, where X is any smooth projective variety and G is a connected complex reductive group acting on X. The equivalent of PKron_{n_1,n_2} is then the cone $\{\mathcal{L} \text{ s.t. } \exists N \in \mathbb{N}^*,\ \mathrm{H}^0(X,\mathcal{L}^{\otimes N})^G \neq \{0\}\}$.

Proof. It comes directly from [Res10], Lemma 3. □

Lemma 6.2.4. *If $P=B$ or $\hat{P}=\hat{B}$, and if (C_1,τ_1) and (C_2,τ_2) are two well-covering pairs such that $\mathcal{F}(C_1)=\mathcal{F}(C_2)$, then there exists $g \in G$ such that $g.C_2 = C_1$.*

Proof. This comes from [Res11], Lemma 6.5. □

Ressayre also proved, in [Res10], that any regular face of PKron_{n_1,n_2} is given by a well-covering pair[2]. Finally, another consequence of [Res11] is that, if C is a singleton and (C,τ) is well-covering, then the face $\mathcal{F}(C)$ is a regular face of minimal dimension of PKron_{n_1,n_2} (i.e. n_1n_2).

6.3 Application to obtain stable triples

6.3.1 Link between well-covering pairs and stability

Theorem 6.3.1. *Assume that (C,τ) is well-covering. Then, for all G-linearised line bundles $\mathcal{L}$ on X such that $\mu^{\mathcal{L}}(C,\tau)=0$,*

$$\mathrm{H}^0(X,\mathcal{L})^G \simeq \mathrm{H}^0(C,\mathcal{L}|_C)^{G^\tau}.$$

Proof. It is Theorem 4 of [Res10]. □

Remark 6.3.2. If we only make the hypothesis that (C,τ) is dominant (and $\mu^{\mathcal{L}}(C,\tau)=0$), we still have that

$$\mathrm{H}^0(X,\mathcal{L})^G \hookrightarrow \mathrm{H}^0(C,\mathcal{L}|_C)^{G^\tau}.$$

Corollary 6.3.3. *Assume that (C,τ) is well-covering, and that G^τ has a dense orbit in C. Then the face $\mathcal{F}(C)$ contains only almost stable triples.*

Proof. It is an immediate consequence of the previous theorem: let $(\alpha,\beta,\gamma) \in \mathcal{F}(C)$. Then $\mu^{\mathcal{L}_{\alpha,\beta,\gamma}}(C,\tau)=0$ and therefore

$$\forall d \in \mathbb{N}^*,\ g_{d\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X,\mathcal{L}_{\alpha,\beta,\gamma})^G = \dim \mathrm{H}^0(C,\mathcal{L}|_C)^{G^\tau} \leq 1$$

since G^τ has a dense orbit in C. □

[2] He even proved it in a much more general setting than PKron_{n_1,n_2}.

Remark 6.3.4. If we only make the hypothesis that (C,τ) is dominant, Remark 6.3.2 tells us that we still have almost stable triples: for all $(\alpha,\beta,\gamma)\in\mathcal{F}(C)$, for all $d\in\mathbb{N}^*$,

$$g_{d(\alpha,\beta,\gamma)}\leq 1.$$

But note that $\mathcal{F}(C)$ can be reduced to zero.

Remark 6.3.5. There is an important particular case when τ is dominant, regular (i.e. for all $\alpha\in\Phi$, $\langle\alpha,\tau\rangle\neq 0$), and $\hat{G}$-regular (i.e. for all $\hat{\alpha}\in\hat{\Phi}$, $\langle\hat{\alpha},\tau\rangle\neq 0$). Then $G^\tau=T$ and $\hat{G}^\tau=\hat{T}$. As a consequence, C is a singleton – say $\{x_0\}$ –, and the condition "G^τ has a dense orbit in C" is automatic. Moreover one then has:

$$\dim \mathrm{H}^0(C,\mathcal{L}|_C)^{G^\tau}=1\Longleftrightarrow T\text{ acts trivially on }\mathcal{L}_{x_0}\Longleftrightarrow\forall\sigma\in X_*(T),\ \mu^{\mathcal{L}}(C,\sigma)=0.$$

All the previous results and remarks lead directly to the following main result:

Theorem 6.3.6. *Assume that (C,τ) is well-covering and that τ is dominant, regular, and $\hat{G}$-regular (then C is a singleton). Then $\mathcal{F}(C)$ is a regular face of minimal dimension (i.e. n_1n_2) of the Kronecker cone* PKron_{n_1,n_2} *and contains only stable triples.*

Remark 6.3.7. When X has this form, there is a characterisation of the dominant and covering pairs in terms of a Schubert condition. It can be found in [Res10]. Let us explain it quickly here: we use the cohomology ring, $\mathrm{H}^*(G/P(\tau),\mathbb{Z})$, of $G/P(\tau)$ and we denote, for any closed subvariety Y of $G/P(\tau)$, by $[Y]\in\mathrm{H}^*(G/P(\tau),\mathbb{Z})$ its cycle class in cohomology. We also use, with the same notations, $\mathrm{H}^*(\hat{G}/\hat{P}(\tau),\mathbb{Z})$.
Then, since $P(\tau)=G\cap\hat{P}(\tau)$, $G/P(\tau)$ identifies with the G-orbit of $\hat{P}(\tau)/\hat{P}(\tau)$ in $\hat{G}/\hat{P}(\tau)$, which gives a closed immersion $\iota: G/P(\tau)\hookrightarrow\hat{G}/\hat{P}(\tau)$. It induces a map ι^* in cohomology:

$$\iota^*:\mathrm{H}^*(\hat{G}/\hat{P}(\tau),\mathbb{Z})\longrightarrow\mathrm{H}^*(G/P(\tau),\mathbb{Z}).$$

Then the result from [Res10] (Lemma 14) is:

Lemma 6.3.8. *(i) The pair (C,τ) is dominant if and only if*

$$[\overline{PvP(\tau)/P(\tau)}]\cdot\iota^*([\overline{\hat{P}\hat{v}\hat{P}(\tau)/\hat{P}(\tau)}])\neq 0.$$

(ii) It is covering if and only if

$$[\overline{PvP(\tau)/P(\tau)}]\cdot\iota^*([\overline{\hat{P}\hat{v}\hat{P}(\tau)/\hat{P}(\tau)}])=[\mathrm{pt}],$$

i.e. if and only if the intersection between two generic translates in $\hat{G}/\hat{P}(\tau)$ of $\overline{PvP(\tau)/P(\tau)}$ and $\overline{\hat{P}\hat{v}\hat{P}(\tau)/\hat{P}(\tau)}$ contains exactly one point.

6.3.2 A sufficient condition to get dominant pairs

We now want to see that we can indeed obtain dominant or well-covering pairs (C,τ). For this we consider respective bases of the vector spaces V_1 and V_2: $(e_1,\dots,e_{n_1})$ and $(f_1,\dots,f_{n_2})$. They give also a basis of $V_1\otimes V_2$: $(e_i\otimes f_j)_{i,j}$ ordered lexicographically (i.e. $(e_1\otimes f_1, e_1\otimes f_2,\dots,e_{n_1}\otimes f_{n_2})$), sometimes denoted $(\hat{e}_1,\dots,\hat{e}_{n_1n_2})$[3]. Thanks to these bases we will often identify G_1, G_2, and $\hat{G}$ respectively with $\mathrm{GL}_{n_1}(\mathbb{C})$, $\mathrm{GL}_{n_2}(\mathbb{C})$, and $\mathrm{GL}_{n_1n_2}(\mathbb{C})$. We finally take B and $\hat{B}$ the respective Borel subgroups of G and $\hat{G}$ formed by the upper-triangular matrices, and set from now on $X = G/B\times\hat{G}/\hat{B}$.

Start now from a one-parameter subgroup τ of T which is supposed to be dominant, regular, and even $\hat{G}$-regular. In particular, τ has the form

$$\begin{array}{rccl}\tau: & \mathbb{C}^* & \longrightarrow & T\\ & t & \longmapsto & \left(\begin{pmatrix}t^{x_1} & & \\ & \ddots & \\ & & t^{x_{n_1}}\end{pmatrix},\begin{pmatrix}t^{y_1} & & \\ & \ddots & \\ & & t^{y_{n_2}}\end{pmatrix}\right)\end{array},$$

with non-negative integers $x_1>\dots>x_{n_1}$, $y_1>\dots>y_{n_2}$. We then create the matrix $M=(x_i+y_j)_{i,j}\in\mathcal{M}_{n_1,n_2}(\mathbb{R})$. Since τ was taken $\hat{G}$-regular, it has pairwise distinct coefficients. From this we define what we will call the "order matrix" of τ: it is a matrix having the same size as M but whose coefficient at position (i,j) is the ranking of the coefficient x_i+y_j when one orders the coefficients of M decreasingly (we will usually circle that ranking when we write the order matrix in order to highlight the difference with the coefficient x_i+y_j).

Giving such an order matrix is equivalent to giving a flag $\hat{w}.\hat{B}/\hat{B}$ fixed by $\hat{T}$ (and then a well-defined $\hat{w}\in\hat{W}$): to each matrix position $(i,j)\in[\![1,n_1]\!]\times[\![1,n_2]\!]$ we associate the element $e_i\otimes f_j$ of the basis of $V_1\otimes V_2$. Then we create a $\hat{T}$-stable complete flag in $V_1\otimes V_2$ by ordering the elements $e_i\otimes f_j$ according to the numbers in the order matrix.

Example: For the one-parameter subgroup

$$\begin{array}{rccl}\tau: & \mathbb{C}^* & \longrightarrow & T\\ & t & \longmapsto & \left(\begin{pmatrix}t^{4} & & \\ & t^2 & \\ & & 1\end{pmatrix},\begin{pmatrix}t^{3} & \\ & 1\end{pmatrix}\right)\end{array},$$

[3]The ordering of the basis of $V_1\otimes V_2$ gives in particular an explicit bijection between $[\![1,n_1]\!]\times[\![1,n_2]\!]$ and $[\![1,n_1n_2]\!]$, which we will regularly use to identify the two in what follows.

the matrix M is $\begin{pmatrix} 7 & 4 \\ 5 & 2 \\ 3 & 0 \end{pmatrix}$, and so the order matrix of τ is

$$\begin{pmatrix} ① & ③ \\ ② & ⑤ \\ ④ & ⑥ \end{pmatrix}.$$

Then the flag $\hat{w}.\hat{B}/\hat{B}$ happens to be

$$\begin{aligned}
&\big(\mathbb{C}e_1 \otimes f_1 \\
&\subset \mathbb{C}e_1 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_1 \\
&\subset \mathbb{C}e_1 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_1 \oplus \mathbb{C}e_1 \otimes f_2 \\
&\subset \mathbb{C}e_1 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_1 \oplus \mathbb{C}e_1 \otimes f_2 \oplus \mathbb{C}e_3 \otimes f_1 \\
&\subset \mathbb{C}e_1 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_1 \oplus \mathbb{C}e_1 \otimes f_2 \oplus \mathbb{C}e_3 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_2 \\
&\subset \mathbb{C}e_1 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_1 \oplus \mathbb{C}e_1 \otimes f_2 \oplus \mathbb{C}e_3 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_2 \oplus \mathbb{C}e_3 \otimes f_2\big),
\end{aligned}$$

which we denote by

$$\mathrm{fl}(e_1 \otimes f_1, e_2 \otimes f_1, e_1 \otimes f_2, e_3 \otimes f_1, e_2 \otimes f_2, e_3 \otimes f_2) \in \mathcal{F}\ell(V_1 \otimes V_2).$$

This corresponds to

$$\hat{w} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix} = \underbrace{\big(2\ \ 3\big)\big(4\ \ 5\big)}_{\text{notation as a product of transpositions}} \in \mathfrak{S}_6 \simeq \hat{W}.$$

As usual with such a one-parameter subgroup, we get two parabolic subgroups $P(\tau)$ and $\hat{P}(\tau)$ of G and $\hat{G}$ respectively. According to the hypotheses made on τ, $P(\tau) = B$ in that case, and $\hat{P}(\tau)$ is a Borel subgroup denoted instead $\hat{B}(\tau)$. The set of positive (resp. negative) roots of $\hat{G}$ for this choice of Borel subgroup is denoted by $\hat{\Phi}^+(\tau)$ (resp. $\hat{\Phi}^-(\tau)$). The unipotent radicals of B, $\hat{B}$, and $\hat{B}(\tau)$ are respectively denoted U, $\hat{U}$, and $\hat{U}(\tau)$, while those of the respective opposite Borel subgroups will be U^-, $\hat{U}^-$, $\hat{U}^-(\tau)$.

