locale Index Theory for Certain Fourier Integral Operators on Lie Groupoids

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December 31, 2013

Abstract
We develop a local index theory, in the sense of non-commutative geometry, for operators associated to non-proper and non-isometric actions of Lie groupoids on smooth submersions.

Keywords: extensions, K-theory, cyclic cohomology.
MSC 2000: 19D55, 19K56.

1 Introduction
This article concerns the index theory of a certain class of operators associated to smooth actions of Lie groupoids on manifolds. When a Lie groupoid $G$ acts properly on a submersion of smooth manifolds $M \to B$, the index theory of a $G$-equivariant family of elliptic pseudodifferential operators on $M$ is rather well-understood [7, 5]. In his fundamental article [3], Connes introduced cyclic cohomology techniques in order to deal with the $K$-theoretic index of a $G$-equivariant elliptic family. His result and subsequent generalizations use in a crucial manner the properness of the action, or in other circumstances, the fact that the action is isometric with respect to some Riemannian data on $M$. Much less is known about improper or non-isometric actions. Already in the simple case of a discrete group $G$ acting by diffeomorphisms on a closed manifold $M$ one wishes to study operators of the form

$$
\sum_{g \in G} P_g U_g : C^\infty(M) \to C^\infty(M),
$$

where $P_g$ is a pseudodifferential operator (say of order $\leq 0$) for any $g$, and $U_g$ is the representation of $g$ as a diffeomorphism on $M$. Such an operator is not pseudodifferential, and belongs to the larger class of Fourier-integral operators [14]. Its principal symbol is not a smooth function on the cosphere bundle $S^*M$. Instead, it defines an element of the non-commutative crossed product algebra $C^\infty(S^*M) \rtimes G$, or equivalently, the smooth convolution algebra of the étale groupoid $S^*M \rtimes G$. When $G$ is infinite this groupoid is not proper and the associated convolution algebra can be highly non-commutative. In the case of
the group $G = \mathbb{Z}$, Savin and Sternin recently computed the index of an elliptic operator like (1) as a pairing between its leading symbol and a cyclic cohomology class of the algebra $C^\infty(S^*M) \rtimes G$, see [26] and references therein. In the present article we want to unify these various index theorems and generalize them in two directions:

- Develop an index theory for operators whose non-commutative leading symbol belongs to the smooth convolution algebra of a not necessarily proper groupoid such as the crossed product $S^*M \rtimes G$ above,

- Evaluate the $K$-theoretical index of such operators on a wide range of cyclic cohomology classes, not necessarily localized at units.

The general situation considered in this article is the following. Let $G \rightarrow B$ be a Lie groupoid acting smoothly on a (surjective) submersion $\pi : M \rightarrow B$ of smooth manifolds. This basically means that any morphism $g \in G$ with source $s(g) \in B$ and range $r(g) \in B$ induces a diffeomorphism from the fiber $M_{r(g)} = \pi^{-1}(r(g))$ to the fiber $M_{s(g)} = \pi^{-1}(s(g))$, in a way compatible with the composition of morphisms in $G$. We do not impose any restriction on this action (except smoothness), in particular, it is not necessarily proper nor isometric. Denote by $S^*_\pi M$ the bundle over $M$ whose fiber at a point $x \in M$ is the cotangent sphere of the submanifold $M_{\pi(x)}$ at $x$. Hence, $S^*_\pi M$ is the “vertical” cosphere bundle over the submersion $M$. It is still endowed with a smooth action of $G$, and we consider the action groupoid $S^*_\pi M \rtimes G$. Its smooth convolution algebra

$$\mathcal{A} = C^\infty_c(S^*_\pi M) \rtimes G$$

is highly non-commutative in general. It may naturally be identified with the algebra of leading symbols of “vertical non-commutative pseudodifferential operators” on $M$. Let us explain what are these operators. Denote by $CL^0(M) \rightarrow B$ the bundle with base $B$, whose fiber over a point $b \in B$ is the algebra of compactly supported classical pseudodifferential operators of order $\leq 0$ on the manifold $M_b = \pi^{-1}(b)$. This bundle carries a natural action of $G$. Hence the algebra of smooth compactly supported sections of vertical classical pseudodifferential operators $C^\infty_c(B, CL^0(M))$ can be twisted by the action of $G$. This leads to the crossed-product algebra of non-commutative pseudodifferential operators

$$\mathcal{E} = C^\infty_c(B, CL^0_c(M)) \rtimes G.$$  

Clearly the projection of classical pseudodifferential operators of order $\leq 0$ onto the homogeneous component of degree 0 of their symbol extends to a surjective homomorphism of algebras $\mathcal{E} \rightarrow \mathcal{A}$, whose kernel is the two-sided ideal $\mathcal{B} = C^\infty_c(B, CL^{-1}_c(M)) \rtimes G$ of operators of order $\leq -1$ in $\mathcal{E}$. One thus gets a short exact sequence (extension) of algebras

$$(E) : 0 \rightarrow \mathcal{B} \rightarrow \mathcal{E} \rightarrow \mathcal{A} \rightarrow 0.$$  

We say that an operator in the algebra $\mathcal{E}$ (unitalized) is elliptic if its leading symbol is invertible in the algebra $\mathcal{A}$ (unitalized). This is a purely algebraic notion. An invertible leading symbol naturally defines an algebraic $K$-theory class in $K_1(\mathcal{A})$. The index map of the extension $(E)$ is the morphism

$$\text{Ind}_E : K_1(\mathcal{A}) \rightarrow K_0(\mathcal{B}).$$  

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induced on algebraic $K$-theory in low degrees [15]. Thus, if $[u] \in K_1(\mathcal{A})$ is represented by the non-commutative symbol $u \in GL_\infty(\mathcal{A})$ of an elliptic operator, its index $\text{Ind}_{\mathcal{E}}([u])$ is a $K$-theory element represented by an idempotent (matrix) in $\mathcal{B}$. Our goal is to evaluate this index on genuine cyclic cohomology classes of $\mathcal{B}$. In fact, as shown by Cuntz and Quillen in [11], periodic cyclic cohomology satisfies excision in full generality. This means that the extension $(\mathcal{E})$ leads to a cohomology long exact sequence, with connecting map

$$E^* : HP^*(\mathcal{B}) \to HP^{*+1}(\mathcal{A}).$$

Then Nistor [17] (see also [21]) remarked that the index map $\text{Ind}_{\mathcal{E}}$ in algebraic $K$-theory is adjoint to the excision map $E^*$ with respect to the Chern-Connes pairing. Hence for any $[u] \in K_1(\mathcal{A})$ and $[\tau] \in HP^0(\mathcal{B})$ one has the equality of pairings

$$\langle [\tau], \text{Ind}_{\mathcal{E}}([u]) \rangle = \langle E^*([\tau]), [u] \rangle.$$  

The left-hand side of this equality is a number, or “higher index” characterizing the $K$-theoretic index class of an elliptic operator. The right-hand side computes the higher index by means of a “formula” involving the leading symbol of this operator. The difficulty is then to compute the image of cyclic cocycles $[\tau]$ under the excision map, which is actually not easy at all. We will use the general theory developed in [21, 22] for the explicit computation of the excision map in terms of local formulas. The first step in this direction is to find which classes $[\tau]$ are “good enough” to allow this computation. In our case, the algebra $\mathcal{B} = C_c^\infty(B, CL_c^{-1}(M)) \rtimes G$ is a finitely-summable thickening of the smooth convolution algebra of the pullback groupoid $\pi^*G \rightrightarrows M$. The latter is Morita equivalent to the groupoid $G \rightrightarrows B$, and the topological cyclic cohomology of their respective convolution algebras $C_c^\infty(M) \rtimes \pi^*G$ and $C_c^\infty(B) \rtimes G$ are isomorphic. The topological cyclic cohomology of a smooth convolution algebra is defined according to its natural locally convex topology. We construct a canonical map (Proposition 4.6)

$$\tau_* : HP^*_{\text{top}}(C_c^\infty(B) \rtimes G) \to HP^*(C_c^\infty(B, CL_c^{-1}(M)) \rtimes G).$$

By doing so we need to establish not so well-known properties of localization and Morita invariance of the periodic cyclic cohomology of smooth convolution algebras for general Lie groupoids. To our knowledge this was previously done only in the case of foliation groupoids (see [2, 8]), that is, groupoids which are Morita equivalent to étale ones. The following result shows that on the range of (8) the excision map $E^*$ factors through the topological cyclic cohomology of the convolution algebra $\mathcal{A} = C_c^\infty(S_\pi^*M) \rtimes G$.

**Theorem 4.7** Let $G \rightrightarrows B$ be a Lie groupoid and $\pi : M \to B$ a $G$-equivariant surjective submersion. Then one has a commutative diagram

$$
\begin{array}{ccc}
HP^*(C_c^\infty(B, CL_c^{-1}(M)) \rtimes G) & \xrightarrow{E^*} & HP^{*+1}(C_c^\infty(S_\pi^*M) \rtimes G) \\
\tau_* & & \\
HP^*_{\text{top}}(C_c^\infty(B) \rtimes G) & \xrightarrow{\pi_{\text{top}}^*} & HP^{*+1}_{\text{top}}(C_c^\infty(S_\pi^*M) \rtimes G)
\end{array}
$$

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It remains to compute the map $\pi_G^1$. Let $O \subset G$ be an Ad-invariant isotropic submanifold of $G$. This means that the range and source maps $O \rightrightarrows B$ coincide, and $O$ is globally invariant under the adjoint action of $G$. Then one has the notion of topological cyclic cohomology $H_{\text{top}}(C^\infty_c(B) \rtimes G)|_O$ localized at $O$, together with a forgetful map from the localized to the delocalized cohomology. Under suitable non-degeneracy hypotheses concerning the action of $O$ on $M$, we are able to calculate the map $\pi_G^1$ by means of explicit formulas involving residues of zeta-functions. The use of zeta-function is motivated by the approach of Connes and Moscovici to the local index formula in non-commutative geometry [6]. These residues generalize the well-known Wodzicki residue for classical pseudodifferential operators [28]. They are given by integrals, over the cosphere bundle of the fixed point set for $O$, of certain local expressions in the complete symbol of the operators involved.

As a refinement of Theorem 4.7 we obtain the following result.

**Theorem 5.6** Let $G \rightrightarrows B$ be a Lie groupoid and let $O$ be an Ad-invariant isotropic submanifold of $G$. Let $\pi : M \to B$ be a $G$-equivariant surjective submersion and assume the action of $O$ on $M$ non-degenerate. Then one has a commutative diagram

$$
\begin{array}{ccc}
\text{HP}^\bullet (C^\infty_c(B, CL^{-1}_e(M)) \rtimes G) & \xrightarrow{E^*} & \text{HP}^{\bullet+1}(C^\infty_c(S^*_e \pi M) \rtimes G) \\
\tau_* & & \downarrow \\
\text{HP}_{\text{top}}^\bullet (C^\infty_c(B) \rtimes G)|_O & \xrightarrow{\pi_G^1} & \text{HP}_{\text{top}}^{\bullet+1}(C^\infty_c(S^*_e \pi M) \rtimes G)|_{[\pi^* O]} 
\end{array}
$$

where the isotropic submanifold $\pi^* O \subset S^*_e \pi M \rtimes G$ is the pullback of $O$ by the submersion $S^*_e \pi M \to B$. The map $\pi_G^1$ is given by an explicit residue formula.

Although the residue formula is an explicit local expression in the complete symbols of the operators, it is extremely hard to compute. The reason is that when the dimension $n$ of the fibers of the submersion $\pi : M \to B$ is “large” (typically $n \geq 2$) the residues involve an asymptotic expansion of symbols up to order $n$, which produces a huge quantity of terms. However, in the case of cyclic cohomology localized at units ($O = B$), using the techniques of [22] we are able to give a close expression for the map $\pi_G^1$ in terms of the usual characteristic classes entering the index theorem, namely the $G$-equivariant Todd class of the vertical tangent bundle of $M$. To show this we focus on a class of elements in $H_{\text{top}}^\bullet (C^\infty_c(B) \rtimes G)|_O$ represented by geometric cocycles: the latter are quadruples $(N, E, \Phi, c)$ where $N, E$ are smooth manifolds, $\Phi$ is a flat connection on a certain groupoid, and $c$ is a cocycle in a certain complex. See section 6 for the precise definitions. This geometric construction of localized cyclic cohomology classes is of independent interest since it works for any choice of Ad-invariant isotropic submanifold $O \subset G$. Moreover in the case of the localization at units $O = B$, one recovers all the previously known constructions of cyclic cohomology in special cases, including: the cohomology of the classifying space for $G$ and Gelfand-Fuchs cohomology (étale groupoids or foliation groupoids [3, 5]), differentiable groupoid cohomology ([24]), etc. The next result (theorem 7.5) is a refinement of Theorem 5.6 in the case of geometric cocycles localized at units.
Theorem 7.5 Let $G \rightrightarrows B$ be a Lie groupoid acting on a surjective submersion $\pi: M \to B$. The excision map localized at units

$$\pi_G^1 : HP_{\top}^\bullet(C^\infty_c(B) \rtimes G)[[B]] \to HP_{\top}^\bullet(C^\infty_c(S^*_\pi M \rtimes G))[[S^*_\pi M]]$$

sends the cyclic cohomology class of a proper geometric cocycle $(N, E, \Phi, c)$ to the cyclic cohomology class

$$\pi_G^1([N, E, \Phi, c]) = [N \times_B S^*_\pi M, E \times_B S^*_\pi M, \pi^{-1}_*(\Phi), \text{Td}(T_{\pi} M \otimes \mathbb{C}) \wedge \pi^*(c)]$$

where $\text{Td}(T_{\pi} M \otimes \mathbb{C})$ is the Todd class of the complexified vertical tangent bundle in the invariant leafwise cohomology of $E \times_B S^*_\pi M$.

Once specialized to the case of a discrete group $G$ acting by any diffeomorphisms on a manifold $M$, this theorem completely solves the problem of evaluating the $K$-theoretical index of a Fourier integral operator like (1) on cyclic cohomology classes localized at units, in terms of the non-commutative leading symbol of this operator in $K_1(C^\infty(S^*M) \rtimes G)$. Let us mention that a generalization of this result to the hypoelliptic calculus of Connes and Moscovici [6] would give a direct proof of the index theorem for transversally hypoelliptic operators on foliations. This ongoing research will appear elsewhere [23].

To end this article we also present an application of the residue formula of Theorem 5.6 in a case not localized at units. We prove an equivariant longitudinal index theorem for a codimension one foliation endowed with a transverse action of the group $\mathbb{R}$. Choosing a complete transversal for the foliation, this geometric situation can be reduced to the action of a Lie groupoid on a submersion.

The groupoid possesses a canonical trace, and we show that the pairing of the corresponding cyclic cocycle with the index of any leafwise elliptic equivariant pseudodifferential operator is given by a formula localized at the periodic orbits of the transverse flow on the foliation. This gives an interesting interpretation of the results of Alvarez-Lopez, Kordyukov [1] and Lazarov [13] in the context of the $K$-theory/cyclic cohomology of Lie groupoids.

Let us now give a brief description of the article. Section 2 recalls elementary notions about Lie groupoids, convolution algebras, and pseudodifferential operators. We introduce the basic extension $(E) : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$ associated to the action of a Lie groupoid on a submersion and describe the index map. In section 3 we define the cyclic cohomology of the smooth convolution algebra of a Lie groupoid, both in the algebraic and topological case, and establish the properties of localization and Morita invariance. In section 4 we recall the computation of the excision map from [21] and prove Theorem 4.7. In section 5 we use zeta-function renormalization techniques to prove the residue Theorem 5.6. Then section 6 describes the construction of cyclic cohomology classes from geometric cocycles. By localization at units, this construction is used in section 7 where Theorem 7.5 is proved. Finally section 8 contains the example of a transverse flow on a codimension one foliation.

Acknowledgements: This work partially supported by CNRS emerged from a stay at the Institut de Mathématiques de Jussieu, Université Paris 7 for the academic year 2012-2013. The author is very grateful to all members of the team “Algèbres d’Opérateurs” for their warm hospitality and the excellent working conditions, and especially to Georges Skandalis for many helpful discussions.
2 Convolution algebras of Lie groupoids

In this section we recall some basic facts about Lie groupoids, convolution algebras, pseudodifferential operators, and define the index of a non-commutative elliptic symbol as an algebraic $K$-theory class.

Definition 2.1 A Lie groupoid $G \rightrightarrows B$ consists of:

a) Two smooth manifolds $B = G^{(0)}$ (the set of units) and $G = G^{(1)}$ (the set of morphisms). We will assume that $B$ and $G$ are Hausdorff and without boundary;

b) Two submersions $r, s : G \to B$ called the range and source map respectively;

c) A smooth map $m : G^{(2)} \to G$, where $G^{(2)} = \{(g_1, g_2) \in G \times G \mid s(g_1) = r(g_2)\}$ is the set of composable arrows, called the product map. We usually write $m(g_1, g_2) = g_1g_2$;

d) A smooth embedding $u : B \hookrightarrow G$ and a diffeomorphism $i : G \to G$ called the unit and inverse map respectively. We usually write $u(b) = b$ for any $b \in B$ and $i(g) = g^{-1}$ for any $g \in G$.

These data are subject to compatibility conditions: for all composable morphisms $g_1, g_2, g_3 \in G$ one has

i) $r(g_1 g_2) = r(g_1)$ and $s(g_1 g_2) = s(g_2)$;

ii) $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ (associativity of the product);

iii) $gs(g) = g$ and $r(g)g = g$ (units);

iv) $gg^{-1} = r(g)$ and $g^{-1}g = s(g)$ (inverse).

Following the standard convention we denote by $G^{(n)}$ the submanifold of composable $n$-tuples of morphisms $(g_1, \ldots, g_n)$ in $G^n$. One should keep in mind that a morphism $g \in G$ may be represented by a left-oriented arrow

\[ r(g) \leftarrow s(g) \]

and the product of morphisms is the concatenation of arrows. For any unit $b \in B$, we denote by $G_b = \{g \in G \mid s(g) = b\}$ the fiber of the source map over $b$, and by $G^b = \{g \in G \mid r(g) = b\}$ the fiber of the range map. These are submanifolds in $G$. At any point $b \in B$, the intersection $G^b \cap G^b$ is a Lie group called the isotropy group at $b$. The isotropy subset $I = \{g \in G \mid r(g) = s(g)\}$ is the union of all isotropy groups. It is a closed subset in $G$ but generally not a submanifold; the isotropy groups $G^b_b$ may have jumps when $b$ varies. Let us recall some basic constructions.

Definition 2.2 Let $G \rightrightarrows B$ be a Lie groupoid and $\pi : M \to B$ a submersion of smooth manifolds. The pullback groupoid of $G$ by $\pi$ is the Lie groupoid

\[ \pi^*G \rightrightarrows M \]

where $\pi^*G = \{(x, g, y) \in M \times G \times M \mid \pi(x) = r(g), \pi(y) = s(g)\}$, with range map $(x, g, y) \mapsto x$, source map $(x, g, y) \mapsto y$, and product of composable morphisms $(x, g_1, y) \cdot (y, g_2, z) = (x, g_1g_2, z)$. 

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Definition 2.3 Two Lie groupoids $G_1 \rightrightarrows B_1$ and $G_2 \rightrightarrows B_2$ are Morita equivalent if there exists a smooth manifold $M$ and two surjective submersions

$$B_1 \xrightarrow{\pi_1} M \xrightarrow{\pi_2} B_2$$

together with an isomorphism $\pi_1^* G_1 \cong \pi_2^* G_2$ between the pullback groupoids. Morita equivalence is an equivalence relation among Lie groupoids.

Definition 2.4 Let $M$ be a smooth manifold and $G \rightrightarrows B$ a Lie groupoid. A right $G$-action on $M$ is given by

a) A smooth submersion $\pi : M \to B$;

b) A map from the fibered product $M \times (\pi, r) G = \{(x, g) \in M \times G \mid \pi(x) = r(g)\}$ to $M$, sending a pair $(x, g)$ to the element $x \cdot g$ such that $\pi(x) = s(g)$ and $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ whenever $g_1$ and $g_2$ can be composed.

We call such $\pi : M \to B$ a $G$-equivariant submersion.

Note that the fibered product manifold $M \times (\pi, r) G$ endowed with the composition law $(x, g_1) \cdot (y, g_2) = (x, g_1 g_2)$ whenever $x \cdot g_1 = y$, is a Lie groupoid with range map $(x, g) \mapsto x$ and source map $(x, g) \mapsto x \cdot g$. We usually denote this action groupoid by $M \times G \rightrightarrows M$. It should not be confused with the pullback groupoid $\pi^* G$.

Let $M_b = \pi^{-1}(b)$ be the preimage of a point $b \in B$. Since $\pi$ is a submersion $M_b$ is a submanifold of $M$. Any element $g \in G$ induces a diffeomorphism $M_{r(g)} \to M_{s(g)}$ by $x \mapsto x \cdot g$. We extend this to a diffeomorphism $T^* M_{r(g)} \to T^* M_{s(g)}$ between the cotangent bundles of the respective submanifolds in $M$. One has a natural action of the multiplicative group $\mathbb{R}^*_+$ on the fibers of the cotangent bundle and the quotient $S^* M_b = T^* M_b / \mathbb{R}^*_+$ defines the cosphere bundle over $M_b$. Moreover the diffeomorphism induced by $g$ commutes with the action of $\mathbb{R}^*_+$, hence descends to a diffeomorphism $S^* M_{r(\gamma)} \to S^* M_{s(\gamma)}$. The collection

$$S^*_b M = \bigcup_{b \in B} S^* M_b \quad (12)$$

of vertical cosphere bundles is also clearly a $G$-equivariant submersion with base $B$.

We now define the smooth convolution algebra of a Lie groupoid $(r, s) : G \rightrightarrows B$. The kernel $\text{Ker} \ s_*$ of the tangent map $s_* : TG \to TB$ is the vector bundle over $G$ whose vectors are tangent to the fibers $G_b = s^{-1}(b)$ of the source map. These vectors are the infinitesimal generators of the left multiplication of $G$ on itself. Since left and right multiplication commute, the fibers $(\text{Ker} \ s_*)_g$ and $(\text{Ker} \ s_*)_g$ are canonically isomorphic whenever $r(g_1) = r(g_2)$. The restriction of $\text{Ker} \ s_*$ to the submanifold of units $B \subset G$ yields a vector bundle $A^* G$ over $B$ called the Lie algebroid of $G$, and one has a canonical identification

$$r^* (A^* G) \cong \text{Ker} \ s_*$$

of vector bundles over $G$. Hence, any section of $A^* G$ over $B$ can be pulled back by the rank map $r$ to a “right-invariant” section of $\text{Ker} \ s_*$ over $G$. Passing to the dual bundle $A^* G$ of the Lie algebroid and taking the maximal exterior power,
one thus gets an isomorphism between the line bundle $r^*([\Lambda^\text{max}^*A^*G])$ and the line bundle $[\Lambda^\text{max}(\text{Ker } s_\eta^*)^*]$ of 1-densities along the submanifolds $G_b = s^{-1}(b)$. The smooth convolution algebra of $G$ is then defined as the $C^\infty$-vector space

$$C^\infty_c(B) \times G := C^\infty_c(G, r^*([\Lambda^\text{max}^*A^*G]))$$

(13)
of smooth (complexified) sections of this line bundle. The product of two sections $a_1, a_2 \in C^\infty_c(B) \times G$ evaluated on a point $g \in G$ is given by an integral over all possible decompositions of $g$ into products $g_1g_2$,

$$(a_1a_2)(g) = \int_{g_1g_2 = g} a_1(g_1) a_2(g_2),$$

(14)

where $a_1(g_1)a_2(g_2) \in [\Lambda^\text{max}^*A^*G]_{r(g_1)} \otimes [\Lambda^\text{max}^*A^*G]_{r(g_2)}$. The integral makes sense because $r(g_1) = r(g)$ is fixed, and when the point $g_2$ varies in $G_{s(g)}$ the line $[\Lambda^\text{max}^*A^*G]_{r(g_2)}$ runs over the fibers of the 1-density bundle over $G_{s(g)}$. This product is associative but not commutative in general. The convolution algebra is not unital unless $G$ is étale (i.e. $r$ and $s$ are local diffeomorphisms, equivalently the Lie algebroid is reduced to its zero-section $B$) and $B$ is compact. In the latter case the unit $e$ of the convolution algebra is $e(g) = 0$ for $g \notin B$ and $e(b) = 1$ for $b \in B$.

By choosing a trivialisation of $[\Lambda^\text{max}^*A^*G]$, that is a nowhere vanishing section, one obtains by pullback a right-invariant section of the 1-density bundle, or equivalently smooth Haar system on $G$. A choice of Haar system allows one to identify the convolution algebra of $G$ with the space of complex-valued functions with compact support on $G$,

$$C^\infty_c(B) \times G \cong C^\infty_c(G)$$

and transfer the convolution product on the latter. Since a choice of Haar system is non-canonical, it is sometimes more convenient to use the completely canonical definition of the product given above.

Let $R \to B$ is a $G$-equivariant associative algebra bundle, that is, a bundle of associative algebras over $B$ endowed with an action by isomorphisms $U_g : R_{s(g)} \to R_{r(g)}$, $\forall g \in G$, compatible with the products in $G$. We suppose that the sections of this bundle are endowed with some “smooth” structure ensuring that the subsequent constructions make sense. Then we define the convolution algebra of $G$ twisted by the bundle $R$ as the space of compactly supported sections

$$C^\infty_c(B, R) \times G := C^\infty_c(G, r^*(R \otimes [\Lambda^\text{max}^*A^*G]))$$

(15)

endowed with a slight generalization of the above convolution product. For two sections $a_1, a_2 \in C^\infty_c(B, R) \times G$ we set

$$(a_1a_2)(g) = \int_{g_1g_2 = g} a_1(g_1) \cdot U_g a_2(g_2),$$

(16)

where the isomorphism $U_{g_1} : R_{s(g_1)} \otimes [\Lambda^\text{max}^*A^*G]_{s(g_1)} \to R_{r(g_1)} \otimes [\Lambda^\text{max}^*A^*G]_{s(g_1)}$ acts on the fiber of $R$ but not on the fiber of the density bundle. As an example let $\pi : M \to B$ be a $G$-equivariant submersion, and take $R$ as the bundle whose fiber over $b \in B$ is the commutative algebra of smooth functions with compact support $C^\infty_c(M_b)$. Then

$$C^\infty_c(B, R) \times G \cong C^\infty_c(M) \times G$$
is canonically isomorphic to the smooth convolution algebra of the action groupoid \( M \rtimes G \). We will sometimes use the notation \( C^\infty_p(B, \mathbb{R}) \rtimes G \) for the crossed product algebra of \emph{properly supported} sections of the bundle \( r^*(\mathbb{R} \otimes |A^{\max}A^G|) \) over \( G \).

