AN EQUIVARIANT INDEX THEOREM FOR HYPOELLIPTIC OPERATORS

DENIS PERROT, RUDY RODSPHON

March 2, 2015

ABSTRACT. Let M be a foliated manifold and G a discrete group acting on M by diffeomorphisms mapping leaves to leaves. Then G naturally acts by automorphisms on the algebra of Heisenberg pseudodifferential operators on the foliation. Our main result is an index theorem for hypoelliptic-type operators which belong to the crossed product of the Heisenberg pseudodifferential operators with the group G. As a corollary we get an explicit formula, in terms of characteristic classes of equivariant vector bundles over M, for the Chern-Connes character associated to the hypoelliptic signature operator constructed by Connes and Moscovici.

KEYWORDS. Cyclic cohomology, K-theory, Index theory, Pseudodifferential operators

MSC. 19D55, 19K56, 58J42, 46L87

1. INTRODUCTION

This paper deals with the index theory of certain equivariant hypoelliptic operators on foliated manifolds. Our motivation partly comes from the Connes-Moscovici *transverse* index theorem in [3]. We let (M, V) be a (possibly non-compact) foliated manifold. To such a foliation is canonically associated the algebra $\Psi_{H,c}(M)$ of (compactly-supported) *Heisenberg* pseudodifferential operators. The latter is a modification of the classical algebra of pseudodifferential operators, in which the vector fields tangent to the leaves of the foliation are of order 1, while the transverse vector fields are of order $\leqslant 2$. Let $G \subset \mathrm{Diff}(M)$ denote a discrete group of diffeomorphisms on M mapping leaves to leaves. Then G naturally acts on the algebra $\Psi_{H,c}(M)$ by automorphisms, and the crossed-product $\Psi_{H,c}(M) \rtimes G$ is defined. We write any element $P \in \Psi_{H,c}(M) \rtimes G$ as a finite sum

$$P = \sum_{g \in G} P_g \otimes U_g \ .$$

Remark that P is naturally represented as a linear operator $\sum P_g \circ U_g : C_c^\infty(M) \longrightarrow C_c^\infty(M)$, where U_g acts as the shift operator $U_g(f)(x) = f(x \cdot g)$, for every $f \in C_c^\infty(M)$ and $x \in M$. Thus P is far from being pseudodifferential in general, and belongs to the larger class of Fourier integral operators. At least at the algebraic level (see however the work of Savin and Sternin [13]), the index theory of such operators amounts to describe the K-theory/cyclic homology excision maps associated to the short exact sequence of algebras

$$0 \longrightarrow \Psi_c^{-\infty}(M) \rtimes G \longrightarrow \Psi_{H,c}^0(M) \rtimes G \longrightarrow S_{H,c}^0(M) \rtimes G \longrightarrow 0,$$

where $\Psi_c^{-\infty}(M) \subset \Psi_{H,c}^0(M)$ is the two-sided ideal of smoothing operators in the algebra of order $\leqslant 0$ Heisenberg pseudodifferential operators, and $S_{H,c}^0(M) = \Psi_{H,c}^0(M)/\Psi_c^{-\infty}(M)$ is the quotient algebra of formal Heisenberg symbols. Let $\mathrm{Tr}_{[1]}$ be the trace on $\Psi_c^{-\infty}(M) \rtimes G$ obtained from the usual trace on $\Psi_c^{-\infty}(M)$ by localization at the unit of G :

$$\mathrm{Tr}_{[1]}\left(\sum_{g \in G} P_g U_g\right) = \mathrm{Tr}(P_1)$$

We will mainly be interested in the image of $\mathrm{Tr}_{[1]}$ under the excision map in periodic cyclic cohomology $\partial : \mathrm{HP}^0(\Psi_c^{-\infty}(M) \rtimes G) \to \mathrm{HP}^1(\mathcal{S}_{H,c}(M) \rtimes G)$.

THEOREM 1.1. *The boundary of the localized operator trace* $\partial[\mathrm{Tr}_{[1]}] \in \mathrm{HP}^1(\mathcal{S}_{H,c}(M) \rtimes G)$ *is represented by the* equivariant Radul cocycle

$$\phi(a_0, a_1) = \int\!\!\!\!-\, \left(a_0[\log \Delta_H^{1/4}, a_1]\right)_{[1]}$$

where $\Delta_H^{1/4}$ *is the sub-elliptic sub-laplacian (Example 2.2) associated to* M*, the subscript* ${}_{[1]}$ *denotes the term localized at the unit, and the integral denotes the Connes-Moscovici residue over the algebra of formal symbols* $S_H^0(M)$.

We refer to section 2 for a description of the Connes-Moscovici residue. The cocycle ϕ may be viewed as a *local* formula in the sense that it only involves the formal Heisenberg symbol of pseudodifferential operators, and is given in terms of an integral over the *Heisenberg cosphere bundle* S_H^*M over M. A variant of this cocycle may also be obtained by taking the boundary of the localized operator trace under the extension

$$0 \longrightarrow \Psi_{H,c}^{-1}(M) \rtimes G \longrightarrow \Psi_{H,c}^{0}(M) \rtimes G \longrightarrow C_c^\infty(S_H^*M) \rtimes G \longrightarrow 0\ ,$$

leading to a cocycle $\partial[\tau] \in \mathrm{HP}^1(C_c^\infty(S_H^*M) \rtimes G)$ over the algebra of Heisenberg principal symbols, which may easily be related to ϕ. The main result of this paper (Theorem 7.3) is a geometric realization of this cocycle :

THEOREM 1.2. *Let* M *be a foliated manifold and* G *be a discrete group of foliated diffeomorphisms. Let* EG *be the universal bundle over the classifying space* BG *de* G*. Let*

$$0 \longrightarrow \Psi_{H,c}^{-1}(M) \rtimes G \longrightarrow \Psi_{H,c}^{0}(M) \rtimes G \longrightarrow C_c^\infty(S_H^*M) \rtimes G \longrightarrow 0$$

be the equivariant Heisenberg pseudodifferential extension. Then, the image of the localized trace at the unit $\partial[\tau] \in \mathrm{HP}^1(C_c^\infty(S_H^*M) \rtimes G)$ *by excision is given by*

$$\partial([\tau]) = \Phi(\mathrm{Td}(TM \otimes \mathbb{C}))$$

where $\Phi : H^{\mathrm{ev}}(EG \times_G S_H^*M) \to \mathrm{HP}^1(C_c^\infty(S_H^*M) \rtimes G)$ *is Connes' characteristic map from equivariant cohomology to cyclic cohomology, and* $\mathrm{Td}(TM \otimes \mathbb{C})$ *is the equivariant Todd class of the complexified tangent bundle of* M.

We stress that this result holds for *any* group of foliated diffeomorphisms G. In particluar, we do not assume that G preserves a metric or a conformal structure on M. To prove this, the idea is to give an explicit homotopy between the Radul cocycle and the one above. In the framework of cyclic cohomology, a recipe to obtain transgression cochains between two representatives of a given cohomology class is to use a *JLO formula* [7]. A first observation is that our cocycles are defined on algebras of formal Heisenberg pseudodifferential symbols. Thus, we have to adapt the usual ingredients of the JLO cocycle to this context. We find an answer by adapting the formalism developed in [9] and [10] to the Heisenberg calculus. The constructions are not easy but really flexible because the use of residues has the consequence that these are purely algebraic. Besides, this applies to operators which are not of Dirac type, and more generally to cases where Getzler rescaling does not apply. Though, it should be noted that this Dirac operator is only an intermediary, contrary to the classical JLO situation where it is the object of study. Another crucial feature of the formalism in [9, 10] is to be *invariant under diffeomorphisms.*

These steps being accomplished, we compute the JLO cocycle for two different Dirac operators, the first one gives the Radul cocycle of the pseudodifferential extension, the second one gives the

Poincaré dual of the equivariant Todd class. The usual homotopy arguments from the original JLO formula apply verbatim there, thus giving the theorem.

We then apply Theorem 1.2 to the Connes-Moscovici index problem for *transversally* hypoelliptic operators on foliations. After reduction to a complete transversal W, the holonomy groupoid of a given foliation is Morita equivalent to an étale groupoid $W \rtimes G$ where $G \subset \mathrm{Diff}(W)$ is a discrete (pseudo)group of diffeomorphisms. The main object of study is therefore the crossed product algebra $C_c^\infty(W) \rtimes G$ where, for notational simplicity, we treat G as a group. A first problem occuring is that G does not preserve any measure on W in general, and even less a Riemannian metric, so that G-invariant elliptic differential operators do not exist even at the leading symbol level. The idea of Connes in [1] is to pass by a Thom isomorphism to the bundle of Riemannian metrics M over W. This fibration is in particular a foliation, the leaves being the fibers. *This will be the foliation of interest for us.* The action of G on W lifts to M, mapping leaves to leaves. Connes and Moscovici then construct a hypoelliptic signature operator on M almost invariant under the G-action, in the sense that its Heisenberg leading symbol is G-invariant. This yields a regular spectral triple over the algebra $C_c^\infty(M) \rtimes G$, whose Chern-Connes character may be computed by means of the Connes-Moscovici residue formula ([3]). However, this does not directly provide a characteristic class formula, since the actual calculations give thousands of terms already for very low-dimensional manifolds W. To overcome this difficulty in higher dimensions, Connes and Moscovici developed cyclic cohomology for Hopf algebras in [4]. They defined such an algebra $\mathcal{H}$, which acts like a symmetry group allowing to reorganize the calculations, and built a characteristic map $\chi : HP^\bullet_{\mathrm{Hopf}}(\mathcal{H}) \longmapsto HP^\bullet(C_c^\infty(M) \rtimes G)$. Then they prove

THEOREM 1.3. (Connes-Moscovici, [4]) *If the lifted action of* G *on* M *has no fixed points, then the Chern-Connes character of the hypoelliptic signature operator lies in the range of the characteristic map* χ.

In some sense, the group $HP^\bullet_{\mathrm{Hopf}}(\mathcal{H})$ contains the geometric cocycles : Connes and Moscovici showed that it is isomorphic to the Gel'fand-Fuchs cohomology, which contains e.g characteristic classes of equivariant vector bundles over M. It then remains to actually compute the preimage of the Chern-Connes character. Explicit calculations are made in [4] in dimension 1, giving (twice) the transverse fundamental class of [1]. In dimension 2, the authors show that the coefficient of the first Pontryagin class does not vanish.

Our Theorem 1.2 allows to short-cut the calculation with Hopf algebras and gives direct answer to the problem of computing the Chern-Connes character of the hypoelliptic signature operator in terms of equivariant characteristic classes, for manifolds W of arbitrary dimension.

THEOREM 1.4. *Let* G *be a discrete group of orientation-preserving diffeomorphisms on a manifold* W. *Let* M *be the bundle of Riemannian metrics over* W. *If the lifted action of* G *has no fixed points on* M, *then the Chern-Connes character of the Connes-Moscovici hypoelliptic signature operator is*

$$\pi_* \circ \Phi(L'(M)) \ \in HP^1(C_c^\infty(M) \rtimes G)\ ,$$

where $L'(M)$ *is the modified* L*-genus,* $\Phi : H^{\mathrm{ev}}(EG \times_G S^*_H M) \to HP^1(C_c^\infty(S^*_H M) \rtimes G)$ *is Connes' characteristic map, and* $\pi_* : HP^1(C_c^\infty(S^*_H M) \rtimes G) \to HP^1(C_c^\infty(M) \rtimes G)$ *is induced by the canonical projection* $\pi : S^*_H M \to M$.

Organization of the paper. Section 2 is a self-contained introduction to the equivariant Heisenberg pseudodifferential calculus on foliated manifolds, the pseudodifferential extension, and the Connes-Moscovici residue. Then Theorem 1.1 is proved.

In sections 3, 4 and 5 we adapt the formalism of [9] to the Heisenberg calculus. We introduce various spaces of operators acting on formal Heisenberg symbols, and the universal Dirac operators which will be used in our algebraic JLO formula.

Section 6 introduces the required objects to carry this formalism to the equivariant setting. In particular, we recall the point of view we need to construct Connes' characteristic map from the equivariant cohomology $H^\bullet(EG \times_G M)$ to the periodic cyclic cohomology of the crossed product $HP^\bullet(C^\infty(M) \rtimes G)$. From the technical side we use the X-complex of Cuntz and Quillen [5].

Section 7 finally gives the algebraic JLO formula on the algebra of formal (equivariant) Heisenberg symbols, leading to Theorem 1.2. This is again an adaptation of the formalism developed in [10] for the non-Heisenberg case.

Section 8 shows how to deduce Theorem 1.4 from Theorem 1.2.

2. Equivariant local index formula

2.1. **Heisenberg pseudodifferential calculus on foliations.** Let M be a foliated manifold of dimension n, and let $\mathcal{F}$ be the integrable sub-bundle of the tangent bundle TM of M which defines the foliation. Denote ν the dimension of the leaves and $h = n - \nu$ their codimension.

The fundamental idea of the Heisenberg calculus is that *longitudinal* vector fields (with respect to to the foliation) have *order 1*, whereas *transverse* vector fields have *order* $\leqslant$ *2*. We shall now describe the symbolic calculus allowing to do so, following Connes and Moscovici [3].

Let $(x_1, \ldots, x_n)$ be a foliated local coordinate system of M, i.e, the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\nu}$ locally span $\mathcal{F}$, so that $\frac{\partial}{\partial x_{\nu+1}}, \ldots, \frac{\partial}{\partial x_n}$ are transverse to the leaves of the foliation. Then, we set

$$|p|' = (p_1^4 + \ldots + p_\nu^4 + p_{\nu+1}^2 + \ldots + p_n^2)^{1/4}$$
$$\langle\alpha\rangle = \alpha_1 + \ldots + \alpha_\nu + 2\alpha_{\nu+1} + \ldots 2\alpha_n$$

for every $p \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$.

Definition 2.1. A smooth function $\sigma(x,p) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_p)$ is a *Heisenberg symbol of order* $m \in \mathbb{R}$ if over any compact subset $K \subset \mathbb{R}^n_x$ and for every multi-index α, β, one has the following estimate

$$|\partial_x^\beta \partial_p^\alpha \sigma(x,p)| \leqslant C_{K,\alpha,\beta}(1 + |p|')^{m - \langle\alpha\rangle}$$

We shall focus on the smaller class of *classical Heisenberg symbols.* For this, we first define the *Heisenberg dilations*

$$\lambda \cdot (p_1, \ldots, p_\nu, p_{\nu+1}, \ldots, p_n) = (\lambda p_1, \ldots, \lambda p_\nu, \lambda^2 p_{\nu+1}, \ldots, \lambda^2 p_n)$$

for any non-zero $\lambda \in \mathbb{R}$ and non-zero $p \in \mathbb{R}^n$.
Then, a Heisenberg pseudodifferential symbol σ of order m is called *classical* if it has an asymptotic expansion when $|p|' \to \infty$

$$\sigma(x,p) \sim \sum_{j \geqslant 0} \sigma_{m-j}(x,p) \tag{2.1}$$

where $\sigma_{m-j}(x,p)$ are *Heisenberg homogeneous* functions, that is, for any non zero $\lambda \in \mathbb{R}$,

$$\sigma_{m-j}(x, \lambda \cdot p) = \lambda^{m-j}\sigma_{m-j}(x,p)$$

The *Heisenberg principal symbol* is the symbol of higher degree in the expansion (2.1).

To such a symbol σ of order m, one associates its left-quantization as the linear map:

$$P : C^\infty_c(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \qquad Pf(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot p}\sigma(x,p)\hat{f}(p)dp$$

where $\hat{f}$ denotes the Fourier transform of the function f. We shall say that P is a *classical Heisenberg pseudodifferential operator of order* m. If P is properly supported, then it actually defines a linear map $C^\infty_c(\mathbb{R}^n) \to C^\infty_c(\mathbb{R}^n)$. We denote by $\Psi^m_H(\mathbb{R}^n)$ the vector space of such properly-supported operators and by $\Psi^m_{H,c}(\mathbb{R}^n)$ its subspace of *compactly* supported operators. Since properly-supported operators can be composed, the unions of all-orders operators

$$\Psi_H(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} \Psi^m_H(\mathbb{R}^n)\ , \qquad \Psi_{H,c}(\mathbb{R}^n) = \bigcup_{m\in\mathbb{R}} \Psi^m_{H,c}(\mathbb{R}^n)$$

are associative algebras over $\mathbb{C}$. The ideals of *regularizing operators*

$$\Psi^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} \Psi^m_H(\mathbb{R}^n)\ , \qquad \Psi^{-\infty}_c(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} \Psi^m_{H,c}(\mathbb{R}^n)$$

correspond respectively to the algebras of operators with properly- and compactly-supported smooth Schwartz kernel.

If $P_1, P_2 \in \Psi_H(\mathbb{R}^n)$ are Heisenberg pseudodifferential operators of symbols σ_1 and σ_2, P_1P_2 is a Heisenberg pseudodifferential operator whose symbol σ is given by the *star-product* of symbols :

$$\sigma(x,p) = \sigma_1 \star \sigma_2(x,p) \sim \sum_{|\alpha|\geqslant 0} \frac{(-\mathbf{i})^{|\alpha|}}{\alpha!}\partial^\alpha_p\sigma_1(x,p)\partial^\alpha_x\sigma_2(x,p) \tag{2.2}$$

Note that the order of each symbol in the sum is decreasing while $|\alpha|$ is increasing.

We define the *algebra of Heisenberg formal classical symbols* $\mathcal{S}_H(\mathbb{R}^n)$ and its compactly-supported subalgebra $\mathcal{S}_{H,c}(\mathbb{R}^n)$ as quotients

$$\mathcal{S}_H(\mathbb{R}^n) = \Psi_H(\mathbb{R}^n)/\Psi^{-\infty}(\mathbb{R}^n)\ , \qquad \mathcal{S}_{H,c}(\mathbb{R}^n) = \Psi_{H,c}(\mathbb{R}^n)/\Psi^{-\infty}_c(\mathbb{R}^n)$$

Their elements are formal sums given in (2.1), and the product is the star product (2.2).

A Heisenberg formal symbol is said *Heisenberg elliptic* if it is invertible in $\mathcal{S}_H(\mathbb{R}^n)$. This is equivalent to say that its *Heisenberg principal symbol* is invertible on $\mathbb{R}^n_x \times \mathbb{R}^n_p \setminus \{0\}$. The corresponding pseudodifferential operators are in general not elliptic, but only hypoelliptic.

Example 2.2. The sub-elliptic Laplacian is the differential operator

$$\Delta_H = \partial^4_{x_1} + \ldots + \partial^4_{x_\nu} - (\partial^2_{x_{\nu+1}} + \ldots + \partial^2_{x_n})$$

It has Heisenberg principal symbol $\sigma(x,p) = |p|'^4$, and is therefore Heisenberg elliptic. However, its usual principal symbol, as an ordinary differential operator, is $(x,p) \mapsto \sum_{i=1}^{\nu} p_i^4$, so Δ_H is clearly not elliptic.

Heisenberg pseudodifferential operators are compatible with foliated coordinate changes. Therefore, the Heisenberg calculus can be defined globally on foliations by using a partition of unity. Then, for a foliated manifold M, we denote by $\Psi_H(M)$ the algebra of properly-supported Heisenberg pseudodifferential operators on M, and by $\Psi_{H,c}(M)$ its subalgebra of compactly-supported operators.