Consider now two elements $v \in W$ and $\hat{v} \in \hat{W}$. They give

$$C = \{x_0\} = \{(v^{-1}B/B, \hat{v}^{-1}\hat{B}/\hat{B})\},$$

an irreducible component of X^T. As usual we then have

$$C^+ = Bv^{-1}B/B \times \hat{B}(\tau)\hat{v}^{-1}\hat{B}/\hat{B}$$

and

$$\begin{array}{rccc} \eta: & G \times_B C^+ & \longrightarrow & X \\ & [g:x] & \longmapsto & g.x \end{array}.$$

On root spaces for non-negative weights (i.e. on $\mathrm{T}_{x_0}C^+ = \mathrm{T}_{x_0}X_{\geq 0}$), $\mathrm{T}_{[e:x_0]}\eta$ is just the identity (cf Theorem 6.2.1). As a consequence,

$$\begin{array}{ll} & \mathrm{T}_{[e:x_0]}\eta \text{ is an isomorphism} \\ \Longleftrightarrow & \mathrm{T}_{[e:x_0]}\eta|_{\mathfrak{u}^-} : \mathfrak{u}^- \longrightarrow \mathfrak{u}^- \cap v^{-1}\mathfrak{u}^- v \oplus \hat{\mathfrak{u}}^-(\tau) \cap \hat{v}^{-1}\hat{\mathfrak{u}}^-\hat{v} \text{ is an isomorphism.} \end{array}$$

Then, if we define

$$\begin{array}{rccc} \mathrm{orb}: & U^- & \longrightarrow & X \\ & u & \longmapsto & u.x_0 \end{array},$$

we have $\mathrm{T}_{[e:x_0]}\eta|_{\mathfrak{u}^-} = \mathrm{T}_e\mathrm{orb}$. Moreover $\mathrm{T}_e\mathrm{orb}$ is an isomorphism if and only if it is injective, i.e. if and only if the isotropy subgroup $U^-_{x_0}$ of x_0 in U^- is finite. As a consequence, $\mathrm{T}_e\mathrm{orb}$ is an isomorphism if and only if the Lie algebra of $U^-_{x_0}$ is $\{0\}$. Therefore,

$$\begin{array}{ll} & \mathrm{T}_{[e:x_0]}\eta \text{ is an isomorphism} \\ \Longleftrightarrow & \mathfrak{u}^- \simeq \mathfrak{u}^- \cap v^{-1}\mathfrak{u}^- v \oplus \hat{\mathfrak{u}}^-(\tau) \cap \hat{v}^{-1}\hat{\mathfrak{u}}^-\hat{v} \text{ as } T\text{-modules} \\ \Longleftrightarrow & \displaystyle\bigoplus_{\beta\in\Phi^-\cap v^{-1}\Phi^+} \mathfrak{g}_\beta \simeq \bigoplus_{\hat{\beta}\in\hat{\Phi}^-(\tau)\cap\hat{v}^{-1}\hat{\Phi}^-} \mathfrak{g}_{\hat{\beta}} \text{ as } T\text{-modules.} \end{array}$$

Let us denote by ρ the morphism of restriction of roots of $\hat{G}$ (which are morphisms from $\hat{T}$ to $\mathbb{C}$) to characters of T. Moreover, notice that one has $\hat{\Phi}^-(\tau) = \hat{w}\hat{\Phi}^-$, where $\hat{w} \in \hat{W}$ is the element coming from the order matrix, as explained above. Then $\hat{\Phi}^-(\tau)\cap\hat{v}^{-1}\hat{\Phi}^- = \hat{w}\left(\hat{\Phi}^- \cap ((\hat{v}\hat{w})^\vee)^{-1}\hat{\Phi}^+\right)$, denoted[4] by $\hat{w}\hat{\Phi}\big((\hat{v}\hat{w})^\vee\big)$. Thus:

$$\begin{array}{ll} & \mathrm{T}_{[e:x_0]}\eta \text{ is an isomorphism} \\ \Longleftrightarrow & \rho\left(\hat{\Phi}^-(\tau)\cap\hat{v}^{-1}\hat{\Phi}^-\right) \subset \Phi^-\cap v^{-1}\Phi^+ \text{ and } \rho : \hat{\Phi}^-(\tau)\cap\hat{v}^{-1}\hat{\Phi}^- \longrightarrow \Phi^-\cap v^{-1}\Phi^+ \text{ is bijective} \\ \Longleftrightarrow & \rho\left(\hat{w}\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big)\right) \subset \Phi(v^{-1}) \text{ and } \rho : \hat{w}\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big) \longrightarrow \Phi(v^{-1}) \text{ is bijective,} \end{array}$$

where, for all $u \in W$ (resp. $\hat{u} \in \hat{W}$), $\Phi(u) = \Phi^- \cap u\Phi^+$ (resp. $\hat{\Phi}(\hat{u}) = \hat{\Phi}^- \cap \hat{u}\hat{\Phi}^+$).
We sum it up in the following proposition:

Proposition 6.3.9. *If $\rho\left(\hat{w}\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big)\right) \subset \Phi(v^{-1})$ and $\rho : \hat{w}\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big) \longrightarrow \Phi(v^{-1})$ is bijective, then the pair (C,τ) is dominant.*

Remark 6.3.10. It is a classical result that, for $u \in W$ (resp. $\hat{u} \in \hat{W}$), the cardinal of the set $\Phi(u)$ (resp. $\hat{\Phi}(\hat{u})$) corresponds to the length of the element u (resp. $\hat{u}$) of the Coxeter group W (resp. $\hat{W}$), denoted $\ell(u)$ (resp. $\ell(\hat{u})$). As a consequence $\sharp\Phi(v^{-1}) = \ell(v^{-1}) = \ell(v)$.

In this context ($C = \{(v^{-1}B/B, \hat{v}^{-1}\hat{B}/\hat{B})\}$), there is a characterisation of well-covering pairs given by Ressayre in [Res10], Proposition 11:

[4] for any $\hat{u} \in \hat{W}$, $\hat{u}^\vee$ is defined as $\hat{w}_0\hat{u}$, where $\hat{w}_0$ is the longest element of the Weyl group $\hat{W}$

Lemma 6.3.11. *The pair (C,τ) is well-covering if and only if it is covering and*

$$v^{-1}.\left(\sum_{\alpha\in\Phi^-\cap v\Phi^-}\alpha\right)+\rho\left(\hat{v}^{-1}.\sum_{\hat{\alpha}\in\hat{\Phi}^-\cap\hat{v}\hat{\Phi}^-(\tau)}\hat{\alpha}\right)=\sum_{\alpha\in\Phi^-}\alpha.$$

Lemma 6.3.12. *If v and $\hat{v}$ are chosen as in Proposition 6.3.9, then:*

$$(C,\tau)\text{ is covering}\quad\Longrightarrow\quad(C,\tau)\text{ is well-covering.}$$

Proof. Assume that v and $\hat{v}$ are chosen as in Proposition 6.3.9 and that (C,τ) is covering. Then:

$$\begin{aligned}
&v^{-1}.\left(\sum_{\alpha\in\Phi^-\cap v\Phi^-}\alpha\right)+\rho\left(\hat{v}^{-1}.\sum_{\hat{\alpha}\in\hat{\Phi}^-\cap\hat{v}\hat{\Phi}^-(\tau)}\hat{\alpha}\right)-\sum_{\alpha\in\Phi^-}\alpha\\
=\;&\sum_{\alpha\in\Phi^-\cap v^{-1}\Phi^-}\alpha-\sum_{\alpha\in\Phi^-}\alpha+\rho\left(\sum_{\hat{\alpha}\in\hat{\Phi}^-(\tau)\cap\hat{v}^{-1}\hat{\Phi}^-}\hat{\alpha}\right)\\
=\;&-\sum_{\alpha\in\Phi^-\cap v^{-1}\Phi^+}\alpha+\sum_{\alpha\in\Phi^-\cap v^{-1}\Phi^+}\alpha\\
=\;&0
\end{aligned}$$

and, by the previous lemma, (C,τ) is well-covering. □

6.3.3 Element v of length 0: an existing result

The first and simplest case to satisfy the condition of Proposition 6.3.9 is the case when $\sharp\Phi(v^{-1})=0$, i.e. $\ell(v)=0$. This means that $v=1_W$, the unit in W. But then we also need that $\sharp\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big)=0$, i.e. $\hat{w}^{-1}\hat{v}^{-1}\hat{w}_0=((\hat{v}\hat{w})^\vee)^{-1}=1_{\hat{W}}$. Thus $\hat{v}^{-1}=\hat{w}\hat{w}_0$ gives a dominant pair (C,τ).

Moreover the Schubert condition given in Lemma 6.3.8 is not difficult to check here: according to the form of $C=\{(B/B,\hat{w}\hat{w}_0\hat{B}/\hat{B})\}$, the first Schubert variety to consider is just $\overline{B/B}$, which is a single point, whereas the second one is $\overline{\hat{B}\hat{w}_0\hat{w}^{-1}\hat{B}(\tau)/\hat{B}(\tau)}=\overline{\hat{B}\hat{w}_0\hat{B}/\hat{B}}$, which is the whole variety $\hat{G}/\hat{B}$. Hence the product of the two Schubert classes is in fact the class of a point, and then (C,τ) is well-covering by Lemma 6.3.8 and Lemma 6.3.12.

Theorem 6.3.13. *Each order matrix corresponding to a dominant, regular, $\hat{G}$-regular one-parameter subgroup τ of T gives a well-covering pair (C,τ) with:*

$$C=\big\{(B/B,\hat{w}\hat{w}_0\hat{B}/\hat{B})\big\}.$$

As a consequence the corresponding face $\mathcal{F}(C)$ of the Kronecker cone PKron_{n_1,n_2} *contains only stable triples.*

This theorem is actually an already existing result, due independently to L. Manivel (see [Man15a]) and E. Vallejo (see [Val14]). Let us now explain their result and why it is exactly what we have.

We consider a matrix $A = (a_{i,j})_{i,j} \in \mathcal{M}_{n_1,n_2}(\mathbb{R})$ having entries which are non-negative integers. We call λ and μ its 1-marginals, i.e. the finite sequence of integers $(\lambda_1, \dots, \lambda_{n_1})$ and $(\mu_1, \dots, \mu_{n_2})$ given by

$$\lambda_i = \sum_{j=1}^{n_2} a_{i,j} \qquad \text{and} \qquad \mu_j = \sum_{i=1}^{n_1} a_{i,j},$$

and we suppose that λ and μ are partitions (i.e. are non-increasing). Moreover, we denote by ν the π-sequence of A, i.e. the (finite) non-increasing sequence $(\nu_1, \dots, \nu_{n_1 n_2})$ formed by the entries of A.

Definition 6.3.14. The matrix A is said to be additive if there exist integers $x_1 > \cdots > x_{n_1}$ and $y_1 > \cdots > y_{n_2}$ such that, for all $(i,j),(k,l) \in [\![1,n_1]\!] \times [\![1,n_2]\!]$,

$$a_{i,j} > a_{k,l} \Longrightarrow x_i + y_j > x_k + y_l.$$

Theorem 6.3.15 (Manivel, Vallejo)**.** *Assume that the matrix A is additive. Then the triple (λ, μ, ν) of partitions is a stable triple.*

Manivel and Vallejo gave different proofs of this result. What we want to highlight here is that it corresponds to Theorem 6.3.13.

Proof. The parabolic subgroup $\hat{P}$ of $\hat{G}$ we consider this time is the one corresponding to the "shape" of $-\hat{w}_0.\nu$, i.e. the one such that $\mathcal{L}_\nu^*$ is the pull-back of an ample line bundle on $\hat{G}/\hat{P}$. Furthermore, we take $P = B$, and so

$$Y = G/B \times \hat{G}/\hat{P}.$$

The matrix A gives a flag in $\hat{G}/\hat{P}$, similarly to what we explained about the order matrix of a one-parameter subgroup of T: the ordering of the coefficients $a_{i,j}$ in *non-decreasing* order (it is different from before) gives a partial (since some of these coefficients can be equal) ordering of the elements $e_i \otimes f_j$ of the basis of $V_1 \otimes V_2$. Then this ordering corresponds to a $\hat{T}$-stable partial flag in $V_1 \otimes V_2$ that is precisely an element of $\hat{G}/\hat{P}$.

Example: The additive matrix $\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix}$ gives the flag $(\mathbb{C}e_2 \otimes f_2 \subset \mathbb{C}e_2 \otimes f_2 \oplus \mathbb{C}e_1 \otimes f_2 \subset \mathbb{C}e_2 \otimes f_2 \oplus \mathbb{C}e_1 \otimes f_2 \oplus \mathbb{C}e_1 \otimes f_1 \oplus \mathbb{C}e_2 \otimes f_1 = V_1 \otimes V_2) \in \mathcal{F}\ell(1,2; V_1 \otimes V_2)$, which we will denote by $\mathrm{fl}\big(e_2 \otimes f_2, e_1 \otimes f_2, \{e_1 \otimes f_1, e_2 \otimes f_1\}\big)$.

The obtained flag is thus of the form $\hat{u}\hat{P}/\hat{P}$, with $\hat{u} \in \hat{W} \simeq \mathfrak{S}_{n_1 n_2}$. In the previous example, the flag can for instance be written with $\hat{u} = (1\ 4\ 3)$.

Remark: This element $\hat{u}$ is in general not uniquely defined: what is unique is its class in $\hat{W}/\hat{W}_{\hat{P}}$, but it is sufficient to pick one representative $\hat{u} \in \hat{W}$ of this one.

We then set
$$x_0 = (B/B, \hat{u}\hat{P}/\hat{P}) \in Y.$$
The point x_0 is fixed by T, and we can check (this is an easy computation) that, for any one-parameter subgroup τ of T,
$$\mu^{\mathcal{L}_{\lambda,\mu,\nu}}(x_0, \tau) = 0.$$
Since A is additive, there exist integers $x_1 > \cdots > x_m$ and $y_1 > \cdots > y_n$ such that, for all $(i,j),(k,l) \in [\![1,m]\!] \times [\![1,n]\!]$,
$$a_{i,j} > a_{k,l} \Longrightarrow x_i + y_j > x_k + y_l.$$
This means that, if we consider the following one-parameter subgroup τ of T:
$$\begin{array}{rccl} \tau: & \mathbb{C}^* & \longrightarrow & T \\ & t & \longmapsto & \left(\begin{pmatrix} t^{x_1} & & \\ & \ddots & \\ & & t^{x_m} \end{pmatrix}, \begin{pmatrix} t^{y_1} & & \\ & \ddots & \\ & & t^{y_n} \end{pmatrix} \right) \end{array},$$
it is dominant, regular, and verifies that, for all $\hat{\alpha} \in \hat{\Phi}$,
$$\hat{v}^{-1}.\hat{\alpha} \in \hat{\Phi} \setminus \hat{\Phi}_{\hat{\mathfrak{p}}} \Longrightarrow \langle \hat{\alpha}, \tau \rangle > 0. \tag{6.1}$$
Moreover, we can always assume that τ is $\hat{G}$-regular[5]. As a consequence, $C = \{x_0\}$ is an irreducible component of Y^τ (cf Remark 6.3.5).