Now let \( \text{CL}^m_c(M_c) \) be the space of \emph{compactly supported} classical (1-step polynomially homogeneous) pseudodifferential operators of order \( m \in \mathbb{Z} \), acting on the space of smooth functions with compact support on manifold \( M_c \). Such a linear operator on \( C^\infty_c(M_c) \) has distribution kernel with compact support in \( M_c \times M_c \), and in a local coordinate system \((x, p)\) on \( T^*M_c \) its symbol has an asymptotic expansion \( \sigma(x, p) \sim \sum_{j \geq 0} \sigma_{m-j}(x, p) \), with \( \sigma_{m-j}(x, p) \) a positively homogeneous function of degree \( m - j \in \mathbb{Z} \) in the variable \( p \). For any \( g \in G \) and any \( P \in \text{CL}^m_c(M_{s(g)}) \), the pushforward \( U_gPU^{-1}_g \in \text{CL}^m_c(M_{r(g)}) \) is the adjoint action of the linear isomorphism \( U_g: C^\infty_c(M_{s(g)}) \to C^\infty_c(M_{r(g)}) \) induced by the diffeomorphism \( g \). We denote by \( \text{CL}_c(M) \to B \) the bundle over the base manifold \( B \), whose fiber at a point \( b \in B \) is the algebra \( \text{CL}_c(M_b) \). Hence \( \text{CL}_c(M) \) is a \( G \)-bundle. The subbundle \( L^c_{-\infty}(M_c) \) whose fiber is the algebra of smoothing operators with compact support, is a two-sided ideal in \( \text{CL}_c(M) \). The quotient \( \text{CS}^c_0(M_c) = \text{CL}_c(M) / L^c_{-\infty}(M_c) \) defines the algebra bundle of \emph{formal symbols} over \( B \). We will essentially focus on the algebra bundle \( \text{CL}^0_c(M) \) of classical pseudodifferential operators of order \( \leq 0 \), and its two-sided ideal \( \text{CL}^{-1}_c(M) \). The quotient \( L^c_{0}(M) = \text{CL}^0_c(M) / \text{CL}^{-1}_c(M) \) of \emph{leading symbols} is isomorphic to the bundle whose fiber over a point \( b \in B \) is the commutative algebra \( C^\infty_c(S^*M_b) \) of smooth compactly supported functions on the cosphere bundle of \( M_b \). One thus has a commutative diagram of algebra bundles over \( B \)

\[
\begin{array}{cccccc}
0 & \longrightarrow & L^c_{-\infty}(M) & \longrightarrow & \text{CL}^0_c(M) & \longrightarrow & \text{CS}^0_0(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{CL}^{-1}_c(M) & \longrightarrow & \text{CL}^0_c(M) & \longrightarrow & L^c_{0}(M) & \longrightarrow & 0 \\
\end{array}
\]

(17)

where the rows are exact sequences. The left vertical arrow is an injection, whereas the right vertical arrow is a surjection. Since all bundles are \( G \)-bundles, one can form the corresponding convolution algebras by crossed product with the action of \( G \). Notice there are canonical isomorphisms

\[
C^\infty_c(B, L^c_{-\infty}(M)) \rtimes G \cong C^\infty_c(M) \rtimes \pi^*G, \quad C^\infty_c(B, L^c_{0}(M)) \rtimes G \cong C^\infty_c(S^*_cM) \rtimes G
\]

where \( C^\infty_c(M) \rtimes \pi^*G \) is the smooth convolution algebra of the pullback groupoid \( \pi^*G \rightrightarrows M \), and \( C^\infty_c(S^*_cM) \rtimes G \) is the smooth convolution algebra of the action groupoid \( S^*_cM \rtimes G \cong S^*_cM \). All arrows of (17) being \( G \)-equivariant homomorphisms of algebra bundles over \( B \), one gets a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C^\infty_c(M) \rtimes \pi^*G & \longrightarrow & C^\infty_c(B, \text{CL}^0_c(M)) \rtimes G & \longrightarrow & C^\infty_c(B, \text{CS}^0_0(M)) \rtimes G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\infty_c(B, \text{CL}^{-1}_c(M)) \rtimes G & \longrightarrow & C^\infty_c(B, \text{CL}^0_c(M)) \rtimes G & \longrightarrow & C^\infty_c(S^*_cM) \rtimes G & \longrightarrow & 0 \\
\end{array}
\]

where the rows are short exact sequences of associative algebras. The bottom row is our main object of interest. We set \( \mathcal{A} = C^\infty_c(S^*_cM) \rtimes G \), \( \mathcal{B} = \)
$C^\infty_c(B, CL^{-1}_c(M)) \times G$, $\mathcal{E} = C^\infty_c(B, CL^0_c(M)) \times G$ and consider the extension

$$(E) : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0.$$  \hfill (18)

The boundary map of Milnor [15] associates to any algebraic $K$-theory class $[u] \in K_1(\mathcal{A})$ an index $\text{Ind}_E([u]) \in K_0(\mathcal{B})$. Recall that a class $[u]$ is represented by an element $u$ in the group $GL_\infty(\mathcal{A})$ of invertible infinite matrices of the form $u = 1 + v$ with $v \in M_\infty(\mathcal{A})$. We can choose two matrices $P$ and $Q$ with entries in the algebra $\mathcal{E}$ (with a unit adjoined), which project respectively to $u$ and its inverse $u^{-1}$. Then $P$ and $Q$ are inverse to each other modulo the ideal of matrices with entries in $\mathcal{B}$. The index \(\text{Ind}_E([u]) \in K_0(\mathcal{B})\) is represented by the difference of idempotents $[e] - [e_0]$ where

$$e = \begin{pmatrix} 1 - (1 - PQ)^2 & Q(2 - PQ)(1 - PQ) \\ (1 - PQ)P & (1 - PQ)^2 \end{pmatrix}, \quad e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  \hfill (19)

**Definition 2.5** Let $G \rightrightarrows B$ be a Lie groupoid and $\pi : M \to B$ a smooth $G$-equivariant submersion. Set $\mathcal{A} = C^\infty_c(S^*_uM) \times G$ and $\mathcal{B} = C^\infty_c(B, CL^{-1}_c(M)) \times G$. An invertible matrix $u \in GL_\infty(\mathcal{A})$ is called an elliptic symbol. Its index is the $K$-theory class

$$\text{Ind}_E([u]) \in K_0(\mathcal{B}),$$  \hfill (20)

the image of $[u] \in K_1(\mathcal{A})$ under the boundary map associated to the natural extension $(E) : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$.

### 3. Cyclic homology

We recall the basic notions of cyclic homology. Let $\mathcal{A}$ be an associative $C^*$-algebra. The space of noncommutative $n$-forms over $\mathcal{A}$ is $\Omega^n \mathcal{A} = \mathcal{A}^n \otimes \mathcal{A}^\otimes n$ for all $n \geq 1$, where $\mathcal{A}^n = \mathcal{A} \otimes \mathcal{A}$ is the algebra obtained by adjoining a unit. For $n = 0$ one has $\Omega^0 \mathcal{A} = \mathcal{A}$. We write $a_0 a_1 \ldots a_n$ (reps. $da_1 \ldots da_n$) for the generic element $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ (reps. $1 \otimes a_1 \otimes \ldots \otimes a_n$) in $\Omega^n \mathcal{A}$. A differential $d : \Omega^n \mathcal{A} \to \Omega^{n+1} \mathcal{A}$ is defined by $d(a_0 a_1 \ldots a_n) = d(a_0) a_1 \ldots a_n$ and $d(da_1 \ldots da_n) = 0$, and of course $d^2 = 0$. The direct sum $\Omega \mathcal{A} = \bigoplus_{n \geq 0} \Omega^n \mathcal{A}$ is gifted with the unique graded product satisfying the Leibniz rule $d(\omega_1 \omega_2) = d\omega_1 \omega_2 + (-1)^{n_1} \omega_1 d\omega_2$ for all $\omega_1 \in \Omega^{n_1} \mathcal{A}$. This turns $\Omega \mathcal{A}$ into a differential graded algebra. The Hochschild boundary operator $b : \Omega^n \mathcal{A} \to \Omega^{n-1} \mathcal{A}$ is defined by $b(\omega da) = (-1)^{n-1} [\omega, a]$ for all $\omega$ of degree $n - 1$ and $a \in \mathcal{A}$. Equivalently

$$b(a_0 a_1 \ldots a_n) = a_0 a_1 da_2 \ldots da_n + \sum_{i=1}^{n-1} (-1)^i a_0 da_1 \ldots d(a_{i+1}) \ldots da_n$$

$$+ (-1)^n a_n a_0 da_1 \ldots da_{n-1};$$  \hfill (21)

for all $a_0 \in \mathcal{A}^+$ and $a_i \in \mathcal{A}$, $i \geq 1$. Let $\kappa = 1 - (bd + db)$ be the Karoubi operator. One has $\kappa(da) = (-1)^nda \omega$ for all $n$-form $\omega$ and $a \in \mathcal{A}$. The Connes boundary operator $B : \Omega^n \mathcal{A} \to \Omega^{n+1} \mathcal{A}$ is defined by $B = (1 + \kappa + \ldots + \kappa^n)d$, or equivalently

$$B(a_0 a_1 \ldots da_n) = \sum_{i=0}^{n} (-1)^i da_1 \ldots da_n da_{i+1} \ldots da_{n-1}. $$  \hfill (22)
One has $b^2 = bB + Bb = B^2 = 0$, hence $\Omega A$ endowed with the orators $(b, B)$ is a bicomplex. By definition the cyclic homology $HC_n(\mathcal{A})$ is the homology of the following total complex with boundary $b + B$:

\[
\begin{array}{ccccccccc}
\Omega^2 \mathcal{A} & \xrightarrow{B} & \Omega^1 \mathcal{A} & \xrightarrow{B} & \Omega^0 \mathcal{A} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^1 \mathcal{A} & \xrightarrow{B} & \Omega^0 \mathcal{A} \\
\downarrow & & \downarrow \\
\Omega^0 \mathcal{A}
\end{array}
\] (23)

A cyclic homology class in $HC_n(\mathcal{A})$ is therefore represented by an inhomogeneous differential form $\omega_n + \omega_{n-2} + \ldots \in \Omega^n \mathcal{A} \oplus \Omega^{n-2} \mathcal{A} \oplus \ldots$ which is closed in the sense $b\omega_n + B\omega_{n-2} + \ldots = 0$, etc... The obvious shift of degree two obtained by deleting the first column of the cyclic bicomplex gives rise to the periodicity operator $S : HC_n(\mathcal{A}) \to HC_{n-2}(\mathcal{A})$. Now complete the space of differential forms $\Omega A$ by taking direct products instead of direct sums, and write $\hat{\Omega} A = \prod_{n \geq 0} \Omega^n A$. The operator $b + B$ extends to a well-defined boundary operator on the completed space. The periodic cyclic homology of $\mathcal{A}$ is defined as the homology of this complex:

\[
HP_\bullet(\mathcal{A}) = H_\bullet(\hat{\Omega} A, b + B) .
\] (24)

By construction the periodic cyclic homology is $\mathbb{Z}_2$-graded, that is $HP_n(\mathcal{A}) \cong HP_{n+2}(\mathcal{A})$ for all $n$. Since the complex $(\hat{\Omega} A, b + B)$ can be recovered as the projective limit (under the operation $S$) of the cyclic bicomplex of $\mathcal{A}$, the periodic and non-periodic cyclic homologies are related by a Milnor $\lim^1$ exact sequence

\[
0 \to \lim^1_S HC_{\bullet-1}(C_c^\infty(G)) \to HP_\bullet(C_c^\infty(G)) \to \lim_S HC_\bullet(C_c^\infty(G)) \to 0
\] (25)

The cyclic cohomology groups $HC^n(\mathcal{A})$ are defined through the dual complex of (23) over $\mathbb{C}$. One simply has to replace the vector space $\Omega^n \mathcal{A}$ by its dual $\Omega^n \mathcal{A}' = \text{Hom}(\Omega \mathcal{A}, \mathbb{C})$ over $\mathbb{C}$, and transpose the boundaries $(b, B)$. The periodicity operator $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$ now raises the degree by two. Periodic cyclic cohomology is defined as the cohomology of the direct sum $\Omega \mathcal{A}' = \bigoplus_{n \geq 0} \Omega^n \mathcal{A}'$, which is a $\mathbb{Z}_2$-graded complex once endowed with the transposed of the operator $b + B$:

\[
HP_\bullet(\mathcal{A}) = H^\bullet(\Omega \mathcal{A}', b + B) .
\] (26)

The link between periodic and non-periodic cyclic cohomology is simpler than the case of homology, since $HP_\bullet(\mathcal{A})$ is the inductive limit over $S$ of the groups $HC_\bullet(\mathcal{A})$. There are obvious bilinear pairings $HC^n(\mathcal{A}) \times HC_n(\mathcal{A}) \to \mathbb{C}$ and $HP^n(\mathcal{A}) \times HP_n(\mathcal{A}) \to \mathbb{C}$. 

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Let $G \rightrightarrows B$ be a Lie groupoid. For notational convenience we suppose that a smooth Haar system on $G$ has been fixed, so the smooth convolution algebra $A = C^\infty_c(B) \rtimes G$ is isomorphic to $C^\infty_c(G)$. The space of noncommutative $n$-forms $\Omega^r C^\infty_c(G) = C^\infty_c(G)^+ \otimes C^\infty(G)^{\otimes n}$ is a subspace of the smooth functions with compact support on the union of manifolds $G^{n+1} \cup G^n$. Indeed a generic $n$-form $a_0 da_1 \ldots da_n$, with $a_i \in C^\infty_c(G)$, is a function of $n + 1$ points in $G$,

$$ (a_0 da_1 \ldots da_n)(g_0, g_1, \ldots, g_n) = a_0(g_0)a_1(g_1) \ldots a_n(g_n), $$
while a $n$-form $da_1 \ldots da_n$ is a function of $n$ points:

$$ (da_1 \ldots da_n)(g_1, \ldots, g_n) = a_1(g_1) \ldots a_n(g_n). $$

Thus $\Omega^r C^\infty_c(G)$ is actually a (complicated) subspace of the smooth compactly supported functions on the manifold $\bigcup_{n \geq 0} (G^{n+1} \cup G^n)$. Let $I = I^{(1)}$ be the isotropy subset of $G$, i.e. the set of morphisms $g \in G$ such that $r(g) = s(g)$. Following [2], we define the set of loops $I^{(n)} \subset G^n$ for higher $n$ as

$$ I^{(n)} = \{(g_1, \ldots, g_n) \in G^n \mid s(g_i) = r(g_{i+1}) \forall i < n, \; s(g_n) = r(g_1)\} \quad (27) $$

We want to show that the periodic cyclic (co)homology of $C^\infty_c(G)$ can be localized, in the sense that the information of the cyclic bicomplex is entirely contained in the vicinity of the set of loops. To make it precise, let $\Omega^r C^\infty_c(G)$ be the subspace of functions vanishing on some neighborhood of the set of loops $I^{(n+1)} \cup I^{(n)} \subset G^{n+1} \cup G^n$. We define the quotient space of localized forms

$$ \Omega^r C^\infty_c(G)(I) = \Omega^r C^\infty_c(G)/\Omega^r C^\infty_c(G)_0 \quad (28) $$

Clearly the direct sum $\Omega C^\infty_c(G)_0 = \bigoplus_{n \geq 0} \Omega^r C^\infty_c(G)_0$ is stable by the boundary operators $b, B$ hence yields a subcomplex of the cyclic bicomplex. Consequently, the operators $b, B$ descend to the quotient $\Omega^r C^\infty_c(G)(I)$. This leads to a localized cyclic bicomplex. The localized periodic cyclic homology of $C^\infty_c(G)$ is defined accordingly, through the completion $\hat{\Omega^r C^\infty_c(G)}(I) = \varprojlim_{n \geq 0} \Omega^r C^\infty_c(G)(I)$:

$$ HP_\bullet(\hat{\Omega^r C^\infty_c(G)}(I)) = H_\bullet(\hat{\Omega^r C^\infty_c(G)}(I), b + B) \quad (29) $$

The localized periodic cyclic cohomology of $C^\infty_c(G)$ is defined by duality. We let $\Omega^r C^\infty_c(G)^{(I)} = \text{Hom}(\Omega^r C^\infty_c(G)(I), \mathbb{C})$ be the space of $\mathbb{C}$-linear functionals on localized $n$-forms. This is exactly the set of linear maps $\varphi : \Omega^r C^\infty_c(G) \to \mathbb{C}$ vanishing on the subspace $\Omega^r C^\infty_c(G)_0$. Then the direct sum $\Omega^r C^\infty_c(G)^{(I)} = \bigoplus_{n \geq 0} \Omega^r C^\infty_c(G)^{(I)}$ is a $\mathbb{Z}_2$-graded complex with boundary the transpose of $b + B$, whence

$$ HP^\bullet(\Omega^r C^\infty_c(G)^{(I)}) = H^\bullet(\text{Hom}(\Omega^r C^\infty_c(G)(I), \mathbb{C}), b + B) \quad (30) $$

**Proposition 3.1 (Localization: algebraic case)** Let $G$ be any Lie groupoid with isotropy subset $I$. Then the projection of cyclic bicomplexes $\Omega^r C^\infty_c(G) \to \Omega^r C^\infty_c(G)^{(I)}$ induces isomorphisms

$$ HP_\bullet(\Omega^r C^\infty_c(G)) \cong HP_\bullet(\Omega^r C^\infty_c(G)^{(I)}), \quad H P^\bullet(\Omega^r C^\infty_c(G)) \cong H^P^\bullet(\Omega^r C^\infty_c(G)^{(I)}). \quad (31) $$
Proof: Denote by $HC_{\bullet}(C_{c}^{\infty}(G))_0$ and $HP_{\bullet}(C_{c}^{\infty}(G))_0$ the cyclic homologies computed by the subcomplex $\Omega C_{c}^{\infty}(G)_0$ of forms vanishing in the vicinity of loops. Our goal is to prove that $HP_{\bullet}(C_{c}^{\infty}(G))_0 = 0$. Then using the homology six-term exact sequence

$$
\begin{CD}
HP_0(C_{c}^{\infty}(G))_0 @>>> HP_0(C_{c}^{\infty}(G)) @>>> HP_0(C_{c}^{\infty}(G))_{(1)} \\
| @. | @. | \\
HP_1(C_{c}^{\infty}(G))_{(1)} @<<< HP_1(C_{c}^{\infty}(G)) @<<< HP_1(C_{c}^{\infty}(G))_0
\end{CD}
$$

associated to the exact sequence of complexes $0 \to \hat{\Omega} C_{c}^{\infty}(G)_0 \to \hat{\Omega} C_{c}^{\infty}(G) \to \hat{\Omega} C_{c}^{\infty}(G)_{(1)} \to 0$ gives the first isomorphism in (31). We first show that the periodicity operator $S : HC_n(C_{c}^{\infty}(G))_0 \to HC_{n-2}(C_{c}^{\infty}(G))_0$ vanishes. Let $(U_i)_{i \in I}$ be a locally finite open covering of the space $B$ of units in $G$, where $I \subseteq \mathbb{N}$ is an ordered, at most countable set (thus each compact subset of $B$ intersects a finite number of $U_i$’s). Let $(c_i)_{i \in I}$ be a partition of unity relative to this covering, in the sense that $c_i \in C_{c}^{\infty}(U_i)$ for all $i \in I$ and $\sum_{i \in I} c_i(x)^2 = 1$ for all $x \in B$. We view each function $c_i \in C_{c}^{\infty}(B)$ as a multiplier of the algebra $C_{c}^{\infty}(G)$: for any $a \in C_{c}^{\infty}(G)$ and $g \in G$ set

$$(c_ia)(g) := c_i(r(g))a(g), \quad (ac_i)(g) := a(g)c_i(s(g)).$$

We use the partition of unity to build a map $\rho$ from $C_{c}^{\infty}(G)$ to the algebra of infinite matrices $\mathcal{M}_{\infty}(C_{c}^{\infty}(G))$ as follows: for each $a \in C_{c}^{\infty}(G)$, the matrix element of $\rho(a)$ in position $(i,j)$ is $c_ia_cj$. By the compactness of the support of $a$, only a finite number of matrix elements are non-zero. The condition $\sum_i c_i^2 = 1$ shows that $\rho$ is a homomorphism of algebras. Therefore, the induced map $\rho_* : \Omega^n C_{c}^{\infty}(G) \to \Omega^n \mathcal{M}_{\infty}(C_{c}^{\infty}(G))$ composed with the trace of matrices yields a morphism of cyclic bicomplexes $\text{tr} \rho_* : \Omega C_{c}^{\infty}(G) \to \Omega C_{c}^{\infty}(G)$. Explicitly for any $n$-form $a_0 da_1 \ldots d a_n$ we have

$$\text{tr} \rho_*(a_0 da_1 \ldots d a_n) = \sum_{i_0, \ldots, i_n} (c_{i_0} a_0 c_{i_1}) d(c_{i_1} a_1 c_{i_2}) \ldots d(c_{i_n} a_n c_{i_0}),$$

and similarly for $\text{tr} \rho_*(a_1 \ldots d a_n)$. This morphism clearly restricts to a morphism of subcomplexes $\Omega C_{c}^{\infty}(G)_0 \to \Omega C_{c}^{\infty}(G)_0$. Observe also that the $n$-form $(c_{i_0} a_0 c_{i_1}) d(c_{i_1} a_1 c_{i_2}) \ldots d(c_{i_n} a_n c_{i_0})$, viewed as a smooth function on $G^{n+1}$, has compact support consisting of multiplets $(g_0, g_1, \ldots, g_n)$ such that $s(g_0)$ and $r(g_1)$ are in the support of $c_{i_1}$, $s(g_1)$ and $r(g_2)$ are in the support of $c_{i_2}$, and so on. Thus, if the supports of the $c_{i_j}$’s are small enough, the function $\text{tr} \rho_*(a_0 da_1 \ldots d a_n)$ can be localized to an arbitrary small neighborhood of the set of loops $I^{(n+1)}$ in $G^{n+1}$. In particular if $a_0 da_1 \ldots d a_n$ belongs to the subspace $\Omega^n C_{c}^{\infty}(G)_0$, one can always find a suitably fine covering of $B$ together with a partition of unity such that $\text{tr} \rho_*(a_0 da_1 \ldots d a_n)$ vanishes. The next step is to provide a homotopy between the homomorphism $\rho$ and the natural inclusion $C_{c}^{\infty}(G) \hookrightarrow \mathcal{M}_{\infty}(C_{c}^{\infty}(G))$ in the upper left matrix position. Indeed, consider the isomorphism of algebras

$$
\begin{pmatrix}
C_{c}^{\infty}(G) & C_{c}^{\infty} \otimes C_{c}^{\infty}(G) \\
C_{\text{col}} \otimes C_{c}^{\infty}(G) & \mathcal{M}_{\infty}(C_{c}^{\infty}(G))
\end{pmatrix}
$$

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where $\mathbb{C}^\infty_{\text{row}}$ and $\mathbb{C}^\infty_{\text{col}}$ are respectively the spaces of infinite row and column matrices, with finitely many non-zero entries in $\mathbb{C}$. According to this $2 \times 2$ matrix notation one has two relevant homomorphisms $\rho^0, \rho^1 : C^\infty_c(G) \to M_\infty(C^\infty_c(G))$ given by

$$\rho^0(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho^1(a) = \begin{pmatrix} 0 & 0 \\ 0 & \rho(a) \end{pmatrix}. $$

Then $\text{tr} \rho^0$ is the identity map on $\Omega C^\infty_c(G)$, while $\text{tr} \rho^1 = \text{tr} \rho_s$. Let $u = (u_i)_{i \in I}$ be the infinite row with $u_i = c_i$, and $v = (v_i)_{i \in I}$ the infinite column with $v_i = c_i$. Note that $u$ and $v$ may have infinitely many non-zero entries. Nevertheless the scalar product $vw = \sum_i c_i^2 = 1$ is well-defined, so that $uv$ is an idempotent matrix with infinitely many non-zero entries. From this one can produce a matrix $W$ and its inverse $W^{-1}$, which are both multipliers of the algebra $M_\infty(C^\infty_c(G))$:

$$W = \begin{pmatrix} 0 & -u \\ u & 1 - vu \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} 0 & u \\ -v & 1 - vu \end{pmatrix}.$$

One has $\rho(a) = vau$ for all $a \in C^\infty_c(G)$, and the elements $\rho^0(a), \rho^1(a)W, W^{-1}\rho^0(a)$ and $W^{-1}\rho^1(a)W = \rho^1(a)$ are all in $M_\infty(C^\infty_c(G))$. A classical argument using rotation matrices in $M_2(\mathbb{C})$ then shows that the homomorphisms $\rho^0$ and $\rho^1$ are homotopic after tensoring by $M_2(\mathbb{C})$. Hence the morphisms of cyclic bicomplexes $\text{tr} \rho^0 = \text{Id}$ and $\text{tr} \rho^1 = \text{tr} \rho_s$ induce the same maps in cyclic homology after stabilization by the periodicity operator $S$. Applying this to the subcomplex $\Omega C^\infty_c(G)_0$ shows that $S : HC_n(C^\infty_c(G))_0 \to HC_{n-2}(C^\infty_c(G))_0$ coincides with $S \circ \text{tr} \rho_s$. By virtue of the above observation, for any given n-cycle $\omega$ there is a choice of open covering of $B$ with partition of unity so that $\text{tr} \rho_s(\omega) = 0$. Hence $S = 0$ on $HC_0(C^\infty_c(G))_0$ as claimed. Note that we cannot apply this argument directly to the periodic cyclic homology $HP_\bullet(C^\infty_c(G))_0$, because a periodic cycle is an infinite sequence of n-forms so we may find no suitable covering of $B$. Instead, we use the $\lim^1$ exact sequence

$$0 \to \lim^1_S HC_{-1}(C^\infty_c(G))_0 \to PH_\bullet(C^\infty_c(G))_0 \to \lim_S HC_\bullet(C^\infty_c(G))_0 \to 0.$$

Since $S = 0$, one has $\lim^1 S HC_{-1}(C^\infty_c(G))_0 = 0$ and $\lim S HC_\bullet(C^\infty_c(G))_0 = 0$, hence $PH_\bullet(C^\infty_c(G))_0$ vanishes as required.

Passing to cohomology, we observe that, as a vector space over $\mathbb{C}$, the non-periodic cyclic cohomology $HC^n(C^\infty_c(G))_0$ is the algebraic dual of the space $HC_n(C^\infty_c(G))_0$. Hence, by transposition of the above result, the suspension operator $S : HC^n(C^\infty_c(G))_0 \to HC^{n+2}(C^\infty_c(G))_0$ vanishes as well as the inductive limit $HP^\bullet(C^\infty_c(G))_0 = \lim_{\to S} HC^\bullet(C^\infty_c(G))_0$. The second isomorphism in (31) then follows from the six-term exact sequence relating the periodic cyclic cohomology groups $HP^\bullet(C^\infty_c(G))_0$, $HP^\bullet(C^\infty_c(G))$ and $HP^\bullet(C^\infty_c(G))(I)$.  

When an algebra $\mathcal{A}$ comes equipped with a locally convex topology, the algebraic cyclic (co)homologies $HP_\bullet(\mathcal{A})$ and $HP^\bullet(\mathcal{A})$ described above can be replaced by appropriate topological versions. The topological cyclic homology of such an algebra is defined through a space of noncommutative differential forms as in the algebraic case, the only difference is that one has to replace algebraic tensor products by topological ones. The space of topological $n$-forms is thus $\Omega^*_\text{top} \mathcal{A} = \mathcal{A}^+ \hat{\otimes} \mathcal{A}^{\otimes n}$ where $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ is the algebra obtained by adjoining
a unit, and \( \hat{\otimes} \) is an appropriate completion of the algebraic tensor product. In the case of a Lie groupoid \( G \), its smooth convolution algebra \( \mathcal{A} = C^\infty(G) \) has the topology of an \( LF \)-space, which is the inductive limit topology, over all compact subsets \( K \subset G \), of the Fréchet spaces \( C^\infty_K(G) \) of smooth functions with support contained in \( K \). In this case one may choose Grothendieck’s inductive tensor product. Recall that if \( M \) and \( N \) are two manifolds, the inductive tensor product of \( LF \)-spaces \( C^\infty(M) \hat{\otimes} C^\infty(N) \) is isomorphic to \( C^\infty(M \times N) \). The space of noncommutative \( n \)-forms over the convolution algebra is thus isomorphic to

\[
\Omega^n_{\text{top}} C^\infty_c(G) \cong C^\infty_c(G^{n+1} \cup G^n) .
\] (32)

The operators \((b, B)\) extend by continuity to well-defined boundary operators on the direct sum \( \Omega_{\text{top}} C^\infty_c(G) = \bigoplus_{n \geq 0} \Omega^n_{\text{top}} C^\infty_c(G) \) and also on the direct product \( \hat{\Omega}_{\text{top}} C^\infty_c(G) = \prod_{n \geq 0} \Omega^n_{\text{top}} C^\infty_c(G) \). By definition the topological periodic cyclic homology of the convolution algebra is

\[
HF^\bullet_{\text{top}}(C^\infty_c(G)) = H^\bullet(\hat{\Omega}_{\text{top}} C^\infty_c(G), b + B) .
\] (33)

Passing to cohomology one has to consider an appropriate dual space to noncommutative forms \( \Omega^n_{\text{top}} C^\infty_c(G)^\prime = \text{Hom}(\Omega^n_{\text{top}} C^\infty_c(G), \mathbb{C}) \). We take the space of continuous and linear functionals \( \varphi : \Omega^n_{\text{top}} C^\infty_c(G) \to \mathbb{C} \) with bounded singularity order. Such a functional \( \varphi \) is exactly represented by a distribution on the manifold \( G^{n+1} \cup G^n \) whose singularity order is finite, say \( k \): its evaluation on a smooth function \( \omega \in \Omega^n_{\text{top}} C^\infty_c(G) \) formally reads

\[
\varphi(\omega) = \int_{G^{n+1}} \varphi_{0,n}(g_0, \ldots, g_n) \omega(g_0, \ldots, g_n) + \int_{G^n} \varphi_{1,n}(g_1, \ldots, g_n) \omega(g_1, \ldots, g_n)
\]

and can be extended to a continuous linear functional on functions of class \( C^k \). We endow the space \( \Omega^n_{\text{top}} C^\infty_c(G)^\prime \) with the weak-* topology. The transpose of the total operator \( b + B \) acting on the direct sum \( \Omega_{\text{top}} C^\infty_c(G)^\prime = \bigoplus_{n \geq 0} \Omega^n_{\text{top}} C^\infty_c(G) \) thus yields a \( \mathbb{Z}_2 \)-graded topological complex. The topological periodic cyclic cohomology of the convolution algebra is by definition

\[
HF^\bullet_{\text{top}}(C^\infty_c(G)) = H^\bullet(\Omega_{\text{top}} C^\infty_c(G)^\prime, b + B) .
\] (34)

We want to discuss localization of topological periodic cyclic (co)homology. Thus let \( \Omega^n_{\text{top}} C^\infty_c(G)_0 \subset \Omega^n_{\text{top}} C^\infty_c(G) \) be the subspace of functions vanishing on some neighborhood of the set of loops \( I^{(n+1)} \cup I^{(n)} \). This subspace is not closed in \( \Omega^n_{\text{top}} C^\infty_c(G) \). We define the localized noncommutative \( n \)-forms as the (non-Hausdorff) quotient space

\[
\Omega^n_{\text{top}} C^\infty_c(G)_{(I)} = \Omega^n_{\text{top}} C^\infty_c(G)/\Omega^n_{\text{top}} C^\infty_c(G)_0 .
\] (35)

Since the operators \((b, B)\) descend on localized forms, we define the localized topological periodic cyclic homology of the convolution algebra as the homology of the direct product \( \hat{\Omega}_{\text{top}} C^\infty_c(G)_{(I)} = \prod_{n \geq 0} \Omega^n_{\text{top}} C^\infty_c(G)_{(I)} \)

\[
HF^\bullet_{\text{top}}(C^\infty_c(G))_{(I)} = H^\bullet(\hat{\Omega}_{\text{top}} C^\infty_c(G)_{(I)}, b + B) .
\] (36)

The definition of localized topological cyclic cohomology is analogous to algebraic setting. Let \( \Omega^n_{\text{top}} C^\infty_c(G)_{(I)} \) be the set of continuous bounded linear
functionals $\varphi : \Omega^n_{\text{top}} C_c^\infty(G) \to \mathbb{C}$ vanishing on the subspace $\Omega^n_{\text{top}} C_c^\infty(G)_0$ (hence also on its closure). These functionals are characterized by their distribution kernel whose support is entirely contained in the set of loops $I^{(n+1)} \cup I^{(n)}$. The direct sum $\Omega_{\text{top}} C_c^\infty(G)'_I = \bigoplus_{n \geq 0} \Omega^n_{\text{top}} C_c^\infty(G)'_I$ endowed with the transposed of $b + B$ is therefore a $\mathbb{Z}_2$-graded subcomplex of $\Omega_{\text{top}} C_c^\infty(G)'$ and we set

$$H^\bullet_{\text{top}}(C_c^\infty(G))_I = H^\bullet(\Omega_{\text{top}} C_c^\infty(G)'_I, b + B).$$

**Proposition 3.2 (Localization: topological case)** Let $G$ be a Lie groupoid. The projection of cyclic bicomplexes $\Omega_{\text{top}} C_c^\infty(G) \to \Omega_{\text{top}} C_c^\infty(G)_I$ induces an isomorphism in periodic cyclic homology

$$H^\bullet_{\text{top}}(C_c^\infty(G)) \cong H^\bullet_{\text{top}}(C_c^\infty(G))_I.$$  

Moreover if the closure of $\Omega_{\text{top}} C_c^\infty(G)_0$ is a direct summand in $\Omega_{\text{top}} C_c^\infty(G)$ as a topological vector subspace, then the natural map in periodic cyclic cohomology

$$H^\bullet_{\text{top}}(C_c^\infty(G))_I \to H^\bullet_{\text{top}}(C_c^\infty(G))$$

is surjective (with kernel a topological vector space which does not separate zero from any other vector).