For a ($\mathbb{Z}_2$-graded) complex vector bundle E over M, one defines in the same way the algebra of Heisenberg pseudodifferential operators $\Psi_H(M,E)$ acting on the smooth compactly-supported sections $C^\infty_c(M,E)$ of E. One always has an exact sequence

$$0 \to \Psi^{-\infty}(M,E) \to \Psi_H(M,E) \to \mathcal{S}_H(M,E) \to 0$$

and similarly for the algebra of compactly-supported operators $\Psi_{H,c}(M,E)$. Note that for $a \in \mathcal{S}_H(M,E)$, $(x,p) \in T^*_x M$, we have $a(x,p) \in \mathrm{End}(E_x)$. Let $\mathcal{PS}_H(M,E) \subset \mathcal{S}_H(M,E)$ denote the subalgebra of polynomial Heisenberg symbols (with respect to the cotangent coordinate p). The latter is isomorphic to the algebra of differential operators, endowed with the Heisenberg degree.

The vector bundle of interest in this paper will be the exterior algebra $E = \Lambda^\bullet(T^*M \otimes \mathbb{C})$ of the complexified cotangent bundle. In a distinguish coordinate system $(x^1,\ldots,x^n)$ over an open subset $U \subset M$, a local basis of the sections of E is given by $1, dx^{i_1}, dx^{i_1}\wedge dx^{i_2}, \ldots, dx^{i_1}\wedge\ldots\wedge dx^{i_n}$, $1 \leqslant i_1 < \ldots < i_n \leqslant n$. Moreover, the endomorphisms $\mathrm{End}(E_x)$ of the fibre E_x are generated by

$$\psi^i = dx^i \wedge \cdot, \quad \overline{\psi}_i = \iota(\partial_{x^i}) \tag{2.3}$$

for $i = 1,\ldots,n$, where ι stands for the interior product with a vector field. We have the following *anti-commutation relations rules* :

$$[\psi^i, \overline{\psi}_j] = \delta^j_i, \quad [\psi^i, \psi^j] = [\overline{\psi}_i, \overline{\psi}_j] = 0 \tag{2.4}$$

In other words, $\mathrm{End}(E_x)$ is the Clifford algebra of $T_xM \oplus T^*_xM$, the metric is the duality bracket. Let us also recall the commutation relations of symbols : in the coordinate system (x,p) over T^*U, we have :

$$[x^i, p_j] = -\mathbf{i}\delta^i_j, \quad [x^i, x^j] = [p_i, p_j] = 0 \tag{2.5}$$

Thus, every element $a \in \mathcal{S}_H(M,E)$ is locally a formal series

$$a(x,p,\psi,\overline{\psi}) = \sum_{j=0} a_{m-j}(x,p,\psi,\overline{\psi})$$

where the functions a_{m-j} are Heisenberg-homogeneous in p and polynomial in the variables ψ and $\overline{\psi}$.

2.2. **Wodzicki residue on $\Psi_H(M)$.** Let M be a foliated manifold. By choosing a Riemannian metric on M, one can construct a sub-elliptic sub-laplacian Δ as in the flat example 2.2. Its complex powers Δ^{-z} are defined as properly-supported Heisenberg pseudodifferential operators, using the parametrix $(\lambda - \Delta)^{-1}$ and an appropriate Cauchy integral

$$\Delta^{-z} = \frac{1}{2\pi\mathbf{i}} \int \lambda^{-z}(\lambda-\Delta)^{-1}\, d\lambda$$

where the contour is a vertical line pointing downwards.

THEOREM 2.3. (Connes - Moscovici, [3]) *Let M be a foliated manifold of dimension n, ν be the dimensions of the leaves, h their codimension, and $P \in \Psi^m_{H,c}(M)$ be a compactly-supported Heisenberg pseudodifferential operator of order $m \in \mathbb{R}$. Then, for any sub-elliptic sub-laplacian Δ, the zeta function*

$$\zeta_P(z) = \mathrm{Tr}(P\Delta^{-z/4})$$

is holomorphic on the half-plane $\mathrm{Re}(z) > m+\nu+2h$, and extends to a meromorphic function of the whole complex plane, with at most simple poles in the set

$$\{m+\nu+2h, m+\nu+2h-1, \ldots\}$$

The meromorphic extension of the zeta function given by this theorem allows the construction of a Wodzicki-Guillemin trace on $\mathcal{S}_{H,c}(M) = \Psi_{H,c}(M)/\Psi_c^{-\infty}(M)$.

THEOREM 2.4. (Connes - Moscovici, [3]) *The Wodzicki residue functional*

$$⨍ : \mathcal{S}_{H,c}(M) \longrightarrow \mathbb{C}, \quad P \longmapsto \mathrm{Res}_{z=0}\mathrm{Tr}(P\Delta^{-z/4})$$

is a trace. Moreover we have the following formula, only depending on the formal symbol σ *of* P *up to finite order:*

$$\text{(2.6)} \qquad ⨍ P = \frac{1}{(2\pi)^n} \int_{S^*_H M} \iota_L \left(\sigma_{-(\nu+2h)}(x,p) \, \frac{\omega^n}{n!} \right)$$

Here, $S^*_H M$ is the Heisenberg cosphere bundle, that is, the sub-bundle

$$S^*_H M = \{(x,p) \in T^*M\,;\, |p|' = 1\}$$

of the cotangent bundle T^*M, ω denotes the standard symplectic form on T^*M, ι stands for the interior product, and L is the generator of the Heisenberg dilations given by the formula

$$L = \sum_{i=1}^{\nu} p_i \partial_{p_i} + 2 \sum_{i=\nu+1}^{n} p_i \partial_{p_i}$$

All these results still hold for Heisenberg pseudodifferential operators acting on sections of a vector bundle E over M. In this case, the symbol $\sigma_{-(\nu+2n)}(x,p)$ above is at each point (x,p) an endomorphism acting on the fibre E_x, and (2.6) becomes :

$$⨍ P = \frac{1}{(2\pi)^n} \int_{S^*_H M} \iota_L \left(\mathrm{tr}(\sigma_{-(\nu+2n)}(x,p)) \, \frac{\omega^n}{n!} \right)$$

where tr denotes the trace of endomorphisms.

2.3. **Excision and equivariant residue index formula.** Let M be a foliated manifold. Consider a discrete subgroup $G \subset \mathrm{Diff}(M)$ of diffeomorphisms mapping leaves to leaves. By convention we suppose that G acts from the right, so, for any $g \in G$ the induced linear action U_g on the space of functions $C^\infty(M)$ reads

$$(U_g f)(x) = f(x \cdot g) \ , \quad \forall f \in C^\infty(M) \ , \ x \in M$$

Recall that the *algebraic crossed product* $\Psi_{H,c}(M) \rtimes G$ is the universal algebra generated by Heisenberg pseudodifferential operators and group elements, that is,

$$\Psi_{H,c}(M) \rtimes G = \left\{ \sum_{g \in G} P_g U_g \,;\, P_g \in \Psi_{H,c}(M) \right\}$$

where $P_g U_g$ is a short-hand notation for the tensor product $P_g \otimes U_g$, and the sum only contains a finite number of non-zero terms. The multiplication is given by the rule

$$PU_g \cdot QU_h = P(U_g Q U_{g^{-1}}) U_{gh}$$

To this effect, remark that $U_g Q U_{g^{-1}}$ is still a classical Heisenberg pseudodifferential operator, so that the product makes sense. Note also that in general, the representation of $\Psi_{H,c}(M) \rtimes G$ as linear operators on $C^\infty(M)$ does not yield pseudodifferential operators.

Then, one has an *extension*

$$\text{(2.7)} \qquad 0 \to \Psi^{-\infty}_c(M) \rtimes G \to \Psi^0_{H,c}(M) \rtimes G \to S^0_{H,c}(M) \rtimes G \to 0$$

The usual trace on the algebra of regularizing operators $\Psi^{-\infty}_c(M)$, given by

$$\mathrm{Tr}(K) = \int_M k(x,x) \mathrm{dvol}(x)$$

where k stands for the kernel of K, extends to a trace $\mathrm{Tr}_{[1]}$ on $\Psi^{-\infty}_c(M) \rtimes G$ by *localization at the unit* of G

$$\mathrm{Tr}_{[1]} \left(\sum_{g \in G} K_g U_g \right) = \mathrm{Tr}(K_1)$$

That $\mathrm{Tr}_{[1]}$ still remains a trace only comes from the invariance of the ordinary operator trace Tr under conjugation by U_g.

In the same way, the Wodzicki residue on $S_{H,c}(M)$ extends to a trace on $S_{H,c}(M) \rtimes G$ by localization at the unit.

The pseudodifferential extension gives rise to the following commutative diagram involving *algebraic K-theory* and *periodic cyclic homology* ([8])

$$\begin{array}{ccc} K_1^{\mathrm{alg}}(S^0_{H,c}(M) \rtimes G) & \xrightarrow{\mathrm{Ind}} & K_0^{\mathrm{alg}}(\Psi_c^{-\infty}(M) \rtimes G) \\ \downarrow{\scriptstyle \mathrm{ch}_1} & & \downarrow{\scriptstyle \mathrm{ch}_0} \\ HP_1(S^0_{H,c}(M) \rtimes G) & \xrightarrow{\partial} & HP_0(\Psi_c^{-\infty}(M) \rtimes G) \end{array} \tag{2.8}$$

The vertical arrows are respectively the odd and even Chern character.

Denote again $\partial : HP^0(\Psi_c^{-\infty}(M) \rtimes G) \to HP^1(S^0_{H,c}(M) \rtimes G)$ the induced excision map in cohomology. We shall now compute $\partial[\mathrm{Tr}_{[1]}]$. To do this, we lift $\mathrm{Tr}_{[1]}$ on $\Psi_c^{-\infty}(M) \rtimes G$ to a linear map on $\Psi^0_{H,c}(M) \rtimes G$ using a zeta function renormalization

$$\mathrm{Tr}'_{[1]}\left(\sum_{g\in G} P_g U_g\right) = \mathrm{Pf}_{z=0}\mathrm{Tr}\left(P_1 \cdot \Delta^{-z/4}\right)$$

where Δ is a sub-elliptic Laplacian, and $\mathrm{Pf}_{z=0}$ is the constant term in the Laurent series expansion of the zeta-function at $z = 0$. Then, $\partial[\mathrm{Tr}_{[1]}]$ is represented in $HP^1(S^0_H(M) \rtimes G)$ by the cyclic 1-cocycle

$$\phi(aU_g, bU_h) = \mathrm{Tr}'_{[1]}([aU_g, bU_h]) \tag{2.9}$$

for all $a, b \in \Psi^0_{H,c}(M)$ and $g, h \in G$. This expression makes sense as a cocycle over $S^0_H(M) \rtimes G$ because it vanishes whenever a or b belongs to the smooting ideal $\Psi^{-\infty}_{H,c}(M)$. Then, because the trace is localized at units, one finds that $\phi(aU_g, bU_h) = 0$ if $gh \neq 1$, and

$$\phi(aU_g, bU_{g^{-1}}) = \mathrm{Tr}'_{[1]}([aU_g, bU_{g^{-1}}]) = \mathrm{Pf}_{z=0}\mathrm{Tr}\left([aU_g, bU_{g^{-1}}] \cdot \Delta^{-z/4}\right)$$

otherwise. The formula can be made a little more explicit if we gather accurately the relevant terms. This is the aim of the following proposition.

PROPOSITION 2.5. *The cyclic 1-cocycle ϕ given above is given in terms of the Connes-Moscovici residue :*

$$\phi(aU_g, bU_{g^{-1}}) = ⨍ aU_g[\log \Delta^{1/4}, bU_{g^{-1}}]$$

PROOF. Firstly, remark that

$$\begin{aligned}\phi(aU_g, bU_{g^{-1}}) &= \mathrm{Pf}_{z=0}\mathrm{Tr}\left([aU_g, bU_{g^{-1}}] \cdot \Delta^{-z/4}\right) \\ &= \mathrm{Res}_{z=0}\frac{1}{z}\mathrm{Tr}\left((aU_g bU_{g^{-1}} - bU_{g^{-1}})aU_g \cdot \Delta^{-z/4}\right)\end{aligned}$$

Then, work at $z \in \mathbb{C}$ with $\mathrm{Re}(z) \gg 0$, so that $\mathrm{Tr}\left([aU_g, bU_{g^{-1}}] \cdot \Delta^{-z/4}\right)$ is well-defined. Then, the trace property yields

$$\mathrm{Tr}\left([aU_g, bU_{g^{-1}}] \cdot \Delta^{-z/4}\right) = \mathrm{Tr}\left(a(U_g bU_{g^{-1}}\Delta^{-z/4} - U_g\Delta^{-z/4} bU_{g^{-1}})\right)$$

Then, write

$$\begin{aligned} U_g b U_{g^{-1}} \Delta^{-z/4} - U_g \Delta^{-z/4} b U_{g^{-1}} &= U_g b U_{g^{-1}} \Delta^{-z/4} - U_g b \Delta^{-z/4} U_{g^{-1}} - U_g [\Delta^{-z/4}, b] U_{g^{-1}} \\ &= -U_g [\Delta^{-z/4}, b] U_{g^{-1}} + U_g b U_{g^{-1}} [\Delta^{-z/4}, U_g] U_{g^{-1}} \end{aligned}$$

To end the calculations, we need the following lemma, whose proof may be found in [6].

LEMMA 2.6. (Connes-Moscovici's trick, [3, 6]) *For every $z \in \mathbb{C}$, we have the following expansion,*

$$[\Delta^{-z}, b] \sim \sum_{k \geqslant 1} \binom{-z}{k} b^{(k)} \Delta^{-z-k} \qquad U_{g^{-1}} [\Delta^{-z}, U_g] \sim \sum_{k \geqslant 1} \binom{-z}{k} U_{g^{-1}} U_g^{(k)} \Delta^{-z-k}$$

where we denote $T^{(k)} = \mathrm{ad}(\Delta)^k(T)$, $\mathrm{ad}(\Delta) = [\Delta, .]$.

Moreover, note that for every integer $k \geqslant 1$, $b^{(k)}$ and $U_{g^{-1}} U_g^{(k)}$ are classical Heisenberg pseudodifferential operators whose order stricly decrease as k grows. Hence, evaluating the expression under the trace, Theorem 2.3 may be used. We deduce that on the one hand, the sums

$$\mathrm{Res}_{z=0} \frac{1}{z} \sum_{k \geqslant 1} \mathrm{Tr}\left(a \left(U_g \binom{-z/4}{k} b^{(k)} \Delta^{-z/4-k} U_{g^{-1}} \right) \right)$$
$$\mathrm{Res}_{z=0} \frac{1}{z} \sum_{k \geqslant 1} \mathrm{Tr}\left(a \left(U_{g^{-1}} \binom{-z/4}{k} U_g^{(k)} \Delta^{-z/4-k} \right) \right)$$

are finite, since the zeta function is holomorphic on a half-plane $\mathrm{Re}(z) \gg 0$. On the other hand, as the poles of the zeta function are simple, the terms carrying a power of z^2 vanish under the residue, and we are respectively left with

$$-\mathrm{Res}_{z=0} \sum_{k \geqslant 1} \mathrm{Tr}\left(a \left(U_g \frac{(-1)^{k-1}}{4k} b^{(k)} \Delta^{-z/4-k} U_{g^{-1}} \right) \right)$$
$$-\mathrm{Res}_{z=0} \sum_{k \geqslant 1} \mathrm{Tr}\left(a \left(U_{g^{-1}} \frac{(-1)^{k-1}}{4k} U_g^{(k)} \Delta^{-z/4-k} \right) \right)$$

We then recognize the commutator with the logarithm of $\Delta^{1/4}$ (cf. [12]), and we finally obtain

$$\begin{aligned} \phi(aU_g, bU_{g^{-1}}) &= \int\!\!\!\!\!- a \left(U_g [\log \Delta^{1/4}, b] U_{g^{-1}} - U_g b U_{g^{-1}} [\log \Delta^{1/4}, U_g] U_{g^{-1}} \right) \\ &= \int\!\!\!\!\!- a U_g [\log \Delta^{1/4}, b U_{g^{-1}}] \end{aligned}$$

This ends the proof of the proposition. □

The pseudodifferential extension (2.7) is closely related to another extension. Indeed the quotient of $\Psi^0_{H,c}(M)$ by its two-sided ideal $\Psi^{-1}_{H,c}(M)$ of operators of order $\leqslant -1$ is G-equivariantly isomorphic to the commutative algebra of leading symbols $C^\infty_c(S^*_H M)$. The natural inclusion of smoothing operators in $\Psi^{-1}_{H,c}(M)$ and the leading symbol map thus yield a morphism of extensions

$$\begin{tikzcd} 0 \arrow[r] & \Psi^{-\infty}_c(M) \rtimes G \arrow[r] \arrow[d] & \Psi^0_{H,c}(M) \rtimes G \arrow[r] \arrow[d, equal] & \mathcal{S}^0_{H,c}(M) \rtimes G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Psi^{-1}_{H,c}(M) \rtimes G \arrow[r] & \Psi^0_{H,c}(M) \rtimes G \arrow[r] & C^\infty_c(S^*_H M) \rtimes G \arrow[r] & 0 \end{tikzcd} \tag{2.10}$$

The cyclic cohomology class of the operator trace $\mathrm{Tr}_{[1]}$ localized at unit extends in a straightforward manner to a cyclic cohomology class $[\tau] \in HP^0(\Psi^{-1}_{H,c}(M) \rtimes G)$. The latter is represented, for any choice of even integer $k > \nu + 2h$, by the cyclic k-cocycle τ_k defined as follows:

$$\tau_k(a_0, \ldots, a_k) = \mathrm{Tr}_{[1]}(a_0 \ldots a_k) \tag{2.11}$$

for all $a_i \in \Psi^{-1}_{H,c}(M) \rtimes G$. By naturality of excision, the class $\partial[\mathrm{Tr}_{[1]}] \in \mathrm{HP}^1(\mathcal{S}^0_{H,c}(M) \rtimes G)$ is the pullback of $\partial[\tau] \in \mathrm{HP}^1(C^\infty_c(S^*_H M) \rtimes G)$ under the leading symbol homomorphism. Now the computation of $\partial[\tau]$ is fairly analogous to the above computation of $\partial[\mathrm{Tr}_{[1]}]$. We use the generalized Goodwillie theorem of Cuntz and Quillen [5], which states that the periodic cyclic cohomology of an associative algebra $\mathcal{A}$ is isomorphic to the periodic cyclic cohomology of its *completed tensor algebra*

$$\widehat{T}\mathcal{A} = \varprojlim_n T\mathcal{A}/(J\mathcal{A})^n \ , \tag{2.12}$$

where the two-sided ideal $J\mathcal{A} \subset T\mathcal{A}$ is the kernel of the multiplication homomorphism $T\mathcal{A} \to \mathcal{A}$, $a_1 \otimes \ldots \otimes a_n \mapsto a_1 \ldots a_n$. We let $\mathcal{A} = C^\infty_c(S^*_H M) \rtimes G$ and choose any linear splitting $\sigma : \mathcal{A} \to \Psi^0_{H,c}(M) \rtimes G$ of the leading symbol homomorphism. σ is multiplicative up to the ideal $\Psi^{-1}_{H,c}(M) \rtimes G$. By the universal property of the tensor algebra, σ extends to an homomorphism $\sigma_* : T\mathcal{A} \to \Psi^0_H(M) \rtimes G$ respecting the ideals, whence a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & J\mathcal{A} \arrow[r] \arrow[d, "\sigma_*"'] & T\mathcal{A} \arrow[r] \arrow[d, "\sigma_*"'] & \mathcal{A} \arrow[r] \arrow[d, equal] \arrow[dl, dotted, "\sigma"'] & 0 \\
0 \arrow[r] & \Psi^{-1}_{H,c}(M) \rtimes G \arrow[r] & \Psi^0_{H,c}(M) \rtimes G \arrow[r] & C^\infty_c(S^*_H M) \rtimes G \arrow[r] & 0
\end{tikzcd}$$

Observe that the cocycle ϕ, viewed as a cyclic 1-cocycle over $\Psi^0_{H,c}(M) \rtimes G$, vanishes on the large powers of the ideal $\Psi^{-1}_{H,c}(M) \rtimes G$ because it involves the Connes-Moscovici residue. Hence the composite $\phi \circ \sigma_*$, which is a bilinear form on $T\mathcal{A}$, extends to a cyclic 1-cocycle over $\widehat{T}\mathcal{A}$. By [9] Corollary 2.6, this cocycle is precisely a representative of the periodic cyclic cohomology class $\partial[\tau] \in \mathrm{HP}^1(C^\infty_c(S^*_H M) \rtimes G)$. Therefore we obtain

Proposition 2.7. *Let* $0 \to \Psi^{-1}_H(M) \rtimes G \to \Psi^0_H(M) \rtimes G \to C^\infty(S^*_H M) \rtimes G \to 0$ *be the fundamental extension in the Heisenberg pseudodifferential calculus. Then the image* $\partial[\tau]$ *of the canonical trace localized at units under the excision map* $\partial : \mathrm{HP}^0(\Psi^{-1}_{H,c}(M) \rtimes G) \to \mathrm{HP}^1(C^\infty_c(S^*_H M) \rtimes G)$ *is represented by a cyclic 1-cocycle over the completed tensor algebra of* $\mathcal{A} = C^\infty_c(S^*_H M) \rtimes G$,

$$(\phi \circ \sigma_*)(\hat{a}_0, \hat{a}_1) = ⨍ \big(\sigma(\hat{a}_0)[\log \Delta^{1/4}, \sigma(\hat{a}_1)]\big)_{[1]} \ , \qquad \forall\ \hat{a}_0, \hat{a}_1 \in \widehat{T}\mathcal{A} \ , \tag{2.13}$$

for any choice of sub-Laplacian Δ *and linear splitting* $\sigma : \mathcal{A} \to \Psi^0_{H,c}(M) \rtimes G$.