This pair (C, τ) corresponds actually to the same pair as in Theorem 6.3.13: consider the order matrix of the one-parameter subgroup τ of T and $\hat{w} \in \hat{W}$ the well-defined (since τ is dominant, regular, $\hat{G}$-regular) Weyl group element we associated to such an order matrix.
First case: Assume that the coefficients of the matrix A are pairwise distinct. Then the relation between $\hat{w}$ and $\hat{u}$ is simply $\hat{u} = \hat{w}\hat{w}_0$. Then the pair (C, τ) is exactly the one of Theorem 6.3.13 and thus the face $\mathcal{F}(C)$ of PKron_{n_1,n_2} contains only stable triples. Finally, considering what we have written before, $(\lambda, \mu, \nu) \in \mathcal{F}(C)$.
Second case: Assume some of the coefficients of A are equal. Then the relation between $\hat{w}$ and $\hat{u}$ is rather that $\hat{u}$ and $\hat{w}\hat{w}_0$ are the same modulo multiplication by $\hat{W}_{\hat{P}}$ on the right. But this means that the two still define the same partial flag in $\hat{G}/\hat{P}$ and, for the same reasons as in the first case, the triple (λ, μ, ν) is on the face of the Kronecker

[5]The set of one-parameter subgroups of T verifying condition (6.1) is an open convex polyhedral cone and, among those subgroups, the not $\hat{G}$-regular ones are elements of some hyperplanes. Thus the set of dominant $\hat{G}$-regular one-parameter subgroups of T verifying condition (6.1) is not empty.

cone PKron_{n_1,n_2} given by Theorem 6.3.13. As a consequence it is stable. Moreover, the face of $\mathrm{PKron}_{n_1,n_2} \cap \{(\alpha,\beta,\gamma)$ s.t. $\mathcal{L}_\gamma^*$ is a line bundle on $\hat{G}/\hat{P}\}$ given by (C,τ) with $C = \{(B/B, \hat{u}\hat{P}/\hat{P})\} \subset Y$ is simply the intersection of this former face with the subspace $\{(\alpha,\beta,\gamma)$ s.t. $\mathcal{L}_\gamma^*$ is a line bundle on $\hat{G}/\hat{P}\}$. □

The fact that an additive matrix gives in fact a face of minimal dimension (among regular faces) of the cone PKron_{n_1,n_2} was already explained by Manivel in [Man15a].

6.3.4 Second case: length 1

At first we need some more notations: for $i \in [\![1,n_1]\!]$, $j \in [\![1,n_2]\!]$, and $k \in [\![1,n_1n_2]\!]$ (which corresponds to a pair $(i',j') \in [\![1,n_1]\!] \times [\![1,n_2]\!]$, see Footnote 3), ε_i, η_j, and $\hat{\varepsilon}_k = \hat{\varepsilon}_{(i',j')}$ are the characters of T_1 (the set of diagonal matrices in G_1), T_2 (same in G_2), and $\hat{T}$ respectively, defined by

$$\varepsilon_i : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{n_1} \end{pmatrix} \longmapsto t_i,$$

$$\eta_j : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{n_2} \end{pmatrix} \longmapsto t_j,$$

and

$$\hat{\varepsilon}_k : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{n_1n_2} \end{pmatrix} \longmapsto t_k.$$

Assume that $\sharp\Phi(v^{-1}) = 1$, i.e. $\ell(v) = 1$. This means that $v = v^{-1} = s_\alpha$, with α a simple root of G (and then $\Phi(s_\alpha) = \{-\alpha\}$). There are two kinds of such α:

- the roots of $G_1 = \mathrm{GL}(V_1)$, which are the $\varepsilon_i - \varepsilon_{i+1}$, for $i \in [\![1,n_1-1]\!]$,
- the roots of $G_2 = \mathrm{GL}(V_2)$, which are the $\eta_i - \eta_{i+1}$, for $i \in [\![1,n_2-1]\!]$.

In addition, since we also want $\sharp\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big) = 1$, it is necessary that $((\hat{v}\hat{w})^\vee)^{-1} = s_{\hat{\alpha}}$, for $\hat{\alpha}$ a simple root of $\hat{G}$, i.e. a $\hat{\varepsilon}_k - \hat{\varepsilon}_{k+1}$ for $k \in [\![1,n_1n_2-1]\!]$. Then we have

$$\hat{w}.\hat{\Phi}(s_{\hat{\alpha}}) = \{\hat{\varepsilon}_{\hat{w}(k+1)} - \hat{\varepsilon}_{\hat{w}(k)}\}.$$

As a consequence we see that α and $\hat{\alpha}$ will be suitable if and only if

- $\hat{w}(k) = (i,j) \in [\![1,n_1]\!] \times [\![1,n_2]\!] \simeq [\![1,n_1n_2]\!]$ and $\hat{w}(k+1) = (i,j+1)$,
- or $\hat{w}(k) = (i,j)$ and $\hat{w}(k+1) = (i+1,j)$.

In these cases it is easy to then express v^{-1} and $\hat{v}^{-1}$ and we will get the following result:

Theorem 6.3.16. • *As soon as we have $k \in [\![1, n_1n_2 - 1]\!]$ such that $\hat{w}(k) = (i,j) \in [\![1, n_1]\!] \times [\![1, n_2]\!] \simeq [\![1, n_1n_2]\!]$ and $\hat{w}(k+1) = (i, j+1)$, we have a well-covering pair (C, τ) (and hence a regular face $\mathcal{F}(C)$ of the Kronecker cone PKron_{n_1,n_2} containing only stable triples), where*

$$C = \Big\{ \Big(\big(1, (j \;\; j+1)\big).B/B, \hat{w}(k \;\; k+1)\hat{w}_0.\hat{B}/\hat{B} \Big) \Big\}.$$

• *Likewise, if $k \in [\![1, n_1n_2 - 1]\!]$ is such that $\hat{w}(k) = (i,j)$ and $\hat{w}(k+1) = (i+1, j)$, the pair (C, τ), where*

$$C = \Big\{ \Big(\big((i \;\; i+1), 1\big).B/B, \hat{w}(k \;\; k+1)\hat{w}_0.\hat{B}/\hat{B} \Big) \Big\},$$

is well-covering.

Proof. We have already seen why these two kinds of properties for $\hat{w}$ give dominant pairs (thanks to Proposition 6.3.9). Then all that remains to be seen is whether these pairs are in fact well-covering, which will be done by looking at the Schubert condition (see Lemma 6.3.8): recall that we have an injective map

$$\begin{array}{rccc} \iota: & G/B & \hookrightarrow & \hat{G}/\hat{B}(\tau) \\ & gB & \longmapsto & g\hat{B}(\tau) \end{array}$$

(with $\hat{B}(\tau)/\hat{B}(\tau) = \hat{w}\hat{B}/\hat{B}$). We can be a little more precise while describing what ι does on flags:

$$\begin{array}{rccc} \iota: & \mathcal{F}\ell(V_1) \times \mathcal{F}\ell(V_2) & \hookrightarrow & \mathcal{F}\ell(V_1 \otimes V_2) \\ & \big((E_1 \subset \cdots \subset E_{n_1-1}), (F_1 \subset \cdots \subset F_{n_2-1})\big) & \longmapsto & (H_1 \subset \cdots \subset H_{n_1n_2-1}) \end{array},$$

with:

$$\forall k \in [\![1, n_1n_2 - 1]\!], H_k = H_{k-1} + E_i \otimes F_j, \text{ where } (i,j) = \hat{w}(k) \text{ and } H_0 = \{0\}, E_{n_1} = V_1, F_{n_2} = V_2.$$

Then we want to look at the intersection between two generic translates of

$$\hat{X}_{\hat{v}} = \overline{\hat{B}\hat{v}\hat{B}(\tau)/\hat{B}(\tau)}$$

and

$$\iota(X_v) = \iota(\overline{BvB/B})$$

(with the usual notation $C = \big\{(v^{-1}B/B, \hat{v}^{-1}\hat{B}/\hat{B})\big\}$). Here, in both cases, there exists $k_0 \in [\![1, n_1n_2 - 1]\!]$ such that $\hat{X}_{\hat{v}}$ is of the form $\overline{\hat{B}\hat{w}_0 s_{\hat{\alpha}_{k_0}} \hat{w}^{-1}\hat{B}(\tau)/\hat{B}(\tau)} = \overline{\hat{B}\hat{w}_0 s_{\hat{\alpha}_{k_0}} \hat{B}/\hat{B}}$, i.e. is a Schubert variety of codimension 1, and hence a divisor of $\hat{G}/\hat{B} = \mathcal{F}\ell(V_1 \otimes V_2)$. As a consequence it can be rewritten as

$$\hat{X}_{\hat{v}} = \Big\{(H_1 \subset \cdots \subset H_{n_1n_2-1}) \in \mathcal{F}\ell(V_1 \otimes V_2) \text{ s.t. } \dim\big(H_{k_0} \cap \mathrm{Vect}(\hat{e}_1, \ldots, \hat{e}_{n_1n_2-k_0}\big) \geq 1\Big\}.$$

Then all the translates of $\hat{X}_{\hat{v}}$ correspond to

$$\big\{(H_1 \subset \cdots \subset H_{n_1n_2-1}) \in \mathcal{F}\ell(V_1 \otimes V_2) \text{ s.t. } \dim(H_{k_0} \cap S) \geq 1\big\},$$

for all vector subspaces S of $V_1 \otimes V_2$ of dimension $n_1n_2 - k_0$.

First case: $v = \big(1, (j_0 \;\; j_0+1)\big) = s_{\beta_{j_0}}$ (i.e. $\hat{w}(k_0) = (i_0, j_0)$ for some i_0 and $\hat{w}(k_0+1) = (i_0, j_0+1)$). Then

$$\begin{aligned} & X_{s_{\beta_{j_0}}} = \overline{Bs_{\beta_{j_0}}B/B} \\ = \;& \Big\{\Big(\big(\mathrm{Vect}(e_1) \subset \cdots \subset \mathrm{Vect}(e_1, \ldots, e_{n_1-1})\big), \big(\mathrm{Vect}(f_1) \subset \cdots \subset \mathrm{Vect}(f_1, \ldots, f_{j_0-1}) \subset F_{j_0} \\ & \subset \mathrm{Vect}(f_1, \ldots, f_{j_0+1}) \subset \cdots \subset \mathrm{Vect}(f_1, \ldots, f_{n_2-1})\big)\Big) \in \mathcal{F}\ell(V_1) \times \mathcal{F}\ell(V_2) \;;\; F_{j_0} \text{ of dim } j_0\Big\}. \end{aligned}$$

As a consequence, $\iota(X_{s_{\beta_{j_0}}})$ is the set of all flags $(H_1 \subset \cdots \subset H_{n_1n_2-1})$ such that there exists a subspace F_{j_0} of dimension j_0 of V_2 verifying $\mathrm{Vect}(f_1, \ldots, f_{j_0-1}) \subset F_j \subset \mathrm{Vect}(f_1, \ldots, f_{j_0+1})$ and, for all $k \in [\![1, n_1n_2]\!]$ corresponding to some (i, j),

$$\left\{ \begin{array}{ll} H_k = H_{k-1} + \mathrm{Vect}(e_1, \ldots, e_i) \otimes F_{j_0} & \text{when } j = j_0 \\ H_k = H_{k-1} + \mathrm{Vect}(e_1, \ldots, e_i) \otimes \mathrm{Vect}(f_1, \ldots, f_j) & \text{otherwise} \end{array} \right. .$$

Then a generic translate of $\hat{X}_{\hat{v}}$ is

$$\big\{(H_1 \subset \cdots \subset H_{n_1n_2-1}) \in \mathcal{F}\ell(V_1 \otimes V_2) \text{ s.t. } \dim(H_k \cap S) \geq 1\big\}$$

for S of dimension $n_1n_2 - k_0$ which does not intersect $\mathrm{Vect}(\hat{e}_{\hat{w}(1)}, \ldots, \hat{e}_{\hat{w}(k_0)})$. Thus

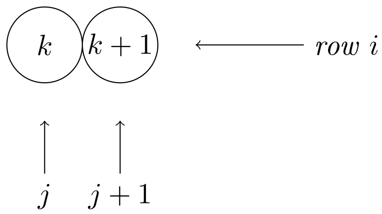
$$\begin{aligned} \iota(X_{s_{\beta_{j_0}}}) \cap Y &= \big\{(H_1 \subset \cdots \subset H_{n_1n_2-1}) \in \iota(X_{s_{\beta_{j_0}}}) \text{ s.t. } H_k \cap S \neq \{0\}\big\} \\ &= \big\{(H_1 \subset \cdots \subset H_{n_1n_2-1}) \in \iota(X_{s_{\beta_{j_0}}}) \text{ s.t. } F_{j_0} = \mathrm{Vect}(f_1, \ldots, f_{j_0-1}, f_{j_0+1})\big\} \end{aligned}$$

(since $\hat{w}(k_0+1)$ is (i_0, j_0+1)). This is a singleton, and the Schubert condition is then verified. As a consequence, the pair (C, τ) is covering, and hence well-covering by Lemma 6.3.12.

Second case: $v = \big((i_0 \;\; i_0+1), 1\big) = s_{\alpha_{i_0}}$. It is sufficient to exchange the roles of V_1 and V_2. □

These kinds of properties of $\hat{w}$ are really easy to "read" on the order matrix, which allows us to reformulate the previous theorem in the following equivalent way:

Theorem 6.3.17. *For any dominant, regular, $\hat{G}$-regular one-parameter subgroup τ of T, each configuration of the following type in the order matrix:*



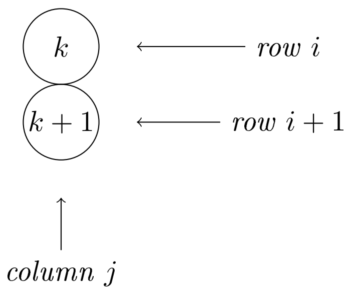
gives a well-covering pair (C, τ), *with*

$$C = \Big\{ \Big(\big(1, (j\ \ j+1)\big).B/B, \hat{w}(k\ \ k+1)\hat{w}_0.\hat{B}/\hat{B} \Big) \Big\}.$$

Hence we obtain a regular face of the Kronecker cone PKron_{n_1,n_2}, *of dimension* $n_1 n_2$ *and containing only stable triples:*

$$\mathcal{F}(C) = \Big\{ (\alpha, \beta, \gamma) \in \mathrm{PKron}_{n_1,n_2} \ s.t.\ \ \big(\hat{w}(k\ \ k+1)\big).\gamma\big|_T = \big(\alpha, (j\ \ j+1).\beta\big) \Big\},$$

where $\big(\alpha, (j\ \ j+1).\beta\big)$ *and* $\big(\hat{w}(k\ \ k+1)\big).\gamma$ *are respectively identified with characters of* T *and* $\hat{T}$.
Likewise, each configuration of the type



gives a well-covering pair (C, τ) *with*

$$C = \Big\{ \Big(\big((i\ \ i+1), 1\big).B/B, \hat{w}(k\ \ k+1)\hat{w}_0.\hat{B}/\hat{B} \Big) \Big\}.$$

Example: The order matrix

$$\begin{pmatrix} \textcircled{1} & \textcircled{3} \\ \textcircled{2} & \textcircled{4} \end{pmatrix}$$

which comes for instance from the one-parameter subgroup

$$\tau : t \longmapsto (\begin{pmatrix} t & \\ & 1 \end{pmatrix}, \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix})$$

and corresponds to $\hat{w} = (2\ \ 3) \in \mathfrak{S}_4 \simeq \hat{W}$, gives two different well-covering pairs according to the previous theorem:

- one with $C_1 = \Big\{\Big(\big((1\ \ 2),1\big).B/B, \underbrace{\hat{w}(1\ \ 2)\hat{w}_0}_{=(1\ 4\ 3)}.\hat{B}/\hat{B}\Big)\Big\}$,
- the other with $C_2 = \Big\{\Big(\big((1\ \ 2),1\big).B/B, \underbrace{\hat{w}(3\ \ 4)\hat{w}_0}_{=(1\ 2\ 4)}.\hat{B}/\hat{B}\Big)\Big\}$.