**Proof:** For cyclic homology, the proof of Proposition 3.1 applies verbatim. A partition of unity $(c_i)_{i \in I}$ relative to an open covering of $B = G^{(0)}$ yields a continuous homomorphism $\rho : C_c^\infty(G) \to M_\infty(C_c^\infty(G))$ by setting $\rho(a)_{ij} = c_i a c_j$. The resulting chain map $\text{tr} \rho_\ast : \Omega_{\text{top}} C_c^\infty(G) \to \Omega_{\text{top}} C_c^\infty(G)$ is explicitly described as follows. Any $n$-form $\omega \in \Omega^n_{\text{top}} C_c^\infty(G)$ may be viewed as a smooth function over $G^{n+1} \cup G^n$. One has

$$(\text{tr} \rho_\ast(\omega))(g_0, \ldots, g_n) = \omega(g_0, \ldots, g_n) \times 
\sum_{i_0, \ldots, i_n} c_{i_0}(r(g_0)) c_{i_1}(s(g_0)) c_{i_2}(s(g_1)) \ldots c_{i_n}(r(g_n)) c_{i_n}(s(g_n)),$$

and similarly on $(g_1, \ldots, g_n)$. This expression vanishes if $\omega$ belongs to the subspace $\Omega^n_{\text{top}} C_c^\infty(G)_0$ and the covering of $B$ is fine enough. The isomorphism $H^\bullet_{\text{top}}(C_c^\infty(G)) \cong H^\bullet_{\text{top}}(C_c^\infty(G))_I$ thus follows from homotopy invariance as before.

The case of cohomology requires some care, because we can no longer use a duality argument as in the algebraic setting. If we assume that the closure of $\Omega_{\text{top}} C_c^\infty(G)_0$ is a topological direct summand in $\Omega_{\text{top}} C_c^\infty(G)$, then at the dual level $\Omega_{\text{top}} C_c^\infty(G)'_I$ endowed with the weak-* topology splits as the direct sum of $\Omega_{\text{top}} C_c^\infty(G)'_I$ and a closed supplementary subspace. In particular the short exact sequence of cyclic bicomplexes $0 \to \Omega_{\text{top}} C_c^\infty(G)'_I \to \Omega_{\text{top}} C_c^\infty(G)' \to \Omega_{\text{top}} C_c^\infty(G)_0 \to 0$ has a *continuous* linear section. This implies the existence of a six-term exact sequence with continuous maps in topological periodic cyclic cohomology:

$$
\begin{array}{cccccc}
\downarrow & & & & \uparrow \\
H^0_{\text{top}}(C_c^\infty(G))_0 & \leftarrow & H^0_{\text{top}}(C_c^\infty(G)) & \leftarrow & H^0_{\text{top}}(C_c^\infty(G))_I & \\
& & \downarrow & & \uparrow & \\
H^1_{\text{top}}(C_c^\infty(G))_I & \rightarrow & H^1_{\text{top}}(C_c^\infty(G)) & \rightarrow & H^1_{\text{top}}(C_c^\infty(G))_0
\end{array}$$

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The quotient complex $\Omega_{\top}\subset C_c^\infty(G)_0$ is the continuous dual of the closure of $\Omega_{\top}\subset C_c^\infty(G)_0$ inside $\Omega_{\top}\subset C_c^\infty(G)$. We will show that the periodic cyclic cohomology $HP_{\top}^\bullet(C_c^\infty(G)_0)$ is degenerate in the sense that its topology does not separate zero from any other element. For this we endow the manifold $B$ with a riemannian metric. Consider a sequence of real numbers $\varepsilon > 0$ with limit $\varepsilon \to 0$. For each $\varepsilon$, we can choose an open covering $(U_\varepsilon^i)_{i \in I}$ of $B$ together with a partition of unity $(\varepsilon_i^\varepsilon)_{i \in I}$ with the following properties: over any compact subset $K \subset B$ hold

i) For all $x \in K$, the number of $U_i$’s containing $x$ is bounded uniformly with respect to $x$ and $\varepsilon$;

ii) The radius of $U_\varepsilon^i \cap K$ is $\leq \varepsilon$ for all $i \in I$ and $\varepsilon$;

iii) The partial derivatives $\partial_i^\varepsilon$ are bounded by $C_{K,\varepsilon}^{-|\alpha|}$ for all $i \in I$, $\varepsilon > 0$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. Here $n = \dim B$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $C_{K,\varepsilon}$ is a constant independent of $\varepsilon$.

These data give rise to a homomorphism $\rho^\varepsilon : C_\infty^\infty(G) \to M_\infty(C_\infty^\infty(G))$ for each $\varepsilon$ of the sequence, together with the associated chain map $\text{tr}\rho^\varepsilon : \Omega_{\top}C_\infty^\infty(G) \to \Omega_{\top}C_\infty^\infty(G)$. If a $n$-form $\omega$ is in the closure of $\Omega_{\top}C_\infty^\infty(G)_0$, then as a smooth function on $C^{n+1} \cup C^n$, $\omega$ vanishes as well as all its partial derivatives on the set of loops $I^{(n+1)} \cup I^{(n)}$. This means that at a distance $\varepsilon$ away from the set of loops, the partial derivatives of $\omega$ grow slower than any power of $\varepsilon$. From points i), ii), iii) above it follows that $\lim_{\varepsilon \to 0} \text{tr}\rho^\varepsilon(\omega) = 0$. Passing to the continuous dual, any periodic cyclic cocycle $\varphi \in \Omega_{\top}C_\infty^\infty(G)_0$ is cohomologous to the sequence of cocycles $\varphi \circ \text{tr}\rho^\varepsilon$ which tends to zero in the weak$^*$ topology. Hence $HP_{\top}^\bullet(C_c^\infty(G)_0)$ is a degenerate topological vector space as claimed. Now let $V$ be the range of the map $p : HP_{\top}^\bullet(C_c^\infty(G)) \to HP_{\top}^\bullet(C_c^\infty(G)_0)$. The six-term exact sequence of periodic cyclic cohomology gives rise to a short exact sequence

$$HP_{\top}^\bullet(C_c^\infty(G)_{(I)}) \to HP_{\top}^\bullet(C_c^\infty(G))_0 \to 0$$

The topology of $V$ as a closed vector subspace of $HP_{\top}^\bullet(C_c^\infty(G)_0)$ coincides with the quotient topology of $HP_{\top}^\bullet(C_c^\infty(G))/\text{Ker} p$. We know that it is degenerate, hence $\text{Ker} p$ must be dense in $HP_{\top}^\bullet(C_c^\infty(G))$. Since $p$ is continuous $\text{Ker} p$ is also necessarily closed. Hence one has $\text{Ker} p = HP_{\top}^\bullet(C_c^\infty(G))$, and the six-term exact sequence reduces to $0 \to HP_{\top}^\bullet(C_c^\infty(G)_0) \to HP_{\top}^\bullet(C_c^\infty(G)_{(I)}) \to HP_{\top}^\bullet(C_c^\infty(G)) \to 0$.

**Remark 3.3** The proof of 3.2 shows that any topological cyclic cohomology class $[\varphi] \in HP_{\top}^\bullet(C_c^\infty(G))$ can be represented by a finite collection of distributions $\varphi : \Omega_{\top}C_c^\infty(G) \to C$ whose supports are contained in an arbitrarily small open neighborhood of the localization set $I^{(n+1)} \cup I^{(n)}$. In general we do not know whether $[\varphi]$ can be represented by distributions with support exactly contained in $I^{(n+1)} \cup I^{(n)}$, unless the closure of $\Omega_{\top}C_c^\infty(G)_0$ admits a topological supplementary subspace in $\Omega_{\top}C_c^\infty(G)$. A sufficient condition for this to be true is that the space of loops $I^{(n)}$ is a smooth submanifold of $G^n$ for all $n$. This happens if the foliation $(B, \mathcal{F})$ induced on the unit space of the groupoid $G \rightrightarrows B$ is non-singular, and is always verified, for example, by étale groupoids. This condition can be slightly relaxed by requiring that $I^{(n)}$ is a union of smooth submanifolds, with “sufficiently nice” crossings.

The fact that the quotient $\Omega_{\top}C_c^\infty(G)_{(I)}$ is a non-Hausdorff space is inconvenient. In order to deal with a much nicer space we introduce a strict local-
ization of differential forms, quotienting by the closure of \( \Omega^n_{\text{top}} C^\infty_c(G) \) inside \( \Omega^n_{\text{top}} C^\infty_c(G) \):

\[
\Omega^n_{\text{top}} C^\infty_c(G)[I] = \Omega^n_{\text{top}} C^\infty_c(G) / \Omega_{\text{top}}^n C^\infty_c(G)[0].
\]  

(40)

We define accordingly the localized periodic cyclic cohomology \( HP^n_{\text{top}}(C^\infty_c(G))[I] \).

Since the distributions vanishing on the subspace \( \Omega^n_{\text{top}} C^\infty_c(G)[0] \) and its closure actually coincide, one always has an isomorphism

\[
HP^n_{\text{top}}(C^\infty_c(G))[I] \cong HP^n_{\text{top}}(C^\infty_c(G))(I).
\]  

(41)

More generally if an isotropic subset \( O \subset I \) is invariant under the adjoint action of \( G \), we define \( O^{(n)} \subset G^n \) as the set of composable arrows \( (g_1, \ldots, g_n) \) with product \( g_1 \cdots g_n \in O \), and \( \Omega_{\text{top}}^n C^\infty_c(G)[O] \) as the functions with compact support on \( G^{n+1} \cup G^n \) vanishing in a neighborhood of \( O^{(n+1)} \cup O^{(n)} \). Then the strict localization at \( O \)

\[
\Omega^n_{\text{top}} C^\infty_c(G)[O] = \Omega^n_{\text{top}} C^\infty_c(G) / \Omega_{\text{top}}^n C^\infty_c(G)[O].
\]  

(42)

is a quotient complex of the cyclic bicomplex. The corresponding localized cyclic cohomology \( HP^n_{\text{top}}(C^\infty_c(G))[O] \) is the cohomology of the complex of distributions with bounded singularity order, whose support is contained in \( O^{(n+1)} \cup O^{(n)} \).

Note that if \( O^{(n)} \) is a submanifold of \( G^n \) for all \( n \), the quotient \( \Omega^n_{\text{top}} C^\infty_c(G)[O] \) is the space of jets of functions to any order at the localization manifold, i.e. the space of Taylor expansions of functions in the direction transverse to \( O^{(n+1)} \cup O^{(n)} \).

We now study the invariance of cyclic homology with respect to Morita equivalences. The following lemma is based on an idea of G. Skandalis.

**Lemma 3.4** Let \( G \supseteq B \) be a Lie groupoid. Let \( V \subset B \) be an open subset which intersects each orbit of \( G \) and denote by \( G_V \supseteq V \) the restriction groupoid. Then one has isomorphisms in algebraic and topological periodic cyclic (co)homology

\[
HP^*_c(C^\infty_c(G_V)) \cong HP^*_c(C^\infty_c(G)), \quad HP^*_c(C^\infty_c(G_V)) \cong HP^*_c(C^\infty_c(G))
\]

\[
H^*_c(C^\infty_c(G_V)) \cong H^*_c(C^\infty_c(G)), \quad H^*_c(C^\infty_c(G_V)) \cong H^*_c(C^\infty_c(G))
\]

induced by the inclusion of convolution algebras \( C^\infty_c(G_V) \hookrightarrow C^\infty_c(G) \).

**Proof:** We first observe that the group of bisections of \( G \) acts by multipliers on the convolution algebra \( C^\infty_c(G) \). Indeed if \( \beta : B \to G \) is a bisection, its left and right actions on an element \( a \in C^\infty_c(G) \) are defined by

\[
(\beta \cdot a)(g) = a(\beta^{-1}(r(g)) \cdot g), \quad (a \cdot \beta)(g) = a(g \cdot \beta(s(g))^{-1})
\]

for all \( g \in G \). One checks that the usual relations \( \beta_1 \cdot (\beta_2 \cdot a) = \beta_1 \beta_2 \cdot a, \) \( (\beta \cdot a_1) a_2 = \beta \cdot a_1 a_2 \) etc... are fulfilled, i.e. the group of bisections acts by multipliers on \( C^\infty_c(G) \). If \( \beta : U \to G \) is only a local bisection over an open subset \( U \subset B \), the actions \( \beta \cdot a \) and \( a \cdot \beta \) are still defined provided that the support of \( a \) satisfies appropriate compatibility conditions with respect to the domain and range of the local diffeomorphism \( \phi_\beta \) associated to \( \beta \).

Now let \( (\beta_i, U_i)_{i \in I} \) be a collection of local bisections \( \beta_i : U_i \to G \) indexed by an at most countable set \( I \), such that: i) the collection \( (U_i)_{i \in I} \) is a locally finite
covering of $B$ and ii) the range of the local diffeomorphism $\phi_{\beta_i} : U_i \to V_i$ is contained in the open subset $V \subset B$ for all $i \in I$. Condition ii) can be satisfied because $V$ intersects each orbit of $G$ by hypothesis. Then choose a partition of unity $(c_i)_{i \in I}$, with $\sum_{i} c_i(x)^2 = 1$, relative to the covering $(U_i)$. One builds an algebra homomorphism

$$
\rho : C^\infty_c(G) \to M_\infty(C^\infty_c(G_V))
$$

by setting the $(i,j)$ entry of the matrix $\rho(a)$ equal to $\rho(a)_{ij} = \beta_i c_i a c_j \beta_j^{-1}$ for all $a \in C^\infty_c(G)$. Here the functions $c_i$ are multipliers of the convolution algebra as in the proof of Proposition 3.1. An explicit computation gives, by evaluating on a point $g \in G$,

$$
\rho(a)_{ij}(g) = c_i(\phi_{\beta_i^{-1}}(r(g))) a(\beta_i^{-1}(r(g)) \cdot g \cdot \beta_j^{-1}(s(g))^{-1}) c_j(\phi_{\beta_j^{-1}}(s(g)))
$$

Since $\text{supp}(c_i) \subset U_i$ and $\text{supp}(c_j) \subset U_j$, the latter expression vanishes unless $\phi_{\beta_i^{-1}}(r(g)) \in U_i$ and $\phi_{\beta_j^{-1}}(s(g)) \in U_j$, that is, unless $r(g) \in V_i$ and $s(g) \in V_j$. This shows that $\rho(a)_{ij}$ is indeed an element of the subalgebra $C^\infty_c(G_V)$. The map induced by the homomorphism $\rho$ on differential forms composed with the trace map yields a morphism of cyclic bicomplexes $\text{tr} \rho : \Omega C^\infty_c(G_V) \to \Omega C^\infty_c(G_V)$. We want to show that the latter is an isomorphism in periodic cyclic homology. The obvious candidate for an inverse comes from the morphism of cyclic bicomplexes $\iota : \Omega C^\infty_c(G_V) \to \Omega C^\infty_c(G)$ induced by the inclusion homomorphism $\iota : C^\infty_c(G_V) \to C^\infty_c(G)$. Hence it remains to prove that $\iota \circ \text{tr} \rho$ and $\text{tr} \rho \circ \iota$ are chain homotopic to the identity maps of the $(b+1)$-complexes $\hat{\Omega} C^\infty_c(G)$ and $\hat{\Omega} C^\infty_c(G_V)$ computing the periodic cyclic homologies $HP_\bullet(C^\infty_c(G))$ and $HP_\bullet(C^\infty_c(G_V))$ respectively. We follow the proof of Proposition 3.1 and introduce infinite row and column matrices $u = (u_i)_{i \in I}$ and $v = (v_i)_{i \in I}$ given by $u_i = c_i \beta_i^{-1}$ and $v_i = \beta_i c_i$. Then $uv = 1$, and $vu$ is an idempotent infinite matrix. Moreover $\rho(a) = vau$ for all $a \in C^\infty_c(G)$ by definition. The invertible matrices

$$
W = \begin{pmatrix} 0 & -u \\ v & 1 - vu \end{pmatrix}, \\
W^{-1} = \begin{pmatrix} 0 & u \\ -v & 1 - vu \end{pmatrix}
$$

are multipliers of the algebras $M_\infty(C^\infty_c(G))$ and $M_\infty(C^\infty_c(G_V))$. Furthermore, the identity $\rho^1(a) = W^{-1} \rho^0(a) W$ holds for all $a \in C^\infty_c(G)$, where $\rho^0, \rho^1 : C^\infty_c(G) \to M_\infty(C^\infty_c(G))$ are the homomorphisms

$$
\rho^0(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \\
\rho^1(a) = \begin{pmatrix} 0 & 0 \\ 0 & \rho(a) \end{pmatrix}.
$$

Then $\iota \circ \text{tr} \rho = \text{tr} \rho^1$ is chain homotopic to $\text{tr} \rho^0 = \text{id}$ on $\hat{\Omega} C^\infty_c(G)$. In the same way $\text{tr} \rho^1 \circ \iota = \text{tr} \rho^0$ is chain homotopic to $\text{tr} \rho^0 = \text{id}$ on $\hat{\Omega} C^\infty_c(G_V)$. The isomorphism

$$
HP_\bullet(C^\infty_c(G)) \cong HP_\bullet(C^\infty_c(G_V))
$$

follows, as well as the isomorphism in periodic cyclic (co)homology. The proof is the same for topological periodic cyclic (co)homology.

\[\text{Lemma 3.5} \quad \text{Let } G \text{ be Lie groupoid. Let } G' \text{ be the direct product of the pair groupoid } \mathbb{R} \times \mathbb{R} \text{ with } G. \text{ Then one has isomorphisms}
$$
HP_\bullet^{op}(C^\infty_c(G)) \cong HP_\bullet^{op}(C^\infty_c(G')), \quad HP_\bullet^{top}(C^\infty_c(G)) \cong HP_\bullet^{top}(C^\infty_c(G')).
$$

(43)
Proof: The convolution algebra of the pair groupoid \(\mathbb{R} \times \mathbb{R}\) acts as smoothing operators on the Hilbert space of square-integrable functions on \(\mathbb{R}\) with respect to the Lebesgue measure: the action of \(k \in C_c^\infty(\mathbb{R} \times \mathbb{R})\) on a function \(f \in L^2(\mathbb{R}, dx)\) reads

\[
(k \cdot f)(x) = \int_\mathbb{R} k(x, y) f(y) \, dy .
\]

Now choose a Hilbert basis \((|e_i\rangle)_{i \in \mathbb{N}}\) of \(L^2(\mathbb{R}, dx)\) with the following properties: i) each \(|e_i\rangle\) is a smooth function with compact support on \(\mathbb{R}\) and ii) the infinite \(\mathbb{N} \times \mathbb{N}\) matrix with scalar coefficients \(k_{ij} = \langle e_i|k|e_j\rangle\) is of rapid decay for all \(a \in C_c^\infty(\mathbb{R} \times \mathbb{R})\). Such a basis can be obtained by modification of the orthonormal basis of Hermite polynomials, in such a way that the unitary matrix passing from the Hermite polynomials to \((|e_i\rangle)\) is of the form \(1 + \alpha\) with \(\alpha\) rapid decay. Thus, \(C_c^\infty(\mathbb{R} \times \mathbb{R})\) is represented as a subalgebra of \(\mathcal{H}\), the algebra of \(\mathbb{N} \times \mathbb{N}\) matrices with rapid decay. There is naturally locally convex topology on \(\mathcal{H}\) defined by the family of norms

\[
||k||_n = \sum_{(i,j) \in [\mathbb{R} \times \mathbb{N}]} (1 + i + j)^n |k_{ij}|
\]

for all non-negative integers \(n\), which turns \(\mathcal{H}\) into a Fréchet algebra containing \(C_c^\infty(\mathbb{R} \times \mathbb{R})\) as a dense subalgebra. In fact the inclusion \(C_c^\infty(\mathbb{R} \times \mathbb{R}) \to \mathcal{H}\), sending an operator \(k\) to the matrix with coefficients \(k_{ij} = \langle e_i|k|e_j\rangle\), is continuous with respect to the LF topology on \(C_c^\infty(\mathbb{R} \times \mathbb{R})\). Remark also that the convolution algebra of the product groupoid \(G' = (\mathbb{R} \times \mathbb{R}) \times G\) is isomorphic to \(C_c^\infty(\mathbb{R} \times \mathbb{R}) \otimes C_c^\infty(G)\), where \(\otimes\) is the inductive tensor product of LF spaces. From this observation one has a continuous inclusion \(\iota : C_c^\infty(G) \to C_c^\infty(G')\) defined by \(\iota(a) = |e_0\rangle \langle e_0| \otimes a\) for all \(a \in C_c^\infty(G)\), where \(|e_0\rangle \langle e_0| \in C_c^\infty(\mathbb{R} \times \mathbb{R})\) is the orthogonal projector associated to the vector \(|e_0\rangle \in L^2(\mathbb{R}, dx)\). The extension \(\iota\) to differential forms yields a morphism of (topological) cyclic bicomplexes

\[
\iota_* : \Omega_{\text{top}} C_c^\infty(G) \to \Omega_{\text{top}} C_c^\infty(G') .
\]

Now write \(\mathcal{H}(C_c^\infty(G'))\) for the completed tensor product \(\mathcal{H} \otimes C_c^\infty(G')\). Then \(\mathcal{H}(C_c^\infty(G'))\) is a completion of the algebra of infinite matrices \(M_\infty(C_c^\infty(G'))\). We construct a continuous homomorphism \(\rho : C_c^\infty(G) \to \mathcal{H}(C_c^\infty(G'))\) by setting the \((i,j)\) entry of the matrix \(\rho(b)\) equal to \(\rho(b)_{ij} = |e_0\rangle \langle e_0| \otimes \langle e_i|b|e_j\rangle\) for all \(b \in C_c^\infty(G)\). Remark that \(|e_i|b|e_j\rangle \in C_c^\infty(G)\), so that \(\rho(b)_{ij}\) lies in the image of \(\iota\). Hence the extension of \(\rho\) to differential forms composed with the matrix trace \(\tau : \mathcal{H} \to \mathbb{C}\) and the operator trace \(\tau' : C_c^\infty(\mathbb{R} \times \mathbb{R}) \to \mathbb{C}\) gives rise to a morphism of cyclic bicomplexes

\[
\tau' \rho_{\text{top}} : \Omega_{\text{top}} C_c^\infty(G') \to \Omega_{\text{top}} C_c^\infty(G) .
\]

It remains to show that \(\iota_* \circ \tau' \rho_{\text{top}}\) and \(\tau' \rho_{\text{top}} \circ \iota_*\) are chain-homotopic to the identity maps on the \((b + B)\)-complexes \(\Omega C_c^\infty(G')\) and \(\Omega C_c^\infty(G)\) respectively. For any \(a \in C_c^\infty(G)\) one has \(\rho(\iota(a))_{ij} = 0\) if \((i,j) \neq (0,0)\) and \(\rho(\iota(a))_{00} = \iota(a)\). Hence the identity \(\tau'(|e_0\rangle \langle e_0|) = 1\) gives \(\tau' \rho_{\text{top}} \circ \iota_* = \text{id}\) on \(\Omega C_c^\infty(G)\). In the converse direction, a simple computation yields the equality \(\iota_* \circ \tau' \rho_{\text{top}} = \rho_{\text{top}}\), hence it is sufficient to show that \(\rho_{\text{top}}\) is chain-homotopic to the identity map on \(\Omega C_c^\infty(G')\). Let \(u = (u_i)_{i \in \mathbb{N}}\) be the infinite row matrix with entries \(u_i = |e_i\rangle \langle e_0| \in C_c^\infty(\mathbb{R} \times \mathbb{R})\), and \(v = (v_i)_{i \in \mathbb{N}}\) be the infinite column matrix with entries
isomorphisms in topological periodic cyclic (co)homology by the Cuntz-Quillen formalism [10, 11]. We recall that the cyclic homology of $v$ sufficies to show that, given a groupoid $\pi$ of $\mathcal{C}_c(G')$ and let $\sigma$ be a submersion $\pi \colon U \to B$ thus give a surjective submersion $\pi_0 B \times \mathbb{R}^n$ such that $\pi_0 G \cong G$ is étale, the isomorphisms also hold in algebraic periodic cyclic (co)homology.

**Proposition 3.6 (Morita invariance)** Let $G_1 \cong B_1$ and $G_2 \cong B_2$ be Lie groupoids and let $B_1 \xrightarrow{\varepsilon_1} M \xrightarrow{\varepsilon_2} B_2$ be a Morita equivalence. Then one has isomorphisms in topological periodic cyclic (co)homology

$$HP^{\top}_\bullet(C_c^\infty(G_1)) \cong HP^{\top}_\bullet(C_c^\infty(G_2)) \quad , \quad HP^{\top}_\bullet(C_c^\infty(G_1)) \cong HP^{\top}_\bullet(C_c^\infty(G_2)) \ .$$

(44)

Moreover, if the submersions $\pi_1$ and $\pi_2$ are étale, the isomorphisms also hold in algebraic periodic cyclic (co)homology.

**Proof:** It suffices to show that, given a groupoid $G \cong B$ and a surjective submersion $\pi : M \to B$, the periodic cyclic (co)homologies of $C_c^\infty(G)$ and $C_c^\infty(\pi^*G)$ are isomorphic. Let $n$ be the dimension of the fibers of $\pi$, and denote by $\pi_0 : B \times \mathbb{R}^n \to B$ the projection onto the first factor. The manifolds $B \times \mathbb{R}^n$ and $M$ have the same dimension. Let $U = M \coprod (B \times \mathbb{R}^n)$ be their disjoint union. $\pi$ and $\pi_0$ thus give a surjective submersion $\sigma : U \to B$. Then $M$ and $B \times \mathbb{R}^n$ are open subsets of $U$ intersecting each orbit of the groupoid $\sigma^*G$. Moreover $\pi^*G$ is the restriction groupoid of $\sigma^*G$ to the subset $M$, and $\pi_0^*G$ is the restriction groupoid of $\sigma^*G$ to the subset $B \times \mathbb{R}^n$. By Lemma 3.4, the periodic cyclic (co)homologies of $C_c^\infty(\pi^*G)$, $C_c^\infty(\pi_0^*G)$ and $C_c^\infty(\pi^*G)$ are isomorphic, in the algebraic as well as topological setting. If $n = 0$, that is when $\pi$ is étale, $\pi_0^*G \cong G$ implies that $C_c^\infty(G)$ and $C_c^\infty(\pi^*G)$ have the same algebraic periodic cyclic (co)homology. If $n$ is arbitrary, then $\pi_0^*G$ is the direct product of $G$ with the pair groupoid $\mathbb{R}^n \times \mathbb{R}^n$. By virtue of Lemma 3.5, the topological periodic cyclic (co)homologies of $C_c^\infty(G)$, $C_c^\infty(\pi_0^*G)$ and $C_c^\infty(\pi^*G)$ coincide.

4 Excision

A convenient way to calculate excision in periodic cyclic cohomology is provided by the Cuntz-Quillen formalism [10, 11]. We recall that the cyclic homology of
an associative algebra $\mathcal{A}$ can be entirely recovered from cyclic $0$-cocycles (traces) and cyclic $1$-cocycles over suitable extensions $0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{A} \to 0$ of $\mathcal{A}$.