We have so far obtained a local index formula for a G-Heisenberg elliptic operator, in the sense that it is not sensitive under perturbations by elements in $\Psi^{-\infty}_c(M) \rtimes G$. However, the formula is really hard to compute explicitely, because we have to calculate the term of degree $-\nu - 2h$ in the expansion of the symbol $aU_g[\log \Delta^{1/4}, bU_{g^{-1}}]$, and adding the fact that the symbolic calculus is not commutative, the task is even more difficult.

The general strategy to handle this problem is the construction of a cohomologous (b, B)-cocycle, in which the leading symbol is exactly of degree $-\nu - 2h$. Then, we will just have to take the principal symbol to obtain a formula as an integral over the Heisenberg cosphere bundle. This will be possible through an "algebraic JLO formula" whose entries are formal (Heisenberg) symbols. This construction includes the following main steps:

- An algebra of operators acting on symbols,
- A related notion of "heat kernel",

- A related notion of trace,
- A related notion of "Dirac operator" D, in the sense that for a symbol $a \in \mathcal{S}_{H,c}(M)$, $[D, a]$ is (up to some lower order terms) the differential da of the symbol a,
- Carrying all this to the equivariant setting.

The constructions required are more tedious than the original JLO setting, however, it is much more flexible and has many advantages. As we will see, the constructions are purely algebraic, and we do not need to appeal to analytic properties of the heat equation, so that we can deal with much more general operators that Dirac type operators. In the same vein, we do not need to consider entire (b, B)-cochains. Thus, what we develop here holds in cases where Getzler's rescaling does not apply.

3. Bimodule of Heisenberg formal symbols

We adapt the framework developed in [9] to Heisenberg calculus on foliations. When the proof of a result is deferred to this paper, this means that it applies verbatim in our case. We shall mainly focus on the changes which occur in the Heisenberg setting.

Let $(M, \mathcal{F})$ be a foliated manifold of dimension n, where $\mathcal{F} \subset TM$ is the sub-bundle of rank ν defining the foliation, and let h denote the codimension of the foliation. We consider the $\mathbb{Z}_2$-graded algebra of formal Heisenberg symbols $\mathcal{S}_H(M, E)$, with $E = \Lambda^\bullet(T^*M \otimes \mathbb{C})$. We view it as a left $\mathcal{S}_H(M, E)$-module and right $\mathcal{PS}_H(M, E)$-module : the left action of $a \in \mathcal{S}_H(M, E)$, and the right action $b \in \mathcal{PS}_H(M, E)$ on $\xi \in \mathcal{S}_H(M, E)$ are given by

$$a_L \cdot \xi = a\xi, \quad b_R \cdot \xi = \pm \xi b$$

Here, the sign $\pm$ depends on the parity of b and ξ : it is $-$ when both are odd and $+$ otherwise. This action defines a $\mathbb{Z}_2$-graded subalgebra of $\mathrm{End}(\mathcal{S}_H(M, E))$

$$\mathcal{L}(M) = \mathrm{span}\{a_L b_L \, ; \, a \in \mathcal{S}_H(M, E), b \in \mathcal{PS}_H(M, E)\}$$

Now, let us have a closer look on the operators contained in this algebra. Let $(x^1, \ldots, x^n)$ be a foliated coordinate system over an open subset $U \subset M$. The function x^i is a symbol of order 0, so that x^i_L, x^i_R may be viewed as elements of $\mathcal{L}(M)$. The conjugate coordinate p_i is a symbol of order 1 if $i = 1, \ldots, \nu$, but of order 2 when $i = \nu + 1, \ldots, n$. As before, this defines elements p_{iL}, p_{iR} of $\mathcal{L}(M)$. Now, observe that :

$$(x^i_L - x^i_R) = \mathbf{i}\partial_{p_i}, \quad (p_{iL} - p_{iR}) = -\mathbf{i}\partial_{x^i} \tag{3.1}$$

The same holds for the odd coordinates ψ^i and $\overline{\psi}_i$:

$$(\psi^i_L - \psi^i_R) = \partial_{\overline{\psi}_i}, \quad (\overline{\psi}_{iL} - \overline{\psi}_{iR}) = \partial_{\psi^i} \tag{3.2}$$

Hence, $\mathcal{L}(M)$ contains all the "elementary operations" on (Heisenberg) symbols.

Little calculations shows that a generic element $a_L b_R \in \mathcal{L}(M)$ reads over U as a series

$$a_L b_R = \sum_{|\alpha|=1}^{k} \sum_{|\beta|=1}^{\infty} \sum_{|\eta|=1}^{n} \sum_{|\theta|=1}^{n} (s_{\alpha,\beta,\eta,\theta})_L (\psi^\eta \overline{\psi}^\theta)_R \partial_x^\alpha \partial_p^\beta \tag{3.3}$$

where $s_{\alpha,\beta,\eta,\theta} \in \mathcal{S}_H(U, E)$ and $k \in \mathbb{N}$. It is not necessarily true that a series of that form comes from an element of $\mathcal{L}(M)$.

Now, consider the $\mathbb{Z}_2$-graded algebra $\mathcal{S}(M) = \mathcal{L}(M)[[\varepsilon]]$ of *formal power series* with coefficients $\mathcal{L}(M)$, and indeterminate ε, which comes with a trivial grading. $\mathcal{S}$ is filtered by the subalgebras $\mathcal{S}_k(M) = \mathcal{S}(M)\varepsilon^k$, for every $k \in \mathbb{N}$. This k counts the minimal power of ε appearing in an element of $\mathcal{S}(M)$. We now define an important subalgebra of $\mathcal{S}(M)$.

DEFINITION 3.1. The subspace $\mathcal{D}^m(M) \subset \mathcal{S}(M)$ consists of elements $s = \sum s_k \varepsilon^k$ such that in any distinguished local chart $U \in M$, we have

$$s_k = \sum_{|\alpha|=1}^{k} \sum_{|\beta|=1}^{\infty} \sum_{|\eta|=1}^{n} \sum_{|\theta|=1}^{n} (s^m_{k,\alpha,\beta,\eta,\theta})_L (\psi^\eta \overline{\psi}^\theta)_R \partial_x^\alpha \partial_p^\beta \tag{3.4}$$

where $s^m_{k,\alpha,\beta,\eta,\theta} \in \mathcal{S}_H(U, E)$ has *Heisenberg order* $\leqslant m + (k + |\beta| - 3|\alpha|)/2$. We also denote $\mathcal{D}^m_k(M) = \mathcal{D}^m(M) \cap \mathcal{S}_k(M)$. We set

$$\mathcal{D}(M) = \bigcup_{m \in \mathbb{R}} \mathcal{D}^m(M)$$

The space $\mathcal{D}(M)$ is a bi-filtered subalgebra of $\mathcal{S}(M)$, that is $\mathcal{D}^m_k(M) \cdot \mathcal{D}^{m'}_{k'}(M) \subset \mathcal{D}^{m+m'}_{k+k'}(M)$, for $m, m' \in \mathbb{R}$ and $k, k' \in \mathbb{N}$ ([9], Lemma 3.1). Using symbols *with compact supports*, we define analogously the subalgebra $\mathcal{D}_c(M) \subset \mathcal{D}(M)$.

DEFINITION 3.2. A *generalized Laplacian* is an operator $\Delta \in \mathcal{D}^{1/2}_1$ of even parity, which can be written, in any local coordinate system over a local distinguished chart $U \in M$:

$$\Delta = \mathbf{i}\varepsilon \partial_{x^i} \partial_{p_i} \bmod \mathcal{D}^0_1(U) \tag{3.5}$$

Throughout the paper, we shall use Einstein summation notation for repeated indices.

That such an operator exists is not obvious, cf. [9], Lemma 3.3. We will see some important examples in the section concerning generalized Dirac operators .

A generalized Laplacian Δ will be our first point of departure towards the construction of a *JLO formula on Heisenberg symbols*. In this type of formula, one needs to know how to deal with an exponential of such an operator in order to have a "heat kernel". As a formal power series in ε this indeed defines an element of $\mathcal{S}(M)$:

$$\exp(t\Delta) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^k, \ \forall t \in \mathbb{R} \tag{3.6}$$

This operator does not belong to $\mathcal{D}(M)$. We define a one parameter group of automorphisms $(\sigma^t_\Delta)_{t \in \mathbb{R}}$ of the algebra $\mathcal{S}(M)$ as follows :

$$\sigma^t_\Delta(s) = \exp(t\Delta) s \exp(-t\Delta), \ \forall s \in \mathcal{D}(M) \tag{3.7}$$

LEMMA 3.3. *For every generalized Laplacian Δ, $(\sigma^t_\Delta)_{t \in \mathbb{R}}$ is actually a one parameter group of automorphisms of $\mathcal{D}(M)$. More precisely, one has, for every $m \in \mathbb{R}$, $k \in \mathbb{N}$:*

$$[\Delta, \mathcal{D}^m_k] \subset \mathcal{D}^m_{k+1}, \quad \sigma^t_\Delta(\mathcal{D}^m_k) \subset \mathcal{D}^m_k$$

PROPOSITION 3.4. *(Duhamel formula) Let $\Delta + s$ be a perturbation of a generalized Laplacian Δ, where $s \in \mathcal{D}^0_1(M)$. Then,*

$$\exp(\Delta + s) = \sum_{k=0}^{\infty} \int_{\Delta_k} \exp(t_0 \Delta) s \exp(t_1 \Delta) \dots s \exp(t_k \Delta) \, dt \tag{3.8}$$

where Δ_k is the standard k-simplex, and $dt = dt_1 \dots dt_k$. Equivalently,

$$\exp(\Delta + s) = \sum_{k=0}^{\infty} \int_{\Delta_k} \sigma^{t_0}_\Delta(s) \sigma^{t_0+t_1}_\Delta(s) \dots \sigma^{t_0+\dots t_{k-1}}_\Delta(s) \exp(\Delta) \, dt \tag{3.9}$$

DEFINITION 3.5. Let Δ be a generalized Laplacian. The *bimodule of trace class operators* is the subspace $\mathcal{T}(M) = \mathcal{D}_c(M) \exp(\Delta)$ of $\mathcal{S}(M)$.

REMARK 3.6. $\mathcal{T}(M)$ is a $\mathcal{D}(M)$-bimodule, and does not depend on the choice of the generalized Laplacian ([9], Proposition 3.6). However, this is not a subalgebra of $\mathcal{S}(M)$.

The terminology will be explained in the next section. Finally note that if $G \subset \mathrm{Diff}(M)$ is a group of foliated diffeomorphisms, the algebra $\mathcal{D}(M)$ and the bimodule $\mathcal{T}(M)$ carry natural G-actions by automorphisms.

4. CANONICAL TRACE ON THE BIMODULE $\mathcal{T}$

4.1. **Construction of the trace.** Let $(M^n, \mathcal{F})$ be a foliated manifold of codimension h, and let ν denote the dimension of the leaves, so that $n = \nu + h$, and take $E = \Lambda^\bullet(T^*M \otimes \mathbb{C})$. The aim of this section is to construct a canonical trace on $\mathcal{T}(M)$ from the Wodzicki residue 2.4.

First, we work locally. Let $U \subset M$ be a distinguished local chart of M . Recall that $\mathcal{T}(U)$ is a bimodule over $\mathcal{D}(U)$. A trace on $\mathcal{T}(U)$ is in this sense a linear map $\mathcal{T}(U) \to \mathbb{C}$ vanishing on the subspace $[\mathcal{T}(U), \mathcal{D}(U)]$ of graded commutators. Choose a coordinate system (x, p) on T^*U adapted to the foliation, and let Δ be the "flat" (generalized) Laplacian

$$\Delta = \mathbf{i}\varepsilon \partial_{x^i}\partial_{p_i}$$

For every multi-indices α and β, we define a bracket operation

$$\langle \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle = \partial_x^\alpha \partial_p^\beta \exp\left(\frac{\mathbf{i}}{\varepsilon}(p_i - q_i)(x^i - y^i)\right)\Big|_{x=y, p=q} \tag{4.1}$$

Remark that this vanishes unless $|\alpha| = |\beta|$.

EXAMPLE 4.1. One has

$$\langle \exp \Delta \rangle = 1, \quad \langle \partial_{x^i} \exp \Delta \rangle = \langle \partial_{p_i} \exp \Delta \rangle = 0, \quad \langle \partial_{x^i}\partial_{p_j} \exp \Delta \rangle = \frac{\mathbf{i}}{\varepsilon}\delta_i^j$$

where δ_i^j denotes the Kronecker symbol. More generally, the formula with a polynomial $\partial_x^\alpha \partial_p^\beta$ involves all the possible contractions between ∂_{x^i} and ∂_{p_j}. For example,

$$\langle \partial_{x^i}\partial_{x^j}\partial_{p_k}\partial_{p_l} \exp \Delta \rangle = \left(\frac{\mathbf{i}}{\varepsilon}\right)^2 (\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)$$

We also define a contraction map for the odd variables : for every multi-indices η and θ, we set

$$\langle (\psi^\eta \overline{\psi}^\theta)_R \rangle = (-1)^n \mathrm{tr}_s(\psi^\eta \overline{\psi}^\theta) \tag{4.2}$$

In particular, from the normalization we chose for tr_s, we have $\langle (\psi^1 \dots \psi^n \overline{\psi}_1 \dots \overline{\psi}_n)_R \rangle = 1$
From this, we construct a linear map

$$\langle\langle\ .\ \rangle\rangle : \mathcal{T}(M) \to \mathcal{S}_H(U, E)[[\varepsilon]] \tag{4.3}$$

as follows. Let $s\exp(\Delta) \in \mathcal{T}(M)$ be a generic element, where $s = \sum_{k\geqslant 0} s_k \varepsilon^k \in \mathcal{D}^m(U)$. The symbol s_k may be written

$$s_k = \sum_{|\alpha|=1}^{k} \sum_{|\beta|=1}^{\infty} \sum_{|\eta|=1}^{n} \sum_{|\theta|=1}^{n} (s_{k,\alpha,\beta,\eta,\theta})_L (\psi^\eta \overline{\psi}^\theta)_R \partial_x^\alpha \partial_p^\beta$$

where $s_{k,\alpha,\beta,\eta,\theta} \in \mathcal{S}_H(U, E)$ has *Heisenberg order* $\leqslant m + (k + |\beta| - 3|\alpha|)/2$. Then, we set

$$\langle\langle s_k \exp \Delta \rangle\rangle = \sum_{|\alpha|=1}^{k} \sum_{|\beta|=1}^{\infty} \sum_{|\eta|=1}^{n} \sum_{|\theta|=1}^{n} s_{k,\alpha,\beta,\eta,\theta} \langle (\psi^\eta \overline{\psi}^\theta)_R \rangle \langle \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle$$

This sum is finite by definition of the even contraction, and is consequently a polynomial of degree at most k in the variable ε^{-1}. Then, $\langle\langle s_k \exp\Delta\rangle\rangle\varepsilon^k$ is a polynomial of degree at most k in ε. Then, we finally define

$$(4.4) \qquad \langle\langle s \exp\Delta\rangle\rangle = \sum_{k\geqslant 0} \langle\langle s_k \exp\Delta\rangle\rangle\varepsilon^k$$

which is an element of $\mathcal{S}_H(U, E)[[\varepsilon]]$ (this is not totally obvious, refer to [9], Lemma 4.1).
We can now pass to the definition of the trace on $\mathcal{T}(M)$.

DEFINITION 4.2. Let $U \in M$ be a distinguished local chart, and denote by $\langle\langle s \exp\Delta\rangle\rangle[n] \in S_H(U, E)$ the coefficient of ε^n in the formal series $\langle\langle s \exp\Delta\rangle\rangle$. Then, we define the following graded trace

$$(4.5) \qquad \mathrm{Tr}_s^U : \mathcal{T}(U) \to \mathbb{C}, \quad \mathrm{Tr}_s^U(s \exp\Delta) = ⨍ \mathrm{Tr}_s^U \langle\langle s \exp\Delta\rangle\rangle[n]$$

This map does not depend on the choice of distinguished coordinates (x, p) on T^*U, so that these maps may be glued together to give a canonical graded trace :

$$(4.6) \qquad \mathrm{Tr}_s : \mathcal{T}(M) \to \mathbb{C}$$

on the $\mathcal{D}(M)$-bimodule of trace class operators.

The proof that this is a trace is the same as that of [9], Lemma 4.2. That we can glue these quantities to get a global functional on the whole foliation M is Proposition 4.3 of the same paper. For the same reason, Tr_s is invariant under the action of any group G of foliated diffeomorphisms on $\mathcal{T}(M)$.