It is not difficult to see that they cannot come from additive matrices: with Lemma 6.2.4 in mind, we can "normalise" these well-covering pairs by the action of G so that C is of the form $\{(B/B, \hat{u}\hat{B}/\hat{B})\}$.

- $\big((1\ \ 2),1\big).C_1 = \Big\{\big(B/B,(1\ \ 2\ \ 4).\hat{B}/\hat{B}\big)\Big\}$,
- $\big((1\ \ 2),1\big).C_2 = \Big\{\big(B/B,(1\ \ 4\ \ 3).\hat{B}/\hat{B}\big)\Big\}$.

Then we see that these $\hat{u}$ cannot be a $\hat{w}\hat{w}_0$, for a $\hat{w}$ coming from an additive matrix. Hence, thanks to Lemma 6.2.4, these two examples give two new faces – $\mathcal{F}(C_1)$ and $\mathcal{F}(C_2)$ – of the Kronecker cone $\mathrm{PKron}_{2,2}$ which contain only stable triples. The equations of the subspaces spanned by these faces in $\big\{(\alpha,\beta,\gamma)$ s.t. $|\alpha| = |\beta| = |\gamma|,\ \ell(\alpha) \leq 2,\ \ell(\beta) \leq 2,\ \ell(\gamma) \leq 4\big\}$ are easy to write:

- $\left\{\begin{array}{l}\alpha_1 = \gamma_1 + \gamma_4\\ \beta_1 = \gamma_1 + \gamma_2\end{array}\right.$ for $\mathcal{F}(C_1)$,
- $\left\{\begin{array}{l}\alpha_1 = \gamma_2 + \gamma_3\\ \beta_1 = \gamma_1 + \gamma_2\end{array}\right.$ for $\mathcal{F}(C_2)$.

We can in addition for instance give a minimal list of inequalities describing the face $\mathcal{F}(C_1)$ inside the previous vector space of dimension 4:

$$\left\{\begin{array}{l}\gamma_2 \geq \gamma_3 \geq \gamma_4\\ \gamma_1 - \gamma_2 \geq \gamma_3 - \gamma_4\end{array}\right. .$$

We can then notice that this face is a simplicial face. It is moreover interesting to note that the previous inequalities are not entirely those saying that γ is dominant. In the case of a face obtained by the theorem of Manivel and Vallejo, this minimal list of inequalities would in fact exactly be $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n_1 n_2}$.

6.3.5 Third case: length 2

We follow exactly the same reasoning as in the second case, and keep the same notations. We assume here that $\sharp\Phi(v^{-1}) = 2$, i.e. $v^{-1} = s_\alpha s_\beta$, for α and β distinct simple roots of G. If s_α and s_β commute, one then has $\Phi(v^{-1}) = \{-\alpha, -\beta\}$. If not, $\Phi(v^{-1}) = \{-\alpha, -\alpha-\beta\}$. According to the different possibilities there are for α and β, we then have seven different types for $\Phi(v^{-1})$:

1. $\Phi(v^{-1}) = \{\varepsilon_{i+1} - \varepsilon_i, \eta_{j+1} - \eta_j\}$, with $i \in [\![1, n_1 - 1]\!]$, $j \in [\![1, n_2 - 1]\!]$,

2. $\Phi(v^{-1}) = \{\varepsilon_{i+1} - \varepsilon_i, \varepsilon_{j+1} - \varepsilon_j\}$, with $i, j \in [\![1, n_1 - 1]\!]$ and $|i - j| \geq 2$,
3. $\Phi(v^{-1}) = \{\eta_{i+1} - \eta_i, \eta_{j+1} - \eta_j\}$, with $i, j \in [\![1, n_1 - 1]\!]$ and $|i - j| \geq 2$,
4. $\Phi(v^{-1}) = \{\varepsilon_{i+1} - \varepsilon_i, \varepsilon_{i+2} - \varepsilon_i\}$, with $i \in [\![1, n_1 - 2]\!]$,
5. $\Phi(v^{-1}) = \{\varepsilon_{i+2} - \varepsilon_{i+1}, \varepsilon_{i+2} - \varepsilon_i\}$, with $i \in [\![1, n_1 - 2]\!]$,
6. $\Phi(v^{-1}) = \{\eta_{i+1} - \eta_i, \eta_{i+2} - \eta_i\}$, with $i \in [\![1, n_2 - 2]\!]$,
7. $\Phi(v^{-1}) = \{\eta_{i+2} - \eta_{i+1}, \eta_{i+2} - \eta_i\}$, with $i \in [\![1, n_2 - 2]\!]$.

Then we must also have $((\hat{v}\hat{w})^\vee)^{-1} = s_{\hat{\alpha}} s_{\hat{\beta}}$, for $\hat{\alpha}$, $\hat{\beta}$ simple roots of $\hat{G}$. As before, this yields three kinds of $\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big)$:

(a) $\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big) = \{\hat{\varepsilon}_{k+1} - \hat{\varepsilon}_k, \hat{\varepsilon}_{k'+1} - \hat{\varepsilon}_{k'}\}$, with $k, k' \in [\![1, n_1 n_2 - 1]\!]$ and $|k - k'| \geq 2$,

(b) $\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big) = \{\hat{\varepsilon}_{k+1} - \hat{\varepsilon}_k, \hat{\varepsilon}_{k+2} - \hat{\varepsilon}_k\}$, with $k \in [\![1, n_1 n_2 - 2]\!]$,

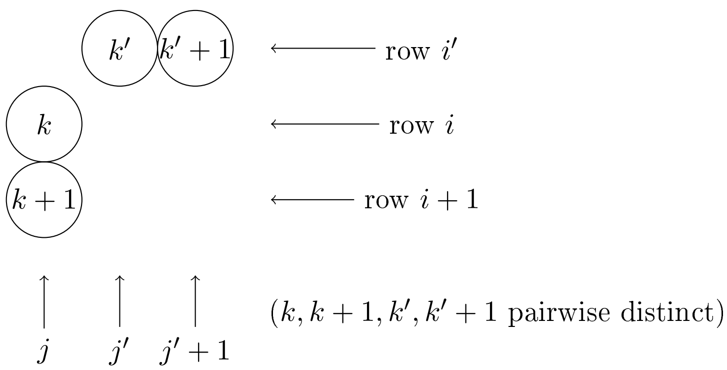
(c) $\hat{\Phi}\big(((\hat{v}\hat{w})^\vee)^{-1}\big) = \{\hat{\varepsilon}_{k+2} - \hat{\varepsilon}_{k+1}, \hat{\varepsilon}_{k+2} - \hat{\varepsilon}_k\}$, with $k \in [\![1, n_1 n_2 - 2]\!]$.

And finally some of the cases 1 to 7 are compatible with some of the cases (a) to (c), and will give – as in Paragraph 6.3.4 – configurations (concerning $\hat{w}$) providing v^{-1} and $\hat{v}^{-1}$ which verify the condition from Proposition 6.3.9. After removing those which are not possible for a $\hat{w}$ coming from a dominant, regular, $\hat{G}$-regular one-parameter subgroup (i.e. coming from an additive matrix), we obtain the following:

Configuration Ⓐ (corresponding to cases 1 and (a)):
There exist $k, k' \in [\![1, n_1 n_2 - 1]\!]$ such that $|k - k'| \geq 2$ and

$$\begin{cases} \hat{w}(k) = (i, j) \\ \hat{w}(k+1) = (i+1, j) \\ \hat{w}(k') = (i', j') \\ \hat{w}(k'+1) = (i', j'+1) \end{cases}.$$

It corresponds to the following situation in the order matrix:



This gives:

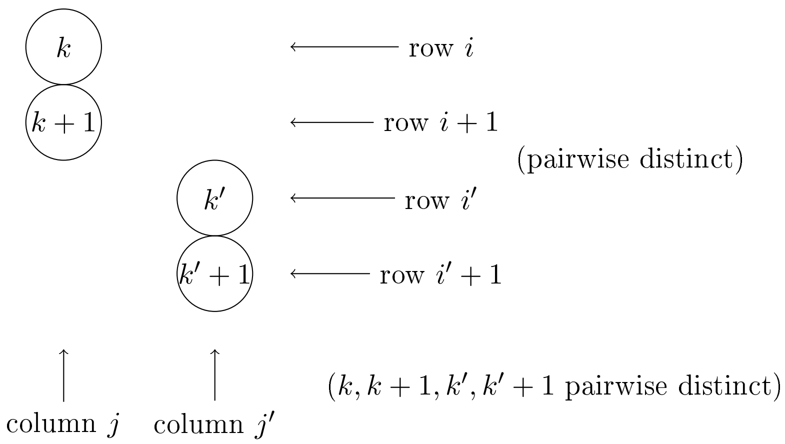
$$v^{-1} = \big((i\ \ i+1), (j'\ \ j'+1)\big) \quad \text{and} \quad \hat{v}^{-1} = \hat{w}(k\ \ k+1)(k'\ \ k'+1)\hat{w}_0$$

(with $|k-k'| \geq 2$).

Configuration Ⓑ (cases 2 and (a)):
There exist $k, k' \in [\![1, n_1 n_2 - 1]\!]$ such that $|k-k'| \geq 2$ and

$$\begin{cases} \hat{w}(k) = (i,j) \\ \hat{w}(k+1) = (i+1,j) \\ \hat{w}(k') = (i',j') \\ \hat{w}(k'+1) = (i'+1,j') \end{cases},$$

with $|i-i'| \geq 2$. It corresponds to the following situation in the order matrix:



This gives

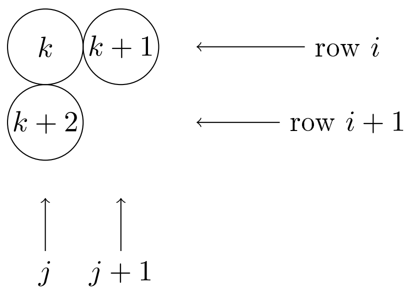
$$v^{-1} = \big((i\ \ i+1)(i'\ \ i'+1), 1\big) \quad \text{and} \quad \hat{v}^{-1} = \hat{w}(k\ \ k+1)(k'\ \ k'+1)\hat{w}_0$$

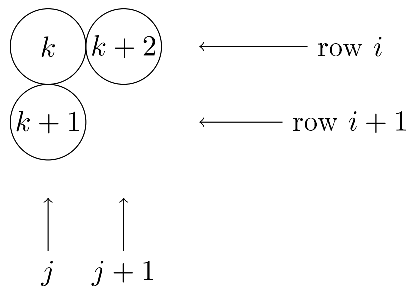
(with $|i-i'|, |k-k'| \geq 2$).

Configuration Ⓒ (cases 1 and (b)):
There exists $k \in [\![1, n_1 n_2 - 2]\!]$ such that

$$\begin{cases} \hat{w}(k) = (i,j) \\ \hat{w}(\{k+1, k+2\}) = \{(i+1,j), (i,j+1)\} \end{cases}.$$

It corresponds to two types of situation in the order matrix:





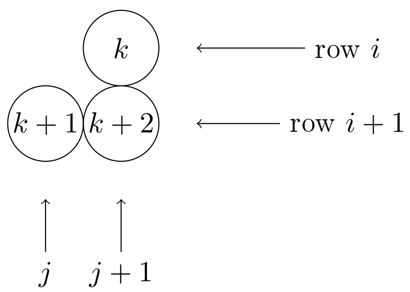
And this gives

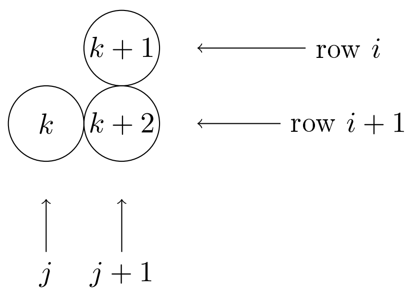
$$v^{-1} = \big((i\ \ i+1),(j\ \ j+1)\big) \quad \text{and} \quad \hat{v}^{-1} = \hat{w}\underbrace{(k\ \ k+1)(k+1\ \ k+2)}_{=(k\ \ k+1\ \ k+2)}\hat{w}_0.$$

Configuration Ⓓ (cases 1 and (c)):
There exists $k \in [\![1, n_1n_2-2]\!]$ such that

$$\begin{cases} \hat{w}(\{k,k+1\}) = \{(i,j+1),(i+1,j)\} \\ \hat{w}(k+2) = (i+1,j+1) \end{cases} .$$

It corresponds to two types of situation in the order matrix:





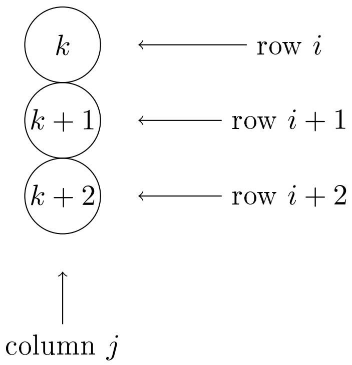
And this gives

$$v^{-1} = \big((i\ \ i+1),(j\ \ j+1)\big) \quad \text{and} \quad \hat{v}^{-1} = \hat{w}\underbrace{(k+1\ \ k+2)(k\ \ k+1)}_{=(k\ \ k+2\ \ k+1)}\hat{w}_0.$$

Configuration Ⓔ (cases 4 and (b) on the one hand, and 5 and (c) on the other):
There exists $k \in [\![1, n_1n_2-2]\!]$ such that

$$\begin{cases} \hat{w}(k) = (i,j) \\ \hat{w}(k+1) = (i+1,j) \\ \hat{w}(k+2) = (i+2,j) \end{cases} .$$

It corresponds to the following situation in the order matrix:



And this configuration gives two different pairs $(v^{-1}, \hat{v}^{-1})$:

$$v^{-1} = \big((i \;\; i+1 \;\; i+2), 1\big) \quad \text{and} \quad \hat{v}^{-1} = \hat{w}(k \;\; k+1 \;\; k+2)\hat{w}_0,$$

and

$$v^{-1} = \big((i \;\; i+2 \;\; i+1), 1\big) \quad \text{and} \quad \hat{v}^{-1} = \hat{w}(k \;\; k+2 \;\; k+1)\hat{w}_0.$$

Furthermore the Configurations Ⓑ and Ⓔ each have a "transposed configuration" in which the roles of the rows and columns are exchanged. Those two transposed configurations will be denoted respectively by ⓑ and ⓔ. They also give a pair $(v^{-1}, \hat{v}^{-1})$ (or two, in the case of Configuration ⓔ) in which the roles of V_1 and V_2 are exchanged. In other words, $\hat{v}^{-1}$ does not change and – for instance – $v^{-1} = \big((i \;\; i+1)(i' \;\; i'+1), 1\big)$ (for configuration Ⓑ) becomes $v^{-1} = \big(1, (j \;\; j+1)(j' \;\; j'+1)\big)$ (for configuration ⓑ).