The basic ingredient is the $X$-complex of $\mathcal{R}$,

$$X(\mathcal{R}) : \mathcal{R} \cong \Omega^1 \mathcal{R}_2,$$

where $\Omega^1 \mathcal{R}_2 = \Omega^1 \mathcal{R} / [\mathcal{R}, \Omega^1 \mathcal{R}]$. For all elements $x, y \in \mathcal{R}$ we write $\xi xdy$ for the class of the one-form $xdy$ mod $[\mathcal{R}, \Omega^1 \mathcal{R}]$. The boundary map $\partial_0 : \mathcal{R} \to \Omega^1 \mathcal{R}_2$ is the non-commutative differential $x \mapsto \xi dx$, while the boundary map $\partial_1 : \Omega^1 \mathcal{R}_2 \to \mathcal{R}$ is the commutator $\xi xdy \mapsto [x, y]$ induced by the Hochschild boundary on $\Omega^1 \mathcal{R}$. One has $\partial_1 \circ \partial_0 = 0 = \partial_0 \circ \partial_1$ hence $X(\mathcal{R})$ is a $\mathbb{Z}_2$-graded complex, with even part $\mathcal{R}$ and odd part $\Omega^1 \mathcal{R}_2$. If $\mathcal{J} \subset \mathcal{R}$ is a two-sided ideal, Cuntz and Quillen define a decreasing filtration of $X(\mathcal{R})$ by the subcomplexes $F^n X(\mathcal{R})$, $n \in \mathbb{Z}$, as follows:

$$F^2\mathcal{J} X(\mathcal{R}) : \mathcal{J}^{n+1} + [\mathcal{J}^n, \mathcal{R}] = \xi \mathcal{J}^1 d\mathcal{R}$$

$$F^{2n+1} X(\mathcal{R}) : \mathcal{J}^{n+1} = \xi (\mathcal{J}^n d\mathcal{R} + \mathcal{J}^n d\mathcal{J})$$

where $\mathcal{J}^n = \mathcal{R}$ and $\mathcal{J}^n \mathcal{J}^+ = \mathcal{J}^+$ for $n \leq 0$. In particular $F^n X(\mathcal{R}) = X(\mathcal{R})$ whenever $n < 0$. The $\mathcal{J}$-adic completions of $\mathcal{R}$ and $X(\mathcal{R})$ are respectively a pro-algebra and a pro-complex given by the projective limits

$$\hat{\mathcal{R}} = \lim_{\leftarrow n} \mathcal{R} / \mathcal{J}^n, \quad X(\hat{\mathcal{R}}) = \lim_{\leftarrow n} X(\mathcal{R}) / F^n X(\mathcal{R}).$$

A cocycle of pro-complex $\tau : X(\hat{\mathcal{R}}) \to \mathbb{C}$ is exactly a cocycle over $X(\mathcal{R})$ vanishing on the sub complex $F^n X(\mathcal{R})$ for some $n$. Thus, a cocycle of even degree is a trace on $\mathcal{R}$ vanishing on the large powers of the ideal $\mathcal{J}$. Similarly, a cocycle of odd degree is a cyclic $1$-cocycle over $\mathcal{R}$ vanishing whenever one of its arguments lies in $\mathcal{J}^n$ for some $n$. The link with the cyclic homology of the quotient algebra $\mathcal{A} = \mathcal{R} / \mathcal{J}$ shows up in the case of the universal free extension $\hat{\mathcal{R}} = T\mathcal{A}$ corresponding to the non-unital tensor algebra of $\mathcal{A}$:

$$T\mathcal{A} = (\mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}) \oplus \ldots$$

The product on $T\mathcal{A}$ is the tensor product, and by definition the two-sided ideal $\mathcal{J} = J\mathcal{A}$ is the kernel of the multiplication homomorphism $m : T\mathcal{A} \to \mathcal{A}$, $a_1 \otimes \ldots \otimes a_n \mapsto a_1 \ldots a_n$. One checks that $J\mathcal{A}$ is generated by the inhomogeneous elements of the form $a_1 a_2 - a_1 \otimes a_2$. As a $\mathbb{Z}_2$-graded vector space, $X(T\mathcal{A})$ is isomorphic to the space of non-commutative differential forms $\Omega^1 \mathcal{A}$. More precisely $T\mathcal{A}$ is isomorphic to the space $\Omega^+ \mathcal{A}$ of differential forms of even degree over $\mathcal{A}$, whereas $\Omega^1 T\mathcal{A} \cong T\mathcal{A}^+ \otimes \mathcal{A}$ is isomorphic to the space $\Omega^- \mathcal{A}$ of differential forms of odd degree. These isomorphisms are given by

$$a_0 da_1 \ldots da_{2n} \in \Omega^{2n} \mathcal{A} \mapsto a_0 \otimes \omega(a_1, a_2) \otimes \ldots \otimes \omega(a_{2n-1}, a_{2n}) \in T\mathcal{A},$$

$$a_0 da_1 \ldots da_{2n+1} \in \Omega^{2n+1} \mathcal{A} \mapsto \xi (a_0 \otimes \omega(a_1, a_2) \otimes \ldots \otimes da_{2n+1}) \in \Omega^1 T\mathcal{A}_2,$$

where $\omega(a_1, a_2) = a_1 a_2 - a_1 \otimes a_2$. The tensor product on $T\mathcal{A}$ corresponds to the Fedosov product on $\Omega^+ \mathcal{A}$, given by $\omega_1 \otimes \omega_2 = \omega_1 \omega_2 - d\omega_1 d\omega_2$ for all differential forms $\omega_1, \omega_2$ of even degree. The ideal $J\mathcal{A}$ corresponds to the space of all forms of even degree $\geq 2$. Cuntz and Quillen show in [10] that the linear isomorphism
\( \Omega \mathcal{A} \cong X(T \mathcal{A}) \) is, up to a rescaling factor, a quasi-isomorphism between the cyclic bicomplex of \( \mathcal{A} \) and the \( X \)-complex of \( T \mathcal{A} \) endowed with the filtration \( F^*_T X(T \mathcal{A}) \). In particular the periodic cyclic homology of \( \mathcal{A} \) is the homology of the pro-complex \( X(\hat{T} \mathcal{A}) \). More generally \( X(\hat{\mathcal{B}}) \) computes the periodic cyclic homology of \( \mathcal{A} \) provided that \( 0 \to \mathcal{J} \to \mathcal{B} \to \mathcal{A} \to 0 \) is a quasi-free extension, see [10].

Let \( (E) : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0 \) be an arbitrary extension. As proved by Cuntz and Quillen in [11], this extension gives rise to an excision map in periodic cyclic cohomology

\[
E^* : HP^*(\mathcal{B}) \to HP^{*+1}(\mathcal{A}) \quad \text{. (50)}
\]

We shall explain how to compute it, following [21]. The universal property of the tensor algebra allows to lift the homomorphism \( E \to A \) to an homomorphism \( T E \to T \mathcal{A} \) sending the ideal \( \mathcal{J} E \) to \( \mathcal{J} A \). We obtain in this way a commutative diagram where all rows and columns are extensions:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \mathcal{J} & J \mathcal{E} \\
0 & \mathcal{B} & T \mathcal{E} \\
0 & \mathcal{E} & \mathcal{A} \\
0 & 0 & 0 \\
\end{array}
\quad \text{. (51)}
\]

By construction \( \mathcal{B} \) is the kernel of the homomorphism \( T \mathcal{E} \to T \mathcal{A} \), and \( \mathcal{J} \) is the kernel of \( J \mathcal{E} \to J \mathcal{A} \). The above diagram allows to calculate the excision map \( HP^*(\mathcal{B}) \to HP^{*+1}(\mathcal{A}) \), once the relevant cyclic cohomology classes of \( \mathcal{B} \) are represented in a suitable form. To this end we note that (51) allows to define two relevant filtrations of the complex \( X(T \mathcal{E}) \). The first filtration is induced by the ideal \( J \mathcal{E} \). We denote the corresponding \( \hat{J} \)-adic completions with a hat:

\[
\hat{T} \mathcal{E} = \lim_{\leftarrow k} T \mathcal{E} / (J \mathcal{E})^k \quad \text{,} \quad X(\hat{T} \mathcal{E}) = \lim_{\leftarrow k} X(T \mathcal{E}) / F^k_{\mathcal{J} \mathcal{E}} X(T \mathcal{E}) \quad \text{. (52)}
\]

The complex \( X(\hat{T} \mathcal{E}) \) computes the periodic cyclic homology of \( \mathcal{E} \). The second filtration is induced by the ideal \( J \mathcal{E} + \mathcal{B} \). We denote the corresponding completions with a tilde. Then we can write

\[
\hat{T} \mathcal{E} = \lim_{\leftarrow n} T \mathcal{E} / (J \mathcal{E} + \mathcal{B})^n \quad \text{,} \quad X(\hat{T} \mathcal{E}) = \lim_{\leftarrow n} X(T \mathcal{E}) / F^n_{\mathcal{J} \mathcal{E}} X(T \mathcal{E}) \quad \text{. (53)}
\]

where \( F^n_{\mathcal{J} \mathcal{E}} X(T \mathcal{E}) = \lim_{\leftarrow k} F^n_{\mathcal{J} \mathcal{E}} X(T \mathcal{E}) / (F^n_{\mathcal{J} \mathcal{E}} X(T \mathcal{E}) \cap F^k_{\mathcal{J} \mathcal{E}} X(T \mathcal{E})) \) is a subcomplex of \( X(\hat{T} \mathcal{E}) \).
Lemma 4.1 ([21]) The \( \mathbb{Z}_2 \)-graded complex of cochains over \( F^n_\mathcal{E} X(\hat{T}\mathcal{E}) \) computes the periodic cyclic cohomology \( HP^*_C(\mathcal{E}) \) for all \( n \geq 1 \).

Hence any periodic cyclic cohomology class over \( \mathcal{E} \) can be represented by a cocycle \( \tau : F^n_\mathcal{E} X(\hat{T}\mathcal{E}) \to \mathbb{C} \) for some \( n \geq 1 \). We will see below interesting examples of such cocycles in the case of Lie groupoids. Let us now describe in full generality the excision map associated to the extension \( 0 \to \mathcal{E} \to \mathcal{A} \to \mathcal{A} \to 0 \), sending a cocycle \( \tau \) over \( F^n_\mathcal{E} X(\hat{T}\mathcal{E}) \) to a cyclic cocycle over \( \mathcal{A} \). This is explained in [21] §2. We first choose any extension of \( \tau \) to a linear map

\[ \tau_R : X(\hat{T}\mathcal{E}) \to \mathbb{C}. \]  

(54)

We call \( \tau_R \) a renormalization of \( \tau \). Of course \( \tau_R \) is generally not a cocycle in \( \text{Hom}(X(\hat{T}\mathcal{E}), \mathbb{C}) \). However its composite \( \tau_R \partial \) with the \( X \)-complex boundary map \( \partial \) is a cocycle vanishing on the subcomplex \( F^n_\mathcal{E} X(\hat{T}\mathcal{E}) \) by construction. Hence \( \tau_R \partial \) defines a cocycle in \( \text{Hom}(X(\hat{T}\mathcal{E}), \mathbb{C}) \). Then observe that the diagonal of (51) leads to an extension of \( \mathcal{A} \) by the algebra \( T\mathcal{E} \), with kernel the ideal \( J\mathcal{E} + \mathcal{R} \). Choose any linear splitting \( \sigma : \mathcal{A} \to T\mathcal{E} \) of this extension. The universal property of the tensor algebra \( T\mathcal{A} \) allows to lift the linear map \( \sigma \) to a homomorphism of algebras \( \sigma_* : T\mathcal{A} \to T\mathcal{E} \) respecting the ideals:

\[ \begin{array}{cccc}
0 & \to & J\mathcal{A} & \to & T\mathcal{A} & \to & \mathcal{A} & \to & 0 \\
\sigma_* & & \sigma_* & & \sigma_* & & \sigma_* & & \\
0 & \to & J\mathcal{E} + \mathcal{R} & \to & T\mathcal{E} & \to & \mathcal{A} & \to & 0
\end{array} \]

(55)

Explicitly \( \sigma_*(a_1 \otimes \ldots \otimes a_n) = \sigma(a_1) \ldots \sigma(a_n) \) in \( T\mathcal{E} \). Since \( \sigma_* \) respects the ideals, it extends to a homomorphism \( \hat{T}\mathcal{A} \to \hat{T}\mathcal{E} \). This in turn induces a chain map still denoted \( \sigma_* : X(\hat{T}\mathcal{A}) \to X(\hat{T}\mathcal{E}) \).

Proposition 4.2 ([21]) The excision map \( HP^*_C(\mathcal{E}) \to HP^{*+1}_C(\mathcal{A}) \) associated to the extension \( 0 \to \mathcal{E} \to \mathcal{A} \to \mathcal{A} \to 0 \) is realized by sending a cocycle \( \tau \in \text{Hom}(F^n_{T(\mathcal{E})}X(\hat{T}\mathcal{E}), \mathbb{C}) \) to the cocycle \( \tau_R \partial \circ \sigma_* \in \text{Hom}(X(\hat{T}\mathcal{A}), \mathbb{C}) \) for any choice of renormalization \( \tau_R \) and linear splitting \( \sigma : \mathcal{A} \to T\mathcal{E} \).

We now apply this formalism to the pseudodifferential extensions obtained for groupoid actions in section 2. Let \( G \rightrightarrows B \) be a Lie groupoid and denote by

\[ \mathcal{E} = C^\infty_c (B) \rtimes G \]

the corresponding convolution algebra. Then any cyclic cohomology class \( [\varphi] \in \text{HP}^*_C(\mathcal{E}) \) is represented by a cocycle \( \varphi \in \text{Hom}(X(\hat{T}\mathcal{E}), \mathbb{C}) \) where \( \hat{T}\mathcal{E} \) is the \( Jm \)-adic completion of the tensor algebra \( T\mathcal{E} \). Taking the locally convex topology of \( \mathcal{E} \) into account, we find that \( \text{HP}^*_\text{top}(\mathcal{E}) \) is the cohomology of the complex \( \text{Hom}(X(\hat{T}\mathcal{E})_{1, \text{top}}, \mathbb{C}) \) of continuous and bounded cochains. A continuous bounded cochain of even degree is a linear map \( \varphi : T\mathcal{E} \to \mathbb{C} \) given by a family of distributions \( \varphi_n^+ \in C^{-\infty} (G^n), n \geq 1 \), with singularity order bounded uniformly in \( n \), such that

\[ \varphi(f_1 \otimes \ldots \otimes f_n) = \int_{G^n} \varphi_n^+(g_1, \ldots, g_n) f_1(g_1) \ldots f_n(g_n) \]

(56)
for all $f_i \in \mathcal{C}$, and $\varphi$ has to vanish on the large powers of $J\mathcal{C}$. In the same way, a continuous cochain of odd degree is a linear map $\varphi : \Omega^1 T\mathcal{C}_n \cong T\mathcal{C}^+ \otimes \mathcal{C} \rightarrow \mathbb{C}$ given by a family of distributions $\varphi^+_n \in C^{-\infty}(G^n)$, $n \geq 1$, with singularity order bounded uniformly in $n$, such that

$$
\varphi(\partial(f_1 \otimes \ldots \otimes f_{n-1} df_n)) = \int_{G^n} \varphi^+_n(g_1, \ldots, g_n) f_1(g_1) \ldots f_n(g_n)
$$

(57)

for all $f_i \in \mathcal{C}$, and $\varphi$ has to vanish whenever its argument lies in a large power of $J\mathcal{C}$. In view of the quasi-isomorphism of complexes $\hat{\Omega}\mathcal{C} \cong X(T\mathcal{C})$, the localization proposition 3.2 shows that any topological cyclic cohomology class can be represented by an $X$-complex cocycle $\varphi = (\varphi^+_n)_{n \geq 1}$ such that each distribution $\varphi^+_n$ has a support contained in an arbitrary small neighborhood of the set of loops $I^{(n)} \subset G^n$.

Now let $\pi : M \rightarrow B$ be a $G$-equivariant submersion. We defined in section 2 the crossed product algebra of $G$ with the sections of the bundle $\mathcal{CL}^0(M) \rightarrow B$ of vertical compactly supported pseudodifferential operators of order $\leq 0$. This leads to an extension $0 \rightarrow \mathcal{B} \rightarrow \mathcal{E} \rightarrow \mathcal{A} \rightarrow 0$ of the convolution algebra of the groupoid $S^* M_B \rtimes G$, where

$$
\mathcal{B} = C_c^\infty(B, \mathcal{CL}_c^{-1}(M)) \rtimes G, \quad \mathcal{E} = C_c^\infty(B, \mathcal{CL}_c^0(M)) \rtimes G, \quad \mathcal{A} = C_c^\infty(S^*_\pi M) \rtimes G.
$$

Recall that the ideal $C_c^\infty(B, \mathcal{L}^\infty_c(M)) \rtimes G \subset \mathcal{B}$ of smoothing operators is canonically isomorphic to the smooth convolution algebra of the pullback groupoid $\pi^* G \rightrightarrows M$. Hence, any topological cyclic cohomology class $[\varphi] \in \mathcal{HP}^\bullet_{\text{top}}(\mathcal{C})$ corresponds to a class in $\mathcal{HP}^\bullet_{\text{top}}(C_c^\infty(B, \mathcal{L}^\infty_c(M)) \rtimes G)$ by virtue of the Morita equivalence of groupoids $G \sim \pi^* G$. Our first goal is to show that $[\varphi]$ can be lifted to a class $[\tau_\varphi] \in \mathcal{HP}^\bullet(\mathcal{B})$. This requires the following notion of connection.

**Definition 4.3** A generalized connection on a submersion $\pi : M \rightarrow B$ is a function $h$ defined on a neighborhood of the diagonal in $B \times B$, sending any pair of neighboring points $(b_1, b_2)$ to a linear operator $h(b_1, b_2) : C_c^\infty(M_{b_2}) \rightarrow C_c^\infty(M_{b_1})$ which may be decomposed as a locally finite sum

$$
h(b_1, b_2) = \sum_i h_i(b_1, b_2) \circ f_i(b_1, b_2)
$$

(58)

where:

- $h_i(b_1, b_2)$ is a local diffeomorphism from an open subset of $M_{b_2}$ to an open subset of $M_{b_1}$, depending smoothly on $(b_1, b_2)$, such that $h_i(b, b) = \text{Id}$ for all $b \in B$;
- $f_i(b_1, b_2)$ is a smooth function on $M_{b_2}$, with support contained in the domain of $h_i(b_1, b_2)$, depending smoothly on $(b_1, b_2)$, such that $\sum_i f_i(b, b) = 1$ for all $b \in B$. The action of $f_i(b_1, b_2)$ on $C_c^\infty(M_{b_2})$ is by pointwise multiplication.

A submersion always has a generalized connection. Indeed let $n = \dim(M/B)$ be the dimension of the fibers of $\pi : M \rightarrow B$ and consider the trivial submersion $\pi_0 : B \times \mathbb{R}^n \rightarrow B$ as in the proof of Proposition 3.6. Then we can find a locally finite open covering $(U_i)_{i \in I}$ of $M$ together with local diffeomorphisms $\beta_i : U_i \rightarrow V_i \subset B \times \mathbb{R}^n$ compatible with the projections, i.e. $\pi_0 \circ \beta_i = \pi|_{U_i}$ for all $i \in I$. Let $(c_i)_{i \in I}$, $\sum_i c_i^2 = 1$, be a smooth partition of unity relative
to the covering \((U_i)\). For all \(b \in B\) let \(\beta_i(b) : U_i \cap M_b \to V_i \cap \{b\} \times \mathbb{R}^n\) and \(c_i(b) \in C^\infty(M_b)\) be respectively the restriction of the diffeomorphism \(\beta_i\) and the function \(c_i\) to the fiber \(M_b\). Since the fibers of \(B \times \mathbb{R}^n\) are canonically diffeomorphic, we can view \(\beta_i(b_1)^{-1} \beta_i(b_2)\) as a local diffeomorphism from \(M_{b_2}\) to \(M_{b_1}\) for all pairs \((b_1, b_2) \in B \times B\). Then

\[
h(b_1, b_2) = \sum_{i \in I} c_i(b_1) \circ \beta_i(b_1)^{-1} \beta_i(b_2) \circ c_i(b_2),
\]

where \(c_i(b_1)\) and \(c_i(b_2)\) act by pointwise multiplication on the vector spaces \(C^\infty_c(M_{b_i})\) and \(C^\infty_c(M_{b_2})\) respectively, defines a generalized connection on \(M\), with local diffeomorphisms \(h_i(b_1, b_2) = \beta_i(b_1)^{-1} \beta_i(b_2)\) and smooth functions \(f_i(b_1, b_2) = (c_i(b_1) \circ h_i(b_1, b_2)) c_i(b_2)\). In this example the function \(h\) is defined on the entire product \(B \times B\) and not only on a neighborhood of its diagonal.

Remark 4.4 In fact one can always find a generalized connection where the local diffeomorphisms \(h_i\) agree pairwise on their common domain in a neighborhood of the diagonal. A possible construction goes as follows. Choose a Riemannian metric on \(B\) and a horizontal distribution on \(M\), that is, a subbundle \(H'\) of the tangent bundle \(TM\) complementary the vertical tangent bundle \(\text{Ker}(\pi_*: TM \to TB)\). That is, one has a decomposition \(TM = H' \oplus \text{Ker}(\pi_*)\). Then on a suitable open subset the diffeomorphism \(h_i(b_1, b_2)\) is obtained by lifting the geodesic between \(b_2\) and \(b_1\) according to the horizontal paths determined by \(H'\).

We now come back to the situation of the groupoid \(G \rightrightarrows B\) acting on the submersion \(\pi : M \to B\), and fix any choice of generalized connection \(h\). Let \(e_1, \ldots, e_n\) be elements of the crossed product \(\mathcal{E} = C^\infty_c\left(B, \text{CL}^{-m}_{1}(M)\right) \rtimes G\), such that each \(e_i\) belongs to the subspace \(C^\infty_c\left(B, \text{CL}^{-m_i}_{1}(M)\right) \rtimes G\) with \(m_1 + \ldots + m_n = \dim(M/B) + k\) for a given integer \(k\). We can map the tensor \(e_1 \otimes \ldots \otimes e_n \in T\mathcal{E}\) to a compactly supported function \(\text{Tr}^h_{e_1, \ldots, e_n}\) of class \(C^k\) on an appropriate neighborhood \(V\) of the set of loops \(I^{(n)} \subset G^n\). This function is defined by evaluation on any point \((g_1, \ldots, g_n) \in V\), by

\[
\text{Tr}^h_{e_1, \ldots, e_n}(g_1, \ldots, g_n) = \text{Tr}(e_1(g_1) U_{g_1} h(s(g_1), r(g_2)) \ldots e_n(g_n) U_{g_n} h(s(g_n), r(g_1))),
\]

where \(\text{Tr}\) is the ordinary trace of operators acting on the space of scalar functions on the manifold \(M_{r(g_1)}\). In order to show that this expression makes sense, recall that \(e_i(g_i) \in \text{CL}^{-m_{i}}_{1}(M_{r(g_i)})\) and \(U_{g_i}\) is the diffeomorphism from \(M_{s(g_i)}\) to \(M_{r(g_i)}\) defined by the action of \(g_i\) on \(M\). Since \(h(s(g_i), r(g_{i+1}))\) is a sum of diffeomorphisms composed with pointwise multiplication by smooth functions, the product \(e_i(g_i) U_{g_i} h(s(g_i), r(g_{i+1}))\) is a compactly supported operator carrying smooth functions on \(M_{r(g_{i+1})}\) to smooth functions on \(M_{r(g_i)}\). Hence the product under the trace in \((60)\) is a sum of pseudodifferential operators in \(\text{CL}^{-m}_{1}(M_{r(g_i)})\) composed with diffeomorphisms of \(M_{r(g_1)}\), with \(m = m_1 + \ldots + m_n\). Its partial derivatives of order \(k\) with respect to the variables \(g_i\) yield a sum of operators in \(\text{CL}^{-m+k}_{1}(M_{r(g_i)})\) composed with diffeomorphisms, which remains in the domain of the trace. Therefore \(\text{Tr}^h_{e_1, \ldots, e_n}\) is a function of class \(C^k\).
Lemma 4.5 As above let \( \mathcal{E} = \mathcal{C}^\infty_\circ(B) \rtimes G, \mathcal{E} = \mathcal{C}^\infty_\circ(B, \text{CL}_c^0(M)) \rtimes G, \mathcal{A} = \mathcal{C}^\infty_\circ(S^*_c M) \rtimes G \) and \( \mathcal{A} = \text{Ker}(T\mathcal{E} \rightarrow T\mathcal{A}) \). Choose any generalized connection \( h ) on the submersion \( \pi : M \rightarrow B \). Then the map sending a bounded continuous cochain \( \varphi \in \text{Hom}(X(\mathcal{T}\mathcal{E})_{\text{top}}, \mathbb{C}) \) to the cochain \( \tau_\varphi \in \text{Hom}(\mathcal{F}_\mathcal{A}^n X(\mathcal{T}\mathcal{E}), \mathbb{C}) \) defined by

\[
\tau_\varphi(e_1 \otimes \ldots \otimes e_n) = \int_{G^n} \varphi^+_\tau(g_1, \ldots, g_n) \text{Tr}_{1,\ldots,e_n}^h (g_1, \ldots, g_n),
\]

\[
\tau_\varphi(\hat{\varphi}(e_1 \otimes \ldots \otimes e_{n-1})(d\epsilon_n)) = \int_{G^n} \varphi^-_\tau(g_1, \ldots, g_n) \text{Tr}_{1,\ldots,e_n}^h (g_1, \ldots, g_n)
\]

for all \( e_i \in \mathcal{E}, n \geq 1 \), is a morphism of complexes provided that \( m \) is sufficiently large. The induced map

\[
\tau_\varphi : HP_{\text{top}}^\bullet(C^\infty_\circ(B) \rtimes G) \rightarrow HP^\bullet(C^\infty_\circ(B, \text{CL}_c^{-1}(M)) \rtimes G)
\]

(61)

does not depend on the choice of connection \( h \).

Proof: The distributions \( \varphi^\pm \) being of finite singularity order, \( \tau_\varphi \) is well-defined provided that the function \( \text{Tr}_{1,\ldots,e_n}^h \) is regular enough, that is, \( m \) sufficiently large. Since \( h = \text{Id} \) on the diagonal of \( B \times B \), one easily checks that the trace map \( e_1 \otimes \ldots \otimes e_n \mapsto \text{Tr}_{1,\ldots,e_n}^h \) commutes with all operators on the cyclic bi-complexes of \( \mathcal{E} \) and \( \mathcal{E} \). Therefore \( \tau_\varphi \) is a cocycle. The fact that two different choices of generalized connections give cohomologous cocycles is a consequence of a classical transgression formula.

Proposition 4.6 For any Lie groupoid \( G \rightrightarrows B \) and any \( G \)-equivariant surjective submersion \( \pi : M \rightarrow B \), one has a commutative diagram

\[
\begin{array}{ccc}
HP_{\text{top}}^\bullet(C^\infty_\circ(M) \rtimes \pi^*G) & \xrightarrow{\tau} & HP^\bullet(C^\infty_\circ(M) \rtimes \pi^*G) \\
\downarrow & & \downarrow \\
HP_{\text{top}}^\bullet(C^\infty_\circ(B) \rtimes G) & \xrightarrow{\tau_\pi} & HP^\bullet(C^\infty_\circ(B, \text{CL}_c^{-1}(M)) \rtimes G)
\end{array}
\]

(62)

where the left equality is the canonical isomorphism accounting for the Morita equivalence of groupoids \( G \sim \pi^*G \), and the right vertical arrow is the restriction morphism induced by the inclusion of the ideal \( C^\infty_\circ(B, \text{CL}_c^{-1}(M)) \rtimes G \) into \( C^\infty_\circ(B, \text{CL}_c^{-1}(M)) \rtimes G \).

Proof: If \( e_1, \ldots, e_n \) are smoothing operators, the trace map \( e_1 \otimes \ldots \otimes e_n \mapsto \text{Tr}_{1,\ldots,e_n}^h \) is precisely the realization of the Morita equivalence \( G \sim \pi^*G \) at the level of topological cyclic cohomology.

As remarked in section 2, there is a canonical morphism between the two extensions \( (E_0) \) and \( (E) \),

\[
\begin{array}{cccccccc}
0 & \rightarrow & C^\infty_\circ(M) \rtimes \pi^*G & \rightarrow & C^\infty_\circ(B, \text{CL}_c^0(M)) \rtimes G & \rightarrow & C^\infty_\circ(B, \text{CL}_c^0(M)) \rtimes G & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C^\infty_\circ(B, \text{CL}_c^{-1}(M)) \rtimes G & \rightarrow & C^\infty_\circ(B, \text{CL}_c^{-1}(M)) \rtimes G & \rightarrow & C^\infty_\circ(S^*_c M) \rtimes G & \rightarrow & 0
\end{array}
\]

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where the left vertical arrow is the inclusion homomorphism and the right vertical arrow is the leading symbol homomorphism. By naturality, the respective excision maps $E_0^*$ and $E^*$ are compatible through the induced morphisms in periodic cyclic cohomology, and this combined with Proposition 4.6 leads to a commutative diagram

\[
\begin{array}{ccc}
HP^*(C_\infty^c(M) \times \pi^* G) & \xrightarrow{E_0^*} & HP^{*+1}(C_\infty^c(B, CS_0^0(M)) \times G)
\\
HP^*(C_\infty^c(B, CL_e^{-1}(M)) \times G) & \xrightarrow{E^*} & HP^{*+1}(C_\infty^c(S_\pi^* M) \times G)
\\
\tau_* & & \\
HP^*_{top}(C_\infty^c(B) \times G)
\end{array}
\]

As a consequence of Morita invariance, the excision map $E_0^*$ restricted to the image of topological cyclic cohomology of $C_\infty^c(M) \times \pi^* G$ thus factors through the cyclic cohomology of the leading symbol algebra $C_\infty^c(S_\pi^* M \times G)$. Hence all the relevant information is carried by the excision map $E^*$.

**Theorem 4.7** Let $G \equiv B$ be a Lie groupoid and $\pi : M \to B$ a $G$-equivariant surjective submersion. Then one has a commutative diagram

\[
\begin{array}{ccc}
HP^*(C_\infty^c(B, CL_e^{-1}(M)) \times G) & \xrightarrow{E^*} & HP^{*+1}(C_\infty^c(S_\pi^* M) \times G)
\\
\tau_* & & \\
HP^*_{top}(C_\infty^c(B) \times G) & \xrightarrow{\pi_{12}} & HP^{*+1}_{top}(C_\infty^c(S_\pi^* M) \times G)
\end{array}
\]  

(63)

**Proof:** It suffices to remark that for any cocycle $\varphi \in \text{Hom}(X(\hat{T}\mathcal{E})_{\text{top}}, \mathbb{C})$, the image $\tau_\varphi \in \text{Hom}(F^m_0 X(\hat{T}\mathcal{E}), \mathbb{C})$ can be renormalized in a continuous and bounded way. For example, by inserting in the trace map (60) a projection operator onto pseudodifferential operators of sufficiently low order.  