4.2. **An algebraic Mehler formula.** In this section, we show how the "Todd series" can be recovered from the contractions we defined in the previous paragraph. The formula may be seen as a pseudodifferential analogue of the Mehler formula for the harmonic oscillator, and will be crucial for obtaining the index theorem. We keep the notations of the previous subsection and work in the distinguished local chart U.

For a $N \times N$ matrix R with coefficients in $\mathbb{C}[[\varepsilon]]$, which has no degree zero term in ε, we can define the following formal power series in $M_N(\mathbb{C}[[\varepsilon]])$ and in $\mathbb{C}[[\varepsilon]]$

$$(4.7) \qquad \frac{R}{e^R - 1} = 1 - \frac{1}{2}R + \frac{1}{12}R^2 + \dots, \quad \mathrm{Td}(R) = \det\left(\frac{R}{e^R - 1}\right)$$

We call $\mathrm{Td}(R)$ the *Todd series* of R.

Now, consider the operator

$$s = p_L \cdot R \cdot \partial_p = p_{iL} \cdot R^i_j \cdot \partial_{p_j}$$

and the perturbation of the flat Laplacian $\Delta + s$, which is not a generalized Laplacian. However, by the Duhamel formula, Proposition 3.4, $\exp(\Delta + s)$ still defines an element of $\mathcal{T}(U)$.

PROPOSITION 4.3. *For every multi-indices α and β, we have*

$$(4.8) \qquad \langle \partial_x^\alpha \partial_p^\beta \exp(\Delta + s)\rangle = \mathrm{Td}(R) s(R, p)$$

where the symbol $s(R, p)$ is polynomial in p and given by the following formula :

$$s(R, p) = \partial_x^\alpha \partial_p^\beta \exp\left(\frac{\mathbf{i}}{\varepsilon} q \cdot R \cdot (x - y) + \frac{\mathbf{i}}{\varepsilon}(p - q) \cdot \frac{R}{1 - e^{-R}} \cdot (x - y)\right)\bigg|_{x=y, p=q}$$

EXAMPLE 4.4. We give some particular cases of the formula which will be useful in the sequel. We have

$$\langle \exp(\Delta + p_L \cdot R \cdot \partial_p) \rangle = \mathrm{Td}(R) \tag{4.9}$$

$$\langle (\mathbf{i}\varepsilon\partial_x + p_L \cdot R)^\alpha \exp(\Delta + p_L \cdot R \cdot \partial_p) \rangle = 0 \tag{4.10}$$

where α is any multi-index.

5. DIRAC OPERATORS

5.1. **Generalities.** The algebra of differential forms $\Omega^\bullet(M)$ on M and the Lie algebra $\mathrm{Vect}(M)$ of vector fields may be seen as elements of the space $\mathcal{PS}^0_H(M,E)$ of polynomials Heisenberg symbols of degree 0 via the following maps :

$$\mu : dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Omega^\bullet(M) \mapsto \psi^{i_1} \ldots \psi^{i_k} \in \mathcal{PS}^0_H(M,E),$$
$$\iota : \partial_{x^i} \in \mathrm{Vect}(M) \mapsto \iota(\partial_{x^i}) \in \mathcal{PS}^0_H(M,E)$$

Then, we shall be interested in various subspaces of $\mathcal{L}(M) = \mathcal{S}_H(M,E)_L \mathcal{PS}_H(M,E)_R$, which are needed to see where lie the generalized Dirac operators.

Let $\mathcal{SPS}^1_H(M,E) \subset \mathcal{PS}^1_H(M,E)$ be the space of differential operators a of order 1, with *scalar Heisenberg principal symbol*, whose local expression reads

$$a(x,p) = a^i(x)p_i + a^i_j(x)\psi^j\overline{\psi}_i + b(x)$$

where the coefficients depending on x are smooth functions. Remark that $\mathcal{SPS}^1_H(M,E)$ is a Lie algebra. We also consider the subspaces $\mathcal{SPS}^1_H(M,E)_L\Omega^1(M)_R \subset \mathcal{D}^1_0(M)$ and $\Omega^0(M)_L\mathrm{Vect}(M)_R \subset \mathcal{D}^0_0(M)$, respectively spanned by elements which are locally given by the series

$$s = \sum_{|\alpha|\geqslant 0} (s^k_{\alpha i}(x)p_k + s^k_{\alpha ij}(x)\psi^i\overline{\psi}_k + s_{\alpha i}(x))_L \psi^i_R \partial^\alpha_p$$
$$r = \sum_{|\alpha|\geqslant 0} (r^i_\alpha)_L \overline{\psi}_{iR} \partial^\alpha_p$$

The coefficients depending on x are again smooth functions.

DEFINITION 5.1. A *generalized Dirac operator* D is an element of $\mathcal{D}(M)$ which writes

$$D = i\varepsilon\nabla + \overline{\nabla} \in \mathcal{D}^1_1(M) + \mathcal{D}^{-1/2}_0(M)$$

where ∇ and $\overline{\nabla}$ are such that

$$\nabla \equiv \psi^i_R \partial_{x^i} \mod \mathcal{SPS}^1_H(M,E)_L\Omega^1(M)_R,$$
$$\overline{\nabla} \equiv \overline{\psi}_{iR}\partial_{p_i} \mod \Omega^0(M)_L\mathrm{Vect}(M)_R \cap \mathcal{D}^{-1}_0(M)$$

Remark that if G is a group of foliated diffeomorphisms on M, it transforms Dirac operators into Dirac operators. The terminology lies in the following important proposition.

PROPOSITION 5.2. *Let D be a generalized Dirac operator. Then, $-D^2$ is a generalized Laplacian.*

The rest of the paragraph gives the two crucial examples of generalized Dirac operators.

5.2. **De Rham - Dirac operators.** The exterior differentiation d acting on the space of diffferential forms $\Omega^\bullet(M)$ defines an element of $\mathcal{PS}^1(M,E)$. Its right action on the bimodule of formal Heisenberg symbols $\mathcal{S}_H(M,E)$ gives an element of odd degree $d_R \in \mathcal{L}(M)$. Locally,

$$d_R = \mathbf{i}(p_i\psi^i)_R = \mathbf{i}\psi^i_R p_{iR} = -\psi^i_R \partial_{x^i} + \mathbf{i}p_{iL}\psi^i_R \tag{5.1}$$

As $\mathbf{i}p_{iL}\psi^i_R \in \mathcal{SPS}^1(M,E)$, we have a generalized Dirac operator

$$D = -\mathbf{i}\varepsilon d_R + \overline{\nabla} \tag{5.2}$$

for any choice of $\overline{\nabla}$ as in Definition 5.1. Such generalized Dirac operators will be called of *de Rham - Dirac type*.

PROPOSITION 5.3. *Let $D = -\mathbf{i}\varepsilon d_R + \overline{\nabla}$ be a de Rham - Dirac operator. Locally, the associated generalized Laplacian is given by the following formula :*

$$\begin{aligned} -D^2 = \mathbf{i}\varepsilon\left(\partial_{x^i}\partial_{p_i} + \sum_{|\alpha|\geqslant 2}(a^i_\alpha)_L\partial_{x^i}\partial^\alpha_p\right) + \varepsilon\left(p_{iL}\partial_{p_i} + \sum_{|\alpha|\geqslant 2}(a^i_\alpha p_i)_L\partial^\alpha_p\right) \\ + \varepsilon\left((\psi^i\overline{\psi}_i)_R + \sum_{|\alpha|\geqslant 1}(b^i_{\alpha j})_L(\psi^j\overline{\psi}_i)_R\partial^\alpha_p\right) \end{aligned} \tag{5.3}$$

where the coefficients $a^i_\alpha, b^i_{\alpha j}$ are smooth functions.

The formula seems to be tough at first sight. Nevertheless, one should retain that in the final calculations, the sums over $|\alpha| \geqslant \ldots$ will be killed for reasons of order.

5.3. **Dirac operators associated to affine connections.** Let Γ be an affine connection without torsion on M, characterized by its Christoffel symbols Γ^k_{ij} in a local coordinate system $(x^1,\ldots,x^n)$ over U. Then, we define a "covariant derivative" operator on $\mathcal{S}_H(U,E)$ given by

$$\nabla^\Gamma_i = \partial_{x^i} + \Gamma^k_{ij}(x)\left(p_{kL}\partial_{p_j} + [\overline{\psi}_k\psi^j, \,.\,]\right) \tag{5.4}$$

This is not properly speaking a covariant derivative, since the coordinates x and p do not commute. However, the action of ∇^Γ_i on the generators $x, p, \psi, \overline{\psi}$ are what we expect from a covariant derivative :

$$\nabla^\Gamma_i(x^k) = \delta^k_i, \qquad \nabla^\Gamma_i(p_j) = \Gamma^k_{ij}p_k, \qquad \nabla^\Gamma_i(\psi^k) = -\Gamma^k_{ij}\psi^j, \qquad \nabla^\Gamma_i(\overline{\psi}_j) = \Gamma^k_{ij}\overline{\psi}_k \tag{5.5}$$

A generalized Dirac operator $D = \mathbf{i}\varepsilon\nabla + \overline{\nabla}$ is called *affiliated to the affine connection* Γ on M if locally over U we have

$$\nabla = \psi^i_R(\nabla^\Gamma_i + s) \tag{5.6}$$

with $s \in \mathcal{SPS}^1_H(M,E)_L\Omega^1(M)_R \cap \mathcal{D}^0_0(M)$.

PROPOSITION 5.4. *For such a Dirac operator D, one has the following analogue of the Lichnerowicz formula :*

$$\begin{aligned} -D^2 = \mathbf{i}\varepsilon\left(\partial_{x^i}\partial_{p_i} + (\Gamma^k_{ij})_L(\psi^i\overline{\psi}_k)_R\partial_{p_j} + u + v\right) \\ + \varepsilon^2\left(\frac{1}{2}(\psi^i\psi^j)_R R^k_{lij}(p_{kL}\partial_{p_l} + (\overline{\psi}_k\psi^l)_L) + w\right) \end{aligned} \tag{5.7}$$

where $R^k_{lij} = \partial_{x^i}\Gamma^k_{jl} - \partial_{x^j}\Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}$ *are the components of the curvature tensor of* Γ, *and*

$$\begin{aligned}
u &= \sum_{|\alpha|\geqslant 2} \big((u_{\alpha i})_L \partial_{x^i} + (u^k_\alpha p_k)_L + (u^k_{\alpha i})_L(\psi^i\overline{\psi}_k)_R + (u_\alpha)_L\big)\,\partial_p^\alpha \\
v &= \sum_{|\alpha|\geqslant 1} (v^k_{\alpha i}\overline{\psi}_k\psi^i)_L\partial_p^\alpha \\
w &= (\psi^i\psi^j)_R\left(\sum_{|\alpha|\geqslant 2}(w^k_{\alpha ij}p_k)_L\partial_p^\alpha + \sum_{|\alpha|\geqslant 1}(w^k_{\alpha lij}\overline{\psi}_k\psi^l + w_{\alpha ij})_L\partial_p^\alpha\right)
\end{aligned}$$

where the coefficients are smooth functions on M.

As in the de Rham - Dirac case, the terms u, v, w will be killed in the final calculations.

6. Equivariant cohomology

Let G be a discrete group acting by orientation-preserving diffeomorphisms on a smooth oriented manifold M. Following [10], we shall explain an alternative construction of Connes' characteristic map from the G-equivariant cohomology of M to the periodic cyclic cohomology of the crossed product algebra $C^\infty_c(M) \rtimes G$ which differs slightly from the original construction given by Connes in [1], but is particularly well-adapted to the proof of the equivariant index theorem.

6.1. **Classifying spaces.** We recall that the nerve of a discrete group G is the simplicial set $NG_\bullet$ with $NG_n = G^n$ for all $n \geqslant 0$. The face maps $\delta_i : NG_n \to NG_{n-1}$ and degeneracy maps $\sigma_i : NG_n \to NG_{n+1}$ are given by

$$\begin{aligned}
&\delta_0(g_1,\dots,g_n) = (g_2,\dots,g_n)\\
&\delta_i(g_1,\dots,g_n) = (g_1,\dots,g_ig_{i+1},\dots,g_n) \qquad 1\leqslant i\leqslant n-1\\
&\delta_n(g_1,\dots,g_n) = (g_1,\dots,g_{n-1})\\
&\sigma_i(g_1,\dots,g_n) = (g_1,\dots,g_i,1,g_{i+1},\dots,g_n) \qquad 0\leqslant i\leqslant n\ .
\end{aligned}\tag{6.1}$$

Let $\Delta_n = \{(s_0,\dots,s_n) \in [0,1]^{n+1} \mid s_0+\dots+s_n = 1\}$ be the standard n-simplex in $\mathbb{R}^{n+1}$, with $\delta^i : \Delta_n \to \Delta_{n+1}$, $(s_0,\dots,s_n) \mapsto (s_0,\dots,s_{i-1},0,s_i,\dots,s_n)$ the inclusion of the i-th face, and $\sigma^i : \Delta_n \to \Delta_{n-1}$, $(s_0,\dots,s_n) \mapsto (s_0,\dots,s_i+s_{i+1},\dots,s_n)$ the collapse of the i-th edge. The classifying space of G is the geometric realization of the simplicial set $NG_\bullet$, defined as the quotient

$$BG = \Big(\bigcup_{n\geqslant 0} NG_n\times\Delta_n\Big)/\sim \tag{6.2}$$

where the equivalence relation $\sim$ identifies a point $(g,\delta^i s) \in NG_n\times\Delta_n$ (resp. $(g,\sigma^i s)\in NG_n\times\Delta_n$) with the point $(\delta_i g, s) \in NG_{n-1}\times\Delta_{n-1}$ (resp. $(\sigma_i g, s)\in NG_{n+1}\times\Delta_{n+1}$). Let $\Omega(\Delta_n)$ denote the DG algebra of (complex) smooth differential forms over Δ_n which are extendable over the hyperplane $\{(s_0,\dots,s_n)\in\mathbb{R}^{n+1} \mid s_0+\dots+s_n=1\}$. Let $\Omega(NG_m\times\Delta_n)$ be the space of functions from the discrete set NG_m to $\Omega(\Delta_n)$. This is naturally a DG algebra. A differential form ω of degree k over BG is a collection of k-forms $\omega_n \in \Omega^k(NG_n\times\Delta_n)$, $n\in\mathbb{N}$, subject to the constraints

$$(\mathrm{Id}\times\delta^i)^*\omega_n = (\delta_i\times\mathrm{Id})^*\omega_{n-1}\ ,\qquad (\mathrm{Id}\times\sigma^i)^*\omega_n = (\sigma_i\times\mathrm{Id})^*\omega_{n+1}\ ,$$

for all $i=0,\dots,n$ and $n\geqslant 0$. The space $\Omega(BG)$ of differential forms over BG is a DG algebra. The de Rham cohomology of BG, which is the cohomology of the complex $\Omega(BG)$ endowed with the exterior differential d, is known to be canonically isomorphic to the group cohomology of G with complex coefficients:

$$H^\bullet(BG) \cong H^\bullet(G,\mathbb{C}) \ . \tag{6.3}$$

The universal G-bundle over the nerve $NG_\bullet$ is the simplicial set $N\overline{G}_\bullet$ with $N\overline{G}_n = G^{n+1}$ for all n, and the face and degeneracy maps are

$$\begin{aligned} &\delta_i(g_0,\dots,g_n) = (g_0,\dots,\check{g}_i,\dots,g_n) \qquad 0\leqslant i\leqslant n \\ &\sigma_i(g_0,\dots,g_n) = (g_0,\dots,g_i,g_i,\dots,g_n) \qquad 0\leqslant i\leqslant n \ , \end{aligned} \tag{6.4}$$

where the symbol $\check{}$ denotes omission. The projection $N\overline{G}_\bullet \to NG_\bullet$ defined by $(g_0,\dots,g_n) \mapsto (g_0g_1^{-1}, g_1g_2^{-1},\dots,g_{n-1}g_n^{-1})$ is a simplicial map. Its fibers are in one to one correspondence with the orbits of the (free) G-action $(g_0,\dots,g_n)\cdot g = (g_0g,\dots,g_ng)$, which is also a simplicial map for all $g\in G$. The geometric realization

$$EG = \Big(\bigcup_{n\geqslant 0} N\overline{G}_n \times \Delta_n\Big)/\sim \tag{6.5}$$

is therefore a G-bundle over BG. The DG algebra of differential forms $\Omega(EG)$ defined as above carries a natural action of G. By pullback, $\Omega(BG)$ is isomorphic to the DG subalgebra of $\Omega(EG)$ consisting of G-invariant differential forms. Hence $H^\bullet(BG)$ is also the cohomology of the complex of G-invariant differential forms on EG.

Now let $G\subset \mathrm{Diff}(M)$ be a discrete group of diffeomorphisms on a smooth manifold M. The product $EG\times M$, endowed with the diagonal G-action, is a G-bundle over the quotient $EG\times_G M$. The DG algebra $\Omega(EG\times M)$, defined as the collection of differential forms over the manifold $N\overline{G}_n\times \Delta_n\times M$ with gluing constraints as above, inherits an action of G by pullback. The DG subalgebra of G-invariant differential forms is isomorphic to $\Omega(EG\times_G M)$. We define the G-equivariant cohomology of M (with complex coefficients) as the corresponding de Rham cohomology $H^\bullet(EG\times_G M)$.

The Chern-Weil theory of characteristic classes for vector bundles carries easily to the equivariant case. Let V be a G-equivariant (complex) vector bundle over M, and choose a connection ∇_0 on V. Of course ∇_0 is not G-invariant in general, and we denote by $\mathrm{Ad}_g(\nabla_0)$ its image under the adjoint action of an element $g\in G$. The set of all connections being an affine space, at any point $(g_0,\dots,g_n)(s_0,\dots,s_n)\in N\overline{G}_n\times\Delta_n$ we can build a new connection ∇ on V by means of the barycentric formula

$$\nabla(g_0,\dots,g_n)(s_0,\dots,s_n) = \sum_{i=0}^n s_i\,\mathrm{Ad}_{g_i}^{-1}(\nabla_0) \ . \tag{6.6}$$

Let $W = EG\times V$ be the pullback of the vector bundle V over $EG\times M$. If d denotes the exterior differential over EG, then $d+\nabla$ is a connection on W, whose curvature 2-form $R = [d,\nabla]+\nabla^2\in\Omega^2(EG\times M,\mathrm{End}(W))$ reads

$$R(g_0,\dots,g_n)(s_0,\dots,s_n) = \sum_{i=0}^n ds_i\,\mathrm{Ad}_{g_i}^{-1}(\nabla_0) + \sum_{i,j} s_is_j\,\mathrm{Ad}_{g_i}^{-1}(\nabla_0)\,\mathrm{Ad}_{g_j}^{-1}(\nabla_0) \ .$$

Observe that R is always the sum of a form of bidegree $(1,1)$ and a form of bidegree $(0,2)$ with respect to the product manifold $EG\times M$. Since R is G-equivariant by construction, any Ad-invariant polynomial in the curvature yields a closed G-invariant differential form on $EG\times M$. In particular the Chern character $\mathrm{ch}(V)$ and the Todd class $\mathrm{Td}(V)$ are represented by

$$\mathrm{ch}(iR/2\pi) = \mathrm{tr}(\exp(iR/2\pi)) \ , \qquad \mathrm{Td}(iR/2\pi) = \det\left(\frac{iR/2\pi}{e^{iR/2\pi}-1}\right) \ , \tag{6.7}$$

and a classical homotopy argument shows that their respective cohomology classes in $H^\bullet(EG\times_G M)$ do not depend on the particular choice of connection ∇_0 on V.