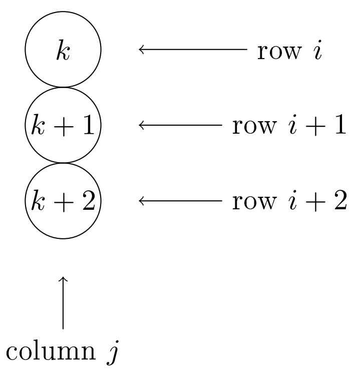
Theorem 6.3.18. *Let τ be a dominant, regular, $\hat{G}$-regular one-parameter subgroup of T. Let*

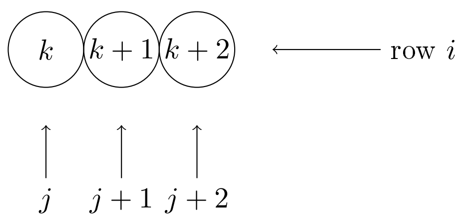
$$C = \big\{(v^{-1}B/B, \hat{v}^{-1}\hat{B}/\hat{B})\big\},$$

with v^{-1} and $\hat{v}^{-1}$ coming from one of the Configurations Ⓐ to Ⓔ or one of their transposed configurations. Then the pair (C, τ) is dominant and, as a consequence, gives a face – not necessarily regular and possibly reduced to zero – of the Kronecker cone PKron_{n_1,n_2} *which contains only almost stable triples:*

$$\mathcal{F}(C) = \big\{(\alpha, \beta, \gamma) \in \mathrm{PKron}_{n_1,n_2} \ s.t. \ \ (\hat{v}^{-1}\hat{w}_0).\gamma\big|_T = v^{-1}.(\alpha, \beta)\big\}.$$

Remark 6.3.19. As we wrote, the two configurations





(Ⓔ for the former and ⓔ for the latter) each give two – a priori different – dominant pairs . We will see later (cf Paragraph 6.4.2) that such configurations can indeed give two different faces of PKron_{n_1,n_2}.

Example: When one applies Theorem 6.3.18 to the same order matrix $\begin{pmatrix} ① & ③ \\ ② & ④ \end{pmatrix}$ as in Paragraph 6.3.4, one gets two new dominant pairs: a Configuration Ⓒ and a Configuration Ⓓ can be observed, which give respectively

$$C_3 = \Big\{ \Big(\big((1\ 2),(1\ 2)\big).B/B, \underbrace{\hat{w}(1\ 2\ 3)\hat{w}_0}_{=(1\ 4\ 3\ 2)}.\hat{B}/\hat{B} \Big) \Big\}$$

and

$$C_4 = \Big\{ \Big(\big((1\ 2),(1\ 2)\big).B/B, \underbrace{\hat{w}(2\ 4\ 3)\hat{w}_0}_{=(1\ 2\ 3\ 4)}.\hat{B}/\hat{B} \Big) \Big\}.$$

Once again we can normalise these C's:

- $\big((1\ 2),(1\ 2)\big).C_3 = \Big\{ \big(B/B, (2\ 4).\hat{B}/\hat{B}\big) \Big\}$,
- $\big((1\ 2),(1\ 2)\big).C_4 = \Big\{ \big(B/B, (1\ 3).\hat{B}/\hat{B}\big) \Big\}$.

The equations defining the subspaces spanned by the corresponding faces – possibly reduced to zero – $\mathcal{F}(C_3)$ and $\mathcal{F}(C_4)$ of the Kronecker cone $\mathrm{PKron}_{2,2}$ are respectively:

$$\begin{cases} \alpha_1 = \gamma_1 + \gamma_4 \\ \beta_1 = \gamma_2 + \gamma_4 \end{cases} \quad \text{and} \quad \begin{cases} \alpha_1 = \gamma_2 + \gamma_3 \\ \beta_1 = \gamma_2 + \gamma_4 \end{cases}.$$

One can then check that $\mathcal{F}(C_3)$ and $\mathcal{F}(C_4)$ are indeed not reduced to zero: $\big((5,5),(5,5),(3,3,2,2)\big) \in \mathcal{F}(C_3) \cap \mathcal{F}(C_4)$ (it is really a non-stable almost stable triple). But in fact, $\mathcal{F}(C_3)$ and $\mathcal{F}(C_4)$ are equal: the equations of the subspace that they span can be rewritten as

$$\begin{cases} \gamma_1 = \gamma_2 \\ \gamma_3 = \gamma_4 \\ \alpha_1 = \gamma_1 + \gamma_3 \\ \beta_1 = \gamma_1 + \gamma_3 \end{cases}.$$

So we have actually found only one face of $\mathrm{PKron}_{2,2}$, which is not regular and contains only almost stable triples.

6.4 Application to all cases of size 2×2, 3×2, and 3×3

6.4.1 All order matrices of size 2×2

For a dominant, regular, $\hat{G}$-regular one-parameter subgroup τ of T, there are only two possible order matrices (i.e. types of additive matrices) in this case:

$$\begin{pmatrix} ① & ② \\ ③ & ④ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} ① & ③ \\ ② & ④ \end{pmatrix},$$

corresponding – for instance – respectively to the one-parameter subgroups

$$\tau_1 : t \mapsto (\begin{pmatrix} t^2 & \\ & 1 \end{pmatrix}, \begin{pmatrix} t & \\ & 1 \end{pmatrix}) \qquad \text{and} \qquad \tau_2 : t \mapsto (\begin{pmatrix} t & \\ & 1 \end{pmatrix}, \begin{pmatrix} t^2 & \\ & 1 \end{pmatrix}),$$

which we will from now on denote by $\tau_1 = (2,0|1,0)$ and $\tau_2 = (1,0|2,0)$. Each one of these order matrices gives one face of $\mathrm{PKron}_{2,2}$, that we will call "additive", coming from the result of Manivel and Vallejo (i.e. Theorem 6.3.13). They are respectively given by

$$C_0^{(1)} = \left\{ \left(B/B, (1\ \ 4)(2\ \ 3)\hat{B}/\hat{B}\right) \right\}$$

and

$$C_0^{(2)} = \left\{ \left(B/B, (1\ \ 4)\hat{B}/\hat{B}\right) \right\}.$$

Then Theorems 6.3.17 and 6.3.18 enable us to find other faces (all those coming from the first order matrix will be denoted with an exponent (1) and all others with an exponent (2)). While giving them, we will at the same time "normalise" the singletons C's as we did in the previous examples: they will all have the form $\left\{(B/B, \hat{u}\hat{B}/\hat{B})\right\}$ with $\hat{u} \in \hat{W}$. (It allows to apply Lemma 6.2.4 when the pairs are well-covering).

Theorem 6.3.17 applied to the first possible order matrix gives two well-covering pairs, with

$$\begin{aligned} C_1^{(1)} &= \left\{ \left(B/B, (1\ \ 4\ \ 2\ \ 3)\hat{B}/\hat{B}\right) \right\}, \\ C_2^{(1)} &= \left\{ \left(B/B, (1\ \ 3\ \ 2\ \ 4)\hat{B}/\hat{B}\right) \right\}. \end{aligned}$$

Theorem 6.3.18 gives two dominant ones, with

$$\begin{aligned} C_3^{(1)} &= \left\{ \left(B/B, (2\ \ 4\ \ 3)\hat{B}/\hat{B}\right) \right\}, \\ C_4^{(1)} &= \left\{ \left(B/B, (1\ \ 2\ \ 3)\hat{B}/\hat{B}\right) \right\}. \end{aligned}$$

Let us do the same for the second possible order matrix (it is actually what we did in the examples of the previous section). With Theorem 6.3.17:

$$\begin{aligned} C_1^{(2)} &= \left\{ \left(B/B, (1\ \ 2\ \ 4)\hat{B}/\hat{B}\right) \right\}, \\ C_2^{(2)} &= \left\{ \left(B/B, (1\ \ 4\ \ 3)\hat{B}/\hat{B}\right) \right\}. \end{aligned}$$

And with Theorem 6.3.18:

$$\begin{aligned} C_3^{(2)} &= \left\{ \left(B/B, (2\ \ 4)\hat{B}/\hat{B}\right) \right\}, \\ C_4^{(2)} &= \left\{ \left(B/B, (1\ \ 3)\hat{B}/\hat{B}\right) \right\}. \end{aligned}$$

These examples being small, it is then not difficult to look in details at every possibly non-regular face (i.e. those coming from Theorem 6.3.18). What we find is that these four dominant pairs actually define all the same non-regular and non-zero face of $\mathrm{PKron}_{2,2}$.

As before, the subspace of $\{(\alpha,\beta,\gamma) \text{ s.t. } |\alpha|=|\beta|=|\gamma|,\ \ell(\alpha)\leq 2,\ \ell(\beta)\leq 2,\ \ell(\gamma)\leq 4\}$ spanned by this face has the following equations:

$$\begin{cases} \gamma_1 = \gamma_2 \\ \gamma_3 = \gamma_4 \\ \alpha_1 = \gamma_1 + \gamma_3 \\ \beta_1 = \gamma_1 + \gamma_3 \end{cases}.$$

In total, we have then found 4 new distinct (by Lemma 6.2.4) regular faces of $\mathrm{PKron}_{2,2}$ which contain only stable triples, whereas 2 others were already known. And we have also found 1 non-regular face containing only almost stable triples.

6.4.2 All order matrices of size 3×2

Let us do exactly as in the previous case. Five order matrices are possible here:

$$\begin{pmatrix} ① & ② \\ ③ & ④ \\ ⑤ & ⑥ \end{pmatrix}, \quad \begin{pmatrix} ① & ④ \\ ② & ⑤ \\ ③ & ⑥ \end{pmatrix}, \quad \begin{pmatrix} ① & ② \\ ③ & ⑤ \\ ④ & ⑥ \end{pmatrix}, \quad \begin{pmatrix} ① & ③ \\ ② & ④ \\ ⑤ & ⑥ \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} ① & ③ \\ ② & ⑤ \\ ④ & ⑥ \end{pmatrix}.$$

We number them in that order from 1 to 5 and will denote accordingly some possible corresponding one-parameter subgroups:

$$\tau_1 = (4,2,0|1,0),\ \tau_2 = (2,1,0|3,0),\ \tau_3 = (4,1,0|2,0),\ \tau_4 = (4,3,0|2,0),\ \tau_5 = (4,2,0|3,0).$$

We have then 5 additive faces with:

$$\begin{aligned} C_0^{(1)} &= \left\{ \big(B/B, (1\ \ 6)(2\ \ 5)(3\ \ 4)\hat{B}/\hat{B}\big) \right\}, \\ C_0^{(2)} &= \left\{ \big(B/B, (1\ \ 6)(2\ \ 4\ \ 5\ \ 3)\hat{B}/\hat{B}\big) \right\}, \\ C_0^{(3)} &= \left\{ \big(B/B, (1\ \ 6)(2\ \ 4\ \ 3\ \ 5)\hat{B}/\hat{B}\big) \right\}, \\ C_0^{(4)} &= \left\{ \big(B/B, (1\ \ 6)(2\ \ 5\ \ 3\ \ 4)\hat{B}/\hat{B}\big) \right\}, \\ C_0^{(5)} &= \left\{ \big(B/B, (1\ \ 6)(2\ \ 4)(3\ \ 5)\hat{B}/\hat{B}\big) \right\}. \end{aligned}$$

Theorem 6.3.17 furthermore gives 15 well-covering pairs: since we normalise them by writing C as $\{(B/B, \hat{u}\hat{B}/\hat{B})\}$, we give in the following table the list of elements $\hat{u}$ obtained, together with the name of the singleton C that they give.

name of C	element $\hat{u}$ giving C
$C_1^{(1)}$	(1 5 2 6)
$C_2^{(1)}$	(1 5)(2 6)(3 4)
$C_3^{(1)}$	(1 6 2 5)
$C_1^{(2)}$	(1 6)(3 4 5)
$C_2^{(2)}$	(1 6 3 2 4 5)
$C_3^{(2)}$	(1 4 5 3 2 6)
$C_4^{(2)}$	(1 6)(2 4 3)
$C_1^{(3)}$	(1 5 2 3 6)
$C_2^{(3)}$	(1 4 3 5 2 6)
$C_3^{(3)}$	(1 6)(2 4 5)
$C_1^{(4)}$	(1 6)(2 5 3)
$C_2^{(4)}$	(1 6 3 4 2 5)
$C_3^{(4)}$	(1 6 2 5 4)
$C_1^{(5)}$	(1 6)(3 5)
$C_2^{(5)}$	(1 6)(2 4)