For completeness let us recall the link between the excision map $E^*$ in periodic cyclic cohomology and the $K$-theoretic index map [17, 21]. The Chern-Connes pairing between an even cyclic cohomology class $[\tau] \in HP^d(\mathcal{R})$, represented by a trace $\tau : \mathcal{R}^m \to \mathbb{C}$, and a $K$-theory class $[e] \in K_0(\mathcal{R})$ is described as follows (see [21] §4). Suppose for simplicity that $[e]$ is represented by a $2 \times 2$ matrix idempotent $e \in M_2(\mathbb{R}^+)$ satisfying the property $e - e_0 \in M_2(\mathbb{R})$, where $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Looking at (51) we see that the intersection $\mathcal{R} \cap (J\mathcal{E})_l^{\text{f}}$ is a two-sided ideal in $\mathcal{R}$ for any $l \geq 1$, and the quotient algebra $\mathcal{R}/(\mathcal{R} \cap (J\mathcal{E})_l^{\text{f}})$ is a nilpotent extension of $\mathcal{R}$. Define the projective limit

\[
\mathcal{R} = \lim_{\rightarrow \ell} \mathcal{R}/(\mathcal{R} \cap (J\mathcal{E})_l^{\text{f}}).
\]  

(64)

It is a classical result ([10]) that $e$ can be lifted to an idempotent $\hat{e} \in M_2(\hat{\mathcal{R}}^+)$, such that $\hat{e} - e_0 \in M_2(\hat{\mathcal{R}})$. Two different liftings give rise to the same $K$-theory class of $\hat{\mathcal{R}}$. Since the trace $\tau$ vanishes on $\mathcal{R}_m \cap (J\mathcal{E})_l^{\text{f}}$ for large $l$, it extends to a
Corollary 4.8 ([21]) Let $G \rightrightarrows B$ be a Lie groupoid with convolution algebra $\mathcal{C} = C^\infty_c(B) \rtimes G$, and let $\pi : M \to B$ be a $G$-equivariant surjective submersion. Let

$$(E) : 0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$$

be the associated extension of the algebra $\mathcal{A} = C^\infty_c(S^*_x M) \rtimes G$ of non-commutative symbols by $\mathcal{B} = C^\infty_c(B, CL^{-1}_c(M)) \rtimes G$. Then for any $[u] \in K_1(\mathcal{A})$ and any $[\varphi] \in HP^0_{top}(\mathcal{C})$, the pairing of the index $\text{Ind}_E([u]) \in K_0(\mathcal{B})$ with the cyclic cohomology class $[\tau_\varphi] \in HP^0_{top}(\mathcal{B})$ is given by the formula

$$\langle [\tau_\varphi], \text{Ind}_E([u]) \rangle = \langle E^*(\tau_\varphi), [u] \rangle$$

where $E^*(\tau_\varphi) \in HP^1(\mathcal{A})$ is represented by the above cocycle $(\tau_\varphi)_R \bar{\partial} \circ \sigma_*$, for any choice of renormalization $(\tau_\varphi)_R$.

---

1In [21] we used different normalization conventions for the Chern-Connes pairing, involving the numerical factor $\sqrt{2\pi}$ in the odd case, which was dictated by the need of compatibility with the bivariant Chern-Connes character. Since we don’t want to discuss these matters here we use simpler conventions.
The residue formula

As before consider a Lie groupoid $G \rightrightarrows B$ and a $G$-equivariant submersion $\pi : M \rightarrow B$. Let $O \subset G$ be an isotropic submanifold invariant under the adjoint action of $G$. We will compute the excision map on the localized cyclic cohomology classes $[\varphi] \in H^{*}_{\text{top}}(C^{\infty}(B) \rtimes G)_{[O]}$ by means of a residue formula. This closely follows (and actually generalizes) the construction of [21]. In order to make everything work, we need to impose some constraints on the structure of the fixed points for $O$. Remark that any $h \in O$ verifies $r(h) = s(h)$ by definition, hence the fiber $M_{r(h)}$ carries an action of $h$ by diffeomorphisms.

**Definition 5.1** Let $\pi : M \rightarrow B$ be a $G$-equivariant submersion and $O \subset G$ an isotropic submanifold. We say that the action of $O$ on $M$ is non-degenerate if the following holds:

i) The set of loops $O^{(n)}$ are submanifolds in $G^{n}$ for all $n$;

ii) For any $h \in O$, the set of fixed points $M^{h}_{r(h)}$ is a union of isolated submanifolds in $M_{r(h)}$ varying smoothly with $h$;

iii) At any point $x \in M^{h}_{r(h)}$, the tangent space $T_{x}M_{r(h)}$ in the ambient manifold $M_{r(h)}$ splits as a direct sum

$$T_{x}M_{r(h)} = T_{x}M^{h}_{r(h)} \oplus N^{h}_{x}$$

of two subspaces globally invariant by the action of the tangent map $h_{*}$ associated to the diffeomorphism. We denote $h'$ the restriction of $h_{*}$ to the normal subspace $N^{h}_{x}$;

iv) The endomorphism $1 - h'$ of $N^{h}_{x}$ is non-singular, that is $\det(1 - h') \neq 0$ at any point $x \in M^{h}_{r(h)}$.

The non-degeneracy condition is automatically satisfied, for example, when $h$ acts isometrically with respect to a Riemannian metric on $M_{r(h)}$. In the latter case the subspace $N^{h}_{x}$ is the fiber of the normal bundle, in the Riemannian sense, of the fixed submanifold $M^{h}_{r(h)}$ at $x$. Note that when $h$ is not an isometry, condition iv) may definitely fail; this happens for example in the situation of conformal mappings considered in [20]. We will not cover such situations in this article. Let us now focus on a connected component of the submanifold $M^{h}_{r(h)}$, say of dimension $r$ and codimension $s$. We fix a local coordinate system $x = (x_{1}, \ldots, x_{r})$ of $M^{h}_{r(h)}$, and complete it with a normal coordinate system $y = (y_{1}, \ldots, y_{s})$ in a neighborhood of the fixed submanifold with the following properties:

- $y = (0, \ldots , 0)$ on the fixed submanifold $M^{h}_{r(h)}$;

- The tangent vectors $\partial / \partial y_{i}$, $i = 1, \ldots, s$, belong to the normal subspace $N^{h}_{x}$ at any point $x \in M^{h}_{r(h)}$.

Such a local coordinate system $(x; y)$ on $M_{r(h)}$ will be called *adapted to the fixed submanifold*. The stability of the subspaces $N^{h}_{x}$ implies the following important fact: if $h^{\ast}x$ denotes the pullback of the coordinate functions $x$ by the diffeomorphism $h$, then the difference $x - h^{\ast}x$ is of order 2 with respect to the variable $y$ near the fixed submanifold $y = 0$, while $y - h^{\ast}y$ is only of order 1. This will be
used for establishing the properties of zeta-functions.

Let $h \in B$ be a point and $Q \in \text{CL}^1(M_h)$ be an elliptic, positive and invertible pseudodifferential operator of order one. We assume that the spectrum of $Q$ is contained in an interval $(\varepsilon, +\infty)$ with $\varepsilon > 0$. For example, we can take $Q = \sqrt{\Delta + 1}$ where $\Delta$ is a laplacian associated to a smooth Riemannian complete metric on $M_h$. Following Seeley [27], the complex powers $Q^{-z}$ are defined for any number $z \in \mathbb{C}$ with $\text{Re}(z) \gg 0$ via an appropriate contour integral

$$Q^{-z} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-z}(\lambda - Q)^{-1} d\lambda,$$

around the positive real axis. Taking subsequent products with $Q$ yields the complex powers $Q^{-z}$ for any $z \in \mathbb{C}$. If $P$ is a compactly supported pseudodifferential operator of order $m$, the product $PQ^{-z}$ is a trace-class operator provided that $\text{Re}(z) > m + \dim M_h$. The same is true for a product $PQ^{-z}$, where $h$ is a diffeomorphism of $M_h$ and $U_h$ the corresponding linear operator on scalar functions.

**Lemma 5.2** Let $O \subset G$ be an isotropic submanifold whose action on $M$ is non-degenerate. For any $h \in O$ let $P_h$ and $Q_h$ be respectively a pseudodifferential operator and an elliptic positive invertible operator of order one, acting along the fiber $M_{r(h)}$ and depending smoothly on the parameter $h$. Then the zeta-function

$$z \mapsto \text{Tr}(P_h U_h Q_h^{-z})$$

defined for $\text{Re}(z) \gg 0$ extends to a meromorphic function with simple poles on the complex plane, whose coefficients depend smoothly on $h$.

**Proof:** The pullback of the submersion $M \to B$ with respect to the map $O \to B$, $h \mapsto r(h)$ is a submersion with base $O$ and fiber $M_{r(h)}$ over any point $h$. Since by hypothesis the submanifold $M_{r(h)}^0$ of fixed points for $h$ varies smoothly with $h$, one can locally choose a system of vertical coordinates $(x;y)$ on this submersion, with the following property: over any point $h$ in a small open set $V \in O$, $x = (x_1, \ldots, x_r)$ provides a coordinate system on the fixed submanifold $M_{r(h)}^0$, and $y = (y_1, \ldots, y_s)$ provides a normal coordinate system compatible with the diffeomorphy $h$. Then we complete $(x;y)$ into the canonical coordinates $(x;p;y,q)$ on the cotangent bundle $TM_{r(h)}$, such that $(x,p)$ are the canonical coordinates on $TM_{r(h)}^0$ for each $h \in V$. The trace $\text{Tr}(P_h U_h Q_h^{-z})$ is then obtained as the integral, over the manifold $M_{r(h)}$, of a density expressed in the local coordinate system by

$$\rho^h_\tau(x;y) = \left( \int \int \sigma^h_\tau(x;p,y,q) e^{i(p,x-h^*x)+i(q,y-h^*y)} \frac{d^p d^q}{(2\pi)^{r+s}} \right) d^p y d^q x,$$

where $\sigma^h_\tau(x;p,y,q)$ denotes the complete symbol of the pseudodifferential operator $Q_h^{-z} P_h$, of order $|P| - z$ if $|P|$ is the order of $P$. For notational simplicity we shall drop the subscript $h$ and keep in mind that all symbols depend smoothly on $h$. Note that the symbol $\sigma^\tau(x;p,y,q)$ is a holomorphic function of $z$. The above integral converges for $\text{Re}(z) \gg 0$ and we want to show that $\rho^\tau(x;y)$ can be extended to a distribution in the variables $(x,y)$ with values in meromorphic functions of $z \in \mathbb{C}$. Hence, using local coordinate charts and a partition.
of unity we get the desired meromorphic extension of the trace. First we perform a change of variables $(x; y) \mapsto (x; u)$ near the submanifold of fixed points, with $u = y - h^* y$. This is allowed because the matrix of partial derivatives $\partial u / \partial y = 1 - \partial (h^* y) / \partial y$ is non-singular by hypothesis. Hence the density becomes

$$\rho^z(x; u) = \left( \int \left( \frac{\sigma^z(x, p; u, q)}{|\det(1 - h^' )|} e^{i(p, x - h^* x)} \frac{d^p p}{(2\pi)^r} e^{i(q, u)} \frac{d^q q}{(2\pi)^s} \right) d^u u d^x x \right)$$

where the matrix $h' = \partial (h^* y) / \partial y$ is a function of $(x, u)$. The next step is to Taylor expand the symbol $\sigma^z(x, p; u, q)$ with respect to $q$, up to a certain order $n$. This involves the sequence of holomorphic symbols $\partial^k \sigma^z(x, p; u, q) / \partial q^k$ of order $|P| - k - z$:

$$\sigma^z(x, p; u, q) = \sum_{k=0}^n \frac{q^k}{k!} \partial^k \sigma^z(x, p; u; 0) + \frac{q^{n+1}}{n!} \int_0^1 (1 - t)^n \partial^{n+1} \sigma^z(x, p; u, t q) dt .$$

Note that the remainder $R^n_q(x, p; u, q) = \int_0^1 (1 - t)^n \partial^{n+1} \sigma^z(x, p; u, t q) dt$ is not a symbol of order $|P| - n - 1 - z$ because of integration near $t = 0$. Plugging the Taylor expansion in the above expression for $\rho^z(x; u)$, one is left with the terms

$$\frac{1}{k!} \left( \int \left( \frac{\partial^k \sigma^z(x, p; u; 0)}{|\det(1 - h^' )|} e^{i(p, x - h^* x)} \frac{d^p p}{(2\pi)^r} q^k e^{i(q, u)} \frac{d^q q}{(2\pi)^s} \right) d^u u d^x x \right)$$

where we performed the integral over $q$, and $\delta^*(u)$ is the Dirac mass localized at $(u_1, \ldots, u_s) = (0, \ldots, 0)$ which corresponds to the submanifold of fixed points. Hence the $k$-th derivative $\partial^k \delta^*(u) / \partial u^k$ is a distribution of order $k$ supported by this submanifold. It follows that we only need to know the Taylor expansion around $u = 0$ of the integral

$$I_k = \frac{1}{k!} \int \frac{\partial^k \sigma^z(x, p; u; 0)}{|\det(1 - h^' )|} e^{i(p, x - h^* x)} \frac{d^p p}{(2\pi)^r}$$

up to order $k$ in the variable $u$, because the higher orders are killed in the product with the $\delta$-distribution. Then the crucial fact is that by the non-degeneracy hypothesis, $x - h^* x$ is of order $u^2$, hence the Taylor expansion of the oscillatory term $\exp(i(p, x - h^* x))$ yields a polynomial in $p$. Therefore $I_k$ reduces to the integral of a classical symbol in the variables $(x, p)$, holomorphic in $z$. By a well-known result it extends to a meromorphic function of $z$ with only simple poles [28]. Finally we still have to look at the remainder term

$$\int \frac{R^n_q(x, p; u, q)}{|\det(1 - h^' )|} e^{i(p, x - h^* x)} q^{n+1} e^{i(q, u)} \frac{d^p p}{(2\pi)^r} \frac{d^q q}{(2\pi)^s}$$

Here we cannot simply perform the integral over $q$ because $R^n_q$ depends on $q$. By the way, this integral will not yield a distribution localized at $u = 0$. Instead
we shall move the derivatives $\partial/\partial u$ and rewrite the integral as a sum of terms like
\[
\left( \frac{\partial}{\partial u} \right)^j \left( \int \left( \frac{\partial}{\partial u} \right)^m \left( \frac{R^h_n(x, p; u, q)}{\det(1 - h')} \right) \frac{\partial^k e^{i(p.x - h^* x)}}{(-i\partial u)^k} e^{i(q.u)} \right) \frac{dt}{(2\pi)^r} \frac{d^q}{(2\pi)^s}
\]
with $j + m + k = n + 1$. We have $\partial^k e^{i(p.x - h^* x)}/\partial u^k = f(x, p; u)e^{i(p.x - h^* x)}$, where the function $f(x, p; u)$ is a polynomial of degree at most $k$ in $p$. Moreover $x - h^*x$ is of order $u^2$ near $u = 0$, hence the derivative $\partial(x - h^*x)/\partial u$ is of order $u$, and the coefficient of $p^j$ in $f(x, p; u)$ is of order $u^{2l - k}$, with $2l - k$ non-negative. But a power of $u$ amounts to a derivative $\partial/\partial q$ against $e^{i(q.u)}$. Thus we may replace $f$ by a sum of differential operators $p^l(\frac{\partial}{\partial q})^{2l-k}$ with coefficients smooth functions of $(x; u)$. Each operator amounts to raise the order of $R^h_n(x, p; u, q)$ by $l -(2l - k) = k - l$, which is $\leq k/2$ and hence $\leq (n + 1)/2$.

Explicitly
\[
p^l \left( \frac{\partial}{\partial q} \right)^{2l-k} R^z_n(x, p; u, q) = \int_0^1 (1 - t)^{n+1} t^{2l-k} p^l \frac{\partial^{2l-k+n+1} e^{i z^2}}{\partial q^{2l-k+n+1}} (x, p; u, tq) dt
\]
where $p^l \frac{\partial^{2l-k+n+1} e^{i z^2}}{\partial q}$ is a symbol of order $|P| + k - n - 1 - z \leq |P| - \frac{n+1}{2} - z$. Finally, one is left with integrals of the form
\[
J = \left( \frac{\partial}{\partial u} \right)^j \left( \int \int \int \int \frac{S^z(x, p; u, tq)e^{i(p.x - h^* x) + i(q.u)}}{L(t) dt} \frac{dt}{(2\pi)^r} \frac{d^q}{(2\pi)^s} \right)
\]
where $L(t)$ is a polynomial in $t$, and $S^z(x, p; u, q)$ is a holomorphic symbol of order at most $|P| - \frac{n+1}{2} - z$. Thus $S^z$ is dominated by the symbol $(p^2 + q^2 + 1)^{-Z/2}$ for $Z = |P| + \frac{n+1}{2} + z$. Moreover the integral
\[
\int \int \int \int \frac{dt}{(2\pi)^r} \frac{d^q}{(2\pi)^s}
\]
converges to a holomorphic function of $Z$ provided $\text{Re}(Z)$ is large enough. Hence the integral $J$ converges is a holomorphic function of $z$, provided $n$ is chosen sufficiently large. We conclude that $\text{Tr}(PU_h Q^{-z})$ extends to a meromorphic function with simple poles.

It remains to show the smoothness with respect to the parameter $h$. In fact it is clear from the integral expression of the density $\rho_h$ and in all the subsequent calculations, that derivating with respect to $h$ simply amounts to replace the holomorphic symbol $\sigma_h^*$ of order $|P| - z$ by a new holomorphic symbol of order $|P| - z + 1$. One concludes that the meromorphic function $\text{Tr}(PU_h Q^{-z})$ is infinitely differentiable with respect to the parameter $h$.

We now compute the residue at $z = 0$ of a zeta-function of type $\text{Tr}(PU_h Q^{-z})$ with $P$ a compactly supported pseudodifferential operator on the manifold $M_h$, $h$ a diffeomorphism of $M_h$ and $Q$ an elliptic positive operator of order one. Not surprisingly, the residue is given by an explicit local formula involving the complete symbol $\sigma_P$ of $P$. In particular when $U_h$ is the identity, one recovers the well-known Wodzicki residue [28], which can be written as an integral, over the cosphere bundle, of a certain homogeneous component of $\sigma_P$. In the general situation the residue is localized at the set of fixed points for $U_h$.  

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Hence for high values of \( k \) function of \( z \), a symbol with respect to the variables \((x, p)\), we have the expansion of \( z^\sigma \) in the canonical variables. The remainder of the expansion can be represented as

\[
\text{Res}_{z=0} \text{Tr}(PU_h Q^{-z}) = \int S^*M^h_y \left[ e^{i(p, x - h^* y)} \frac{\partial^k \sigma^z}{\partial q^k} \right] \frac{\det(1 - h')}{(2\pi)^r} \eta(d\eta)^{-1}
\]

where \( S^*M^h_y \) is the cosphef bundle of the fixed submanifold, \( \eta = \langle p, dx \rangle \) is the canonical one-form on the cosphef bundle \( S^*M^h_y \), \( \sigma^z = \sigma^z(x, p, y, q) \) is the complete symbol of \( P \), \( [ \cdot ]_r \) is the order \(-r\) component of a symbol in the variables \((x, p, 0, 0)\), and \( h' \) is the matrix of partial derivatives \( \partial(h^* y)/\partial y \).

**Proof:** Let \( h \) be a diffeomorphism of the manifold \( M_y \). Assume that the set of fixed points of \( h \) is a non-degenerate smooth submanifold \( M^h_y \subset M_y \) of dimension \( r \). Choose a local coordinate system \((x, y)\) adapted to \( M^h_y \), and complete it into a canonical coordinate system \((x, p, y, q)\) on the cotangent bundle \( T^*M_y \), such that \((x, p)\) is a canonical coordinate system of \( T^*M^h_y \). Then for any pseudodifferential operator \( P \in \mathcal{CL}_c(M_y) \) and any elliptic strictly positive invertible operator \( Q \in \mathcal{CL}_c(M_y) \), one has the localization formula

\[
\text{Res}_{z=0} \text{Tr}(PU_h Q^{-z}) = \int S^*M^h_y \left[ e^{i(p, x - h^* y)} \frac{\partial^k \sigma^z}{\partial q^k} \right] \frac{\det(1 - h')}{(2\pi)^r} \eta(d\eta)^{-1}
\]

where \( S^*M^h_y \) is the cosphef bundle of the fixed submanifold, \( \eta = \langle p, dx \rangle \) is the canonical one-form on the cosphef bundle \( S^*M^h_y \), \( \sigma^z = \sigma^z(x, p, y, q) \) is the complete symbol of \( P \), \( [ \cdot ]_r \) is the order \(-r\) component of a symbol in the variables \((x, p, 0, 0)\), and \( h' \) is the matrix of partial derivatives \( \partial(h^* y)/\partial y \).

**Proof:** Set \( u = y - h^* y \). In the proof of Lemma 5.2 we established that \( \text{Tr}(PU_h Q^{-z}) \) is the integral of a density on \( M_y \) given in the local coordinate chart \((x; u)\) by an expansion

\[
\rho^z(x; u) = \sum_{k=0}^n \frac{1}{k!} \left( \int \frac{\partial^k \sigma^z}{\partial q^k}(x, p, u, 0) \frac{e^{i(p, x - h^* y)}}{\det(1 - h')} \frac{d^r p}{(2\pi)^r} \right) \frac{\partial^k \delta^z(u)}{(i\partial u)^k} d^r u d^r x + H_n(z)
\]

where the remainder \( H_n(z) \) is holomorphic in \( z \) provided \( n \) is sufficiently large, and \( \sigma^z \) is the symbol of \( Q^{-z} P \) of order \( |P| - z \). Hence integrating over \((x; u)\) and taking the residue at \( z = 0 \) will only retain the finite sum over \( k \):

\[
\text{Res}_{z=0} \sum_{k=0}^n \frac{1}{k!} \left( \int \frac{\partial^k \sigma^z}{\partial q^k}(x, p, u, 0) \frac{e^{i(p, x - h^* y)}}{\det(1 - h')} \frac{d^r p}{(2\pi)^r} \right) \frac{\partial^k \delta^z(u)}{(i\partial u)^k} d^r u d^r x
\]

Integrating by parts with respect to the variable \( u \) one sees that the Dirac measure \( \delta \) localizes the residue at the submanifold of fixed points \( u = 0 \):

\[
\text{Res}_{z=0} \left( \sum_{k=0}^n \frac{1}{k!} \left( \frac{\partial^k}{\partial q^k} - \frac{\partial^k}{\partial q^k} \right) \sigma^z e^{i(p, x - h^* y)} \right) \frac{d^r p}{\det(1 - h')} \frac{d^r x}{(2\pi)^r}
\]

Moreover we know that \( x - h^* y \) is of order \( u^2 \) near \( u = 0 \), and the Taylor expansion of \( \partial^k \sigma^z / \partial q^k(x, p, u, 0) e^{i(p, x - h^* y)} \) up to order \( k \) in the variable \( u \) is a symbol with respect to the variables \((x, p)\), of order at most \(|P| - k/2 - z\). Hence for high values of \( k \) the integral over \((x, p)\) converges to a holomorphic function of \( z \) near \( z = 0 \) and the residues vanish. Consequently we may replace the finite sum \( \sum_{k=0}^n \frac{1}{k!} \left( \frac{\partial^k}{\partial q^k} - \frac{\partial^k}{\partial q^k} \right) \) by the exponential \( \exp(i(\partial / \partial q^1 \partial / \partial q^2)) \) viewed as a formal power series. Moreover, the symbol \( \sigma^z \) of the operator \( Q^{-z} P \) has an asymptotic expansion of the form

\[
\sigma^z \sim (\sigma_Q)^{-z} \sigma_P + z \sigma'
\]

where \( \sigma_Q \) is the symbol of \( Q \) and the product is simply the product of functions in the canonical variables. The remainder \( z \sigma' \) will disappear under the residue.
because the integral of a symbol with respect to \((x,p)\) is meromorphic with only simple poles. Hence we can replace \(\sigma^z\) by the product \((\sigma_Q)^{-z}\sigma_P\). In the same way we can move the function \((\sigma_Q)^{-z}\) behind the differential operators \(\left(\partial^k_{\alpha}, \partial^k_{\beta}\right)\) since the latter can only extract symbols proportional to \(z\) when applied to \((\sigma_Q)^{-z}\). Thus the above integral may be rewritten

\[
\text{Res}_{z=0} \int (\sigma_Q)^{-z} e^{i\left(\frac{\sigma_P}{\sigma_Q}, \frac{\sigma_P}{\sigma_Q}\right) + \sigma_P e^{i(p.x - h_\lambda z)}} \left| \frac{\sigma_P e^{i(p.x - h_\lambda z)}}{\det(1 - h_\lambda)} \right|_{q=0} \frac{d^p p}{q^r q^r\pi x}
\]

One recognizes the Wodzicki residue applied to pseudodifferential operator with symbol \(e^{i\left(\frac{\sigma_P}{\sigma_Q}, \frac{\sigma_P}{\sigma_Q}\right) + \sigma_P e^{i(p.x - h_\lambda z)}} / |\det(1 - h_\lambda)|_{q=0, q=0}\) on the cotangent bundle of the fixed submanifold \(M_b^h\). It is expressed as the integral of the order \(-r\) component of the symbol over the cosphere bundle, whence formula (72).

Following [21], the zeta-renormalized trace of an operator of the form \(PU_h\) can be defined as the finite part at \(z = 0\) of the zeta-function \(\text{Tr}(PU_h^Q)^{-z}\), that is, the term of degree zero in the Laurent expansion of the zeta-function at \(z = 0\). Choose a generalized connection \(h\) on the submersion \(\pi : M \to B\) and a smooth section \(Q \in C^\infty(B, CL^1(M))\) of elliptic, positive and invertible pseudodifferential operators as above. Thus at any point \(b \in B\) one has an elliptic positive invertible operator \(Q_b\) acting on the manifold \(M_b = \pi^{-1}(b)\).

Let \(O \subset G\) be an Ad-invariant isotropic submanifold whose action on \(M\) is non-degenerate. Then by inserting the complex power \(Q^{-z}\) inside the trace map (60), we associate to any tensor \(e_1 \otimes \ldots \otimes e_n \in T^E\) the function \(\text{Tr}_{e_1, \ldots, e_n}^Q(z) \in C^k(V)\) defined over an appropriate neighborhood \(V\) of \(O^{(n)} \subset G^n\):

\[
\text{Tr}_{e_1, \ldots, e_n}^Q(z) (g_1, \ldots, g_n) = \text{Tr}(e_1(g_1)U_{g_1} h(s(g_1), r(g_2)) \ldots e_n(g_n)U_{g_n} h(s(g_n), r(g_1)) Q_{r(g_1)}^{-z}(g_1))
\]

The regularity order \(k\) can be as large as wanted, provided that \(\text{Re}(z)\) is large enough. Hence by Lemma 5.2, the function \(\text{Tr}_{e_1, \ldots, e_n}^Q(z)\) projected to the localization space \(C^\infty(V)|_{O}\) of jets to all order at \(O\) extends to a meromorphic function of \(z\) with at most simple poles. This implies that the evaluation of \(\text{Tr}_{e_1, \ldots, e_n}^Q(z)\) on a distribution \(\varphi \in C^\infty(V)\) with support localized at \(O\) and of bounded singularity order, yields a meromorphic function of \(z\). We also introduce the notation

\[
\text{Tr}_{e_1, \ldots, e_n}^Q(z) (g_1, \ldots, g_n-1|g_n) = \text{Tr}(e_1(g_1)U_{g_1} h(s(g_1), r(g_2)) \ldots Q_{r(g_n)}^{-z} e_n(g_n)U_{g_n} h(s(g_n), r(g_1)))
\]

Since we regard \(\text{Tr}_{e_1, \ldots, e_n}^Q(z)\) as the jets of a function at the submanifold \(O^{(n)}\), we can take any coefficient of its Laurent expansion at \(z = 0\) before evaluating it on a localized distribution.

**Definition 5.4** Let \(\mathcal{E} = C^\infty_c(B) \rtimes G\) and \(\mathcal{E}' = C^\infty_c(B, CL^0_c(M)) \rtimes G\). Choose any generalized connection \(h\) on the submersion \(\pi : M \to B\) and any elliptic section \(Q\) as above. Then for any cocycle \(\varphi \in \text{Hom}(\pi(T\mathcal{E}|_{\text{top}})|_{O}, \mathbb{C})\) localized at \(O\), the zeta-renormalized cochain \((\tau_\varphi)_R \in \text{Hom}(\pi(T\mathcal{E}), \mathbb{C})\) is defined on \(T^E\)
and \( \Omega^1 T \mathcal{E}_z \) by

\[
(\tau_c)_R(e_1 \otimes \ldots \otimes e_n) = \int_{G^n} \varphi^+_n(g_1, \ldots, g_n) \text{Pf} \left[ \mathcal{T}^{h, Q}_{e_1, \ldots, e_n}(z)(g_1, \ldots, g_n) \right],
\]

\[
(\tau_c)_R(\zeta(e_1 \otimes \ldots \otimes e_{n-1} \otimes d e_n)) = \int_{G^n} \varphi^-_n(g_1, \ldots, g_{n-1} | g_n) \text{Pf} \left[ \mathcal{T}^{h, Q}_{e_1, \ldots, e_n}(z)(g_1, \ldots, g_{n-1} | g_n) \right].
\]

(75)

where \( \text{Pf}_{z=0} \) denotes the finite part at \( z = 0 \) of the corresponding zeta-functions.