6.2. **Characteristic map.** We now explain our construction of Connes' characteristic map from the G-equivariant cohomology of M to the periodic cyclic cohomology of $\mathcal{A} = C_c^\infty(M) \rtimes G$. The idea is to twist the universal tensor extension of the group ring $\mathbb{C}G$

$$0 \to J\mathbb{C}G \to T\mathbb{C}G \to \mathbb{C}G \to 0$$

by the DG algebra of smooth differential forms on $EG \times M$. Indeed G acts on both manifolds EG and M (from the right), and the induced action (from the left) by pullback on the graded-commutative algebra of differential forms $\Omega(EG \times M)$ commutes with the de Rham differential d. Let $\Omega_p(EG \times M)$ be the subalgebra of differential forms $\alpha \in \Omega(EG \times M)$ which have compact M-support at any point of EG. The crossed product

$$\mathcal{G} = \Omega_p(EG \times M) \rtimes G \tag{6.8}$$

is naturally a (non-commutative) DG algebra. The product of two elements reads

$$(\alpha \otimes U_{g_1})(\beta \otimes U_{g_2}) = \alpha \wedge U_{g_1}(\beta) \otimes U_{g_1 g_2}$$

for all $\alpha, \beta \in \Omega_p(EG \times M)$ and $g_i \in G$, where $U_{g_1}(\beta)$ is the pullback of β by the diffeomorphism g_1. The differential reads $d(\alpha \otimes U_g) = d\alpha \otimes U_g$. An algebra extension of $\mathcal{G}$ is defined as the vector space

$$\mathcal{H} = \Omega_p(EG \times M) \otimes T\mathbb{C}G \ , \tag{6.9}$$

graded by the differential form degree ($T\mathbb{C}G$ is trivially graded), and endowed with the twisted product

$$(\alpha\otimes U_{g_1} \otimes \ldots \otimes U_{g_n})(\beta \otimes U_{g_{n+1}} \otimes \ldots \otimes U_{g_{n+m}}) = \alpha \wedge U_{g_1 \ldots g_n}(\beta) \otimes U_{g_1} \otimes \ldots \otimes U_{g_{n+m}} \ .$$

Also the de Rham differential is extended to $\mathcal{H}$ by $d(\alpha\otimes U_{g_1} \otimes \ldots \otimes U_{g_n}) = d\alpha \otimes U_{g_1} \otimes \ldots \otimes U_{g_n}$. Clearly the obvious multiplication map $\mathcal{H} \to \mathcal{G}$, $\alpha\otimes U_{g_1} \otimes \ldots \otimes U_{g_n} \mapsto \alpha \otimes U_{g_1 \ldots g_n}$ is a morphism of DG algebras, hence its kernel $\mathcal{I} = \Omega_p(EG \times M) \otimes J\mathbb{C}G$ is a two-sided DG ideal. We define a completion of $\mathcal{H}$ as

$$\widehat{\mathcal{H}} = \bigoplus_{k\geqslant 0} \varprojlim_n \left(\Omega_p^k(EG \times M) \otimes T\mathbb{C}G/(J\mathbb{C}G)^n\right) \ . \tag{6.10}$$

The product and differential on $\mathcal{H}$ extend in an obvious way to $\widehat{\mathcal{H}}$. If EG were a finite-dimensional manifold, the sum over the form degree k would be finite and $\widehat{\mathcal{H}} = \varprojlim_n \Omega_p(EG\times M)\otimes T\mathbb{C}G/(J\mathbb{C}G)^n$ would coincide with the $\mathcal{I}$-adic completion of $\mathcal{H}$ as in [10]. This does not hold for our construction of EG and (6.10) is a strictly smaller algebra.

Now we view $C_c^\infty(M) \subset \Omega_p^0(EG \times M)$ as the subalgebra of scalar functions which are *constant* in the direction EG. This identification is G-equivariant, hence extends to a morphism of algebras $\rho : \mathcal{A} \to \mathcal{G}$. The universal property of the tensor algebra thus yields an homomorphism $\rho_* : T\mathcal{A} \to \mathcal{H}$ explicitly given by

$$\rho_*(f_1 U_{g_1} \otimes f_2 U_{g_2} \ldots \otimes f_n U_{g_n}) = f_1 U_{g_1}(f_2) \ldots U_{g_1 \ldots g_{n-1}}(f_n) \otimes U_{g_1} \otimes U_{g_2} \otimes \ldots \otimes U_{g_n}$$

on any n-tensor, $f_i \in C_c^\infty(M)$, $g_i \in G$. One easily checks that ρ_* carries the ideal $(J\mathcal{A})^n$ to $\Omega_p^0(EG \times M) \otimes (J\mathbb{C}G)^n$ for all n, hence extends to an homomorphism of completed algebras

$$\rho_* \ : \ \widehat{T}\mathcal{A} \to \widehat{\mathcal{H}} \ . \tag{6.11}$$

We recall that the (completed) space of non-commutative differential forms $\widehat{\Omega}\widehat{T}\mathcal{A} = \prod_{n=0}^\infty \Omega^n \widehat{T}\mathcal{A}$ endowed with the total differential $(b + B)$ computes the periodic cyclic homology of $\mathcal{A}$.

The next step is a slight generalization of the Cuntz-Quillen X-complex [5] to the DG algebra setting. Indeed the de Rham differential d on $\widehat{\mathcal{H}}$ extends in a unique way to the $\widehat{\mathcal{H}}$-bimodule of universal 1-forms $\Omega^1\widehat{\mathcal{H}}$ by

$$d(\hat{h}_0\mathbf{d}\hat{h}_1) = (d\hat{h}_0)\mathbf{d}\hat{h}_1 + (-1)^{|\hat{h}_0|+1}\hat{h}_0\mathbf{d}(d\hat{h}_1)\ , \qquad \forall\ \hat{h}_0\mathbf{d}\hat{h}_1 \in \Omega^1\widehat{\mathcal{H}}\ ,$$

where $|\hat{h}_0|$ denotes the degree of $\hat{h}_0$. Then d is a differential of odd degree on $\Omega^1\widehat{\mathcal{H}}$ endowed with its total grading, compatible with the bimodule structure, and commutes in the graded sense with the universal differential $\mathbf{d}$. We define the X-complex of the DG algebra $(\widehat{\mathcal{H}}, d)$ as the $\mathbb{Z}_2$-graded supercomplex

$$X(\widehat{\mathcal{H}}, d)\ :\ \widehat{\mathcal{H}} \rightleftarrows \Omega^1\widehat{\mathcal{H}}_\natural\ , \tag{6.12}$$

where $\Omega^1\widehat{\mathcal{H}}_\natural = \Omega^1\widehat{\mathcal{H}}/[\widehat{\mathcal{H}}, \Omega^1\widehat{\mathcal{H}}]$ is the quotient of the bimodule of universal 1-forms by its subspace of graded commutators. We denote by $\natural\hat{h}_0\mathbf{d}\hat{h}_1$ the class of $\hat{h}_0\mathbf{d}\hat{h}_1$. The map $\natural\mathbf{d} : \widehat{\mathcal{H}} \to \Omega^1\widehat{\mathcal{H}}_\natural$ is simply $\hat{h} \mapsto \natural\mathbf{d}\hat{h}$, while $\overline{b} : \Omega^1\widehat{\mathcal{H}}_\natural \to \widehat{\mathcal{H}}$ descends from the graded Hochschild boundary operator $\hat{h}_0\mathbf{d}\hat{h}_1 \mapsto (-1)^{|\hat{h}_0|}[\hat{h}_0, \hat{h}_1]$. One has $\natural\mathbf{d}\circ\overline{b} = 0$, $\overline{b}\circ\natural\mathbf{d} = 0$, and the odd differential $\natural\mathbf{d}\oplus\overline{b}$ commutes in the graded sense with d, so that $X(\widehat{\mathcal{H}}, d)$ endowed with the total differential $(\natural\mathbf{d} \oplus \overline{b}) + d$ is a $\mathbb{Z}_2$-graded complex. The proof of the following lemma is a straightforward computation.

LEMMA 6.1. *The linear map of even degree $\chi(\rho_*, d)$ from $\widehat{\Omega}\widehat{T}\mathcal{A}$ to $X(\widehat{\mathcal{H}}, d)$ given by*

$$\begin{aligned} \chi(\rho_*, d)(\hat{a}_0\mathbf{d}\hat{a}_1 \ldots \mathbf{d}\hat{a}_n) = &\\ \frac{1}{(n+1)!}\sum_{i=0}^{n}&(-1)^{i(n-i)}d\rho_*(\hat{a}_{i+1})\ldots d\rho_*(\hat{a}_n)\,\rho_*(\hat{a}_0)\,d\rho_*(\hat{a}_1)\ldots d\rho_*(\hat{a}_i)\\ +\frac{1}{n!}\sum_{i=1}^{n}&\natural\big(\rho_*(\hat{a}_0)\,d\rho_*(\hat{a}_1)\ldots \mathbf{d}\rho_*(\hat{a}_i)\ldots d\rho_*(\hat{a}_n)\big) \end{aligned} \tag{6.13}$$

for all $\hat{a}_i \in \widehat{T}\mathcal{A}$, is a cocycle in the Hom*-complex* $\mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{H}}, d))$.

Note that a differential form $d\rho_*(\hat{a}) \in \widehat{\mathcal{H}}$ has always degree 0 in the direction EG and degree 1 in the direction M, so that $\chi(\rho_*, d)$ vanishes on $\Omega^n\widehat{T}\mathcal{A}$ whenever $n > \dim M$ and thus extends to the direct product $\widehat{\Omega}\widehat{T}\mathcal{A} = \prod_{n=0}^{\infty}\Omega^n\widehat{T}\mathcal{A}$. This would not be the case if the image of $\mathcal{A}$ in $\mathcal{G}$ consisted in non-constant functions in the direction EG.

The last step associates a cocycle $\lambda'_\omega \in \mathrm{Hom}(X(\widehat{\mathcal{H}}, d), \mathbb{C})$ to any closed G-invariant differential form $\omega \in \Omega(EG \times M)$. To that purpose we define the X-complex *localized at units* as the vector space

$$X(\widehat{\mathcal{H}}, d)_{[EG\times M]} = \bigoplus_{k\geqslant 0}\varprojlim_{n}\big(\Omega^k_p(EG \times M) \otimes \Omega^n\mathbb{C}G_{[1]}\big)\ , \tag{6.14}$$

where $\Omega^n\mathbb{C}G_{[1]}$ is the space of completed universal n-forms localized at the unit $1 \in G$:

$$U_{g_0}dU_{g_1}\ldots dU_{g_n} \in \Omega^n\mathbb{C}G_{[1]} \Leftrightarrow g_0g_1\ldots g_n = 1\ ,$$

$$dU_{g_1}\ldots dU_{g_n} \in \Omega^n\mathbb{C}G_{[1]} \Leftrightarrow g_1\ldots g_n = 1\ .$$

$X(\widehat{\mathcal{H}}, d)_{[EG\times M]}$ is a quotient of $X(\widehat{\mathcal{H}}, d)$. Indeed a projection $c : \mathcal{H} \to \Omega_p(EG \times M) \otimes \Omega^+\mathbb{C}G_{[1]}$ is defined by

$$\begin{aligned} c(\alpha \otimes U_{g_0} \otimes (U_{g_1g_2} - U_{g_1}U_{g_2}) \otimes \ldots \otimes (U_{g_{2n-1}g_{2n}} - U_{g_{2n-1}}U_{g_{2n}}))&\\ = (-1)^n n!\,\alpha \otimes U_{g_0}dU_{g_1}dU_{g_2}\ldots dU_{g_{2n-1}}dU_{g_{2n}}& \end{aligned}$$

if $g_0 g_1 \dots g_n = 1$, and

$$c(\alpha \otimes (U_{g_1 g_2} - U_{g_1} U_{g_2}) \otimes \dots \otimes (U_{g_{2n-1} g_{2n}} - U_{g_{2n-1}} U_{g_{2n}})) = (-1)^n n!\, \alpha \otimes dU_{g_1} dU_{g_2} \dots dU_{g_{2n-1}} dU_{g_{2n}}$$

if $g_1 \dots g_n = 1$, and c vanishes on all other tensors for which the localization condition fails. On the other hand, $c : \Omega^1 \mathcal{H}_\natural \to \Omega_p(EG \times M) \otimes \Omega^- \mathbb{C}G_{[1]}$ is uniquely specified by

$$c(\natural(\alpha \otimes U_{g_0} \otimes (U_{g_1 g_2} - U_{g_1} U_{g_2}) \otimes \dots \otimes (U_{g_{2n-1} g_{2n}} - U_{g_{2n-1}} U_{g_{2n}}))\mathbf{d}(\beta \otimes U_{g_{2n+1}})) = (-1)^{n+|\beta|} n!\, \alpha \wedge U_{g_0 \dots g_{2n}}(\beta) \otimes U_{g_0} dU_{g_1} dU_{g_2} \dots dU_{g_{2n-1}} dU_{g_{2n}} dU_{g_{2n+1}}$$

and

$$c(\natural(\alpha \otimes (U_{g_1 g_2} - U_{g_1} U_{g_2}) \otimes \dots \otimes (U_{g_{2n-1} g_{2n}} - U_{g_{2n-1}} U_{g_{2n}}))\mathbf{d}(\beta \otimes U_{g_{2n+1}})) = (-1)^{n+|\beta|} n!\, \alpha \wedge U_{g_1 \dots g_{2n}}(\beta) \otimes dU_{g_1} dU_{g_2} \dots dU_{g_{2n-1}} dU_{g_{2n}} dU_{g_{2n+1}}$$

if the localization condition holds, and $|\beta|$ is the degree of the differential form β. The map c extends to a well-defined projection $X(\widehat{\mathcal{H}}, d) \to X(\widehat{\mathcal{H}}, d)_{[EG \times M]}$, and the boundary operators $\natural\mathbf{d}$, $\overline{b}$ and d descend to boundary operators on the quotient. Then for any G-invariant differential form $\omega \in \Omega(EG \times M)$ we define a cochain $\lambda_\omega \in \mathrm{Hom}(X(\widehat{\mathcal{H}}, d)_{[EG \times M]}, \mathbb{C})$ by

$$\lambda_\omega(\alpha \otimes U_{g_0} dU_{g_1} \dots dU_{g_n}) = \int_{\widetilde{\Delta}(g_1, \dots, g_n) \times M} \alpha \wedge \omega \tag{6.15}$$

where $\widetilde{\Delta}(g_1, \dots, g_n) \subset EG$ denotes the n-simplex with vertices $g_1 \dots g_n$, $g_2 \dots g_n$, $\dots$, g_n, 1, and $\lambda_\omega(\alpha \otimes dU_{g_1} \dots dU_{g_n}) = 0$. Remark that the integral above is well-defined, because α has compact M-support at any point of EG. Also, by the definition of the completion $\widehat{\mathcal{H}}$, the degree k of the differential form α is fixed while n can be arbitrarily large. This causes no trouble since the r.h.s. of (6.15) vanishes for large n. A direct computation involving Stokes theorem yields

Lemma 6.2. *The map sending any G-invariant differential form $\omega \in \Omega(EG \times M)$ to the cochain $\lambda'_\omega = \lambda_\omega \circ c \in \mathrm{Hom}(X(\widehat{\mathcal{H}}, d), \mathbb{C})$ is a morphism of complexes.*

Recall that the Hom-complex $\mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, \mathbb{C})$ computes the periodic cyclic cohomology of $\mathcal{A}$. Collecting Lemmas 6.1 and 6.2 one thus gets

Proposition 6.3. *Let G be a discrete group acting by orientation-preserving diffeomorphisms on a smooth oriented manifold M, and $\mathcal{A} = C_c^\infty(M) \rtimes G$. The map sending any G-invariant differential form $\omega \in \Omega(EG \times M)$ to the cochain $\lambda'_\omega \circ \chi(\rho_*, d) \in \mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, \mathbb{C})$ is a morphism of complexes. We denote by*

$$\Phi : H^\bullet(EG \times_G M) \to HP^\bullet(\mathcal{A}) \tag{6.16}$$

the corresponding map in cohomology.

7. Algebraic JLO formula

We come back to the situation of section 2, where M is a foliated manifold and $G \subset \mathrm{Diff}(M)$ is a discrete group of foliated diffeomorphisms. We first define some useful algebras. Let EG be the universal G-bundle over the classifying space BG. We take EG as the geometric realization of the simplicial set $N\overline{G}_\bullet$ and consider the induced action of G on the DG algebra of smooth differential forms $(\Omega(EG), d)$. The crossed product

$$\mathcal{G} = \Omega(EG) \rtimes G \tag{7.1}$$

is a particular case of the DG algebra constructed in section 6, where the manifold M is reduced to a point. Hence considering twisted tensor products with the tensor algebra of $\mathbb{C}G$, one gets the DG algebra extension of $\mathcal{G}$ and its completion:

$$(7.2) \qquad \mathcal{H} = \Omega(EG) \otimes T\mathbb{C}G \ , \qquad \widehat{\mathcal{H}} = \bigoplus_{k\geqslant 0} \varprojlim_n \big(\Omega^k(EG) \otimes T\mathbb{C}G/(J\mathbb{C}G)^n\big) \ .$$

Now let $\mathcal{S}(M) = \mathcal{L}(M)[[\varepsilon]]$ be the G-algebra of formal power series constructed in section 3, together with its subalgebras $\mathcal{D}(M)$, $\mathcal{D}_c(M)$, and the corresponding $\mathcal{D}(M)$-bimodule of trace-class operators $\mathcal{T}(M)$. The space $\Omega(EG, \mathcal{S}(M))$ of smooth differential forms on EG with values in $\mathcal{S}(M)$ is a DG algebra, for the pointwise product of differential forms and de Rham differential d on EG. We endow $\Omega(EG, \mathcal{S}(M))$ with the G-action which combines the actions of G on EG and $\mathcal{S}(M)$ respectively. The crossed product

$$(7.3) \qquad \mathcal{U} = \Omega(EG, \mathcal{S}(M)) \rtimes G$$

is therefore a DG algebra, with differential $d(\alpha \otimes U_g) = d\alpha \otimes U_g$ for all $\alpha \in \Omega(EG, \mathcal{S}(M))$ and $g \in G$. Considering twisted tensor products with the tensor algebra of $\mathbb{C}G$, a DG algebra extension of $\mathcal{U}$ is defined as above:

$$(7.4) \qquad \mathcal{V} = \Omega(EG, \mathcal{S}(M)) \otimes T\mathbb{C}G \ , \qquad \widehat{\mathcal{V}} = \bigoplus_{k\geqslant 0} \varprojlim_n \big(\Omega^k(EG, \mathcal{S}(M)) \otimes T\mathbb{C}G/(J\mathbb{C}G)^n\big) \ .$$

The next step is the construction of an homomorphism from the $J\mathcal{A}$-adic completion $\widehat{T}\mathcal{A}$ of the tensor algebra over $\mathcal{A} = C^\infty_c(S^*_H M) \rtimes G$, to $\widehat{\mathcal{V}}$. To this end, choose any linear splitting $C^\infty_c(S^*_H M) \to \mathcal{S}_{H,c}(M)$ of the leading symbol homomorphism $\mathcal{S}_{H,c}(M) \to C^\infty_c(S^*_H M)$. Since the trivial line bundle $M \times \mathbb{C}$ over M can be identified with the zero-degree part of the exterior bundle $E = \Lambda^\bullet(T^*M \otimes \mathbb{C})$, the space of scalar symbols $\mathcal{S}_{H,c}(M)$ is a direct summand in the space $\mathcal{S}_{H,c}(M, E)$. Hence composing the linear map $C^\infty_c(S^*_H M) \to \mathcal{S}_{H,c}(M, E)$ with the representation of symbols $L : \mathcal{S}_{H,c}(M, E) \to \mathcal{D}_c(M)$ given by left multiplication, leads to a linear map $\sigma : \mathcal{A} \to \mathcal{U}$. By the universal property of the tensor algebra, σ extends to an homomorphism of algebras

$$(7.5) \qquad \sigma_* \ : \ \widehat{T}\mathcal{A} \to \widehat{\mathcal{V}} \ .$$

Of course the latter depends on the choice of linear splitting, but two different splittings lead to homotopic homomorphisms in the sense of Cuntz and Quillen [5].