(The numbers written in exponent between parentheses indicate from which order matrix the considered well-covering pairs come.) Theorem 6.3.18 gives also 20 dominant pairs, written in the same kind of table:

name of C	element $\hat{u}$ giving C
$C_4^{(1)}$	(1 5 2 6 3)
$C_5^{(1)}$	(1 3 4 5)(2 6)
$C_6^{(1)}$	(1 5)(2 6 4 3),
$C_7^{(1)}$	(1 4 6 2 5)
$C_5^{(2)}$	(1 4 5 3 6)
$C_6^{(2)}$	(1 2 6)(3 4 5)
$C_7^{(2)}$	(1 6 5)(2 4 3)
$C_8^{(2)}$	(1 6 3 2 4)
$C_4^{(3)}$	(1 5)(2 4 6)
$C_5^{(3)}$	(1 5 2)(3 6)
$C_6^{(3)}$	(1 4 6 2 3 5)
$C_7^{(3)}$	(1 3 6)(2 5)
$C_8^{(3)}$	(1 5 2 3 4 6)
$C_4^{(4)}$	(1 6 2 5 4 3)
$C_5^{(4)}$	(1 6 4)(2 5)
$C_6^{(4)}$	(1 5 3)(2 6)
$C_7^{(4)}$	(1 5)(2 6 4)
$C_8^{(4)}$	(1 4)(2 5 6)
$C_3^{(5)}$	(1 5 6 2 4)
$C_4^{(5)}$	(1 5 3 6 2)

If we check here one by one whether these dominant pairs are actually well-covering, we find that eight of them indeed are: those given by $C_5^{(2)}$, $C_6^{(2)}$, $C_7^{(2)}$, $C_8^{(2)}$, $C_7^{(3)}$, $C_8^{(3)}$, $C_4^{(4)}$, and $C_5^{(4)}$. Let us give two examples, for $C_5^{(2)}$ and $C_6^{(2)}$: the equations defining the subspaces of $\big\{(\alpha,\beta,\gamma)$ s.t. $|\alpha| = |\beta| = |\gamma|,\ \ell(\alpha) \leq 3,\ \ell(\beta) \leq 2,\ \ell(\gamma) \leq 6\big\}$ spanned respectively by $\mathcal{F}(C_5^{(2)})$ and $\mathcal{F}(C_6^{(2)})$ are respectively

$$\begin{cases} \alpha_1 = \gamma_1 + \gamma_5 \\ \alpha_2 = \gamma_2 + \gamma_6 \\ \beta_1 = \gamma_1 + \gamma_2 + \gamma_3 \end{cases} \quad \text{and} \quad \begin{cases} \alpha_1 = \gamma_1 + \gamma_6 \\ \alpha_2 = \gamma_2 + \gamma_4 \\ \beta_1 = \gamma_1 + \gamma_2 + \gamma_3 \end{cases}.$$

We can for instance notice that $\big((4,3,2),(8,1),(4,3,1,1)\big)$ belongs to $\mathcal{F}(C_5^{(2)})$, whereas $\big((4,3,1),(7,1),(4,2,1,1)\big)$ is in $\mathcal{F}(C_6^{(2)})$. Then, since these faces contain each some triple (α,β,γ) of partitions which is such that either α and β are regular (meaning that the parts of α are pairwise distinct, as are those of β), or γ is regular (likewise), a theorem from [Res10] (Theorem 12) ensures that these two dominant pairs are in fact well-covering. Moreover, Lemma 6.2.4 proves that the two (thus regular) faces are distinct. This is interesting because they come from the same Configuration Ⓔ, appearing in the first column of the order matrix number 2. Hence this kind of configuration can indeed give two different regular faces of PKron_{n_1,n_2} (cf Remark 6.3.19).

Lemma 6.2.4 furthermore ensures that all the regular faces corresponding to the 28 well-covering pairs that we presented are pairwise distinct. Looking in more details at the 12 other dominant pairs, which are not well-covering, we can see that they in fact give only two distinct non-regular faces of $\mathrm{PKron}_{3,2}$ containing almost stable triples: the equations of the subspaces that they respectively span are

$$\begin{cases} \gamma_1 = \gamma_2 \\ \gamma_3 = \gamma_4 \\ \gamma_5 = \gamma_6 \\ \alpha_1 = \alpha_2 = \gamma_1 + \gamma_3 \\ \beta_1 = \gamma_1 + \gamma_3 + \gamma_5 \end{cases} \quad \text{and} \quad \begin{cases} \gamma_1 = \gamma_2 \\ \gamma_3 = \gamma_4 \\ \gamma_5 = \gamma_6 \\ \alpha_1 = 2\gamma_1 \\ \alpha_2 = \gamma_3 + \gamma_5 \\ \beta_1 = \gamma_1 + \gamma_3 + \gamma_5 \end{cases}.$$

In total we then obtained 23 new regular faces of $\mathrm{PKron}_{3,2}$ which contain only stable triples, whereas 5 others were already known. We also got 2 other non-regular faces, containing only almost stable triples.

6.4.3 All order matrices of size 3×3

For this case the numbers begin to become much larger: there are 36 possible order matrices (i.e. 36 types of additive matrices) of size 3×3. As a consequence we do not write all of them here, but they can be found in Appendix A, along with the number of well-covering and dominant pairs that each provides. First Manivel and Vallejo's theorem yields 36 additive faces of $\mathrm{PKron}_{3,3}$ with this 36 additive matrices. But then if we look in details at these matrices, we find that Theorem 6.3.17 gives 144 well-covering pairs, i.e. 144 regular faces containing only stable triples. Moreover, Theorem 6.3.18 adds 232 dominant pairs to this, i.e. 232 faces of $\mathrm{PKron}_{3,3}$ – possibly non-regular and not necessarily pairwise distinct – containing only almost stable triples. Considering what happened in the two previous cases, we can hope that some of those dominant pairs are in fact well-covering. It would be of course possible to check whether this is true, but it is far too tedious to do it here.

Let us nevertheless give one detailed example of a new face of $\mathrm{PKron}_{3,3}$ that we can obtain with our results: look at the order matrix

$$\begin{pmatrix} ① & ② & ⑦ \\ ③ & ⑤ & ⑧ \\ ④ & ⑥ & ⑨ \end{pmatrix}$$

coming for instance from the dominant, regular, $\hat{G}$-regular one-parameter subgroup $\tau = (4,1,0|7,5,0)$ of T. Theorem 6.3.18 tells us that the pair with

$$C = \Big\{\big(B/B, (1\ \ 9\ \ 4\ \ 2\ \ 6)(5\ \ 8)\hat{B}/\hat{B}\big)\Big\}$$

is dominant (it comes from a Configuration Ⓔ in the third column of the matrix). And we can compute the equations of the subspace spanned by $\mathcal{F}(C)$ in $\big\{(\alpha,\beta,\gamma)$ s.t. $|\alpha| = |\beta| = |\gamma|,\ \ell(\alpha) \leq 3,\ \ell(\beta) \leq 3,\ \ell(\gamma) \leq 9\big\}$:

$$\begin{cases} \alpha_1 = \gamma_4 + \gamma_6 + \gamma_7 \\ \alpha_2 = \gamma_1 + \gamma_2 + \gamma_8 \\ \beta_1 = \gamma_1 + \gamma_3 + \gamma_4 \\ \beta_2 = \gamma_2 + \gamma_5 + \gamma_6 \end{cases}.$$

Then one can notice that $\big((6,5,4),(7,6,2),(3,2^6)\big) \in \mathcal{F}(C)$. As a consequence Theorem 12 from [Res10] assures that the pair (C,τ) is well-covering, and $\mathcal{F}(C)$ is indeed a regular face of $\mathrm{PKron}_{3,3}$ containing only stable triples.

6.5 A question on Configurations Ⓐ to Ⓔ

A natural question to ask after Theorem 6.3.18 is the following.

Question 1: For C obtained from a Configuration between Ⓐ and Ⓔ, is the face $\mathcal{F}(C)$ regular?

According to what we explained earlier right at the end of Section 6.2, it is equivalent to the following question.

Question 2: Are the pairs (C, τ) obtained from Configurations Ⓐ to Ⓔ well-covering (or simply covering, since by Lemma 6.3.12 it is equivalent in that context)?

In the previous examples, we noticed that the faces obtained from Configurations Ⓐ and Ⓔ were regular every time. On the contrary, those coming from Configurations Ⓒ and Ⓓ were never regular. Note that the examples that we considered were too small to observe a Configuration Ⓑ. But our guess would be that this configuration should behave similarly to Configuration Ⓐ.

Question 3: Is it true that a dominant pair (C, τ) coming from Theorem 6.3.18 is well-covering if and only if it comes from a Configuration Ⓐ, Ⓑ, or Ⓔ?

A first step to answer this question would be to consider the following one:

Question 4: Is the answer to Question 2 independent from (n_1, n_2)?

Chapter 7

About zeroes in the Kronecker cone

By a zero in the Kronecker cone PKron_{n_1,n_2} we mean a triple $(\alpha,\beta,\gamma) \in \mathrm{PKron}_{n_1,n_2}$ such that $g_{\alpha,\beta,\gamma} = 0$. The existence of such triples corresponds to the fact that the Kronecker coefficients do not have the saturation property, and the problem of understanding these zeroes is an important and difficult one. In this chapter we look at the half-line $\mathbb{N}^*(\alpha,\beta,\gamma)$ for such a zero (α,β,γ), and more precisely at the set $\Lambda(\alpha,\beta,\gamma) = \{d \in \mathbb{N}^* \text{ s.t. } g_{d\alpha,d\beta,d\gamma} \neq 0\}$. One can notice that in most examples this set is of the form $d_0\mathbb{N}^*$ (for a positive integer d_0), and we prove in the first section that, for almost stable triples, it is always the case. Nevertheless this result is not true if the triple is not almost stable: there is a family of counter-examples due to Briand-Orellana-Rosas in [BOR09]. Therefore we study in details these known counter-examples in order to try to replicate them, which we have not managed thus far.

7.1 Almost stable triples

It is of course obvious that a stable triple of partitions is almost stable, since we have explained that any stable triple (α,β,γ) verifies: for all positive integer d, $g_{d\alpha,d\beta,d\gamma} = 1$ (recall that the definition of almost stable is the same condition with ≤ 1). But one can easily see that the converse is not true: there exist almost stable triples which are not stable.

Example: For $\alpha = \beta = \gamma = (1,1)$ and all $d \in \mathbb{N}^*$,

$$g_{d\alpha,d\beta,d\gamma} = \begin{cases} 0 & \text{if } d \text{ is odd} \\ 1 & \text{if } d \text{ is even} \end{cases}.$$

So (α,β,γ) is almost stable, but not stable.

We can notice, in the previous example, that $\Lambda\big((1,1),(1,1),(1,1)\big) = 2\mathbb{N}^*$ is a semi-group. Then a reasonable question to ask would be: is $\Lambda(\alpha,\beta,\gamma)$ always of this form for almost stable triples? And for any triple?

Remark 7.1.1. The interpretation of the Kronecker coefficients as dimensions of spaces of sections of line bundles shows that the Kronecker semigroup $\{(\alpha,\beta,\gamma) \text{ s.t. } g_{\alpha,\beta,\gamma} \neq 0\}$ is indeed a semigroup, as we said before, but it has also a slightly stronger consequence: as soon as $g_{\alpha,\beta,\gamma} \neq 0$, $g_{\lambda+\alpha,\mu+\beta,\nu+\gamma} \geq g_{\lambda,\mu,\nu}$ for any triple (λ,μ,ν). Indeed, $g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X,\mathcal{L})^G$ and $g_{\lambda,\mu,\nu} = \dim \mathrm{H}^0(X,\mathcal{M})^G$ for some reductive group G acting on some projective variety X and some G-linearised line bundles $\mathcal{L}$ and $\mathcal{M}$ on X. Then, if $g_{\alpha,\beta,\gamma} \neq 0$, there exists a non-zero G-invariant section σ of $\mathcal{L}$. And thus, if $(\tau_1,\dots,\tau_r)$ is a basis of $\mathrm{H}^0(X,\mathcal{M})^G$, then $(\sigma\otimes\tau_1,\dots,\sigma\otimes\tau_r)$ is a linearly independent family in $\mathrm{H}^0(X,\mathcal{L}\otimes\mathcal{M})^G$. Hence $\dim \mathrm{H}^0(X,\mathcal{L}\otimes\mathcal{M})^G \geq \dim \mathrm{H}^0(X,\mathcal{M})^G$.

Remark 7.1.2. A straightforward consequence of the previous remark is that the sequence $(g_{d\alpha,d\beta,d\gamma})_{d\in\mathbb{N}^*}$ is non-decreasing when $g_{\alpha,\beta,\gamma} \neq 0$. Furthermore, if $d_0 \in \Lambda(\alpha,\beta,\gamma)$, then $d_0\mathbb{N}^* \subset \Lambda(\alpha,\beta,\gamma)$.

Theorem 7.1.3. *Let (α,β,γ) be an almost stable triple of partitions. Then there exists $d_0 \in \mathbb{N}^*$ such that, for all $d \in \mathbb{N}^*$,*

$$d \in \Lambda(\alpha,\beta,\gamma) \quad \Longleftrightarrow \quad d_0 | d$$

Proof. Let $d_0 = \min\{d \in \mathbb{N}^* \text{ s.t. } g_{d\alpha,d\beta,d\gamma} \neq 0\}$. The fact that, if d is a multiple of d_0, $g_{d\alpha,d\beta,d\gamma} \neq 0$ has already been explained in Remark 7.1.2. We now prove the converse: We can as before write $g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X,\mathcal{L})^G$, with a product of flag varieties X on which acts a reductive group G and a G-linearised line bundle $\mathcal{L}$ on X. Then,

$$\forall d \in \mathbb{N}^*,\ g_{d\alpha,d\beta,d\gamma} = \dim \mathrm{H}^0(X,\mathcal{L}^{\otimes d})^G$$

Moreover, X can be taken so that $\mathcal{L}$ is an ample line bundle. In that case, the line bundle $\mathcal{L}$ is even very ample (see e.g. [Bri04], Proposition 1.4.1). As a consequence,

$$S = \bigoplus_{d\geq 0} \mathrm{H}^0(X,\mathcal{L}^{\otimes d})$$

is integrally closed, by [Har77], Chapter II, Exercise 5.14(a). Take now $d \in \mathbb{N}^*$ which is no multiple of d_0. Then $d = d_0 q + r$ with $q \in \mathbb{N}$ and $0 < r < d_0$. By contradiction, let us assume that $g_{d\alpha,d\beta,d\gamma} \neq 0$, i.e. $\mathrm{H}^0(X,\mathcal{L}^{\otimes d})^G \neq \{0\}$. Consider $\sigma \in \mathrm{H}^0(X,\mathcal{L}^{\otimes d})^G \setminus \{0\}$. Independently, since $g_{d_0\alpha,d_0\beta,d_0\gamma} \neq 0$, we can also take $\sigma_0 \in \mathrm{H}^0(X,\mathcal{L}^{\otimes d_0})^G \setminus \{0\}$. Then there exists $c \in \mathbb{C}$ such that:

$$\left(\frac{\sigma}{\sigma_0^{\otimes q}}\right)^{d_0} - c\sigma_0^{\otimes r} = 0.$$

Indeed, $\sigma^{\otimes d_0}/(\sigma_0^{\otimes q d_0} \otimes \sigma_0^{\otimes r}) \in \mathbb{C}$, since $d_0 q + r = d$ and $\sigma^{\otimes d_0}$ and $\sigma_0^{\otimes d}$ are both in $\mathrm{H}^0(X,\mathcal{L}^{\otimes d d_0})^G$, which is of dimension 1. Thus $\sigma/\sigma_0^{\otimes q}$ is a root of $T^{d_0} - c\sigma_0^{\otimes r} \in S[T]$, for some $c \in \mathbb{C}$. And then

$$\frac{\sigma}{\sigma_0^{\otimes q}} \in S$$

since S is integrally closed. Hence $\sigma/\sigma_0^{\otimes q} \in \mathrm{H}^0(X,\mathcal{L}^{\otimes r})^G \setminus \{0\}$. This is a contradiction because $r < d_0$. □

Let us signal that the previous result can also be seen as a direct consequence of the works of P.-E. Paradan in [Par17]: it follows from the first statement in Theorem B.