Observe that \((\tau_c)_R\) is well-defined on the pro-complex \( X(\hat{T} \mathcal{E}) \) because evaluation on the distributions \( \varphi^\pm \) kills the high powers of the ideal \( J \mathcal{E} \). The zeta-function also allows to define a residue morphism

\[
\text{Res} : X(\hat{T} \mathcal{E}) \rightarrow X(\hat{T} \mathcal{E}_{\text{top}})[O] \tag{76}
\]

by selecting the poles of (73) and (74) at \( z = 0 \) instead of the finite part. The restriction of (76) to the even subspace of the \( X \)-complex is a linear map \( \hat{T} \mathcal{E} \rightarrow (\hat{T} \mathcal{E}_{\text{top}})[O] \), sending any \( n \)-tensor \( e_1 \otimes \ldots \otimes e_n \in T \mathcal{E} \) to the jets of a function of the variables \( (g_1, \ldots, g_n) \) at the localization submanifold \( O^n \subset G^n \):

\[
(\text{Res}(e_1 \otimes \ldots \otimes e_n))(g_1, \ldots, g_n) = \text{Res}_{z=0} \left[ \mathcal{T}^{h, Q}_{e_1, \ldots, e_n}(g_1, \ldots, g_n)(z) \right].
\]

Notice that the ideal \( J \mathcal{E} \subset T \mathcal{E} \) is sent to \((J \mathcal{E}_{\text{top}})[O]\). Moreover, for any fixed \( k \) the \( k \)-jet of \( \text{Res}(e_1 \otimes \ldots \otimes e_n) \) vanishes whenever \( e_1 \otimes \ldots \otimes e_n \) belongs to a sufficiently high power of the ideal \( \mathcal{R} = \text{Ker}(T \mathcal{E} \rightarrow T \mathcal{A}) \). This is due to the fact that the zeta-function (73) has no pole at \( z = 0 \) in this case. Hence the residue morphism indeed extends to a well-defined linear map \( \hat{T} \mathcal{E} \rightarrow (\hat{T} \mathcal{E}_{\text{top}})[O] \). In the odd case (76) is a linear map \( \Omega^1 \hat{T} \mathcal{E}_z \rightarrow (\Omega^1 \hat{T} \mathcal{E}_{\text{top}})[O] \) defined in an analogous way:

\[
(\text{Res}(\zeta(e_1 \otimes \ldots \otimes e_{n-1} \otimes d e_n)))(g_1, \ldots, g_{n-1} | g_n) = \text{Res}_{z=0} \left[ \mathcal{T}^{h, Q}_{e_1, \ldots, e_n}(g_1, \ldots, g_{n-1} | g_n)(z) \right].
\]

The crucial point is that (76) is a morphism of \( X \)-complexes. This is an easily consequence of the fact that the zeta-functions have only simple poles: the residues at \( z = 0 \) do not depend on the actual place of \( Q^{-z} \) in formulas (73) and (74). This property would fail in the presence of double poles. Following [21] we now introduce the logarithm

\[
\ln Q = -\frac{d}{dz} Q^{-z} |_{z=0} .
\]

(77)

The latter is no longer a section of the classical pseudodifferential operators \( \text{CL}(M) \), but belongs to the larger class of \( \text{log-polyhomogeneous} \) pseudodifferential operators \( \text{CL}(M)_{\text{log}} \). However, the difference \( \ln Q - \ln Q' \) of two such logarithms \( is \) a section of \( \text{CL}(M) \), as well as the commutator \([\ln Q, P]\) with any section \( P \in C_c^\infty(B, \text{CL}(M)) \). We shall enlarge the algebra \( \mathcal{E} \) by adding the log-polyhomogeneous operators. Define

\[
\mathcal{E}_{\text{log}} = C_c^\infty(B, \text{CL}(M)_{\text{log}}) \rtimes G .
\]

Thus the elements of \( \mathcal{E}_{\text{log}} \) are products of logarithms by elements of \( \mathcal{E} \). In particular the section \( \ln Q \) can be viewed as a left multiplier of \( \mathcal{E}_{\text{log}} \), as follows:
(ln \(Q \cdot e\))\((g) = \ln Q_{\mathcal{C}(g)}e(g)\) for all \(e \in \mathcal{E}\) and \(g \in G\). We can mimic the construction of the complex \(X(\check{T}\mathcal{E})\), replacing everywhere classical pseudodifferential operators by log-polyhomogeneous ones. This leads to a complex \(X(\check{T}\mathcal{E})_\log^1\) and its subcomplex

\[X(\check{T}\mathcal{E})_\log^1 \subset X(\check{T}\mathcal{E})_\log,\]  

where the superscript \(1\) means that we retain only the tensors having logarithmic degree at most 1. Thus, the elements of \((T\mathcal{E})_\log^1\) are of the form \(e_1 \otimes \ldots \otimes e_n\) or \(e_1 \otimes \ldots \otimes \ln Q \cdot e_1 \otimes \ldots \otimes e_n\), where all \(e_j\)'s are in \(\mathcal{E}\). Similarly in odd degree, the elements of \((\Omega^1T\mathcal{E})_\log^1\) are of the form \(\{e_1 \otimes \ldots \otimes e_{n-1}d\ln e_n\} \cup \{e_1 \otimes \ldots \ln Q \cdot e_1 \otimes \ldots \otimes e_n \cdot d\ln e_n\}\). Since the difference of logarithms \(\ln Q_{\mathcal{E}(g)} - U_g \ln Q_{\mathcal{E}(g)} U_g^{-1}\) is always a classical pseudodifferential operator on the manifold \(M_{\mathcal{E}(g)}\), one sees that the residue map (76) can be extended to a chain map

\[X(\check{T}\mathcal{E})_\log^1 \cap \text{Dom(Res)} \rightarrow X(\check{T}\mathcal{E}_{\text{top}})_{[G]}\]

where the domain \(\text{Dom(Res)}\) is the linear span of differences of chains where only the place of \(\ln Q\) changes. For example in even degree, the chains in \((T\mathcal{E})_\log^1\) are linearly generated by differences

\[e_1 \otimes \ldots \ln Q \cdot e_1 \otimes \ldots e_n = e_1 \otimes \ldots e_1 \otimes \ldots \ln Q \cdot e_1 \otimes \ldots \otimes e_n\]

For notational convenience we introduce the convention that a right multiplication of a factor \(e_i\) by \(\ln Q\) amounts to the left multiplication of the following factor \(e_{i+1}\) by \(\ln Q\) in a tensor product. In particular

\[e_1 \otimes \ldots \otimes \ln Q \cdot e_1 \otimes e_{i+1} \otimes \ldots \otimes e_n := e_1 \otimes \ldots \otimes \ln Q \cdot e_1 \otimes e_{i+1} \otimes e_i \otimes \ln Q \cdot e_{i+1} \otimes \ldots \otimes e_n\]

\[e_1 \otimes \ldots \otimes e_n \cdot \ln Q := \ln Q \cdot e_1 \otimes \ldots \otimes e_n\]

**Proposition 5.5** Let \(\mathcal{E} = C^\infty_c(B) \times G\) and \(\mathcal{E}' = C^\infty_c(B, CL^0(M)) \times G\). Choose any generalized connection \(h\) on the submersion \(\pi : M \rightarrow B\) and any elliptic section \(Q\) as above. Then for any cocycle \(\varphi \in \text{Hom}(X(\check{T}\mathcal{E}_{\text{top}})_{[G]}, \mathbb{C})\) localized at \(O\), the boundary of the zeta-renormalized cochain \((\tau_\varphi)_R\) is the cocycle \((\tau_\varphi)_R \partial \in \text{Hom}(X(\check{T}\mathcal{E}), \mathbb{C})\) given by

\[(\tau_\varphi)_R \partial (\varphi(e_1 \otimes \ldots \otimes e_{n-1} d\ln e_n)) = \varphi \circ \text{Res}(e_1 \otimes \ldots \otimes e_{n-1} \otimes [\ln Q, e_n])\]  

\[(\tau_\varphi)_R \partial (e_1 \otimes \ldots \otimes e_n) = \sum_{1 \leq i < j \leq n} \varphi \circ \text{Res}(e_1 \otimes \ldots \otimes [\ln Q, e_i] \ldots d e_j \otimes \otimes e_n)\]

**Proof:** By definition one has \(\partial (\varphi(e_1 \otimes \ldots \otimes e_{n-1} d\ln e_n)) = e_1 \otimes \ldots \otimes e_{n-1} \otimes e_n - e_n \otimes e_1 \otimes \ldots \otimes e_{n-1}\). For all \(i\) write \(h_{i+1} = h(s(g_i), r(g_{i+1}))\). Then (75) gives

\[\text{PF}_z \text{Tr}(e_1(g_1)U_{g_1} h_1^{g_1} \ldots e_{n-1}(g_{n-1})U_{g_{n-1}} h_n^{g_n-1} e_n(n_n)U_{g_n} h_n^{g_n} Q^{\tau_{z}}_{\mathcal{E}(g_1)})\]

In the same way

\[\text{PF}_z \text{Tr}(e_n(g_1)U_{g_1} h_1^{g_1} e_1(g_2)U_{g_2} h_2^{g_2} \ldots e_{n-1}(g_n)U_{g_n} h_n^{g_n} Q^{\tau_{z}}_{\mathcal{E}(g_1)})\]

37
Let \( \varphi \) be an \( X \)-complex cocycle, \( \varphi_n^+(g_1, \ldots, g_n) \) is invariant under cyclic permutations of \( (g_1, \ldots, g_n) \). This and the cyclicity of the operator trace implies

\[
(\tau_\varphi)_R e_n \otimes e_1 \otimes \ldots \otimes e_{n-1} = \int_{G^n} \varphi_n^+(g_1, \ldots, g_n) \times \\
\text{Pf Tr}(e_1(g_1)U_{g_1}h_1^1 \ldots e_{n-1}(g_{n-1})U_{g_{n-1}}h_{n-1}^1 Q_{r(g_n)}^- e_n(g_n)U_{g_n}h_n^1)
\]

Thus one can write

\[
(\tau_\varphi)_R \partial(\varphi_1 \otimes \ldots \otimes e_{n-1} \otimes e_n) = \int_{G^n} \varphi_n^+(g_1, \ldots, g_n) \times \\
\text{Pf Tr}(e_1(g_1)U_{g_1}h_1^1 \ldots e_{n-1}(g_{n-1})U_{g_{n-1}}h_{n-1}^1 Q_{r(g_n)}^- e_n(g_n)U_{g_n}h_n^1, Q^{-z})
\]

with the “commutator”

\[
[e_n(g_n)U_{g_n}h_n^1, Q^{-z}] = e_n(g_n)U_{g_n}h_n^1 Q_{r(g_n)}^- - Q_{r(g_n)}^- e_n(g_n)U_{g_n}h_n^1 = e_n(g_n)(U_{g_n}h_n^1 Q_{r(g_n)}^- (U_{g_n}h_n^1))^{-1} - Q_{r(g_n)}^- U_{g_n}h_n^1 - [Q_{r(g_n)}^- , e_n(g_n)]U_{g_n}h_n^1
\]

Now observe that \( U_{g_n}h_n^1 Q_{r(g_n)}^- (U_{g_n}h_n^1))^{-1} - Q_{r(g_n)}^- \) and \([Q_{r(g_n)}^- , e_n(g_n)]\) are pseudodifferential operators on the manifold \( M_{r(g_n)} \). They have an asymptotic expansion in powers of \( z \),

\[
U_{g_n}h_n^1 Q_{r(g_n)}^- (U_{g_n}h_n^1))^{-1} - Q_{r(g_n)}^- \sim -z[U_{g_n}h_n^1 \ln(Q_{r(g_n)})Q_{r(g_n)}^- (U_{g_n}h_n^1))^{-1} - \ln(Q_{r(g_n)})Q_{r(g_n)}^-] + O(z^2)
\]

up to order \( z^2 \). Hence with obvious notations

\[
[e_n(g_n)U_{g_n}h_n^1, Q^{-z}] \sim z[\ln Q, e_n(g_n)U_{g_n}h_n^1]Q_{r(g_n)}^- + O(z^2).
\]

Because the zeta-functions have only simple poles, \( \text{Pf}_{z=0}(z \text{Tr}(\ldots Q^{-z})) \) is the residue of \( \text{Tr}(\ldots Q^{-z}) \) and the terms of order \( z^2 \) are killed. Finally

\[
(\tau_\varphi)_R \partial(\varphi_1 \otimes \ldots \otimes e_{n-1} \otimes e_n) = \int_{G^n} \varphi_n^+(g_1, \ldots, g_n) \times \\
\text{Res Tr}(e_1(g_1)U_{g_1}h_1^1 \ldots e_{n-1}(g_{n-1})U_{g_{n-1}}h_{n-1}^1 [\ln Q, e_n(g_n)U_{g_n}h_n^1]Q_{r(g_n)}^-)
\]

which is precisely (79). One proceeds similarly with the second formula.

Collecting the preceding results one gets the following refinement of Theorem 4.7 which computes the excision map by means of a residue formula.

**Theorem 5.6** Let \( G \to B \) be a Lie groupoid and let \( O \) be an \( Ad \)-invariant isotropic submanifold of \( G \). Let \( \pi : M \to B \) be a \( G \)-equivariant surjective submersion and assume the action of \( O \) on \( M \) non-degenerate. Then one has a commutative diagram

\[
\begin{array}{ccc}
HP^\bullet(C^\infty_c(B, CL^{-1}_c(M)) \rtimes G) & \xrightarrow{E^*} & HP^\bullet + 1(C^\infty_c(S^*_2 M) \rtimes G) \\
\tau_\pi & \uparrow & \\
HP^\bullet_{top}(C^\infty_c(B) \rtimes G)[O] & \xrightarrow{\pi^1_{G}} & HP^\bullet_{top + 1}(C^\infty_c(S^*_2 M) \rtimes G)[\pi \circ O]
\end{array}
\]

(80)
where the isotropic submanifold $\pi^*O \subset S^*\pi M \times G$ is the pullback of $O$ by the submersion $S^*\pi M \to B$.

Let $\mathcal{A} = C^\infty_c(S^*\pi M) \rtimes G$, $\mathcal{E} = C^\infty_c(B, \text{C}^1_c(M)) \rtimes G$, and choose any continuous linear splitting $\sigma : \mathcal{A} \to \mathcal{E}$ of the projection homomorphism. Then the image of an even class $[\varphi] \in \text{HP}^0_{\text{top}}(C^\infty_c(B \times G)|O]$ is represented by the odd cyclic cocycle $\pi^*_G(\varphi) \in \text{Hom}(\Omega^1\tilde{T}\mathcal{A}, \mathbb{C})$ over the algebra $\mathcal{A} = C^\infty_c(S^*\pi M \times G)$, given by the residue

$$\pi^*_G(\varphi)(\tilde{\varphi}(a_1 \otimes \ldots \otimes a_n - da_n)) = \varphi \circ \text{Res}(\sigma(a_1) \otimes \ldots \otimes \sigma(a_{n-1}) \otimes [\ln Q, \sigma(a_n)])$$

for all $\tilde{\varphi}(a_1 \otimes \ldots \otimes a_{n-1}da_n) \in \Omega^1T\mathcal{A}$. In a similar way, the image of an odd class $[\varphi] \in \text{HP}^1_{\text{top}}(C^\infty_c(B \times G)|O]$ is represented by the cyclic cocycle of even degree $\pi^*_G(\varphi) \in \text{Hom}(\tilde{T}\mathcal{A}, \mathbb{C})$ given by the residue

$$\pi^*_G(\varphi)(a_1 \otimes \ldots \otimes a_n) = \sum_{1 \leq i<j \leq n} \varphi \circ \text{Res}(\sigma(a_i) \otimes \ldots \otimes [\ln Q, \sigma(a_j)] \ldots \otimes \sigma(a_n))$$

for all $a_1 \otimes \ldots \otimes a_n \in T\mathcal{A}$.

Theorems 4.8 and 5.6 allow to compute explicitly the pairing $\langle [\tau_{\varphi}], \text{Ind}_E([u]) \rangle$ for any elliptic symbol $u \in \text{GL}_1(\mathcal{A})$. For clarity we suppose $u \in \text{GL}_1(\mathcal{A})$ but the general case follows easily. Hence let $[\tau_{\varphi}]$, $[\tau_{\varphi}] \in \text{HP}^0_{\text{top}}(C^\infty_c(B \times G))|O]$ be a cyclic cohomology class localized at $O$. Then

$$\langle [\tau_{\varphi}], [u] \rangle = \varphi \circ \text{Res}\text{tr}(\sigma_*([\tilde{u}^{-1}][\ln Q, \sigma_*]([u])))$$

for all $\tilde{u} \in \text{GL}_1(\tilde{T}\mathcal{A})$. In a similar way, the image of an odd class $[\varphi] \in \text{HP}^1_{\text{top}}(C^\infty_c(B \times G))|O]$ is represented by the cyclic cocycle of even degree $\pi^*_G(\varphi) \in \text{Hom}(\tilde{T}\mathcal{A}, \mathbb{C})$ given by the residue

$$\pi^*_G(\varphi)(a_1 \otimes \ldots \otimes a_n) = \sum_{1 \leq i<j \leq n} \varphi \circ \text{Res}(\sigma(a_i) \otimes \ldots \otimes [\ln Q, \sigma(a_j)] \ldots \otimes \sigma(a_n))$$

for all $a_1 \otimes \ldots \otimes a_n \in T\mathcal{A}$.

**Corollary 5.7** Under the hypotheses of Theorem 5.6, let $[u] \in K_1(\mathcal{A})$ be an elliptic symbol class element represented by an invertible matrix $u \in M_\infty(\mathcal{A})^+$. Let $[\varphi] \in \text{HP}^0_{\text{top}}(C^\infty_c(B \times G))|O]$ be a cyclic cohomology class localized at $O$. Then

$$\langle [\tau_{\varphi}], [u] \rangle = \varphi \circ \text{Res}\text{tr}(\sigma_*([\tilde{u}^{-1}][\ln Q, \sigma_*]([u])))$$

for all $Q \in \text{C}^\infty_c(B, \text{C}^1_c(M))$ is any section of elliptic positive invertible pseudodifferential operators of order one, and $\tilde{u} \in M_\infty(\tilde{T}\mathcal{A})^+$ is the above invertible lifting of $u$.

### 6 Geometric cocycles

For any Lie groupoid $G \rightrightarrows B$ we shall construct in a geometric way cyclic cohomology classes $[\varphi] \in \text{HP}^0_{\text{top}}(C^\infty_c(B \times G)|O]$ localized at Ad-invariant isotropic submanifolds $O \subset G$. Our approach is intended to combine all the already known cohomologies giving rise to cyclic cocycles localized at the unit submanifold $O = B$: the cohomology $H^*(BG)$ of the classifying space of an étale groupoid [5], the differentiable cohomology $H^*_d(G)$ of any Lie groupoid [24], the Gelfand-Fuchs cohomology appearing in the transverse geometry of a foliation [3], etc., etc., and generalize them order to include cyclic cohomology classes localized at any isotropic submanifold $O$. We start with a definition extending the
notion of Cartan connection on a Lie groupoid; note that all the formalism developed here could certainly be reinterpreted into the language of multiplicative forms of [9].

**Definition 6.1** A connection on a Lie groupoid \((r, s) \colon \Gamma \rightrightarrows E\) is a Lie subgroupoid \(\Phi \rightrightarrows F\) of the tangent groupoid \((r_*, s_*) : T\Gamma \rightrightarrows TE\) such that

- \(\Phi\) a vector subbundle of \(T\Gamma\) simultaneously transverse to \(Ker r_*\) and \(Ker s_*\), in the sense that \(\Phi \cap Ker r_* = \Phi \cap Ker s_*\) is the zero section of \(T\Gamma\),

- \(F\) is a vector subbundle of \(TE\), of the same rank as \(\Phi\).

The connection is flat if \((\Phi, F)\) are integrable as subbundles of the tangent bundles \((T\Gamma, TE)\).

By the transversality hypothesis, the tangent maps \(r_*\) and \(s_*\) send the fibers of \(\Phi\) isomorphically onto the fibers of \(F\). If the connection is flat, \((\Phi, F)\) define regular foliations on the manifolds \((\Gamma, E)\) respectively. The leaves of \(\Gamma\) are locally diffeomorphic to the leaves of \(E\) under both maps \(r\) and \(s\).

When \(\Phi\) has maximal rank equal to the dimension of \(E\), then \(\Phi \oplus Ker r_* = \Phi \oplus Ker s_* = T\Gamma\) and \(F = TE\). In that case, a flat connection defines a foliation of the manifold \(\Gamma\) which is simultaneously transverse to the submersions \(r\) and \(s\) and of maximal dimension. This foliation in turn determines a set of local bisections of \(\Gamma\). Since \(\Phi\) is a subgroupoid of \(T\Gamma\), one can check that these local bisections form a Lie pseudogroup with respect to the composition product on \(\Gamma\). This pseudogroup acts on the manifold \(E\) by local diffeomorphisms: any small enough local bisection provides a diffeomorphism from its source set in \(E\) to its range set in \(E\). Hence, any morphism \(\gamma \in \Gamma\) can be extended to a local diffeomorphism from an open neighborhood of \(s(\gamma)\) to an open neighborhood of \(r(\gamma)\), in a way compatible with the composition of morphisms in \(\Gamma\).

When \(\Phi\) has rank \(< \dim E\), the corresponding local bisections are only defined above each leaf of the foliation \(F\) on \(E\). This means that any morphism \(\gamma \in \Gamma\) can be extended to a local diffeomorphism from a small open subset of the leaf containing \(s(\gamma)\) to a small open subset of the leaf containing \(r(\gamma)\). In general there are topological or geometric obstructions to the existence of a connection (flat or not) on a Lie groupoid. Here are some basic examples:

**Example 6.2** An étale groupoid \(\Gamma \rightrightarrows B\) has a unique connection of maximal rank \(T\Gamma\) which is always flat. The corresponding pseudogroup of local bisections is thus the pseudogroup of all local bisections of \(\Gamma\). Hence, any morphism \(\gamma \in \Gamma\) in an étale groupoid determines a local diffeomorphism from a neighborhood of \(s(\gamma)\) to a neighborhood of the range \(r(\gamma)\) in a unique way.

**Example 6.3** Let \(G\) be a Lie group acting on a manifold \(B\) by global diffeomorphisms. Then the action groupoid \(\Gamma = B \ltimes G\) is endowed with the canonical flat connection \(Ker (pr_*) \subset T\Gamma\), where \(pr : \Gamma \to G\) is the projection. In this case the pseudogroup of local bisections determined by the connection is precisely the group \(G\).

A flat connection \(\Phi \rightrightarrows F\) on a Lie groupoid \(\Gamma \rightrightarrows E\) leads to the \(\Gamma\)-equivariant leafwise cohomology of \(E\). Indeed at any point \(\gamma \in \Gamma\), the fiber \(\Phi_\gamma\) is canonically isomorphic, as a vector space, to the fibers \(s_*(\Phi_\gamma) = F_{s(\gamma)}\).
and \( r_\ast(\Phi_\gamma) = F_\gamma \). Hence any \( \gamma \) yields a linear isomorphism from the fiber \( F_\gamma(\gamma) \) to the fiber \( F_\gamma(\gamma) \), in a way compatible with the composition law in \( \Gamma \). This means that the vector bundle \( F \) over \( E \) is a \( \Gamma \)-bundle. Here the flatness of the connection is not used. In the same way, the dual bundle \( F^\ast \) is also a \( \Gamma \)-bundle. In general the cotangent bundle \( T^\ast E \) may not carry any action of \( \Gamma \), but the line bundle \( \Lambda^\max \Lambda^\Gamma \otimes \Lambda^\max T^\ast E \) always does. As usual \( \Lambda^\Gamma \) is the Lie algebroid of \( \Gamma \). If \( F_\perp = TE/F \) denotes the normal bundle to \( F \), then one has a canonical isomorphism \( \Lambda^\max T^\ast E \cong \Lambda^\max F_\perp \otimes \Lambda^\top F^\ast \). Since \( \Gamma \) acts on the line bundle \( \Lambda^\max F \) through the connection \( \Phi \), one sees that the tensor product \( \Lambda^\max \Lambda^\Gamma \otimes \Lambda^\max T^\ast E \otimes \Lambda^\max F \cong \Lambda^\max \Lambda^\Gamma \otimes \Lambda^\max F_\perp \) is a \( \Gamma \)-bundle. We define the bicomplex of \( \Gamma \)-equivariant leafwise differential forms

\[
C^n(\Gamma, \Lambda^\ast F) = (C^n(\Gamma) \otimes \Lambda^\max F^\ast)(\gamma)_0 \geq m \geq 0
\]

is the space of smooth sections of the vector bundle \( |\Lambda^\max \Lambda^\Gamma| \otimes |\Lambda^\max F^\ast| \otimes \Lambda^\max F^\ast \), pulled back on the manifold \( \Gamma^n \) of composable \( n \)-tuples by the source map \( s : \Gamma^n \to E, (\gamma_1, \ldots, \gamma_n) \mapsto s(\gamma_n) \). In particular for \( n = 0 \), \( C^0(\Gamma, \Lambda^\ast F) = C^\infty(\Gamma, |\Lambda^\max \Lambda^\Gamma| \otimes |\Lambda^\max F^\ast| \otimes \Lambda^\max F^\ast) \) is the space of leafwise \( m \)-forms on \( E \). Twisted by the line bundle \( |\Lambda^\max \Lambda^\Gamma| \otimes |\Lambda^\max F^\ast| \). Note that \( |\Lambda^\max F^\ast| \) is the bundle of 1-densities transverse to the foliation \( F \) on \( E \). The first differential \( d_1 : C^n(\Gamma, \Lambda^\ast F) \to C^{n+1}(\Gamma, \Lambda^\ast F) \) on this bicomplex is the usual differential computing the groupoid cohomology with coefficients in a \( \Gamma \)-bundle. It is given on any cochain \( c \in C^n(\Gamma, \Lambda^\ast F) \) by

\[
(d_1 c)(\gamma_1, \ldots, \gamma_{n+1}) = c(\gamma_1, \ldots, \gamma_{n+1}) \quad + \sum_{i=1}^n (-1)^i c(\gamma_1, \ldots, \gamma_{i\gamma_{i+1}}, \ldots, \gamma_{n+1}) \end{equation}

for all \((\gamma_1, \ldots, \gamma_{n+1}) \in \Gamma^{n+1}\). The last term of the r.h.s. denotes the action of the linear isomorphism \( U^{-1}_n : (|\Lambda^\max \Lambda^\Gamma| \otimes |\Lambda^\max F^\ast| \otimes \Lambda^\max F^\ast)(\gamma_1, \ldots, \gamma_n) \to (|\Lambda^\max \Lambda^\Gamma| \otimes |\Lambda^\max F^\ast| \otimes \Lambda^\max F^\ast)(\gamma_1, \ldots, \gamma_n) \). The second differential \( d_2 : C^n(\Gamma, \Lambda^\ast F) \to C^n(\Gamma, \Lambda^\ast F^\ast) \) comes from the leafwise de Rham differential \( d_F : C^\infty(\Gamma, \Lambda^\ast F) \to C^\infty(\Gamma, \Lambda^\ast F^\ast) \) on the foliated manifold \( E \). Indeed we first observe that the foliation \( F \) on \( E \) defines the sheaf of holonomy-invariant sections of the line bundle \( \Lambda^\max F^\ast \), which in turn induces a canonical foliated connexion \( d_{F_\perp} : C^\infty(\Gamma, |\Lambda^\max F^\ast|) \to C^\infty(\Gamma, |\Lambda^\max F^\ast|) \). This connexion is flat in the usual sense \((d_{F_\perp})^2 = 0\). In the same way, viewing \( r^\ast \Lambda^\Gamma \) as a subbundle of \( TT \), the foliation \( \Phi \) on \( \Gamma \) defines the sheaf of holonomy-invariant sections of the foliated connection \( r^\ast |\Lambda^\max \Lambda^\Gamma| \) which descends to the line bundle \( \Lambda^\max \Lambda^\Gamma \) over \( E \), and subsequently defines a flat foliated connection \( d_{\Lambda^\Gamma} : C^\infty(\Gamma, |\Lambda^\max \Lambda^\Gamma|) \to C^\infty(\Gamma, |\Lambda^\max \Lambda^\Gamma|) \). The sum \( d_A + d_{\Lambda^\Gamma} \) is a \( \Gamma \)-equivariant operator on the space of sections of the vector bundle \( |\Lambda^\max \Lambda^\Gamma| \otimes |\Lambda^\max F^\ast| \otimes \Lambda^\ast F^\ast \) which squares to zero. Finally we extend the foliation \( \Phi \) on \( \Gamma \) to a foliation \( \Phi^{(n)} \) on \( \Gamma^{(n)} \) in such a way that all projection maps \( p_i : \Gamma^{(n)} \to \Gamma \), \( (\gamma_1, \ldots, \gamma_n) \mapsto \gamma_i \) induce vector space isomorphisms \( \Phi^{(n)}(\gamma_1, \ldots, \gamma_n) \mapsto \Phi_{\gamma_i} \). Since \( \Phi \) is a subgroupoid of \( \Gamma \) the foliation \( \Phi^{(n)} \) exists and is unique. Moreover the source map \( s : \Gamma^{(n)} \to E \) is a local diffeomorphism from the leaves of \( \Phi^{(n)} \) to the leaves of \( F \). We use this local identification to lift
\[ d_{AF} + d_{F_s} + df \] to an operator \( s^*(d_{AF} + d_{F_s} + df) \) on the space of sections of the vector bundle \( s^*(|A^{\max}\Lambda^1 \otimes \Lambda^{\max}F^*_\perp \otimes \Lambda^\bullet F^*) \) over \( \Gamma^{(n)} \). For any cochain \( c \in C^n(\Gamma, \Lambda^m F) \) we set

\[ (d_2 c) = (-1)^n s^*(d_{AF} + d_{F_s} + df) c. \]

Since \( d_{AF} + d_{F_s} + df \) is \( \Gamma \)-equivariant, the two differentials on \( C^*(\Gamma, \Lambda^\bullet F^*) \) anticommute: \( d_1 d_2 + d_2 d_1 = 0 \). The \( \Gamma \)-equivariant leafwise cohomology of \( E \) is by definition the cohomology of the total complex obtained from this bicomplex.