We now discuss superconnections [11]. Remark that the DG algebra $\Omega(EG, \mathcal{D}(M))$ acts by left and right multipliers on $\mathcal{U}$, $\mathcal{V}$ and $\widehat{\mathcal{V}}$. Let $D_0 \in \mathcal{D}(M)$ be a generalized Dirac operator as defined in section 5) and consider the function $D \in \Omega^0(EG, \mathcal{D}(M))$ on the classifying space, with values in Dirac operators, given by

$$D(g_0, \ldots, g_n)(s_0, \ldots, s_n) = \sum_{i=0}^n s_i \operatorname{Ad}^{-1}_{g_i}(D_0)$$

for all $(g_0, \ldots, g_n) \in N\overline{G}_n$ and $(s_0, \ldots, s_n) \in \Delta^n$. Since the action of an element $g \in G$ on EG carries $(g_0, \ldots, g_n)$ to $(g_0 g, \ldots, g_n g)$, the function D is G-invariant by construction. The superconnection (ε is the formal parameter in $\mathcal{S}(M)$)

$$(7.6) \qquad \mathbf{D} = \mathbf{i}\varepsilon d + D \ ,$$

acting on $\widehat{\mathcal{V}}$ by graded commutators, is a graded derivation. Its curvature is the inhomogeneous differential form

$$\mathbf{D}^2 = D^2 + \mathbf{i}\varepsilon dD \ \in \Omega^0(EG, \mathcal{D}(M)) \oplus \Omega^1(EG, \mathcal{D}(M))$$

where $dD(g_0, \ldots, g_n)(s_0, \ldots, s_n) = \sum_{i=0}^n ds_i \operatorname{Ad}^{-1}_{g_i}(D_0)$. Choose a positive Heisenberg-elliptic symbol $q \in \mathcal{S}^1_H(M)$ of order one on M. Extend it to a Heisenberg-elliptic symbol $\widetilde{q} \in \mathcal{S}^1_H(M, E)$, requiring that the leading symbol of $\widetilde{q}$ remains of scalar type. Using the left representation of symbols $L : \mathcal{S}_H(M) \to \mathcal{D}(M)$, one gets a constant function $\widetilde{q}_L \in \Omega^0(EG, \mathcal{D}(M))$ over EG. Let κ be an "infinitesimal" odd parameter: $\kappa^2 = 0$. The new superconnection

$$\nabla = \mathbf{D} + \kappa \ln \widetilde{q}_L \ , \tag{7.7}$$

acting on the DG algebra $\widehat{\mathcal{V}}[\kappa] = \widehat{\mathcal{V}} \oplus \kappa \widehat{\mathcal{V}}$ by graded commutators, is a graded derivation. Its curvature is the inhomogeneous differential form

$$\nabla^2 = \mathbf{D}^2 + \kappa [\ln \widetilde{q}_L, \mathbf{D}] \ \in \Omega^0(EG, \mathcal{D}(M)[\kappa]) \oplus \Omega^1(EG, \mathcal{D}(M)[\kappa]) \ .$$

The homomorphism σ_* and the superconnection ∇ are the main ingredients of a JLO-type formula for a cocycle of odd degree in $\mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{H}}, d)_{[EG]})$, where $X(\widehat{\mathcal{H}}, d)_{[EG]}$ is the X-complex *localized at units* defined in section 6. We introduce as intermediate step a cochain of even degree $\chi^{\mathrm{Tr}_s}(\sigma_*, \nabla) \in \mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{H}}[\kappa], d))$ defined on any n-form $\hat{a}_0 \mathbf{d}\hat{a}_1 \ldots \mathbf{d}\hat{a}_n \in \Omega^n \widehat{T}\mathcal{A}$ by

$$\begin{aligned} \chi^{\mathrm{Tr}_s}(\sigma_*, \nabla)(\hat{a}_0 \mathbf{d}\hat{a}_1 \ldots \mathbf{d}\hat{a}_n) = \\ \sum_{i=0}^{n} (-)^{i(n-i)} \int_{\Delta_{n+1}} \mathrm{Tr}_s\big(e^{-t_{i+1}\nabla^2}[\nabla, \sigma_{i+1}] \ldots e^{-t_{n+1}\nabla^2} \sigma_0 \, e^{-t_0\nabla^2}[\nabla, \sigma_1] \ldots e^{-t_i\nabla^2}\big) dt \\ + \sum_{i=1}^{n} \int_{\Delta_n} \mathrm{Tr}_s\big(\natural \sigma_0 \, e^{-t_0\nabla^2}[\nabla, \sigma_1] \ldots e^{-t_{i-1}\nabla^2} \mathbf{d}\sigma_i \, e^{-t_i\nabla^2} \ldots [\nabla, \sigma_n] e^{-t_n\nabla^2}\big) dt \end{aligned} \tag{7.8}$$

where $\sigma_i = \sigma_*(\hat{a}_i) \in \widehat{\mathcal{V}}$ for all i and $\mathrm{Tr}_s : \mathcal{T}(M) \to \mathbb{C}$ is the graded trace of section 4. This expression is well-defined, because it involves a Duhamel-type expansion of the heat operator $\exp(-\nabla^2)$ which belongs to the domain of the trace. In fact $\chi^{\mathrm{Tr}_s}(\sigma_*, \nabla)$ composed with the projection onto the X-complex localized at units $c : X(\widehat{\mathcal{H}}[\kappa], d) \to X(\widehat{\mathcal{H}}[\kappa], d)_{[EG]}$ is a cocycle in the Hom-complex $\mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{H}}[\kappa], d)_{[EG]})$. This crucially depends on the formal identities $\mathbf{d}D = 0$ and $\mathbf{d} \ln \widetilde{q}_L = 0$ which hold in the localized complex. Since $\kappa^2 = 0$, this cocycle is actually a polynomial of degree one with respect to κ. Define

$$\chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln \widetilde{q}_L) = \frac{\partial}{\partial \kappa} \chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D} + \kappa \ln \widetilde{q}_L) \ . \tag{7.9}$$

The latter yields a cocycle of odd degree in $\mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{H}}, d)_{[EG]})$. By a classical homotopy argument, its cohomology class does not depend on any choice regarding the linear map $\sigma : \mathcal{A} \to \mathcal{U}$, the superconnection $\mathbf{D}$, and the elliptic symbol $\widetilde{q}$. The following proposition identifies the composition of this canonical class with the class of the chain map $\lambda'_1 \in \mathrm{Hom}(X(\widehat{\mathcal{H}}, d), \mathbb{C})$ of Lemma 6.2, for $\omega = 1$.

PROPOSITION 7.1. *Let $D_0 \in \mathcal{D}(M)$ be a de Rham-Dirac operator, $D \in \Omega^0(EG, \mathcal{D}(M))$ the associated G-invariant function on the universal bundle, and $\mathbf{D} = d + D$ the corresponding superconnection. Then $\lambda'_1 \circ \chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln \widetilde{q}_L)$ is the cocycle of Proposition 2.7.*

PROOF. We work in a local foliated chart. First, we must observe that D is still a de Rham - Dirac type operator, essentially because $d_R = \mathbf{i} p_i \psi^i$ is G-invariant. Thus, D is of the form

$$D = -\mathbf{i}\varepsilon d_R + \overline{\psi}_{iR} \left(\partial_{p_i} + \sum_{|\alpha| \geqslant 2} (r^i_\alpha)_L \partial^\alpha_p \right)$$

The r^i_α are scalar functions on $EG \times M$. With d the exterior differential on EG, one has

$$dD = \sum_{|\alpha| \geqslant 2} \overline{\psi}_{iR} (dr^i_\alpha)_L \partial^\alpha_p$$

Moreover, D^2 reads

$$(7.10)\quad -D^2 = \Delta + \varepsilon(p_{iL}\partial_{p_i} + (\psi^i\overline{\psi}_i)_R) + \varepsilon\left(\sum_{|\alpha|\geqslant 2}((a^i_\alpha)_L\partial_{x^i} + (a^i_\alpha p_i)_L)\partial^\alpha_p + \sum_{|\alpha|\geqslant 1}(b^i_{\alpha j})_L(\psi^j\overline{\psi}_i)_R\partial^\alpha_p\right)$$

where $\Delta = \mathbf{i}\varepsilon\partial_{x^i}\partial_{p_i}$, and the coefficients in the sums over $|\alpha|\geqslant \ldots$ are also scalar functions on $EG\times M$. Recall also that

$$\mathbf{D} = d + D,\qquad \mathbf{D}^2 = dD + D^2,\qquad \nabla = \mathbf{D} + \kappa\ln\widetilde{q}_L,\qquad \nabla^2 = \mathbf{D}^2 + \kappa[\ln\widetilde{q}_L, \mathbf{D}]$$

We know that dD is proportional to $\overline{\psi}_R$. The symbol q being constant in the direction EG, one has $d\ln\widetilde{q}_L = 0$. Moreover $\ln\widetilde{q}_L$ commutes with p_R, hence the commutator $[\ln\widetilde{q}_L, \mathbf{D}]$ is also proportional to $\overline{\psi}_R$. Finally the elements $\sigma = \sigma_*(\hat{a}) \in \Omega^0(EG, \mathcal{D}_c(M))\otimes\widehat{T}\mathbb{C}G$ are constant in the direction EG, hence we have

$$[\nabla, \sigma] = [\overline{\psi}_{iR}(\partial_{p_i} + \ldots), \sigma] + \kappa[\ln\widetilde{q}_L, \sigma]$$

Now observe that the graded trace Tr_s selects the term proportional to $(\psi^1\overline{\psi}_1\ldots\psi^n\overline{\psi}_n)_R$. The generalized Laplacian D^2 already brings terms proportional to 1 or $(\psi\overline{\psi})_R$ in the right sector. Thus the terms proportional to $\overline{\psi}_R$ in dD, $[\ln\widetilde{q}_L, \mathbf{D}]$ and $[\nabla, \sigma]$ break the balance between the ψ_R and the $\overline{\psi}_R$ and must give a zero contribution to the cocycle. Hence we can consider that

$$\nabla^2 \simeq D^2\ ,\qquad [\nabla, \sigma] \simeq \kappa[\ln\widetilde{q}_L, \sigma]\ .$$

A first consequence, taking into account $\kappa^2 = 0$, is that the cocycle $\chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln\widetilde{q}_L)$ should contain exactly one commutator $[\nabla, \sigma]$. Thus the only non-zero contributions to this cocycle are :

$$\begin{aligned}\chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln\widetilde{q}_L)(\hat{a}_0\mathbf{d}\hat{a}_1) &= \int_{\Delta_2}\mathrm{Tr}_s\left(e^{-t_1D^2}[\ln\widetilde{q}_L, \sigma_1]e^{-t_2D^2}\sigma_0 e^{-t_0D^2}\right)dt\\ &\quad + \int_{\Delta_2}\mathrm{Tr}_s\left(e^{-t_2D^2}\sigma_0 e^{-t_1D^2}[\ln\widetilde{q}_L, \sigma_1]e^{-t_0D^2}\right)dt\end{aligned}$$

$$\begin{aligned}\chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln\widetilde{q}_L)(\hat{a}_0\mathbf{d}\hat{a}_1\mathbf{d}\hat{a}_2) &= \int_{\Delta_2}\mathrm{Tr}_s\left(\natural\sigma_0 e^{-t_0D^2}[\ln\widetilde{q}_L, \sigma_1]e^{-t_1D^2}\mathbf{d}\sigma_2 e^{-t_2D^2}\right)dt\\ &\quad + \int_{\Delta_2}\mathrm{Tr}_s\left(\natural\sigma_0 e^{-t_0D^2}\mathbf{d}\sigma_1 e^{-t_1D^2}[\ln\widetilde{q}_L, \sigma_2]e^{-t_2D^2}\right)dt\end{aligned}$$

A second consequence is that the images of these quantities under the projection $c : X(\widehat{\mathcal{H}}, d) \to X(\widehat{\mathcal{H}}, d)_{[EG]}$ belong to the subspace $\Omega^0(EG)\otimes\widehat{\Omega}\mathbb{C}G_{[1]}$ of the localized X-complex, in other words they are scalar functions over EG. Thus, their evaluation on the cocycle λ_1 drops all the components in $\Omega^0(EG)\otimes\Omega^k\mathbb{C}G$ for $k\geqslant 1$, and the remaining components in $\Omega^0(EG)\otimes\Omega^0\mathbb{C}G$ are simply localized at the unit. In particular

$$\lambda_1'\circ\chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln\widetilde{q}_L)(\hat{a}_0\mathbf{d}\hat{a}_1\mathbf{d}\hat{a}_2) = 0$$

and $\lambda_1'\circ\mathrm{Tr}_s$ *behaves like a graded trace* in the only remaining term :

$$\begin{aligned}\lambda_1'\circ\chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln\widetilde{q}_L)(\hat{a}_0\mathbf{d}\hat{a}_1) &= \int_{\Delta_2}\mathrm{Tr}_s\left(e^{-t_1D^2}[\ln\widetilde{q}_L, \sigma_1]e^{-t_2D^2}\sigma_0 e^{-t_0D^2}\right)_{[1]}dt\\ &\quad + \int_{\Delta_2}\mathrm{Tr}_s\left(e^{-t_2D^2}\sigma_0 e^{-t_1D^2}[\ln\widetilde{q}_L, \sigma_1]e^{-t_0D^2}\right)_{[1]}dt\\ &= \int_0^1\mathrm{Tr}_s\left(\natural\sigma_0 e^{-tD^2}[\ln\widetilde{q}_L, \sigma_1]e^{-(1-t)D^2}\right)_{[1]}dt\end{aligned}$$

The integrand $\mathrm{Tr}_s(\natural\sigma_0 e^{-tD^2}[\ln \widetilde{q}_L, \sigma_1]e^{-(1-t)D^2})_{[1]}$ does not depend on $t \in [0,1]$. Indeed

$$\frac{d}{dt}\mathrm{Tr}_s\left(\natural\sigma_0 e^{-tD^2}[\ln \widetilde{q}_L, \sigma_1]e^{-(1-t)D^2}\right)_{[1]} = -\mathrm{Tr}_s\left(\natural\sigma_0 e^{-tD^2}[D^2,[\ln \widetilde{q}_L, \sigma_1]]e^{-(1-t)D^2}\right)_{[1]}$$
$$= \mathrm{Tr}_s\left(\natural[D,\sigma_0] e^{-tD^2}[D,[\ln \widetilde{q}_L, \sigma_1]]e^{-(1-t)D^2}\right)_{[1]}$$

The last equality comes from $[D^2, X] = D[D,X] + [D,X]D$ and the graded trace property. The above quantity vanishes, because the commutators with D only bring terms proportional to $\overline{\psi}_R$. Therefore, the integrand may be replaced by its value at $t=0$, and we are left with

$$\lambda_1' \circ \chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln \widetilde{q}_L)(\hat{a}_0 \mathbf{d}\hat{a}_1) = \mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]e^{-D^2}\right)_{[1]}$$

Seeing $-D^2$ as a perturbation of the flat Laplacian $\Delta + u$, and using a Duhamel expansion, one gets

$$\mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]e^{-D^2}\right) = \sum_{k\geqslant 0}\int_{\Delta_k}\mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]\sigma_\Delta^{t_0}(u)\dots\sigma_\Delta^{t_0+\dots+t_{k-1}}(u)\exp(\Delta)\right)dt$$

Then, we want to move the operators ∂_x and ∂_p to the right in each term of the sum above, and look at when we have an exact balance in their powers. Otherwise, it will vanish under the graded trace Tr_s by definition. A ∂_p can be absorbed with a p_L by commutation, and ∂_x may appear in $\sigma_\Delta^t(p_L) = p_L + t[\Delta, p_L]) = p_L + it\varepsilon\partial_x$. With this elements at hand, we conclude that the presence of the sums over $|\alpha|$ in (7.10) prevent an exact balance between ∂_x and ∂_p. So, we can neglect these parts in D^2 and get

$$\begin{aligned}\mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]e^{-D^2}\right)_{[1]} &= \mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]\exp(\Delta + \varepsilon p_L\cdot\partial_p + \varepsilon(\psi^i\overline{\psi}_i)_R)\right)_{[1]}\\ &= \mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]\varepsilon^n(\psi^1\overline{\psi}_1\dots\psi^n\overline{\psi}_n)_R\exp(\Delta + \varepsilon p_L\cdot\partial_p)\right)_{[1]}\\ &= ⨍(\sigma_0[\ln \widetilde{q}_L, \sigma_1])_{[1]}\langle\langle\varepsilon^n(\psi^1\overline{\psi}_1\dots\psi^n\overline{\psi}_n)_R\exp(\Delta + \varepsilon p_L\cdot\partial_p)\rangle\rangle[n]\\ &= ⨍(\sigma_0[\ln \widetilde{q}_L, \sigma_1])_{[1]}\langle\exp(\Delta + \varepsilon p_L\cdot\partial_p)\rangle\end{aligned}$$

where in the second equality, we split the exponential. Using the result of Example 4.4, applied to the scalar matrix $R = \varepsilon$, we get

$$\mathrm{Tr}_s\left(\sigma_0[\ln \widetilde{q}_L, \sigma_1]e^{-D^2}\right)_{[1]} = ⨍(\sigma_0[\ln \widetilde{q}_L, \sigma_1])_{[1]}$$

which is the equivariant Radul cocycle of Proposition 2.7. □

Finally consider the diagonal action of G on the product $EG \times S_H^*M$, and its induced action on the space of differential forms $\Omega(EG \times S_H^*M)$ with total de Rham differential d. As in section 6 we form the DG algebra

(7.11) $\quad \mathcal{X} = \Omega_p(EG \times S_H^*M) \rtimes G\ ,$

and its DG extension

(7.12) $\quad \mathcal{Y} = \Omega_p(EG \times S_H^*M) \otimes T\mathbb{C}G\ , \qquad \widehat{\mathcal{Y}} = \bigoplus_{k\geqslant 0}\varprojlim_n\left(\Omega_p^k(EG \times S_H^*M)\otimes T\mathbb{C}G/(J\mathbb{C}G)^n\right)\ .$

Viewing the algebra $C_c^\infty(S_H^*M)$ as constant functions over EG leads to an homomorphism $\rho : \mathcal{A} \to \mathcal{X}$ which extends to

(7.13) $\quad \rho_*\ :\ \widehat{T}\mathcal{A} \to \widehat{\mathcal{Y}}\ .$

Let $\chi(\rho_*, d) \in \mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{Y}}, d))$ be the cocycle of Lemma 6.1. The integration of differential forms along the fibers of the projection $EG \times S^*_H M \to EG$ yields a morphism of complexes

$$(7.14) \quad \int_{S^*_H M} : X(\widehat{\mathcal{Y}}, d) \to X(\widehat{\mathcal{H}}, d) \; .$$

For the Proposition below we choose a positive Heisenberg-elliptic symbol $q_0 \in \mathcal{S}^1_H(M)$ of order one on M, and extend it to a Heisenberg-elliptic symbol $\widetilde{q}_0 \in \mathcal{S}^1_H(M, E)$, requiring that the leading symbol of $\widetilde{q}_0$ remains of scalar type. Then the function $\widetilde{q} \in \Omega^0(EG, \mathcal{S}^1_H(M, E))$ defined on the universal G-bundle by

$$\widetilde{q}(g_0, \ldots, g_n)(s_0, \ldots, s_n) = \sum_{i=0}^{n} s_i \,\mathrm{Ad}^{-1}_{g_i}(\widetilde{q}_0)$$

is G-invariant. The corresponding function $\widetilde{q}_L \in \Omega^0(EG, \mathcal{D}(M))$ is therefore also G-invariant.