Remark 7.1.4. This result does not hold for any triple of partitions: Theorem 2.4 in [BOR09] yields that $g_{(6,6),(7,5),(6,4,2)} = 0$ whereas, for all $d \geq 2$, $g_{d(6,6),d(7,5),d(6,4,2)} > 0$.

We will study this example in more details in the next section. Before this we take a moment to finally prove a fact that we have been using the whole time, because it has already been proven by J. Stembridge in [Ste14]. So forget only for now that stability and weak-stability are two equivalent notions.

Lemma 7.1.5. *If a triple (α, β, γ) of partitions is stable, then it is almost stable.*

Proof. Let us consider a triple (α, β, γ) which is not almost stable and write, exactly as in the proof of the previous theorem:

$$g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X, \mathcal{L})^G$$

with X a product of (partial) flag varieties, G a connected reductive group, and $\mathcal{L}$ an ample G-linearised line bundle on X. Consider the projection to the GIT-quotient: $X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L}) /\!\!/ G$. It is then known that some power of $\mathcal{L}$ descends to $X^{ss}(\mathcal{L}) /\!\!/ G$ (see e.g. [Tel00], §3). Up to replacing $\mathcal{L}$ by a power, we can consequently consider that $\mathcal{L}$ descends to an ample line bundle $\mathcal{M}$ on the GIT-quotient. Then (by [Tel00], Theorem 3.2(a)), for all $d \in \mathbb{N}^*$,

$$\mathrm{H}^0(X, \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(X^{ss}(\mathcal{L}) /\!\!/ G, \mathcal{M}^{\otimes d}).$$

Moreover, by Proposition 8.1 of [Dol03], since (α, β, γ) is not almost stable, $X^{ss}(\mathcal{L}) /\!\!/ G$ is not reduced to a point. So $\dim(\mathrm{H}^0(X^{ss}(\mathcal{L}) /\!\!/ G, \mathcal{M}^{\otimes d})) \xrightarrow[d\to\infty]{} \infty$. Thus,

$$\lim_{d\to\infty} g_{d\alpha,d\beta,d\gamma} = \infty,$$

and (α, β, γ) is not stable. □

Proposition 7.1.6. *If a triple of partitions (α, β, γ) is stable, then it is weakly stable.*

Proof. If (α, β, γ) is stable, then it is almost stable (see the previous proposition). Moreover, since $g_{\alpha,\beta,\gamma} \neq 0$ and using Remark 7.1.2,

$$\forall d \in \mathbb{N}^*,\ g_{d\alpha,d\beta,d\gamma} = 1.$$

□

7.2 Geometric study of the examples of Briand-Orellana-Rosas

The exact theorem in [BOR09] giving the whole family of counter-examples that we have mentioned is:

Theorem 7.2.1 (Briand-Orellana-Rosas). *Let i, j, and k be integers such that $i > j > 0$ and $k > 2i + j$. Set $\alpha = (k, k)$, $\beta = (k + 1, k - 1)$, and $\gamma = (2k - 2i - 2j, 2i, 2j)$. Then, for all $d \in \mathbb{N}^*$,*

$$g_{d\alpha,d\beta,d\gamma} = \begin{cases} \dfrac{d-1}{2} & \text{if } d \text{ is odd} \\ \dfrac{d}{2} + 1 & \text{if } d \text{ is even} \end{cases}.$$

In particular, $g_{\alpha,\beta,\gamma} = 0$ and, if $d \geq 2$, $g_{d\alpha,d\beta,d\gamma} > 0$.

We would like to understand geometrically these examples. Take then integers i, j, k, and partitions α, β, γ of $2k$ as in the theorem. We know that, if we set $G = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$, $X = \{\mathrm{pt}\} \times \mathbb{P}(\mathbb{C}^2) \times \mathcal{F}\ell\big((\mathbb{C}^2)^* \otimes \mathbb{C}^2\big)$, and $\mathcal{L} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta^* \otimes \mathcal{L}_\gamma$, then:

$$g_{\alpha,\beta,\gamma} = \dim \mathrm{H}^0(X, \mathcal{L})^G.$$

Lemma 7.2.2. *Let us denote by Q the quaternionic group, seen as a subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *(of cardinal 8). Then, for any positive integer d, there is a natural embedding*

$$\mathrm{H}^0(X, \mathcal{L}^{\otimes d})^G \hookrightarrow \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q.$$

Let us set some notations concerning the group of quaternions Q:

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and then $Q = \{\pm \mathrm{I}_2, \pm I, \pm J, \pm K\}$, with $IJ = K = -JI$, $JK = I = -KJ$, and $KI = J = -IK$.

Proof. Let d be a positive integer. We first set $Y = \mathbb{P}(\mathbb{C}^2) \times \mathcal{F}\ell\big(\mathcal{M}_2(\mathbb{C})\big)$ and $\mathcal{L}'$ the line bundle on Y which is $\mathcal{L}_{\beta^*} \otimes \mathcal{L}_\gamma$ (with $\beta^* = (-k+1, -k-1)$) on which the action of G is twisted by the character $\chi_k : (g_1, g_2) \mapsto (\det g_1)^{-k}$. Then

$$\mathrm{H}^0(X, \mathcal{L}^{\otimes d})^G \simeq \mathrm{H}^0(Y, (\mathcal{L}')^{\otimes d})^G.$$

The action of G on Y is:

$$\begin{aligned} & (g_1, g_2).\big(\mathbb{C}v, (\mathbb{C}M_1 \subset \mathbb{C}M_1 \oplus \mathbb{C}M_2 \subset \mathbb{C}M_1 \oplus \mathbb{C}M_2 \oplus \mathbb{C}M_3)\big) \\ = \; & \big(\mathbb{C}g_2.v, (\mathbb{C}g_2M_1g_1^{-1} \subset \mathbb{C}g_2M_1g_1^{-1} \oplus \mathbb{C}g_2M_2g_1^{-1} \subset \mathbb{C}g_2M_1g_1^{-1} \oplus \mathbb{C}g_2M_2g_1^{-1} \oplus \mathbb{C}g_2M_3g_1^{-1})\big), \end{aligned}$$

and one can see that $\mathbb{C}^* \mathrm{I}_2$ acts trivially. So it is actually an action of $G' = \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$. Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$x_0 = \big(\mathbb{C}\,\mathrm{I}_2 \subset \mathbb{C}\,\mathrm{I}_2 \oplus \mathbb{C}H \subset \mathbb{C}\,\mathrm{I}_2 \oplus \mathbb{C}H \oplus \mathbb{C}(E+F)\big) \in \mathcal{F}\!\ell\big(\mathcal{M}_2(\mathbb{C})\big).$$

Then we look at the isotropy subgroup G_{x_0} of x_0 in G: let $(g_1, g_2) \in G_{x_0}$. Therefore $g_2\,\mathrm{I}_2\,g_1^{-1} \in \mathbb{C}\,\mathrm{I}_2$, i.e. there exists $\lambda \in \mathbb{C}$ such that $g_1 = \lambda g_2$. Denote from now on g_2 by g. We have moreover

$$\lambda^{-1} g H g^{-1} \in \mathbb{C}\,\mathrm{I}_2 \oplus \mathbb{C}H$$

and then, since the action by conjugation preserves the trace, $gHg^{-1} \in \mathbb{C}H$. But this action preserves also the eigenvalues, and then $gHg^{-1} = \pm H$.

- First case: $gHg^{-1} = H$. Then g is diagonal and can then be written $g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ with $t \in \mathbb{C}^*$. Furthermore

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t^2 \\ t^{-2} & 0 \end{pmatrix}$$

  has to be symmetric. Hence $t^4 = 1$, i.e. $t \in \{\pm 1, \pm i\}$, i.e. $g \in \{\pm\,\mathrm{I}_2, \pm K\}$.

- Second case: $gHg^{-1} = -H$. Then $(gI)H(gI)^{-1} = H$, and thus $g = g_0.I^{-1}$ with g_0 as in the first case. This gives $g \in \{\pm I, \pm J\}$.

Finally we find that $G_{x_0} \subset \big\{(\lambda g, g)\,;\, \lambda \in \mathbb{C}^*, g \in Q\big\}$, and conversely it is not difficult to check that $G_{x_0} = \big\{(\lambda g, g)\,;\, \lambda \in \mathbb{C}^*, g \in Q\big\}$. In particular, the isotropy subgroup G'_{x_0} of x_0 in G' is finite and, since $\dim G' = 6 = \dim \mathcal{F}\!\ell\big(\mathcal{M}_2(\mathbb{C})\big)$, the ($G$- or G'-, since it is the same action) orbit of x_0 is open in $\mathcal{F}\!\ell\big(\mathcal{M}_2(\mathbb{C})\big)$. As a consequence,

$$\begin{aligned}
\mathrm{H}^0(Y, (\mathcal{L}')^{\otimes d})^G &\hookrightarrow \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}) \times G.x_0, (\mathcal{L}')^{\otimes d}\right)^G \\
&\simeq \mathrm{H}^0\left(G \times_{G_{x_0}} \left(\mathbb{P}^1(\mathbb{C}) \times \{x_0\}\right), (\mathcal{L}')^{\otimes d}\right)^G \\
&\simeq \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}) \times \{x_0\}, (\mathcal{L}')^{\otimes d}\right)^{G_{x_0}} \\
&\simeq \mathrm{H}^0(\mathbb{P}^1(\mathbb{C}), \mathcal{L}_{\beta^*}^{\otimes d})^Q \otimes \mathrm{H}^0(\{x_0\}, (\mathcal{L}'')^{\otimes d})^{G_{x_0}},
\end{aligned}$$

where $\mathcal{L}''$ is the line bundle $\mathcal{L}_\gamma$ on $\mathcal{F}\!\ell\big(\mathcal{M}_2(\mathbb{C})\big)$ on which the action of G_{x_0} is twisted by the character $\chi_k|_{G_{x_0}} : (\lambda g, g) \mapsto \lambda^{-2k}$. Then one can check without problem that G_{x_0} acts trivially on the fibre over x_0 in $(\mathcal{L}'')^{\otimes d}$, and therefore the dimension of the second factor of that last tensor product is 1. Finally,

$$\mathrm{H}^0(X, \mathcal{L}^{\otimes d})^G \hookrightarrow \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q.$$

□

Proposition 7.2.3. *For any positive integer d,*

$$g_{d\alpha,d\beta,d\gamma} = \dim \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q = \dim\left(\mathbb{C}[x,y]_{2d}\right)^Q,$$

where $\mathbb{C}[x,y]_{2d}$ denotes the vector space of homogeneous polynomials in two variables x and y, of degree $2d$, on which $Q \subset \mathrm{SL}_2(\mathbb{C})$ acts by its natural action on (x,y).

Proof. Let d be a positive integer. By the previous lemma,

$$g_{d\alpha,d\beta,d\gamma} \leq \dim \mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q.$$

But we are actually going to prove the equality by computing directly the dimension of $\mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right)^Q$ and comparing it to the result given by Briand-Orellana-Rosas about $g_{d\alpha,d\beta,d\gamma}$. Indeed, by standard algebraic geometry,

$$\mathrm{H}^0\left(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(2d)\right) \simeq \mathrm{Sym}^{2d}(\mathbb{C}^2)^* \simeq \left(\mathbb{C}[x,y]_{2d}\right).$$

Hence we obtain the second equality stated in the proposition. Now the dimension of the space of Q-invariants in this space of homogeneous polynomials can be computed. Since $Q = \langle I, J\rangle$, it is sufficient to look at the action of I and J:

- The element $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts by: if $p+q = 2d$, $x^p y^q \mapsto (-1)^p x^q y^p$. Therefore,

$$\begin{aligned} \sum_{p=0}^{2d} c_p x^p y^{2d-p} \in \left(\mathbb{C}[x,y]_{2d}\right)^{\langle I\rangle} \quad &\Longleftrightarrow \quad \sum_{p=0}^{2d} c_p x^p y^{2d-p} = \sum_{p=0}^{2d} (-1)^p c_p x^{2d-p} y^p \\ &\Longleftrightarrow \quad \forall p \in [\![0,2d]\!],\ c_p = \begin{cases} c_{2d-p} & \text{if } p \text{ is even} \\ -c_{2d-p} & \text{if } p \text{ is odd} \end{cases}. \end{aligned}$$

- The element $J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ acts by: if $p+q = 2d$, $x^p y^q \mapsto (-1)^d x^q y^p$. Therefore,

$$\begin{aligned} \sum_{p=0}^{2d} c_p x^p y^{2d-p} \in \left(\mathbb{C}[x,y]_{2d}\right)^{\langle J\rangle} \quad &\Longleftrightarrow \quad \sum_{p=0}^{2d} c_p x^p y^{2d-p} = (-1)^d \sum_{p=0}^{2d} c_p x^{2d-p} y^p \\ &\Longleftrightarrow \quad \forall p \in [\![0,2d]\!],\ c_p = \begin{cases} c_{2d-p} & \text{if } d \text{ is even} \\ -c_{2d-p} & \text{if } d \text{ is odd} \end{cases}. \end{aligned}$$

As a consequence,

- if d is even:

$$\sum_{p=0}^{2d} c_p x^p y^{2d-p} \in \left(\mathbb{C}[x,y]_{2d}\right)^Q \quad \Longleftrightarrow \quad \begin{cases} \forall p \in [\![0,2d]\!] \text{ even, } c_p = c_{2d-p} \\ \forall p \in [\![0,2d]\!] \text{ odd, } c_p = 0 \end{cases}$$

  and $\dim\left(\mathbb{C}[x,y]_{2d}\right)^Q = \dfrac{d}{2} + 1 = g_{d\alpha,d\beta,d\gamma}$ (by Theorem 7.2.1);

- if d is odd:

$$\sum_{p=0}^{2d} c_p x^p y^{2d-p} \in (\mathbb{C}[x,y]_{2d})^Q \iff \begin{cases} \forall p \in [\![0,2d]\!] \text{ even, } c_p = 0 \\ \forall p \in [\![0,2d]\!] \text{ odd, } c_p = -c_{2d-p} \end{cases}$$

and $\dim (\mathbb{C}[x,y]_{2d})^Q = \dfrac{d-1}{2} = g_{d\alpha,d\beta,d\gamma}$.