If \( \eta : E \to N \) is a submersion, we define \( C^*_\eta(\Gamma, \Lambda^\bullet F^*) \) as the subcomplex of cochains having proper support with respect to \( \eta \).

**Definition 6.4** Let \( G \rightrightarrows B \) be a Lie groupoid and let \( O \) be an \( Ad^G \)-invariant isotropic submanifold of \( G \). A geometric cocycle localized at \( O \) is a quadruple \((N, E, \Phi, c)\) where

- \( N \twoheadrightarrow B \) is a surjective submersion. Hence the pullback groupoid \( \nu^*G \rightrightarrows N \) acts on the isotropic submanifold \( \nu^*O \subset \nu^*G \) by the adjoint action.
- \( E \twoheadrightarrow \nu^*O \) is a \( \nu^*G \)-equivariant submersion. Any element \( \gamma \in \nu^*O \), viewed in \( \nu^*G \), is required to act by the identity on its own fiber \( E_\gamma \).
- \( \Phi \rightrightarrows F \) is a flat connection on \( \Gamma = E \ltimes \nu^*G \), with \( F \) oriented. The canonical section \( E \to \nu^*O \to \Gamma \), restricted to a leaf of \( F \), is a leaf of \( \Phi \).
- \( c \in C^*_\eta(\Gamma, \Lambda^\bullet F^*) \) is a total cocycle with proper support relative to the submersion \( E \twoheadrightarrow \nu^*O \), assumed normalized in the sense that

\[ c(\gamma_1, \ldots, \gamma_n) = 0 \quad \text{whenever} \quad \gamma_1 \ldots \gamma_n = \eta(s(\gamma_n)) . \]

\((N, E, \Phi, c)\) is called proper if \( E \) is a proper \( \nu^*G \)-manifold.

From now on let \( G \rightrightarrows B \) be a fixed Lie groupoid. We shall associate to any geometric cocycle \((N, E, \Phi, c)\) localized at an isotropic submanifold \( O \subset G \), a periodic cyclic cohomology class of \( C^\infty_p(G) \) localized at \( O \). This requires a number of steps. On the action groupoid \( \Gamma = E \ltimes \nu^*G \) we first define the convolution algebra

\[ \mathcal{G} = C^\infty_p(E, \Lambda^\bullet F^*) \rtimes \Gamma \]

which is the space \( C^\infty_p(\Gamma, r^*(\Lambda^\bullet F^* \otimes |\Lambda^{\max} A^* \Gamma|)) \) of properly supported sections of the vector bundle \( |\Lambda^{\max} A^* \Gamma| \otimes \Lambda^\bullet F^* \) pulled back by the rank map \( r : \Gamma \to E \). By definition the groupoid \( \Gamma \) acts on the vector bundle \( \Lambda^\bullet F^* \), but there is no such action on the density bundle \( |\Lambda^{\max} A^* \Gamma| \). For any \( \gamma \in \Gamma \) we thus define the linear isomorphism \( U_\gamma : \Lambda^\bullet F^*_s(\gamma) \to \Lambda^\bullet F^*_{r(\gamma)} \) leaving the space \( |\Lambda^{\max} A^* \Gamma|_{s(\gamma)} \) untouched. The convolution product of two elements \( \alpha_1, \alpha_2 \in \mathcal{G} \) is then given by

\[ (\alpha_1 \alpha_2)(\gamma) = \int_{\gamma_1, \gamma_2 = \gamma} \alpha_1(\gamma_1) \wedge U_{\gamma_1} \alpha_2(\gamma_2), \quad \forall \gamma \in \Gamma, \]

where \( \alpha_1(\gamma_1) \wedge U_{\gamma_1} \alpha_2(\gamma_2) \in \Lambda^\bullet F^*_s(\gamma_1) \otimes |\Lambda^{\max} A^* \Gamma|_{r(\gamma_1)} \otimes |\Lambda^{\max} A^* \Gamma|_{r(\gamma_2)} \) involves the exterior product of leafwise differential forms, and the integral is taken against the 1-density \( |\Lambda^{\max} A^* \Gamma|_{r(\gamma)} \) while \( r(\gamma_1) = r(\gamma) \) remains fixed. Hence \( (\alpha_1 \alpha_2)(\gamma) \) really defines an element of the fiber \( \Lambda^\bullet F^*_{r(\gamma)} \otimes |\Lambda^{\max} A^* \Gamma|_{r(\gamma)} \).

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Note that the subalgebra of \( \mathcal{G} \) consisting only in the zero-degree foliated differential forms coincides with the usual convolution algebra \( C^\infty(G) \times \Gamma \cong C^\infty(\Gamma, r^*|A^{max}A^\Gamma|) \) of (properly supported) scalar functions over the groupoid \( \Gamma \). Now let \( d_{A, \Gamma} : C^\infty(E, |A^{max}A^\Gamma|) \to C^\infty(E, F^* \otimes |A^{max}A^\Gamma|) \) be the flat foliated connection dual to \( d_{A\Gamma} \), induced as before by the connection \( \Phi \). Combining \( d_{A, \Gamma} \) with the leafwise de Rham differential \( d_F : C^\infty(E, \Lambda^m F^*) \to C^\infty(E, \Lambda^{m+1} F^*) \), we obtain a total differential \( d_\Phi \) on \( \mathcal{G} \),

\[
d_\Phi \alpha = r^*(d_F + d_{A, \Gamma}) \alpha , \quad \alpha \in \mathcal{G},
\]

where \( d_F + d_{A, \Gamma} \), acting leafwise on \( E \), is lifted to a leafwise operator on \( \Gamma \) through the local identification between the leaves of \( F \) and the leaves of \( \Phi \) provided by the rank map. Then \( d_\Phi \) satisfies the graded Leibniz rule \( d_\Phi(\alpha_1 \alpha_2) = (d_\Phi \alpha_1) \alpha_2 + (-1)^{\deg \alpha_1} \alpha_1 (d_\Phi \alpha_2) \), where \( \deg \alpha_1 \) is the degree of the differential form \( \alpha_1 \). Hence \( (\mathcal{G}, d_\Phi) \) is a differential graded (DG) algebra.

The periodic cyclic cohomology of a DG algebra is defined in complete analogy with the usual case, simply by adding the extra differential and taking care of the degrees of the elements in the algebra. Hence the space of noncommutative differential forms \( \Omega\mathcal{G} \) is the same as in the ungraded case, but the degree of an \( n \)-form \( \omega = \alpha_0 d\alpha_1 \ldots d\alpha_n \) is now \( \deg \omega = n + |\alpha_0| + \ldots + |\alpha_n| \). The differential \( d_\Phi \) is uniquely extended to a differential on the algebra \( \Omega\mathcal{G} \), in such a way that it anticommutes with \( d \). Hence we have

\[
d_\Phi(\alpha_0 d\alpha_1 \ldots d\alpha_n) = (d_\Phi \alpha_0) d\alpha_1 \ldots d\alpha_n - (-1)^{|\alpha_0|} \alpha_0 (d_\Phi \alpha_1) \ldots d\alpha_n + \ldots
\]

The Hochschild boundary is as usual \( b(\omega) = (-1)^{|\omega|} [\omega, \alpha] \) for any \( \omega \in \Omega\mathcal{G} \) and \( \alpha \in \mathcal{G} \), where the commutator is the graded one. Then \( d_\Phi \) anticommutes with \( b \), with the Karoubi operator \( \kappa = 1 - (bd + db) \), and with Connes' operator \( B = (1 + \kappa + \ldots + \kappa^n) d \) on \( \Omega^n \mathcal{G} \). The periodic cyclic cohomology of \( \mathcal{G} \) is therefore defined as the cohomology of the complex \( \text{Hom}(\Omega\mathcal{G}, \mathbb{C}) \) with boundary map the transposed of \( b + B + d_\Phi \).

We now define a linear map \( \lambda : C^*_\eta(\Gamma, \Lambda^* F^*) \to \text{Hom}(\hat{\Omega\mathcal{G}}, \mathbb{C}) \) which, once restricted to normalized cochains, will behave like a chain map. For any cochain \( c \in C^*_\eta(\Gamma, \Lambda^* F^*) \) set

\[
\lambda(c)(\alpha_0 d\alpha_1 \ldots d\alpha_n) = \int_{(\gamma_0, \ldots, \gamma_n) \in \Gamma^{(n)}} \alpha_0(\gamma_0) \wedge U_{\gamma_0} \alpha_1(\gamma_1) \ldots \wedge U_{\gamma_0 \ldots \gamma_{n-1}} \alpha_n(\gamma_n) \wedge c(\gamma_1, \ldots, \gamma_n)
\]

where \( \gamma_0 \) is defined as a function of the \( n \)-tuple \((\gamma_1 \ldots \gamma_n)\) by the localization condition \( \gamma_0 \ldots \gamma_n = \eta(s(\gamma_0)) \). We use the vector space isomorphism \( |A^{max}A^\Gamma|_{r(\gamma_0)} \otimes |A^{max}A^\Gamma|_{s(\gamma_0)} \cong \mathbb{C} \) to view the wedge product under the integral as an element of the fiber \( |A^{max}A^\Gamma|_{r(\gamma_1)} \otimes \ldots \otimes |A^{max}A^\Gamma|_{r(\gamma_n)} \otimes \Lambda^{|\alpha|} F_{s(\gamma_0)}^* \), with \( |\alpha| = |\alpha_0| + \ldots + |\alpha_n| \). Since \( F \) is oriented, the integrand defines a 1-density which can be integrated over the manifold \( \Gamma^{(n)} \) when the leafwise degree \( m + |\alpha| \) matches the rank of \( F \); otherwise the integral is set to zero.

**Lemma 6.5** Let \( c \in C^*_\eta(\Gamma, \Lambda^* F^*) \) be a normalized cochain in the sense that \( c(\gamma_1, \ldots, \gamma_n) = 0 \) whenever \( \gamma_1 \ldots \gamma_n = \eta(s(\gamma_0)) \). Then the periodic cyclic cochain \( \lambda(c) \in \text{Hom}(\Omega\mathcal{G}, \mathbb{C}) \) verifies the identities

\[
\lambda(c) \circ b = \lambda(d_1 c) , \quad \lambda(c) \circ d = 0 , \quad \lambda(c) \circ d_\Phi = \lambda(d_2 c).
\]
Hence if $c$ is a normalized total cocycle, $\lambda(c)$ is a $\kappa$-invariant periodic cyclic cocycle over the DG algebra $\mathcal{G}$.

**Proof:** The three identities are routine computations. Since $\lambda(c) \circ d = 0$ for any normalized $c$, one has $\lambda(c) \circ B = 0$. As a consequence $(d_1 + d_2)c = 0$ implies $\lambda(c) \circ (b + B + d_\mathfrak{f}) = 0$, thus $\lambda(c)$ is a periodic cyclic cocycle of the DG algebra $\mathcal{G}$. Moreover for any normalized cocycle $c$,

$$\lambda(c) \circ (1 - \kappa) = \lambda(c) \circ (db + bd) = \lambda(d_1c) \circ d = -\lambda(d_2c) \circ d = -\lambda(c) \circ d_\mathfrak{f}d = 0$$

since $d_\mathfrak{f}$ and $d$ anticommute, which shows that the periodic cyclic cocycle $\lambda(c)$ is $\kappa$-invariant. 

The $\kappa$-invariance of the cocycle $\lambda(c)$ means that the latter can as well be interpreted as an $X$-complex cocycle for certain DG algebra extensions of $(\mathcal{G}, d_\mathfrak{f})$. The $X$-complex of any associative DG algebra $(\mathcal{H}, d)$ is defined in analogy with the usual case by

$$X(\mathcal{H}, d) : \mathcal{H} \rightleftarrows \Omega^1 \mathcal{H}^2,$$

where $\Omega^1 \mathcal{H}^2$ is the quotient of $\Omega^1 \mathcal{H}$ by the subspace of graded commutators $[\mathcal{H}, \Omega^1 \mathcal{H}] = b\Omega^2 \mathcal{H}$. Since $d$ acts on $\Omega \mathcal{H}$ and anticommutes with the Hochschild operator $b$, it descends to a well-defined differential on $X(\mathcal{H}, d)$ and anticommutes with the usual $X$-complex boundary maps $\partial : \mathcal{H} \rightarrow \Omega^1 \mathcal{H}$ and $\partial : \Omega^1 \mathcal{H} \rightarrow \mathcal{H}$. We always endow the $X$-complex of a DG algebra with the total boundary operator $(\partial d \oplus \partial) + d$. Now take $\mathcal{H}$ as the direct sum $\mathcal{H} = \bigoplus_{n \geq 1} \mathcal{H}_n$, where

$$\mathcal{H}_n = C^\infty_p(\Gamma^{(n)}, r_1^n \Lambda^* F^* \otimes r_1^n |\Lambda^{\text{max}} A^* \Gamma| \otimes \ldots \otimes r_n^n |\Lambda^{\text{max}} A^* \Gamma|)$$

and $r_1 : \Gamma^{(n)} \rightarrow E$ is the rank map $(\gamma_1, \ldots, \gamma_n) \mapsto \tau(\gamma_i)$. The component $\mathcal{H}_1$ is isomorphic, as a vector space, to $\mathcal{G}$. The product of two homogeneous elements $\alpha_1 \in \mathcal{H}_1$ and $\alpha_2 \in \mathcal{H}_n$ is the element $\alpha_1 \alpha_2 \in \mathcal{H}_{1+n}$ defined by

$$(\alpha_1 \alpha_2)(\gamma_1, \ldots, \gamma_{1+n}) = \alpha_1(\gamma_1, \ldots, \gamma_n) \wedge U_{\gamma_1} \ldots U_{\gamma_n} \alpha_2(\gamma_{n+1}, \ldots, \gamma_{1+n})$$

We equip $\mathcal{H}$ with a grading by saying that an element $\alpha \in C^\infty_p(\Gamma^{(n)}, r_1^n \Lambda^m F^* \otimes r_1^n |\Lambda^{\text{max}} A^* \Gamma| \otimes \ldots \otimes r_n^n |\Lambda^{\text{max}} A^* \Gamma|)$ has degree $||\alpha|| = m$. A differential $d$ of degree +1 on $\mathcal{H}$ is then defined by combining the leafwise differential $d_F$ with the flat connections on the density bundles $\Lambda^{\text{max}}|A^* \Gamma|:

$$da = (r_1^n(d_F) + r_1^n(d_{A^*}) + \ldots + r_n^n(d_{A^*}))$$

for any such $\alpha \in \mathcal{H}_n$. Obviously $d$ satisfies the graded Leibniz rule. By construction $(\mathcal{H}, d)$ is an extension of $(\mathcal{G}, d_\mathfrak{f})$. Indeed a surjective DG algebra morphism $m : \mathcal{H} \rightarrow \mathcal{G}$ is defined as follows: the image of any $\alpha \in \mathcal{H}_n$ is the element $m(\alpha) \in \mathcal{G}$ given by the $(n-1)$-fold integral

$$(m(\alpha))(\gamma) = \int_{\gamma_{n+1}=\gamma} \alpha(\gamma_1, \ldots, \gamma_n), \quad \forall \gamma \in \Gamma$$

where the integral is taken against the density bundle $r_2^n |\Lambda^{\text{max}} A^* \Gamma| \otimes \ldots \otimes r_n^n |\Lambda^{\text{max}} A^* \Gamma|$. Let $\mathcal{I}$ be the kernel of the multiplication map. In fact $\mathcal{H}$ is a
kind of localized version of the tensor algebra extension $T\mathcal{I}$ of $\mathcal{I}$. The latter is a graded algebra, the degree of a tensor $\alpha_1 \otimes \ldots \otimes \alpha_n$ being the sum of the degrees of the factors. $T\mathcal{I}$ is endowed with a differential $d_{\mathcal{I}}$ making the multiplication map $T\mathcal{I} \to \mathcal{I}$ a morphism of DG algebras:

$$d_{\mathcal{I}}(\alpha_1 \otimes \ldots \otimes \alpha_n) = (d_{\mathcal{I}}\alpha_1) \otimes \ldots \otimes \alpha_n + (-1)^{|\alpha_1|} \alpha_1 \otimes (d_{\mathcal{I}}\alpha_2) \otimes \ldots \otimes \alpha_n + \ldots$$

A straightforward adaptation of the Cuntz-Quillen theory to the DG case shows that the $X$-complex $X(T\mathcal{I}, d_{\mathcal{I}})$ is quasi-isomorphic to the $(b + B + d_{\mathcal{I}})$-complex of noncommutative differential forms $\hat{\Omega}\Phi$, under the identification of pro-vector spaces $X(T\mathcal{I}) \cong \Omega\Phi$ taking the rescaling factors $(-1)^{|n/2|[n/2]}$ into account.

Since the periodic cyclic cocycle $\lambda(c)$ is $\kappa$-invariant, it can as well be viewed as an $X$-complex cocycle:

$$\lambda'(c) \in \text{Hom}(X(\hat{T}\mathcal{I}, d_{\mathcal{I}}), \mathbb{C}) \, .$$

In fact this cocycle descends to a cocycle over $X(\hat{\mathcal{H}}, d)$, where $\hat{\mathcal{H}}$ denotes the $\mathcal{I}$-adic completion of $\mathcal{H}$. Indeed we note that the canonical linear inclusion $\mathcal{I} \to \mathcal{H}$, which identifies $\mathcal{I}$ and the vector subspace $\mathcal{H}$, commutes with the differential. The universal property of the tensor algebra then implies the existence of a DG algebra homomorphism $T\mathcal{I} \to \mathcal{H}$, mapping the DG ideal $J\mathcal{I}$ to $\mathcal{I}$, whence a chain map $X(T\mathcal{I}, d_{\mathcal{I}}) \to X(\hat{\mathcal{H}}, d)$. Observe that the homomorphism $T\mathcal{I} \to \mathcal{H}$ has dense range. The following lemma is obvious.

**Lemma 6.6** Let $c \in C^*_\mathcal{I}(\Gamma, \Lambda^*F^*)$ be a normalized cocycle. Then $\lambda'(c)$, viewed as a cocycle in $\text{Hom}(X(\hat{T}\mathcal{I}, d_{\mathcal{I}}), \mathbb{C})$, factors through a unique continuous cocycle $\lambda'(c) \in \text{Hom}(X(\hat{\mathcal{H}}_{\text{top}}, d), \mathbb{C})$.

The last step is the construction of an homomorphism from the convolution algebra of compactly supported functions $C^\infty_c(B) \rtimes G$ to the convolution algebra of properly supported functions $C^\infty_p(E) \rtimes \Gamma$. Indeed, the pullback groupoid $\nu^*G \cong N$ is Morita equivalent to $G \cong B$, so using a cut-off function on $N$ we realize the equivalence by an homomorphism

$$C^\infty_c(B) \rtimes G \xrightarrow{\sim} C^\infty_c(N) \rtimes \nu^*G \, .$$

Next, by definition the submersion $E \xrightarrow{\nu} \nu^*O \xrightarrow{\nu} N$ is $\nu^*G$-equivariant. Hence any $\gamma \in \Gamma = E \rtimes \nu^*G$ determines a unique $g \in \nu^*G$, and the resulting map $\Gamma \to \nu^*G$ is a smooth morphism of Lie groupoids. The latter identifies the preimage of the source map $\Gamma_{s(\gamma)}$ in $\Gamma$ with the preimage of the source map $\nu^*G_{s(g)}$ in $\nu^*G$, whence a canonical vector space isomorphism between the fibers $A\Gamma_{r(\gamma)}$ and $A(\nu^*G)_{r(g)}$ of the corresponding Lie algebroids. We conclude that any smooth compactly supported section of $[\Lambda^{\text{max}}A(\nu^*G)]$ over $\nu^*G$ can be canonically pulled back to a smooth properly supported section of $[\Lambda^{\text{max}}A\Gamma]$ over $\Gamma$, and this results in an homomorphism of convolution algebras $C^\infty_c(N) \rtimes \nu^*G \to C^\infty_p(E) \rtimes \Gamma$. By composition with the Morita equivalence, we obtain the desired homomorphism

$$\rho : C^\infty_c(B) \rtimes G \to C^\infty_p(E) \rtimes \Gamma \, . \quad (86)$$

The latter depends on the choice of homomorphism realizing the Morita equivalence. However this dependence will disappear in cohomology. Now put
\( \mathcal{C} = C^\infty_c(B) \rtimes G \). Since \( C^\infty_c(E) \rtimes \Gamma \) is the subalgebra of \( \mathcal{G} \) consisting of zero-degree foliated differential forms, we regard \( \rho \) as an algebra homomorphism \( \mathcal{C} \to \mathcal{G} \). The latter lifts to a linear map \( \iota : \mathcal{C} \to \mathcal{H} \) after composition by the canonical linear inclusion \( \mathcal{G} = \mathcal{H}_1 \hookrightarrow \mathcal{H} \). The diagram of extensions

\[
\begin{array}{cccc}
0 & \longrightarrow & J\mathcal{C} & \longrightarrow & T\mathcal{C} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
\rho_* & & \downarrow & & \downarrow & & \rho & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}
\]

thus allows to extend \( \rho_* \) to an homomorphism of pro-algebras \( \hat{T}\mathcal{C} \to \hat{\mathcal{H}} \).

**Lemma 6.7** The linear map \( \chi(\rho_*, d) \in \text{Hom}(\hat{T}\mathcal{C}, X(\hat{\mathcal{H}}_\text{top}, d)) \) defined on any \( n \)-form \( \hat{c}_0 \hat{d}_1 \ldots \hat{d}_n \) by

\[
\chi(\rho_*, d)(\hat{c}_0 \hat{d}_1 \ldots \hat{d}_n) =
\]

\[
\frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{i(n-i)} d\rho_*(\hat{c}_{i+1}) \ldots d\rho_*(\hat{c}_n) \rho_*(\hat{c}_0) d\rho_*(\hat{c}_1) \ldots d\rho_*(\hat{c}_{i})
\]

\[
+ \frac{1}{n!} \sum_{i=1}^{n} \sum_{j=1}^{i} \rho_*(\hat{c}_0) d\rho_*(\hat{c}_1) \ldots d\rho_*(\hat{c}_i) \ldots d\rho_*(\hat{c}_{n})
\]

is a chain map from the \((b + B)\)-complex of noncommutative differential forms to the DG \( X \)-complex. Moreover the cohomology class of \( \chi(\rho_*, d) \) in the Hom-complex \( \text{Hom}(\hat{T}\mathcal{C}, X(\hat{\mathcal{H}}_\text{top}, d)) \) is independent of any choice concerning the homomorphism \( \rho \).

**Proof:** A routine computation shows that \( \chi(\rho_*, d) \) is a chain map. The independence of its cohomology class upon the choice of homomorphism \( \rho \) is a classical homotopy argument using \( 2 \times 2 \) rotation matrices. \( \blacksquare \)

**Proposition 6.8** Let \( G \rightrightarrows B \) be a Lie groupoid and let \( O \subset G \) be an Ad-invariant isotropic submanifold. Any geometric cocycle \((N, E, \Phi, c)\) localized at \( O \) defines a class \([N, E, \Phi, c] \in HP_\text{top}(C^\infty_c(B) \rtimes G|_O)\), represented by the composition of chain maps

\[
\begin{array}{cccc}
X(\hat{T}\mathcal{C}) & \gamma & \longrightarrow & \hat{T}\mathcal{C} & \chi(\rho_*, d) & \longrightarrow & X(\hat{\mathcal{H}}_\text{top}, d) & \lambda(c) & \longrightarrow & \mathcal{C}
\end{array}
\]

where \( \gamma \) is the generalized Goodwillie equivalence, \( \mathcal{C} = C^\infty_c(B) \rtimes G \) is the convolution algebra of \( G \), and \( (\hat{\mathcal{H}}_\text{top}, d) \) is the DG pro-algebra constructed above from the geometric cocycle. \( \blacksquare \)

## 7 Localization at units

Let \( G \rightrightarrows B \) be a Lie groupoid and \((N, E, \Phi, c)\) a geometric cocycle localized at units. Hence \( \nu : N \to B \) is a surjective submersion, \( \eta : E \to N \) is a \( \nu^* G \)-equivariant submersion, \( \Phi \rightrightarrows F \) is a flat connection on the action groupoid \( \Gamma = E \rtimes \nu^* G \), and \( c \in C^\infty_0(\nu^* F^*) \) is a normalized cocycle. Throughout this
section we assume that $E$ is a proper $\nu^*G$-manifold. Let $\pi: M \to B$ be a $G$-equivariant submersion. We make no properness hypothesis about the action of $G$ on $M$. Then the algebra bundle of vertical symbols $CS_\nu(M)$ over $B$ is a $G$-bundle. Its pullback $CS_\nu(N \times B)$ under the submersion $\nu$ is a bundle over $N$, whose fibers are isomorphic to the same algebras of vertical symbols. The pullback groupoid $\nu^*G \rightrightarrows N$ acts naturally on $CS_\nu(N \times B)$. By hypothesis the action of $\nu^*G$ on $N$ also lifts to $E$, hence the pullback of $CS_\nu(N \times B)$ under the submersion $\eta$ yields a $\Gamma$-bundle $CS_\nu(E \times B)$ over $E$. The vector bundle $F \subset TE$ being also a $\Gamma$-bundle, we can form the convolution algebra

$$\mathcal{O} = C^\infty_p(E, \Lambda^* F^* \otimes CS_\nu(E \times B M)) \rtimes \Gamma,$$  \hspace{1cm} (89)$$

which is a symbol-valued generalization of the algebra $\mathcal{O}$ of section 6. The product on $\mathcal{O}$ is formally identical to (84), involving the algebra structure of the bundle $\Lambda^* F^* \otimes CS_\nu(E \times B M)$ together with the linear isomorphism $U_\gamma: (\Lambda^* F^* \otimes CS_\nu(E \times B M))_{\pi(\gamma)} \to (\Lambda^* F^* \otimes CS_\nu(E \times B M))_{\pi(\gamma)}$ for all $\gamma \in \Gamma$. The algebra $\mathcal{O}$ is naturally graded by the form degree in $\Lambda^* F^*$. Let $(id, \pi)_*$ be the tangent map of the submersion $(id, \pi): E \times B M \to E$. The preimage of the integrable subbundle $F \subset TE$ is an integrable subbundle $(id, \pi)^{-1}(F)$ of $T(E \times B M)$ defining a foliation on $E \times B M$. Choose an horizontal distribution $H$ in this subbundle, that is, a decomposition $(id, \pi)^{-1}(F) = H \oplus \ker(id, \pi)_*$. By construction the groupoid $H$ acts on $(id, \pi)^{-1}(F)$, and by properness we can assume that $H$ is $\Gamma$-invariant if necessary. Let $C^\infty_c(E \times B M) \to E$ be the bundle over $E$ whose fibers are smooth vector fields with compact support. We can identify this bundle with $PS^0(E \times B M) \to E$, the polynomial vertical symbols of order 0. Combining the distribution $H$ with the leafwise de Rham differential $d_F: C^\infty_c(E, \Lambda^m F^*) \to C^\infty_c(E, \Lambda^{m+1} F^*)$ yields a “foliated” connection on this bundle, in the sense of a linear map

$$d_H: C^\infty_c(E, \Lambda^m F^* \otimes C^\infty_c(E \times B M)) \to C^\infty_c(E, \Lambda^{m+1} F^* \otimes C^\infty_c(E \times B M))$$

which is a derivation of $C^\infty_c(E)$-modules. In general the subbundle $H$ is not integrable and $d_H$ does not square to zero. Its curvature

$$(d_H)^2 = 0 \in C^\infty_c(E, \Lambda^2 F^* \otimes CS^1_c(E \times B M))$$

is a $\Gamma$-invariant leafwise 2-form over $E$ with values in vertical vector fields. By definition the bundle of vertical symbols $CS_\nu(E \times B M)$ acts by endomorphisms on the bundle $C^\infty_c(E \times B M)$. Hence the graded commutator $d_H = [d_H, ]$ is a graded derivation on the algebra of sections $\Lambda^\infty_c(E, \Lambda^* F^* \otimes CS_\nu(E \times B M))$, with curvature $(d_H)^2 = [\theta, ]$. Combining further $d_H$ with the flat connection

$$d_{A^* \Gamma}: C^\infty_c(E, |\Lambda^{max} A^* \Gamma|) \to C^\infty_c(E, F^* \otimes |\Lambda^{max} A^* \Gamma|)$$

as in section 6, we get a derivation (still denoted by $d_H$) on the algebra $\mathcal{O}$. Then $(d_H)^2$ still acts by the commutator $[\theta, ]$, where $\theta$ is viewed as a multiplier of $\mathcal{O}$. In order to deal with cyclic cohomology we construct an extension of this algebra. Define the vector space $\mathcal{P} = \bigoplus_{n \geq 1} \mathcal{P}_n$, where

$$\mathcal{P}_n = C^\infty_p(\Gamma^{(n)}, r_1^* (\Lambda^* F^* \otimes CS_\nu(E \times B M)) \otimes r_n^* |\Lambda^{max} A^* \Gamma| \otimes \ldots \otimes r_n^* |\Lambda^{max} A^* \Gamma|)$$

and $r_1: \Gamma^{(n)} \to E$ is the rank map $(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_i$. The component $\mathcal{P}_1$ is isomorphic, as a vector space, to $\mathcal{O}$. The product of two elements $\alpha_1 \in \mathcal{P}_n$, 

\[47\]
and $\alpha_2 \in \mathcal{P}_{n_2}$ is the element $\alpha_1 \alpha_2 \in \mathcal{P}_{n_1 + n_2}$ defined by
\[
(\alpha_1 \alpha_2)(\gamma_1, \ldots, \gamma_{n_1 + n_2}) = \alpha_1(\gamma_1, \ldots, \gamma_{n_1}) \wedge U_{\gamma_{n_1+1}, \ldots, \gamma_{n_1+n_2}} \alpha_2(\gamma_{n_1+1}, \ldots, \gamma_{n_1+n_2})
\]
We equip $\mathcal{P}$ with the grading induced by $\Lambda^*F$. Then $\mathcal{P}$ is a symbol-valued generalization of the algebra $\mathcal{H}$ of section 6. One has a multiplication homomorphism $m : \mathcal{P} \to \mathcal{O}$, and the derivation $\tilde{d}_H$ on $\mathcal{O}$ extends in a unique way to a derivation on the algebra $\mathcal{P}$. We let $\mathcal{D}$ be the ideal $\text{Ker}(m)$ and denote as usual by $\hat{\mathcal{P}}$ the $\mathcal{D}$-adic completion of $\mathcal{P}$. Hence $\hat{\mathcal{P}}$ is a graded pro-algebra, endowed with a derivation $d_H$ of degree 1.