Proposition 7.2. *Let $D_0 \in \mathcal{D}(M)$ be a generalized Dirac operator affiliated to a Levi-Civita connection on M, $D \in \Omega^0(EG, \mathcal{D}(M))$ the associated G-invariant function on the universal bundle, and $\mathbf{D} = d + D$ the corresponding superconnection. Then one has the equality of cochains in $\mathrm{Hom}(\widehat{\Omega}\widehat{T}\mathcal{A}, X(\widehat{\mathcal{H}}, d))$:*

$$(7.15) \quad \chi^{\mathrm{Tr}_s}(\sigma_*, \mathbf{D}, \ln \widetilde{q}_L) = \int_{S^*_H M} \mathrm{Td}(\mathbf{i}R/2\pi) \wedge \chi(\rho_*, d) \; ,$$

where R is the equivariant curvature two-form of the Levi-Civia connection, and $\mathrm{Td}(\mathbf{i}R/2\pi)$ is the G-invariant closed differential form on $EG \times S^*_H M$ representing the equivariant Todd class of $TM \otimes \mathbb{C}$.

Proof. Firstly, note that D, as a function over EG, takes its values in the set of generalized Dirac operators affiliated to affine connections. This comes from the fact that Christoffel symbols behave in a suitable way under coordinates change. More precisely, in a foliated local chart, we have

$$D = \mathbf{i}\varepsilon\psi^i_R(\partial_{x^i} + {}^g\Gamma^k_{ij}(p_k\partial_{p_j} + (\overline{\psi}_j\psi^j)_R) + \overline{\psi}_{iR}\partial_{p_i} + r$$

where ${}^g\Gamma^k_{ij}$ is the function on EG given by

$${}^g\Gamma^k_{ij}(s_0, \ldots, s_m)(g_0, \ldots, g_m) = \sum_{l=0}^{m} s_l g_l(\Gamma^k_{ij})$$

and r is a remainder of the form

$$r = \mathbf{i}\varepsilon\psi^i_R \left(\sum_{|\alpha|\geqslant 2} (s^k_{\alpha i}p_k)_L \partial^\alpha_p + \sum_{|\alpha|\geqslant 1} (s^k_{\alpha ij}\overline{\psi}_k\psi^j + s_{\alpha i})_L \partial^\alpha_p \right) + \overline{\psi}_{iR} \sum_{|\alpha|\geqslant 2} (r^i_\alpha)_L \partial^\alpha_p$$

the coefficients are scalar functions on $EG \times M$.

Recall that ∇ is given by

$$\nabla = \mathbf{D} + \kappa \ln \widetilde{q}_L = d + D + \kappa \ln \widetilde{q}_L$$

and its square by

$$\nabla^2 = dD + D^2 + \kappa[\ln \widetilde{q}_L, \mathbf{D}] = \mathbf{D}^2 + \kappa\delta\mathbf{D}$$

where δ denotes the commutator with $\ln \widetilde{q_L}$. One important point is that $-\mathbf{D}^2$ can be seen as a perturbation of the flat Laplacian $\Delta = \mathbf{i}\varepsilon \partial_{x^i}\partial_{p_i}$

$$
\begin{aligned}
-\mathbf{D}^2 &= \Delta + ({}^g\Gamma^k_{ij})_L(\psi^i\overline{\psi}_k)_R\partial_{p_j} \\
&\qquad + (-\mathbf{i}\varepsilon d({}^g\Gamma^l_{il})_L\psi^i_R + \frac{\varepsilon^2}{2}({}^g\Omega^k_{lij})_L(\psi^i\psi^j)_R)(p_k\partial_{p_l} + (\overline{\psi}_k\psi^l)_L) + \ldots
\end{aligned}
$$

where the ${}^g\Omega^k_{lij}$ are the components of the curvature tensor of the connection ${}^g\Gamma$. Note that the coefficient of the third term is the equivariant curvature

$$
R^k_l = -\mathbf{i}\varepsilon d({}^g\Gamma^l_{il})_L\psi^i_R + \frac{\varepsilon^2}{2}({}^g\Omega^k_{lij})_L(\psi^i\psi^j)_R)
$$

Moreover, the dots are of the same type, but involve strictly higher powers of ∂_p.

We also have

$$
[\nabla,\sigma] = \mathbf{i}\varepsilon\psi^i_R(\partial_{x^i}\sigma + ({}^g\Gamma^k_{ij}p_k)_L\partial_{p_j}\sigma + \ldots) + \overline{\psi}_{iR}(\partial_{p_i}\sigma + \ldots) + \kappa[\ln\widetilde{q}_L,\sigma]
$$

where the term proportional to $\varepsilon\psi$ is of order 0, with the dots of order -1, and the term proportional to $\overline{\psi}_{iR}$ is of order -1 when $1 \leqslant i \leqslant \nu$, of order -2 when $\nu+1 \leqslant i \leqslant n$, the dots here are of stricly smaller order.

We now study *the first sum* of the cochain $\chi^{\mathrm{Tr}_s}(\sigma_*,\nabla)$.

Looking at $-\nabla^2 = \Delta + u$ as a perturbation of the flat Laplacian Δ, and performing a Duhamel expansion of the exponentials appearing leads to a study of terms of the form

$$
\mathrm{Tr}_s\left(\sigma_0\sigma^{t_0}_\Delta(X_1)\ldots\sigma^{t_0+\ldots+t_{k-1}}_\Delta(X_k)\exp\Delta\right)
$$

where $X_i = u$ or $[\nabla,\sigma_j]$. The action of σ_Δ does not modify the fact that X_i is of order 0, because

$$
[\Delta, X_i] = \mathbf{i}\varepsilon(\partial_{x^j}X_i\cdot\partial_{p_j} + \partial_{p_j}X_i\cdot\partial_{x^j} + \partial_{x^j}\partial_{p^j}X_i)
$$

Now, we observe that in the formulas above, $\varepsilon\psi_R$ is always proportional to an operator of order $\leqslant 0$, $\overline{\psi}_{iR}$ to an operator of order $\leqslant -1$ or -2 depending on i. The graded trace Tr_s selects the term proportional to $(\psi^1\overline{\psi}_1\ldots\psi^n\overline{\psi}_n)_R$ which is therefore of order $\leqslant -(\nu+2h)$. This means that only the leading terms are involved in these quantities. In particular, the dots in the formulas above give terms of order $< -(\nu+2h)$ and will vanish under Tr_s. For a similar reason, the derivatives $\partial_x X_i$ may be neglected in calculations. In other words, *all functions of x can be considered as constants.* From the previous discussion, if we choose a coordinate system around a point x_0 such that ${}^g\Gamma^k_{ij}(x_0) = 0$, we have

$$
\begin{aligned}
\mathbf{D} &\simeq d + \mathbf{i}\varepsilon\psi^i_R\partial_{x^i} + \overline{\psi}^i_R\partial_{p_i} \\
-\mathbf{D}^2 &\simeq \Delta + R^k_l(p_k\partial_{p_l} + (\overline{\psi}_k\psi^l)_L))
\end{aligned}
$$

as we can ignore the x-derivatives of ${}^g\Gamma$. Actually, the term

$$
R^k_l(\overline{\psi}_k\psi^l)_L = (-\mathbf{i}\varepsilon d({}^g\Gamma^l_{il})_L\psi^i_R + \frac{\varepsilon^2}{2}({}^g\Omega^k_{lij})_L(\psi^i\psi^j)_R)(\overline{\psi}_k\psi^l)_L
$$

can also be neglected, because it will act by commutators on the σ_i. The latter involve the projection operator $\Pi = \overline{\psi}_1\psi^1\ldots\overline{\psi}_n\psi^n$ onto scalar symbols, and the Bianchi identities of the Levi-Civita connection imply ${}^g\Omega^k_{lij}\overline{\psi}_k\psi^l\Pi = {}^g\Omega^k_{lij}\delta^k_l = {}^g\Omega^k_{kij} = 0$. Moreover, the relation $\Pi\psi^l = 0$ kills the other term, and we may consider that

$$
-\mathbf{D}^2 \simeq \Delta + R^k_l p_k\partial_{p_l}
$$

Performing another Duhamel expansion on $\exp(-t_i\nabla^2)$, and using that $\kappa^2 = 0$, the term

$$
\int_{\Delta_{r+1}} \mathrm{Tr}_s\left(e^{-t_{i+1}\nabla^2}[\nabla,\sigma_{i+1}]\ldots e^{-t_{r+1}\nabla^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\ldots e^{-t_i\nabla^2}\right)dt
$$

is the sum of the following terms :

$$\int_{\Delta_{r+1}} \mathrm{Tr}_s\big(e^{-t_{i+1}\mathbf{D}^2}[\nabla,\sigma_{i+1}]\dots e^{-t_{r+1}\mathbf{D}^2}\sigma_0\, e^{-t_0\mathbf{D}^2}[\nabla,\sigma_1]\dots e^{-t_i\mathbf{D}^2}\big)dt$$
$$\int_{\Delta_{r+2}} \mathrm{Tr}_s\big(e^{-t_{i+1}\mathbf{D}^2}\kappa\delta\mathbf{D}e^{-t_{i+2}\mathbf{D}^2}[\nabla,\sigma_{i+1}]\dots e^{-t_{r+2}\mathbf{D}^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\dots e^{-t_i\mathbf{D}^2}\big)dt$$
$$\int_{\Delta_{r+2}} \mathrm{Tr}_s\big(e^{-t_{i+1}\mathbf{D}^2}[\nabla,\sigma_{i+1}]e^{-t_{i+2}\mathbf{D}^2}\kappa\delta\mathbf{D}e^{-t_{i+3}\mathbf{D}^2}\dots e^{-t_{r+2}\mathbf{D}^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\dots e^{-t_i\mathbf{D}^2}\big)dt$$
$$\vdots$$

i.e, in the quantities $\int_{\Delta_{r+2}}\dots$ one of the $\exp(-t_i\nabla^2)$ is replaced by $e^{-t_i\mathbf{D}^2}\kappa\delta\mathbf{D}e^{-t_{i+1}\mathbf{D}^2}$, and the others by $\exp(-t_i\mathbf{D}^2)$. Note that the dimension of the simplex on which we integrate is risen by one. This also can be rewritten with the action of $\sigma_{-\mathbf{D}^2}$:

$$\int_{\Delta_{r+1}} \mathrm{Tr}_s\big(\sigma_{-\mathbf{D}^2}^{t_{i+1}}([\nabla,\sigma_{i+1}])\sigma_{-\mathbf{D}^2}^{t_{i+1}+t_{i+2}}([\nabla,\sigma_{i+2}])\dots\exp(-\mathbf{D}^2)\big)dt$$
$$\int_{\Delta_{r+2}} \mathrm{Tr}_s\big(\sigma_{-\mathbf{D}^2}^{t_{i+1}}(\kappa\delta\mathbf{D})\sigma_{-\mathbf{D}^2}^{t_{i+1}+t_{i+2}}([\nabla,\sigma_{i+1}])\dots\exp(-\mathbf{D}^2)\big)dt$$
$$\vdots$$

For $X=\kappa\delta\mathbf{D},\sigma_0$ or a commutator $[\nabla,\sigma]$, we have

$$\sigma_{-\mathbf{D}^2}^{t}(X)=X+\sum_{k\geqslant 1}\frac{(-t)^k}{k!}\mathrm{ad}_{-\mathbf{D}^2}^{k}(X)$$

an so on. Then, recalling that we may retain only the leading terms for the calculations, we observe that :

$$-[\mathbf{D}^2,X]=\big[\Delta+R_l^k p_{kL}\partial_{p_l},X\big]\simeq\partial_{p_i}X(\mathbf{i}\varepsilon\partial_{x^i}+R_i^k p_{kL})$$
$$[\mathbf{D}^2,[\mathbf{D}^2,X]]\simeq\partial_{p_i}\partial_{p_j}X(\mathbf{i}\varepsilon\partial_{x^i}+R_i^k p_{kL})(\mathbf{i}\varepsilon\partial_{x^j}+R_j^l p_{lL})+R_i^j\partial_{p_i}X(\mathbf{i}\varepsilon\partial_{x^j}+R_j^l p_{lL})$$

and continuing the process by induction, we finally get

$$\sigma_{-\mathbf{D}^2}^{t}(X)\simeq X+\sum_{k\geqslant 1}\frac{t^k}{k!}\sum_{|\alpha|=1}^{k}P_\alpha(X)(\mathbf{i}\varepsilon\partial_x+p_L\cdot R)^\alpha$$

where $P_\alpha(X)$ is a linear combination of the p-partial derivatives of X. The operators $(i\varepsilon\partial_x+p_L\cdot R)^\alpha$ may be moved to the right in front of $\exp(-\mathbf{D}^2)$ when $|\alpha|\geqslant 1$, because functions of x behave like constants. Then, using Example 4.4, we find that these quantities does not contribute for the calculations, in other words,

$$\sigma_{-\mathbf{D}^2}^{t}(X)\simeq X$$

As a consequence, we may drop the action of σ_{-D^2} in the above calculations and obtain

$$\begin{aligned}
&\int_{\Delta_{r+1}} \mathrm{Tr}_s\big(e^{-t_{i+1}\nabla^2}[\nabla,\sigma_{i+1}]\dots e^{-t_{r+1}\nabla^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\dots e^{-t_i\nabla^2}\big)dt\\
&=\frac{1}{(r+1)!}\mathrm{Tr}_s\big([\nabla,\sigma_{i+1}]\dots[\nabla,\sigma_r]\sigma_0[\nabla,\sigma_1]\dots[\nabla,\sigma_i]\exp(-\mathbf{D}^2)\big)\\
&\quad+\frac{1}{(r+2)!}\big[\mathrm{Tr}_s\big(\kappa\delta\mathbf{D}[\nabla,\sigma_{i+1}]\dots[\nabla,\sigma_r]\sigma_0[\nabla,\sigma_1]\dots[\nabla,\sigma_i]\exp(-\mathbf{D}^2)\big)\\
&\qquad+\mathrm{Tr}_s\big([\nabla,\sigma_{i+1}]\kappa\delta\mathbf{D}\dots[\nabla,\sigma_r]\sigma_0[\nabla,\sigma_1]\dots[\nabla,\sigma_i]\exp(-\mathbf{D}^2)\big)+\dots\\
&\qquad+\mathrm{Tr}_s\big([\nabla,\sigma_{i+1}]\dots[\nabla,\sigma_r]\sigma_0[\nabla,\sigma_1]\dots[\nabla,\sigma_i]\kappa\delta\mathbf{D}\exp(-\mathbf{D}^2)\big)\big]
\end{aligned}$$

the factors $\frac{1}{(r+1)!}$ or $\frac{1}{(r+2)!}$ comes from the volume of the standard simplex. We will now retain only the terms proportional to κ in the latter equality. Then, knowing that $\mathbf{D}^2$ may be replaced by $\Delta + p_L \cdot R \cdot \partial_p$ using Example 4.4, and having in mind that the symbols σ_i are constant in the direction EG, we get

$$\begin{aligned}
&\frac{d}{d\kappa}\int_{\Delta_{r+1}} \mathrm{Tr}_s\big(e^{-t_{i+1}\nabla^2}[\nabla,\sigma_{i+1}]\dots e^{-t_{r+1}\nabla^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\dots e^{-t_i\nabla^2}\big)dt \\
&= \frac{1}{(r+1)!}⨍ \big(\langle\langle \delta\sigma_{i+1}[D,\sigma_{i+2}]\dots[D,\sigma_r]\sigma_0[D,\sigma_1]\dots[D,\sigma_i]\mathrm{Td}(R)\rangle\rangle[n] \\
&\qquad + \langle\langle [D,\sigma_{i+1}]\delta\sigma_{i+2}\dots[D,\sigma_r]\sigma_0[D,\sigma_1]\dots[D,\sigma_i]\mathrm{Td}(R)\rangle\rangle[n] + \dots \\
&\qquad + \langle\langle [D,\sigma_{i+1}]\dots[D,\sigma_r]\sigma_0[D,\sigma_1]\dots[D,\sigma_{i-1}]\delta\sigma_i\mathrm{Td}(R)\rangle\rangle[n]\big) \\
&\quad + \frac{1}{(r+2)!}⨍ \big(\langle\langle \delta\mathbf{D}[D,\sigma_{i+1}]\dots[D,\sigma_r]\sigma_0[D,\sigma_1]\dots[D,\sigma_i]\mathrm{Td}(R)\rangle\rangle[n] \\
&\qquad - \langle\langle [D,\sigma_{i+1}]\delta\mathbf{D}\dots[D,\sigma_r]\sigma_0[D,\sigma_1]\dots[D,\sigma_i]\mathrm{Td}(R)\rangle\rangle[n] + \dots \\
&\qquad + (-1)^{r+2}\langle\langle [D,\sigma_{i+1}]\dots[D,\sigma_r]\sigma_0[D,\sigma_1]\dots[D,\sigma_i]\delta\mathbf{D}\mathrm{Td}(R)\rangle\rangle[n]\big)
\end{aligned}$$