□

7.3 An attempt at constructing examples

Now that we managed to express the Kronecker coefficients of Theorem 7.2.1 quite simply (as dimensions of invariants in spaces of homogeneous polynomials under the action of a finite group), we would like to do the opposite reasoning and try to produce other such examples of family of triples like the one given in [BOR09]. So we would like to start from a finite group acting on spaces of homogeneous polynomials, which gives interesting dimensions when one looks at the spaces of invariants. But this first step already appears to be quite tricky, and for now we only found one such example of finite group. It concerns the following subgroup H of $\mathrm{SL}_3(C)$: set $\omega = e^{i\pi/3}$,

$$A = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & -\omega \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and

$$H = \langle A, B \rangle = \{A^k B^l \,;\, k \in [\![0,5]\!], l \in [\![0,3]\!]\} \simeq \mathbb{Z}/6\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z}$$

(A and B verify the relations $A^6 = 1$, $B^4 = \mathrm{I}_3$, and $BA = AB^{-1}$). Then, for all $d \in \mathbb{N}^*$, H acts linearly on the vector space $\mathbb{C}[x,y,z]_{3d}$ of homogeneous polynomials of degree $3d$ in three variables x, y, z, via its natural action on (x,y,z). Let $d \in \mathbb{N}^*$. We can compute the dimension of the space $(\mathbb{C}[x,y,z]_{3d})^H$ of invariants.

Proposition 7.3.1. *The dimension of the vector space* $(\mathbb{C}[x,y,z]_{3d})^H$ *is:*

$$f(d) = \begin{cases} \frac{9}{16}d^2 + \frac{3}{2}d + 1 & \text{if } d \equiv 0 \pmod 4 \\ \frac{9}{16}d^2 - \frac{3}{8}d - \frac{3}{16} & \text{if } d \equiv 1 \pmod 4 \\ \frac{9}{16}d^2 + \frac{3}{2}d + \frac{3}{4} & \text{if } d \equiv 2 \pmod 4 \\ \frac{9}{16}d^2 - \frac{3}{8}d + \frac{1}{16} & \text{if } d \equiv 3 \pmod 4 \end{cases}.$$

In particular, $f(1) = 0$ *and* $f(d) > 0$ *as soon as* $d \geq 2$.

Proof. Denote by $\mathcal{P}_{3d}$ the set of all triples of non-negative integers whose sum is $3d$. Then:

- The matrix A acts by: if $(p,q,r) \in \mathcal{P}_{3d}$, $x^p y^q z^r \mapsto (-1)^{d+r} x^p y^q z^r$. Therefore,

$$\sum_{(p,q,r)\in\mathcal{P}_{3d}} c_{p,q,r}x^py^qz^r \in (\mathbb{C}[x,y,z]_{3d})^{\langle A\rangle} \iff \forall(p,q,r)\in\mathcal{P}_{3d},\ c_{p,q,r} = (-1)^{d+r}c_{p,q,r}$$
$$\iff \forall(p,q,r)\in\mathcal{P}_{3d},\ \begin{cases} r \text{ odd} \Rightarrow c_{p,q,r}=0 & \text{if } d \text{ is even} \\ r \text{ even} \Rightarrow c_{p,q,r}=0 & \text{if } d \text{ is odd} \end{cases}.$$

- The matrix B acts by: if $(p,q,r)\in\mathcal{P}_{3d}$, $x^py^qz^r \mapsto (-1)^q x^py^rz^q$. Therefore,

$$\sum_{(p,q,r)\in\mathcal{P}_{3d}} c_{p,q,r}x^py^qz^r \in (\mathbb{C}[x,y,z]_{3d})^{\langle B\rangle} \iff \forall(p,q,r)\in\mathcal{P}_{3d},\ c_{p,q,r} = (-1)^r c_{p,r,q}.$$

As a consequence,

- if d is even:

$$\sum_{(p,q,r)\in\mathcal{P}_{3d}} c_{p,q,r}x^py^qz^r \in (\mathbb{C}[x,y,z]_{3d})^{H}$$
$$\iff \forall(p,q,r)\in\mathcal{P}_{3d},\ \begin{cases} q \text{ or } r \text{ odd} \Rightarrow c_{p,q,r}=0 \\ q \text{ and } r \text{ even} \Rightarrow c_{p,q,r}=c_{p,r,q} \end{cases}$$

  and thus

$$\begin{aligned}\dim(\mathbb{C}[x,y,z]_{3d})^H &= \sharp\big\{(p,q,r)\in\mathcal{P}_{3d} \text{ s.t. } q\le r \text{ and } q,r \text{ are even}\big\} \\ &= \frac{9}{16}d^2+\frac{3}{2}d+\begin{cases} 1 & \text{if } d\equiv 0 \pmod 4 \\ \frac{3}{4} & \text{if } d\equiv 2 \pmod 4\end{cases}\end{aligned}$$

  (after computation).

- if d is odd:

$$\sum_{(p,q,r)\in\mathcal{P}_{3d}} c_{p,q,r}x^py^qz^r \in (\mathbb{C}[x,y,z]_{3d})^{H}$$
$$\iff \forall(p,q,r)\in\mathcal{P}_{3d},\ \begin{cases} q \text{ or } r \text{ even} \Rightarrow c_{p,q,r}=0 \\ q \text{ and } r \text{ odd} \Rightarrow c_{p,q,r}=-c_{p,r,q} \end{cases}$$

  and thus

$$\begin{aligned}\dim(\mathbb{C}[x,y,z]_{3d})^H &= \sharp\big\{(p,q,r)\in\mathcal{P}_{3d} \text{ s.t. } q< r \text{ and } q,r \text{ are odd}\big\} \\ &= \frac{9}{16}d^2-\frac{3}{8}d\begin{cases} -\frac{3}{16} & \text{if } d\equiv 1 \pmod 4 \\ +\frac{1}{16} & \text{if } d\equiv 3 \pmod 4\end{cases}.\end{aligned}$$

□

The simplest thing after this would be to identify H as more or less the isotropy subgroup in $\mathrm{GL}_3(\mathbb{C})$ of a sequence of linear spaces of the right dimension, in order to do the reasoning of the proof of Lemma 7.2.2 in the opposite direction. Unfortunately as of now we were not able to do this. But we could maybe in the future find another way of using this action of H on $\mathbb{C}[x,y,z]_{3d}$ to create a new example like Theorem 7.2.1.

Appendix A

List of all possible order matrices of size 3×3

There are 36 possible order matrices (i.e. 36 types of additive matrices) of this size:

$$\textcircled{1}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{3}\\\textcircled{4}&\textcircled{5}&\textcircled{6}\\\textcircled{7}&\textcircled{8}&\textcircled{9}\end{pmatrix}\qquad\textcircled{2}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{3}\\\textcircled{4}&\textcircled{5}&\textcircled{7}\\\textcircled{6}&\textcircled{8}&\textcircled{9}\end{pmatrix}\qquad\textcircled{3}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{3}\\\textcircled{4}&\textcircled{5}&\textcircled{8}\\\textcircled{6}&\textcircled{7}&\textcircled{9}\end{pmatrix}\qquad\textcircled{4}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{3}\\\textcircled{4}&\textcircled{6}&\textcircled{7}\\\textcircled{5}&\textcircled{8}&\textcircled{9}\end{pmatrix}$$

$$\textcircled{5}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{3}\\\textcircled{4}&\textcircled{6}&\textcircled{8}\\\textcircled{5}&\textcircled{7}&\textcircled{9}\end{pmatrix}\qquad\textcircled{6}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{4}\\\textcircled{3}&\textcircled{5}&\textcircled{6}\\\textcircled{7}&\textcircled{8}&\textcircled{9}\end{pmatrix}\qquad\textcircled{7}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{4}\\\textcircled{3}&\textcircled{5}&\textcircled{7}\\\textcircled{6}&\textcircled{8}&\textcircled{9}\end{pmatrix}\qquad\textcircled{8}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{4}\\\textcircled{3}&\textcircled{5}&\textcircled{8}\\\textcircled{6}&\textcircled{7}&\textcircled{9}\end{pmatrix}$$

$$\textcircled{9}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{4}\\\textcircled{3}&\textcircled{6}&\textcircled{7}\\\textcircled{5}&\textcircled{8}&\textcircled{9}\end{pmatrix}\qquad\textcircled{10}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{4}\\\textcircled{3}&\textcircled{6}&\textcircled{8}\\\textcircled{5}&\textcircled{7}&\textcircled{9}\end{pmatrix}\qquad\textcircled{11}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{5}\\\textcircled{3}&\textcircled{4}&\textcircled{6}\\\textcircled{7}&\textcircled{8}&\textcircled{9}\end{pmatrix}\qquad\textcircled{12}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{5}\\\textcircled{3}&\textcircled{4}&\textcircled{7}\\\textcircled{6}&\textcircled{8}&\textcircled{9}\end{pmatrix}$$

$$\textcircled{13}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{5}\\\textcircled{3}&\textcircled{4}&\textcircled{8}\\\textcircled{6}&\textcircled{7}&\textcircled{9}\end{pmatrix}\qquad\textcircled{14}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{5}\\\textcircled{3}&\textcircled{6}&\textcircled{8}\\\textcircled{4}&\textcircled{7}&\textcircled{9}\end{pmatrix}\qquad\textcircled{15}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{6}\\\textcircled{3}&\textcircled{4}&\textcircled{8}\\\textcircled{5}&\textcircled{7}&\textcircled{9}\end{pmatrix}\qquad\textcircled{16}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{6}\\\textcircled{3}&\textcircled{5}&\textcircled{8}\\\textcircled{4}&\textcircled{7}&\textcircled{9}\end{pmatrix}$$

$$\textcircled{17}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{7}\\\textcircled{3}&\textcircled{4}&\textcircled{8}\\\textcircled{5}&\textcircled{6}&\textcircled{9}\end{pmatrix}\qquad\textcircled{18}\begin{pmatrix}\textcircled{1}&\textcircled{2}&\textcircled{7}\\\textcircled{3}&\textcircled{5}&\textcircled{8}\\\textcircled{4}&\textcircled{6}&\textcircled{9}\end{pmatrix}$$

and exactly all the transposed matrices of these ones (that we number in the same order: the transposed matrix of Matrix ⓚ has number 18+k). Respectively associated dominant, regular, $\hat{G}$-regular one-parameter subgroups are for instance (using once again the notation explained in Section 4.3.2 for such subgroups):

$\tau_1 = (6,3,0|2,1,0)$ $\tau_2 = (10,4,0|5,2,0)$ $\tau_3 = (8,2,0|4,3,0)$ $\tau_4 = (10,3,0|6,2,0)$

$\tau_5 = (7,1,0|4,2,0)$ $\tau_6 = (8,5,0|4,2,0)$ $\tau_7 = (6,3,0|4,2,0)$ $\tau_8 = (12,5,0|10,6,0)$

$\tau_9 = (12,5,0|10,4,0)$ $\tau_{10} = (9,2,0|8,4,0)$ $\tau_{11} = (7,5,0|4,3,0)$ $\tau_{12} = (12,8,0|9,6,0)$

$\tau_{13} = (10,6,0|8,7,0)$ $\tau_{14} = (8,2,0|9,4,0)$ $\tau_{15} = (8,4,0|9,6,0)$ $\tau_{16} = (8,2,0|10,7,0)$

$\tau_{17} = (4,2,0|6,5,0)$ $\tau_{18} = (4,1,0|7,5,0)$

(for the transposed matrices, one simply has to exchange the roles of V_1 and V_2).

Then each one of these matrices gives exactly one "additive face" of PKron_{n_1,n_2} by the result of Manivel and Vallejo, i.e. one well-covering pair. Moreover, by Theorem 6.3.17, they also give other such pairs. Here are the numbers of new well-covering pairs that each one gives:

① 6 ② 4 ③ 5 ④ 5 ⑤ 5 ⑥ 4 ⑦ 2 ⑧ 3 ⑨ 3

⑩ 3 ⑪ 5 ⑫ 3 ⑬ 4 ⑭ 4 ⑮ 3 ⑯ 3 ⑰ 5 ⑱ 5

Each transposed matrix gives furthermore by Theorem 6.3.17 the same number of well-covering pairs as the original one. As a consequence that makes in total 144 new well-covering pairs.

In addition to Theorem 6.3.17, Theorem 6.3.18 provides from these 36 order matrices a certain number of dominant pairs. Among them some are probably well-covering while others do not in fact define a new face of $\mathrm{PKron}_{3,3}$. Here are the numbers of dominant pairs that each order matrix gives (for the transposed matrices, it will be the same):

① 6 ② 4 ③ 9 ④ 9 ⑤ 12 ⑥ 4 ⑦ 2 ⑧ 5 ⑨ 3

⑩ 6 ⑪ 9 ⑫ 3 ⑬ 6 ⑭ 6 ⑮ 6 ⑯ 5 ⑰ 12 ⑱ 9

(232 dominant pairs in total.)

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