The bracket selects only the operators proportional to $(\overline{\psi}_1\dots\overline{\psi}_n)_R$, which is of order $\leqslant -(\nu+2h)$. However, in the quantities proportional to $\frac{1}{(r+1)!}$, these operators gain an extra factor $\delta\sigma$, which is of order -1. So, the Wodzicki residue kills this part. We can now make the identifications $\varepsilon\psi^i_R \leftrightarrow dx^i$ and $\overline{\psi}_{iR} \leftrightarrow dp_i - {}^g\Gamma^k_{ij}dx^j \simeq dp_i$, consistent with coordinate changes. Moreover, recall that the symbol of $\ln\widetilde{q}(x,p)$ is of the form

$$\ln\widetilde{q}(x,p) = \ln|p|' + q_0(x,p)$$

where q_0 is a Heisenberg symbol of order 0. Then,

$$\begin{aligned}
\delta\mathbf{D} &\simeq (-\mathbf{i}\varepsilon\psi^i_R(\partial_{x^i}q_0)_L - \overline{\psi}_{iR}(\partial_{p_i}q_0)_L) + \overline{\psi}_{iR}\left(\sum_{i=1}^{\nu}\frac{p_i^3}{|p|'^4} + \sum_{i=\nu+1}^{n}\frac{p_i}{|p|'^4}\right) + d\ln\widetilde{q}_L \\
&\leftrightarrow -dq_0 + dp_i\left(\sum_{i=1}^{\nu}\frac{p_i^3}{|p|'^4} + \sum_{i=\nu+1}^{n}\frac{p_i}{|p|'^4}\right) + d\ln\widetilde{q}_L
\end{aligned}$$

using Theorem 2.6, and we have the following equality

$$\begin{aligned}
&\frac{d}{d\kappa}\int_{\Delta_{r+1}} \mathrm{Tr}_s\big(e^{-t_{i+1}\nabla^2}[\nabla,\sigma_{i+1}]\dots e^{-t_{r+1}\nabla^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\dots e^{-t_i\nabla^2}\big)dt \\
&= \frac{1}{(r+2)!}\int_{S^*_H M} \iota_L \cdot \big[\big(\delta\mathbf{D}d\sigma_{i+1}\dots d\sigma_r\sigma_0 d\sigma_1\dots d\sigma_i - d\sigma_{i+1}\delta\mathbf{D}d\sigma_r\sigma_0 d\sigma_1\dots d\sigma_i + \dots \\
&\qquad + (-1)^{r+2}d\sigma_{i+1}\dots d\sigma_r\sigma_0 d\sigma_1\dots d\sigma_i\delta\mathbf{D}\big)\wedge \mathrm{Td}(R)\Pi\big]_{\mathrm{vol}}
\end{aligned}$$

where L is the generator of the Heisenberg dilations given by the formula

$$L = \left(\sum_{i=1}^{\nu} p_i\partial_{p_i} + \sum_{i=\nu+1}^{n} 2p_i\partial_{p_i}\right)$$

Actually, the terms containing dp_0 have order $< -(\nu+2h)$, and does not contribute to the Wodzicki residue. Indeed, these terms bring n partial derivatives with respect to the variables $(p_1,\dots,p_n)$, and writing the leading symbol of the involved quantity in polar coordinates $(|p|',\theta_1,\dots,\theta_n)$, the latter is then proportional to $|p|^{-(\nu+2h)+1}$ times a partial derivative $\partial\sigma/\partial|p|'$, which is of order

-2. We finally get

$$\frac{d}{d\kappa}\int_{\Delta_{r+1}} \mathrm{Tr}_s\big(e^{-t_{i+1}\nabla^2}[\nabla,\sigma_{i+1}]\dots e^{-t_{r+1}\nabla^2}\sigma_0\, e^{-t_0\nabla^2}[\nabla,\sigma_1]\dots e^{-t_i\nabla^2}\big)dt$$
$$= \frac{1}{(r+1)!}\int_{S^*_H M} d\sigma_{i+1}\dots d\sigma_r\sigma_0 d\sigma_1\dots d\sigma_i$$

Analogous manipulations on the second sum of the cochain $\chi^{\mathrm{Tr}_s}(\sigma_*,\nabla)$ give the final answer. □

Thus combining the previous results one gets

THEOREM 7.3. *Let* M *be a foliated manifold,* $G\subset \mathrm{Diff}(M)$ *a discrete group of diffeomorphisms mapping leaves to leaves,* $0\to \Psi^{-1}_{H,c}(M)\rtimes G\to \Psi^0_{H,c}(M)\rtimes G\to C^\infty_c(S^*_H M)\rtimes G\to 0$ *the extension of equivariant Heisenberg pseudodifferential operators. Then the image of the canonical trace localized at unit* $[\tau]\in HP^0(\Psi^{-1}_H(M)\rtimes G)$ *under the excision map is*

(7.16) $\quad \partial([\tau]) = \Phi(\mathrm{Td}(TM\otimes\mathbb{C}))$

where $\Phi : H^{ev}(EG\times_G S^*_H M)\to HP^1(C^\infty_c(S^*_H M)\rtimes G)$ *is Connes' characteristic map from equivariant cohomology to cyclic cohomology, and* $\mathrm{Td}(TM\otimes\mathbb{C})$ *is the equivariant Todd class of the complexified tangent bundle of* M*.*

8. The transverse index theorem of Connes and Moscovici

Let $\mathcal{A}$ be an associative algebra and (H,F) a (trivially graded) p-summable Fredholm module. Hence, $\mathcal{A}$ is represented by bounded operators on a separable Hilbert space H, and F is a bounded self-adjoint operator on H such that the operators $a(F^2-1)$ and $[F,a]$ are in some Schatten class $\ell^p(H)$ for all $a\in A$. In addition, we suppose given an extension of "abstract pseudodifferential operators"

(8.1) $\quad 0\to\Psi^{-1}\to\Psi^0\to\Psi^0/\Psi^{-1}\to 0$

where

- Ψ^0 is an algebra of bounded operators on H containing the representation of $\mathcal{A}$,
- Ψ^{-1} is a two-sided ideal consisting of p-summable operators on H,
- F is a multiplier of Ψ^0 and $[F,\Psi^0]\subset\Psi^{-1}$.

Let $P=\frac{1}{2}(1+F)$. Then $[P,a]\in\Psi^{-1}$ and $aP^2\equiv aP \mod \Psi^{-1}$ for all $a\in A$. The linear map

(8.2) $\quad \rho_F : \mathcal{A}\to\Psi^0/\Psi^{-1}\ , \qquad \rho_F(a)\equiv aP \mod \Psi^{-1}\ ,$

is an algebra homomorphism since $a_1Pa_2P\equiv a_1a_2P \mod \Psi^{-1}$ for all $a_1,a_2\in\mathcal{A}$.

LEMMA 8.1. *The Chern-Connes character of the Fredholm module* (H,F) *is given by the odd cyclic cohomology class over* $\mathcal{A}$

(8.3) $\quad \mathrm{ch}(H,F)=\rho_F^*\circ\partial([\mathrm{Tr}])$

where $[\mathrm{Tr}]\in HP^0(\Psi^{-1})$ *is the class of the operator trace,* $\partial : HP^0(\Psi^{-1})\to HP^1(\Psi^0/\Psi^{-1})$ *is the excision map associated to extension (8.1), and* $\rho_F^* : HP^1(\Psi^0/\Psi^{-1})\to HP^1(\mathcal{A})$ *is induced by the homomorphism* ρ_F*.*

PROOF. Consider the algebra $\mathcal{E}=\{(Q,a)\in\Psi^0\oplus\mathcal{A} \mid Q\equiv aP \mod \Psi^{-1}\}$. The homomorphism $\mathcal{E}\to\mathcal{A}$, $(Q,a)\mapsto a$ yields an extension

$$0\to\Psi^{-1}\to\mathcal{E}\to\mathcal{A}\to 0\ .$$

By definition ([2]), the Chern-Connes character $\mathrm{ch}(H,F) \in HP^1(\mathcal{A})$ is the image of the operator trace under the excision map associated to this extension. On the other hand, the homomorphism $\mathcal{E} \to \Psi^0$, $(Q,a) \mapsto Q$ yields a commutative diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & \Psi^{-1} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d, "\rho_F"] & 0 \\
0 \arrow[r] & \Psi^{-1} \arrow[r] & \Psi^0 \arrow[r] & \Psi^0/\Psi^{-1} \arrow[r] & 0
\end{tikzcd}$$

The conclusion then follows from the naturality of excision. $\square$

We apply this to the hypoelliptic operators constructed by Connes and Moscovici in [3]. Let M be an oriented foliated manifold and $G \subset \mathrm{Diff}(M)$ a discrete group of orientation-preserving diffeomorphisms mapping leaves to leaves. We make the hypothesis that G has *no fixed points* on M. We denote by $V \subset TM$ the subbundle tangent to the leaves and by $N = TM/V$ the normal bundle; both are equivariant G-bundles by construction. Assume that V and N are provided with G-invariant euclidean structures, called G-invariant *triangular structures* in [3]. Then the hermitean vector bundle

$$E = \Lambda^\bullet(V^* \otimes \mathbb{C}) \otimes \Lambda^\bullet(N^* \otimes \mathbb{C}) \tag{8.4}$$

is G-equivariant, and the euclidean structures on V, N determine a G-invariant volume form on M via the canonical isomorphism of top-degree forms $\Lambda^{\max}V \otimes \Lambda^{\max}N \cong \Lambda^{\max}M$. Let $H = L^2(M,E)$ be the Hilbert space of square-integrable sections of E with respect to the hermitean structure and volume form. The crossed-product algebra

$$\mathcal{A} = C_c^\infty(M) \rtimes G \tag{8.5}$$

is represented by bounded operators on H as follows: a function $f \in C_c^\infty(M)$ acts on the sections of E by pointwise multiplication, while $g \in G$ is represented by the unitary operator coming from the action of G on the manifold M and the vector bundles V, N. Denote by $d_V : C^\infty(M,E) \to C^\infty(M,E)$ the leafwise de Rham differential. Choose an isomorphism of N with a vector subbundle of TM transverse to V, and denote by d_N the corresponding transverse de Rham differential. Then Connes and Moscovici consider the *hypoelliptic signature operator* acting on $C^\infty(M,E)$

$$Q = \pm(d_V d_V^* - d_V^* d_V) + (d_N + d_N^*)\ , \tag{8.6}$$

where the sign $+1$ is taken on $\Lambda^{\mathrm{ev}}N^*$ and -1 on $\Lambda^{\mathrm{odd}}N^*$. This is a formally self-adjoint, hypoelliptic differential operator of order two. Q is not quite invariant under the action of G because the isomorphism $TM \cong V \oplus N$ requires a choice. However, in the Heisenberg pseudodifferential calculus associated to the foliation on M, the operator Q is Heisenberg-elliptic and its leading symbol is exactly G-invariant. From this one builds a properly supported Heisenberg pseudodifferential operator

$$F = \frac{Q}{|Q|} \tag{8.7}$$

which is defined only up to addition of a smoothing operator. Again the Heisenberg leading symbol of F is G-invariant. Now we turn to the geometric example of [3], where M is the bundle of Riemannian metrics over a smooth G-manifold W. Here the foliation on M corresponds to the fibration $M \to W$, and has a tautological triangular structure. The action of G by diffeomorphisms on W canonically lifts to an action on M mapping fibers to fibers, and preserving the triangular structure. In this situation, the results of [3] show that F is a bounded operator on H, $a(F^2-1)$ is smoothing for all $a \in \mathcal{A}$, and the pair (H,F) defines a p-summable Fredholm module over the algebra $\mathcal{A}$, for any $p > \dim V + 2\dim N$. Its Chern-Connes character may thus be computed by means of the above lemma. We let $\Psi_{H,c}(M,E)$ be the algebra of compactly supported Heisenberg pseudodifferential operators acting on the smooth sections of E. The representation of the crossed-product $\Psi_{H,c}(M,E)\rtimes G$ on the Hilbert space H leads to subalgebras of bounded operators

$$\Psi^0=\mathrm{Im}(\Psi^0_{H,c}(M,E)\rtimes G)\ ,\qquad \Psi^{-1}=\mathrm{Im}(\Psi^{-1}_{H,c}(M,E)\rtimes G)\ .$$

Note that these representations are not faithful. Let $\pi:S^*_HM\to M$ be the projection from the Heisenberg cosphere bundle. The pullback π^*E is naturally a G-equivariant vector bundle over S^*_HM. Since G has no fixed points by hypothesis, the leading symbol map $\Psi^0_{H,c}(M,E)\to C^\infty_c(S^*_HM,\mathrm{End}(\pi^*E))$ yields a canonical isomorphism of algebras

$$\Psi^0/\Psi^{-1}\cong C^\infty_c(S^*_HM,\mathrm{End}(\pi^*E))\rtimes G\ .$$

Under this identification the homomorphism $\rho_F:\mathcal{A}\to\Psi^0/\Psi^{-1}$ is given by

$$\rho_F(fU_g)=\pi^*(f)eU^E_g\qquad\forall\ f\in C^\infty_c(M)\ ,\ g\in G$$

where $e\in C^\infty(S^*_HM,\mathrm{End}(\pi^*E))$ is the leading symbol of the operator $P=\frac{1}{2}(1+F)$, and U^E_g is the represntation of g as a linear operator on the space of sections of π^*E. Since $P^2\equiv P$ and $PU^E_g\equiv U^E_gP$ modulo operators of order -1, one has $e^2=e$ and $eU^E_g=U^E_ge$ for all $g\in G$. Hence e is a G-invariant idempotent section of the bundle $\mathrm{End}(\pi^*E)$. Its range is the G-equivariant subbundle E_+ of π^*E consisting in the positive eigenvectors for the leading symbol of F. By usual Chern-Weil theory, the equivariant Chern character $\mathrm{ch}(E_+)$ is represented by a closed G-invariant differential form on the homotopy quotient $EG\times_G S^*_HM$. Taking its product with the equivariant Todd class of the complexified tangent bundle yields a class

$$L'(M)=\mathrm{Td}(TM\otimes\mathbb{C})\cup\mathrm{ch}(E_+)\in H^{\mathrm{ev}}(EG\times_G S^*_HM)\ . \tag{8.8}$$

THEOREM 8.2. *Let G be a discrete group of orientation-preseving diffeomorphisms on a smooth oriented manifold W. Let M be the bundle of Riemannian metrics over W and $\mathcal{A}=C^\infty_c(M)\rtimes G$. If G has no fixed points, then the Chern-Connes character of the Fredholm module (H,F) associated to the hypoelliptic signature operator of Connes and Moscovici is*

$$\mathrm{ch}(H,F)=\pi_*\circ\Phi(L'(M))\ \in HP^1(\mathcal{A})\ , \tag{8.9}$$

where $\Phi:H^{\mathrm{ev}}(EG\times_G S^*_HM)\to HP^1(C^\infty_c(S^*_HM)\rtimes G)$ is Connes' characteristic map from equivariant cohomology to cyclic cohomology, and $\pi_*:HP^1(C^\infty_c(S^*_HM)\rtimes G)\to HP^1(\mathcal{A})$ is the map induced by the projection $\pi:S^*_HM\to M$.

PROOF. One has to compare the two extensions

$$\begin{tikzcd}
0 \arrow[r] & \Psi^{-1}_{H,c}(M,E)\rtimes G \arrow[r]\arrow[d] & \Psi^0_{H,c}(M,E)\rtimes G \arrow[r]\arrow[d] & C^\infty_c(S^*_HM,\mathrm{End}(\pi^*E))\rtimes G \arrow[r]\arrow[d, equal] & 0\\
0 \arrow[r] & \Psi^{-1} \arrow[r] & \Psi^0 \arrow[r] & \Psi^0/\Psi^{-1} \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are the representations as bounded operators in the Hilbert space H. We consider two different cyclic cohomology classes on the ideals. The first one is the operator trace $[\mathrm{Tr}]\in HP^0(\Psi^{-1})$, and the second is the trace *localized at units* $[\tau]\in HP^0(\Psi^{-1}_{H,c}(M,E)\rtimes G)$. Of course $[\tau]$ is not the pullback of $[\mathrm{Tr}]$ under the representation. We use a zeta-function renormalization in order to compute the image $\partial([\mathrm{Tr}])\in HP^1(\Psi^0/\Psi^{-1})$ of the operator trace under the excision map of the bottom extension, as in section 2. Then, *since G has no fixed points*, only the part of the operator trace which is localized at units contributes to the residues. This means that one has the equality

$$\partial([\mathrm{Tr}])=\partial([\tau])$$

in $HP^1(C^\infty_c(S^*_H M, \mathrm{End}(\pi^* E)) \rtimes G)$. A choice of local trivializations of the vector bundle E and a partition of unity allows to identify $C^\infty_c(S^*_H M, \mathrm{End}(\pi^* E)) \rtimes G$ with a subalgebra of the algebra of matrices $M_\infty(C^\infty_c(S^*_H M) \rtimes G)$. Under this identification Theorem 7.3 implies the equality

$$\partial([\mathrm{Tr}]) = \mathrm{tr}\#\Phi(\pi^*\mathrm{Td}(TM \otimes \mathbb{C}))$$

where tr denotes the trace on $M_\infty = (\mathbb{C})$ and $\#$ is the cup-product of cyclic cocycles ([2]). Finally the homomorphism ρ_F is multiplication by the G-invariant idempotent $e \in C^\infty(S^*_H M, \mathrm{End}(\pi^* E)) \subset M_\infty(C^\infty(S^*_H M))$, so the composition $\rho_F \circ \partial([\mathrm{Tr}])$ is the above class twisted by the Chern character $\mathrm{ch}(E_+)$. $\square$

References

[1] A. Connes: Cyclic cohomology and the transverse fundamental class of a foliation, *Pitman Res. Notes in Math., Geometric methods in operator algebras (Kyoto)* **123** (1983) 52-144

[2] A. Connes: Non-commutative differential geometry, *Publ. Math. IHES* **62** (1986) 41-144

[3] A. Connes, H. Moscovici: The local index formula in non-commutative geometry, *GAFA* **5** (1995) 174-243

[4] A. Connes and H. Moscovici: Hopf algebras, cyclic cohomology and the transverse index theorem, *Comm. Math. Phys.* **198** (1998) 199-246

[5] J. Cuntz and D. Quillen: Cyclic homology and nonsingularity, *Journal of the AMS* **8** (1995) 373-442

[6] N. Higson: The residue index theorem of Connes and Moscovici, in *Surveys in Noncommutative Geometry*, vol. 6 of *Clay Math. Proc.* pp. 71-126, Amer. Math. Soc., Providence, RI (2006)

[7] A. Jaffe, A. Lesniewski and K. Osterwalder: Quantum K-theory : I. The Chern character, *Comm. Math. Phys* **118** (1988) 1-14

[8] V. Nistor: Higher index theorems and the boundary map in cyclic cohomology, *Doc. Math.* **2** (1997) 263-295 (electronic)

[9] D. Perrot: Pseudodifferential extension and Todd class, *Adv. in Math.* **246** (2013) 265-302

[10] D. Perrot: Local index theory for certain Fourier integral operators on Lie groupoids, *preprint arXiv:14010225.* (2014)

[11] D. Quillen: Algebra cochains and cyclic cohomology, *Publ. Math. IHES* **68** (1988) 139-174

[12] R. Rodsphon: Zeta functions, excision in cyclic cohomology and index problems, *to appear in JFA* (2013)

[13] A. Savin and B. Sternin: Elliptic theory for operators associated with diffeomorphisms of smooth manifolds, *preprint arXiv:1207.3017.*

Université de Lyon, CNRS UMR 5208, Université Lyon 1, Institut Camille Jordan, 43, Bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France

E-mail address: perrot@math.univ-lyon1.fr, rodsphon@math.univ-lyon1.fr