Let $\mathcal{A} = C^\infty(S^*_p M) \times G$. We want to construct an homomorphism from $\hat{\mathcal{P}} \to \mathcal{A}$. Using a cut-off function on the submersion $\nu : N \to B$, we know that the Morita equivalence between the groupoids $G$ and $\nu^*G$ is realized by an homomorphism $C^\infty_c(B) \times G \to C^\infty_c(N) \times \nu^*G$ of the corresponding convolution algebras. Using the same cut-off function, we get an homomorphism
\[
\mathcal{E} = C^\infty_c(B, CL^0(M)) \times G \to C^\infty_c(N, CL^0_c(N \times B M)) \times \nu^*G .
\]

Then as done in section 6 the $\Gamma$-equivariant map $\eta : E \to N$ induces a pullback homomorphism $C^\infty_c(N, CL^0_c(N \times B M)) \times \nu^*G \to C^\infty_c(E, CL^0_c(E \times B M)) \times \Gamma$. Taking further the projection of classical pseudodifferential operators onto formal symbols one is left with an homomorphism
\[
\mu : \mathcal{E} \to C^\infty_c(E, CS^0(E \times B M)) \times \Gamma \subset \mathcal{O} ,
\]
sending the ideal $\mathcal{B} = C^\infty_c(B, CL^{-1}_c(M)) \times G$ to the convolution algebra of symbols of negative order $C^\infty_c(E, CS^{-1}_c(E \times B M)) \times \Gamma$. By the linear inclusion $\mathcal{O} \hookrightarrow \mathcal{P}_1$ it extends to an homomorphism $\mu_* : T\mathcal{E} \to \mathcal{P}$, sending the ideal $J\mathcal{E}$ to $\mathcal{D}$. Moreover the image of the ideal $I = T(\mathcal{B} : \mathcal{E}) \subset T\mathcal{E}$ contains only symbols of order $\leq -1$ in $\mathcal{P}$. Therefore $\mu_*$ first extends to an homomorphism from $\mathcal{E} = \mathcal{E} / \mathcal{P} \to \mathcal{P}$, and then from $T\mathcal{E}$ to $\hat{\mathcal{P}}$. Composing with the homomorphism $\sigma_* : \hat{\mathcal{P}} \to \hat{\mathcal{E}}$ of section 4, one thus gets a new homomorphism
\[
\sigma'_* = \mu_* \circ \sigma_* : \hat{\mathcal{A}} \to \hat{\mathcal{E}} \to \hat{\mathcal{P}} \tag{90}
\]
Now we twist the algebra $\hat{\mathcal{P}}$ by adding an odd parameter $\epsilon$, with the property $\epsilon^2 = 0$. Let $\hat{\mathcal{P}}[\epsilon]$ be the resulting $\mathbb{Z}_2$-graded algebra: it is linearly spanned by elements of the form $\alpha_0 + \epsilon \alpha_1$ for $\alpha_0, \alpha_1 \in \hat{\mathcal{P}}$, with obvious multiplication rules. Choose a section $Q \in C^\infty_c(E, CL^1(E \times B M))$ of vertical elliptic operators of order one over $E$, with symbol $q \in C^\infty_c(E, CS^1(E \times B M))$. By properness we can even assume that $Q$ and $q$ are $\Gamma$-invariant if necessary. The superconnection
\[
\nabla = d_H + \epsilon \ln q ,
\]
acting by graded commutators, is an odd derivation on $\hat{\mathcal{P}}[\epsilon]$. Indeed $[\nabla, \alpha] = \tilde{d}_H \alpha + \epsilon \ln q, \alpha] \in \hat{\mathcal{P}}[\epsilon]$ for all $\alpha \in \hat{\mathcal{P}}[\epsilon]$. Moreover, any derivative of the logarithmic symbol $\ln q$ being a classical symbol, the curvature of the superconnection
\[
\nabla^2 = \theta - \epsilon \tilde{d}_H \ln q \in C^\infty_c(E, \Lambda^*F^* \otimes CS(E \times B M)[\epsilon])
\]
is a multiplier of $\hat{\mathcal{P}}[\epsilon]$. Let $\hat{\mathcal{H}}$ be the algebra constructed in section 6. From the homomorphism $\sigma'_*$ and the superconnection $\nabla$ we construct a chain map
\( \chi_{\text{Res}}(\sigma', \nabla) \in \text{Hom}(\hat{\Omega}^\ast \mathcal{A}, X(\mathcal{F}_{\text{top}}[\epsilon], d)_{[E]}) \) by means of a JLO-type formula [12]. Since the complex \( X(\mathcal{F}_{\text{top}}[\epsilon], d)_{[E]} \) is localized at units, the Wodzicki residue of vertical symbols gives a linear map
\[
\text{Res} : X(\mathcal{T}[\epsilon]) \to X(\mathcal{F}_{\text{top}}[\epsilon], d)_{[E]}
\]
For any \( n \)-form \( \tilde{a}_0 \tilde{d}a_1 \ldots \tilde{d}a_n \in \Omega^n \mathcal{T} \mathcal{A} \) we set
\[
\chi_{\text{Res}}(\sigma', \nabla)(\tilde{a}_0 \tilde{d}a_1 \ldots \tilde{d}a_n) = \sum_{i=0}^{n} (-1)^{(n-i)} \int_{\Delta_n+1} \text{Res}(e^{-t_i+1} \nabla^2 [\nabla, \sigma'] \ldots e^{-t_n+1} \nabla^2 \sigma' e^{-t_n} [\nabla, \sigma'] \ldots e^{-t_1} \nabla^2) \, dt
\]
where \( \sigma'_i = \sigma'_i(\tilde{a}_i) \) for all \( i \), and \( \Delta_n = \{(t_0, \ldots, t_n) \in [0,1]^n \mid t_0 + \ldots + t_n = 1 \} \) is the standard \( n \)-simplex. This formula makes sense because \( \nabla^2 = \theta - \epsilon d_H \ln q \) is nilpotent as a leafwise differential form of degree \( \geq 1 \) over \( E \). Hence the products under the residue are well-defined elements of \( X(\mathcal{T}[\epsilon]) \), depending polynomially on the simplex variable \( t \). The nilpotency of the curvature also implies that \( \chi_{\text{Res}}(\sigma', \nabla) \) vanishes on \( \Omega^n \mathcal{T} \mathcal{A} \) whenever \( n > \dim F + 2 \). Basic computations show that (93) are the components of a chain map from the \( (b+B) \)-complex \( \hat{\Omega}^\ast \mathcal{A} \) to the \( X \)-complex of the DG algebra \( \mathcal{F}_{\text{top}}[\epsilon] \) localized at \( E \). Now \( \chi_{\text{Res}}(\sigma', \nabla) \) may be expanded as a sum of terms which do not contain \( \epsilon \), plus terms exactly proportional to \( \epsilon \). We define the cocycle \( \chi_{\text{Res}}(\sigma'_0, d_H, \ln q) \in \text{Hom}(\hat{\Omega}^\ast \mathcal{A}, X(\mathcal{F}_{\text{top}}, d)_{[E]}) \) as the coefficient of \( \epsilon \) in the latter expansion, or equivalently as the formal derivative
\[
\chi_{\text{Res}}(\sigma'_0, d_H, \ln q) = \frac{\partial}{\partial \epsilon} \chi_{\text{Res}}(\sigma'_0, d_H + \epsilon \ln q) .
\]
By classical Chern-Weil theory, higher transgression formulas show that the cohomology class of the cocycle \( \chi_{\text{Res}}(\sigma'_0, d_H, \ln q) \) does not depend on the choice of connection \( d_H \) and elliptic symbol \( q \), and is a homotopy invariant of the homomorphism \( \sigma'_0 \).

**Proposition 7.1** Let \( \mathcal{A} \) be the convolution algebra of the action groupoid \( S^+_c M \rtimes G \). Then image of the cyclic cohomology class of a proper geometric cocycle localized at units \( (N, E, \Psi, c) \) under the excision map
\[
\pi^c_G : H^\ast_{\text{top}}(C^\infty_c(B) \rtimes G)_{[B]} \to H^\ast_{\text{top}}(C^\infty_c(S^+_c M \rtimes G)_{[S^+_c M]})
\]
is the cyclic cohomology class over \( \mathcal{A} \) represented by the chain map
\[
\lambda(c) \circ \chi_{\text{Res}}(\sigma'_0, d_H, \ln q) \circ \gamma : X(\mathcal{T}[\epsilon]) \to \hat{\Omega}^\ast \mathcal{A} \to X(\mathcal{F}_{\text{top}}, d)_{[E]} \to \mathbb{C}
\]

**Proof:** Since \( (d_H)^2 = 0 \neq 0 \), \( (\mathcal{T}, \tilde{d}_H) \) is not a differential pro-algebra. We use a trick of Connes ([5] p.229) and add a multiplier \( v \) of \( \mathcal{T} \) of degree 1, with the constraints \( v^2 = 0 = 0 \) \( \alpha_1 \alpha_2 \) 0 for all \( \alpha_1, \alpha_2 \in \mathcal{T} \). Then the algebra \( \mathcal{T}[v] \) generated by \( \mathcal{T} \) and all products with \( v \) is endowed with a canonical differential \( d \) as follows:
\[
da = \tilde{d}_H \alpha + v \alpha + (-1)^{n} \alpha_1 \alpha_2 \, , \quad dv = 0 .
\]
where \( |\alpha| \) is the degree of \( \alpha \in \mathcal{P} \). One easily checks that \( d^2 = 0 \), i.e. \( (\mathcal{P}[v], d) \) is a DG pro-algebra. The homomorphism \( \mu_* : \mathcal{P} \to \mathcal{P} \) and the differential \( d \) on \( \mathcal{P}[v] \) give rise to a cocycle \( \chi(\mu_*, d) \in \text{Hom}(\Omega \mathcal{P}, X(\mathcal{P}[v], d)) \) defined by the same formulas as the cocycle \( \chi(\rho_*, d) \in \text{Hom}(\Omega \mathcal{P}, X(\mathcal{P}, d)) \) of Lemma 6.7. By Remark 4.4, choose a generalized connection on \( \pi : M \to B \) by fixing some horizontal distribution \( H' \subset TM \). Using this generalized connection the residue morphism (76) extends in an obvious fashion to a morphism

\[
Res : \Omega \mathcal{E} \to (\Omega \mathcal{E}_{\text{top}})_{|B}
\]

On the other hand, the pullback of \( H' \) on the submersion \( E \times_B M \to E \) yields a compatible horizontal distribution \( H'' \subset T(E \times_B M) \). Choose \( H = H'' \cap (\text{id}, \pi)^{-1}(F) \) as horizontal distribution defining the foliated connection \( d_H \) (Remark that the latter is generally not \( \Gamma \)-invariant), and thus also the DG algebra \((\mathcal{P}[v], d)\). The residue map (92) extends to a morphism of DG-algebra \( X \)-complexes \( Res : X(\mathcal{P}[v], d) \to X(\mathcal{H}_{\text{top}}, d)_{|E} \) in an obvious fashion. Then a tedious computation shows that one has a commutative diagram of chain maps

\[
\xymatrix{ \Omega \mathcal{E} \ar[r]^{\chi(\mu_*, d)} \ar[rd]_{Res} & X(\mathcal{P}[v], d) \ar[d]^{Res} \\
(\Omega \mathcal{E}_{\text{top}})_{|B} \ar[r]_{\chi(\rho_*, d)} & X(\mathcal{H}_{\text{top}}, d)_{|E} }
\]

Replacing the differential \( d \) by the superconnection \( \nabla_1 = d + v \) acting by commutators on \( \mathcal{P}[v] \), a JLO-type formula as (93) gives a cocycle \( \chi(\mu_*, \nabla_1) \in \text{Hom}(\Omega \mathcal{E}, X(\mathcal{P}[v], d)) \). By a classical transgression formula, the linear homotopy between \( d \) and \( \nabla_1 \) shows that the cocycles \( \chi(\mu_*, d) \) and \( \chi(\mu_*, \nabla_1) \) are cohomologous. Now we proceed as in section 5 and enlarge the complexes \( \Omega \mathcal{E} \) and \( X(\mathcal{P}[v], d) \) by allowing the presence of log-polyhomogeneous pseudodifferential operators. Thus let \( \Omega \mathcal{E}^1_{\text{log}} \) and \( X(\mathcal{P}[v], d)^1_{\text{log}} \) be the complexes containing at most one power of the logarithm \( \ln Q \). The cocycles \( \chi(\mu_*, d) \) and \( \chi(\mu_*, \nabla_1) \) extend to cohomologous cocycles in \( \text{Hom}(\Omega \mathcal{E}^1_{\text{log}}, X(\mathcal{P}[v], d)^1_{\text{log}}) \). Also the above residue morphisms extend to morphisms

\[
\text{Res} : \Omega \mathcal{E}^1_{\text{log}} \cap \text{Dom}(\text{Res}) \to (\Omega \mathcal{E}_{\text{top}})_{|B},
\]

\[
\text{Res} : X(\mathcal{P}[v], d)^1_{\text{log}} \cap \text{Dom}(\text{Res}) \to X(\mathcal{H}_{\text{top}}, d)_{|E},
\]

where the domains \( \text{Dom}(\text{Res}) \) are linearly generated by differences of chains for which only the place of \( \ln Q \) changes. Let \( t \in [0, 1] \) be a parameter, and denote by \( \Omega[0, 1] \) the de Rham complex of differential forms over the interval, with differential \( dt \). Using the superconnection

\[
\nabla_2 = d + dt + v + t \epsilon \ln Q
\]

we view the corresponding JLO cocycle \( \chi(\mu_*, \nabla_2) \) in the complex \( \Omega[0, 1] \otimes \text{Hom}(\Omega \mathcal{E}, X(\mathcal{P}[v][\epsilon], d)^1_{\text{log}}) \). Define the eta-cochain

\[
\eta(\mu_*, \nabla_2) = -\frac{\partial}{\partial \epsilon} \int_{t=0}^{1} \chi(\mu_*, \nabla_2)
\]

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By construction the cocycle $\partial, \eta$ of the residue morphism, and the cocycle $\gamma_{\mu^*,|1}$ Then define a linear map $\chi_{\mu^*,|1}$ in Hom($\tilde{\Omega} T^c, \tilde{\Omega} T^c_{\text{log}}$) of the product with the left multiplier $\ln Q$ as $\ln Q \cdot \hat{c} = (\ln Q \cdot e_1) \otimes \ldots \otimes (\ln Q \cdot e_n)$. Then define a linear map $\psi \in \text{Hom}(\tilde{\Omega} T^c, \tilde{\Omega} T^c_{\text{log}})$ as follows:

$$\psi(\hat{c}_0 d\hat{c}_1 d\hat{c}_2 \ldots d\hat{c}_n) = \hat{c}_0 (\ln Q \cdot \hat{e}_1) d\hat{c}_2 \ldots d\hat{c}_n - \hat{c}_0 \hat{e}_1 d(\ln Q \cdot \hat{e}_2) \ldots d\hat{c}_n$$

on any $n$-form, $n \geq 1$, and $\psi(\hat{c}_0) = \ln Q \cdot \hat{e}_0$. The commutator of $\psi$ with the Hochschild boundary map $b$ on both $\tilde{\Omega} T^c$ and $\tilde{\Omega} T^c_{\text{log}}$ is the degree -1 map

$$[b, \psi](\hat{c}_0 d\hat{c}_1 d\hat{c}_2 \ldots d\hat{c}_n) = \hat{c}_0 (\ln Q \cdot \hat{e}_1) d\hat{c}_2 \ldots d\hat{c}_n - \hat{c}_0 \hat{e}_1 d(\ln Q \cdot \hat{e}_2) \ldots d\hat{c}_n$$

for $n \geq 2$, and $[b, \psi](\hat{c}_0 d\hat{c}_1) = \hat{c}_0 (\ln Q \cdot \hat{e}_1) - (\ln Q \cdot \hat{e}_0) \hat{e}_1$. Similarly $[B, \psi]$ is a simple algebraic expression involving differences of pairs of terms where only $\ln Q$ moves. Hence the range of $[\partial, \psi]$, with $\partial = b + B$ the total boundary of the cyclic bicomplex, is actually contained in the domain of the residue morphism. Therefore $[\partial, \psi] \in \text{Hom}(\tilde{\Omega} T^c, \tilde{\Omega} T^c_{\text{log}} \cap \text{Dom(Res)})$ is a (non-trivial) cocycle. The diagram

$$\begin{array}{c}
\tilde{\Omega} T^c \\
\psi \downarrow \\
\tilde{\Omega} T^c_{\text{log}} \xrightarrow{\chi_{\mu^*,|1}} X(\mathcal{P}[v], d)_{\text{log}}^1
\end{array}$$

is not commutative, but the only difference between $\eta_{\mu^*,|1}$ and $\chi_{\mu^*,|1} \circ \psi$ is that $\ln Q$ does not appear at the same places. Hence the range of the difference $\eta_{\mu^*,|1} - \chi_{\mu^*,|1} \circ \psi$ is contained in the domain $X(\mathcal{P}[v], d)_{\text{log}}^1 \cap \text{Dom(Res)}$ of the residue morphism, and the cocycle $[\partial, \eta_{\mu^*,|1}]$ is homologous to the cocycle $\chi_{\mu^*,|1} \circ [\partial, \psi]$ in $\text{Hom}(\tilde{\Omega} T^c, X(\mathcal{P}[v], d)_{\text{log}}^1 \cap \text{Dom(Res)})$. Finally, the equality $\text{Res} \circ \chi_{\mu^*,|1} = \chi_{\mu^*,|1} \circ \text{Res}$ extends to an equality in $\text{Hom}(\tilde{\Omega} T^c_{\text{log}} \cap \text{Dom(Res)}, X(\mathcal{P}_{\text{top}}, d)_{[E]}).$ Collecting everything, we get a diagram of chain maps

$$\begin{array}{c}
X(\mathcal{T}^c) \\
\xrightarrow{\sigma^* \cdot \gamma} \\
\xrightarrow{[\partial, \psi]} \\
\xrightarrow{[\partial, \eta_{\mu^*,|1}]} \\
\tilde{\Omega} T^c_{\text{log}} \cap \text{Dom(Res)} \\
\xrightarrow{\chi_{\mu^*,|1}} X(\mathcal{P}[v], d)_{\text{log}}^1 \cap \text{Dom(Res)} \\
\xrightarrow{\text{Res}} \\
X(\mathcal{E}_{\text{top}})[E] \\
\xrightarrow{p_X} \\
\tilde{\Omega} T^c_{\text{top}}[E] \\
\xrightarrow{\chi_{\mu^*,|1}} X(\mathcal{E}_{\text{top}}[E])_{\text{log}} \xrightarrow{\lambda(e)} \mathbb{C}
\end{array}$$

which is commutative up to homotopy. The bottom left arrow $p_X$ is a homotopy equivalence, with inverse given by the map $\gamma : X(\mathcal{E}_{\text{top}}[E]) \to \tilde{\Omega} T^c_{\text{top}}[E]$. By construction the cocycle $\varphi \in \text{Hom}(X(\tilde{\mathcal{E}}_{\text{top}}), \mathbb{C})$ representing $[N, E, \Psi, c]$ is the composition of the bottom arrows. By Theorem 5.6, the cocycle $\pi_G(\varphi) \in$
Hom(X(\widehat{T}M), \mathbb{C}) is the path surrounding the diagram via the bottom left corner, while (95) is the upper path.

Let \( \Lambda^* T^*_E M \otimes \mathbb{C} \) be the (complexified) exterior algebra over the vertical cotangent bundle associated to the submersion \( \pi \), and denote by \( V \) its pullback under the projection \( E \times_B M \to E \). Then \( V \) is a \( \mathbb{Z}_2 \)-graded complex vector bundle over \( E \times_B M \), and its algebra of smooth sections, which is a quotient of the algebra of all differential forms over \( E \times_B M \), may be called “vertical” differential forms. The algebra bundle of (non-compactly supported) vertical scalar symbols \( CS(E \times_B M) \) can be enlarged to a \( \mathbb{Z}_2 \)-graded algebra bundle \( CS(E \times_B M, V) \) over \( E \), whose fiber is the algebra of vertical pseudodifferential symbols acting on the smooth sections of \( V \). Let \( PS(E \times_B M, V) \) be the subbundle of polynomial symbols, i.e. the symbols of vertical differential operators acting on the smooth sections of \( V \). Since \( CS(E \times_B M, V) \) is an algebra bundle, there is a left representation \( L : CS(E \times_B M, V) \to \text{End}(CS(E \times_B M, V)) \) and a right representation \( R : CS(E \times_B M, V)^{op} \to \text{End}(CS(E \times_B M, V)) \) as endomorphisms, and the two actions commute in the graded sense. In particular the graded tensor product \( CS(E \times_B M, V) \otimes PS(E \times_B M, V)^{op} \) is naturally represented in the endomorphism bundle. We let

\[
\mathcal{L}(E \times_B M) = \text{Im} \left( CS(E \times_B M, V) \otimes PS(E \times_B M, V)^{op} \to \text{End}(CS(E \times_B M, V)) \right)
\]

be the range of this representation. Hence \( \mathcal{L}(E \times_B M) \) is a \( \mathbb{Z}_2 \)-graded algebra bundle over \( E \), whose fiber is a certain algebra of linear operators acting on vertical symbols. To become familiar with these objects we introduce a local foliated coordinate system \( (z^a, y^\mu) \) on an open subset \( U \subset E \), such that \( (y^\mu)_{\mu=1,2,...} \) are the coordinates along the leaves of the foliation \( F \), and \( (z^a)_{a=1,2,...} \) are transverse coordinates. We complete this system with vertical coordinates \( (x^i)_{i=1,2,...} \) on the fibers of \( M \), so that \( (z^a, y^\mu, x^i) \) is a local foliated coordinate system on \( E \times_B M \). The vertical vectors are generated by the partial derivatives \( \partial/\partial x^i \), and the smooth sections of the vector bundle \( V \) (the vertical differential forms) are generated by products of one-forms \( dx^i \) modulo horizontal forms. Denote by \( ip_i \), with \( i = \sqrt{-1} \), the Lie derivative of vertical differential forms with respect to the vector field \( \partial/\partial x^i \). Viewing the coordinate \( x^i \) as multiplication operator by the function \( x^i \), these operators of even degree fulfill the canonical commutation relations

\[
[x^i, x^j] = 0, \quad [x^i, p_j] = i\delta^i_j, \quad [p_i, p_j] = 0,
\]

and of course all commutators with the basic coordinates \( z^a, y^\mu \) vanish. Let \( \psi^i \) denote the operator of exterior product from the left by \( dx^i \) (modulo horizontal forms) on vertical differential forms, and \( \overline{\psi}_i \) the operator of interior product by the vector field \( \partial/\partial x^i \). Then \( \psi, \overline{\psi} \) are operators of odd degree and fulfill the canonical anticommutation relations

\[
[\psi^i, \psi^j] = 0, \quad [\psi^i, \overline{\psi}_j] = \delta^i_j, \quad [\overline{\psi}_i, \overline{\psi}_j] = 0
\]

where the commutators are graded. Thus in local coordinates a polynomial section \( a \in C^\infty(E, PS(E \times_B M, V)) \) of the bundle of symbols is a smooth function of \( (z, y, x, p, \psi, \overline{\psi}) \) depending polynomially on \( p \). Since \( \psi \) and \( \overline{\psi} \) are odd coordinates any smooth function of them is also automatically polynomial. In the same way, a section \( a \in C^\infty(E, CS(E \times_B M, V)) \) of order \( m \) is locally an
asymptotic expansion \( a \sim \sum_{j \geq 0} a_{m-j} \) where each \( a_{m-j} \) is a smooth function of \((z,y,x,p,\psi,\overline{\psi})\) homogeneous of degree \( m-j \) in \( p \). One then shows ([22]) that the sections \( s \in C^\infty(E, \mathcal{L}(E \times_B M)) \), which act by linear operators on \( C^\infty(E, \mathcal{CS}(E \times_B M, V)) \), are asymptotic expansions of the partial derivatives \( \partial/\partial x \) and \( \partial/\partial p \) of the form

\[
s = \sum_{|\alpha|=0}^k \sum_{|\beta|=0}^\infty \sum_{|\gamma|=0}^\infty \sum_{|\delta|=0}^\infty (s_{\alpha,\beta,\gamma,\delta})_L \gamma^\gamma_R (\overline{\psi})^\alpha_R \left( \frac{\partial}{\partial x} \right)^\alpha \left( \frac{\partial}{\partial p} \right)^\beta \tag{96}
\]

where \( \alpha, \beta, \gamma, \delta \) are multi-indices, and \( s_{\alpha,\beta,\gamma,\delta} \) is a local section of \( \mathcal{CS}(E \times_B M, V) \). Let \( \epsilon \) be an indeterminate (of even parity, not to be confused with the previous odd parameter \( \epsilon \)) and consider the \( \mathbb{Z}_2 \)-graded algebra bundle of formal power series \( \mathcal{D}(E \times_B M) = \mathcal{L}(E \times_B M) [[\epsilon]] \) in \( \epsilon \). One defines a filtered, \( \mathbb{Z}_2 \)-graded algebra sub-bundle

\[
\mathcal{D}(E \times_B M) = \bigcup_{m \in \mathbb{R}} \mathcal{D}^m(E \times_B M) \subset \mathcal{D}(E \times_B M) \tag{97}
\]

as follows: a formal series \( s = \sum_{k \geq 0} s_k \epsilon^k \) is a section of \( \mathcal{D}^m(E \times_B M) \) if each coefficient \( s_k \) is locally an asymptotic expansion (96), where the symbol \( s_{\alpha,\beta,\gamma,\delta} \) has order \( \leq m + (k + |\beta| - 3|\alpha|)/2 \). According to this filtration, the partial derivatives \( \partial/\partial x \) and \( \partial/\partial p \) have degree \( m = 3/2 \), the partial derivatives \( \partial/\partial p \) and \( \epsilon \) both have degree \( m = -1/2 \), and a symbol \( a \in C^\infty(E, \mathcal{CS}^m(E \times_B M, V)) \) in the left representation \( a_L \) has degree \( m \). Following [22] Definition 5.1, a generalized Dirac operator as an odd section \( D \in C^\infty(E, \mathcal{D}(E \times_B M)) \) which in local coordinates reads

\[
D = i \epsilon (\psi^i)_R \left( \frac{\partial}{\partial x^i} + \ldots \right) + (\overline{\psi}_i)_R \left( \frac{\partial}{\partial p_i} + \ldots \right)
\]

where the dots are operators of lower degree (according to the filtration of \( \mathcal{D}(E \times_B M) \)) given by expansions in powers of the partial derivatives \( \partial/\partial p \). Summation over the repeated indices \( i \) is understood. Using a partition of unity one shows that such operators always exist globally on \( E \times_B M \). More importantly, the general form of a Dirac operator above is preserved under any change of local coordinates compatible with the submersion. This makes the present formalism particularly well-adapted to groupoid actions. The square of \( D \) is a generalized Laplacian taking locally the form

\[
\Delta = -D^2 = i \epsilon \frac{\partial}{\partial x^i} \frac{\partial}{\partial p_i} + \ldots
\]

Due to the presence of an overall factor \( \epsilon \) in the Laplacian, the heat operator \( \exp(-D^2) \in C^\infty(E, \mathcal{D}(E \times_B M)) \) is a well-defined formal power series. Moreover, a Duhamel-like expansion holds for the heat operator of perturbed Laplacians [22]. Then let \( \mathcal{D}(E \times_B M) \) be the vector subbundle of \( \mathcal{D}(E \times_B M) \) whose sections are of the form \( s \exp(-D^2) \), for all sections \( s \in C^\infty(E, \mathcal{D}(E \times_B M)) \) with compact vertical support. One shows as in [22] that \( \mathcal{D}(E \times_B M) \) is a \( \mathbb{Z}_2 \)-graded \( \mathcal{D}(E \times_B M) \)-bimodule, and that there exists a canonical graded trace

\[
\text{Tr}_s : C^\infty(E, \mathcal{D}(E \times_B M)) \to C^\infty(E) \tag{98}
\]
coming from the fiberwise Wodzicki residue. \( \mathcal{S}(E \times_B M) \) is called the bimodule of trace-class operators.

In local coordinates, the horizontal distribution \( H \) associated to a choice of decomposition \( (\text{id}, \pi)^{-1}(F) = H \oplus \text{Ker}(\text{id}, \pi) \) is the intersection of the kernels of the collection of 1-forms \( dx^i - \omega^i_\mu dy^\mu \) (summation over repeated indices), where \( \omega^i_\mu \) are scalar functions over \( E \times_B M \). The associated connection \( d_H \) on the bundle of vertical scalar functions is locally expressed by

\[
d_H = dy^\mu \left( \frac{\partial}{\partial y^\mu} + i \omega^i_\mu \partial_i \right) \quad \text{on} \quad C^\infty(E, \Lambda^* \mathcal{O} \otimes C^\infty_c(E \times_B M)).
\]

The curvature of \( d_H \) is the horizontal 2-form \( \theta = \frac{1}{2} dy^\mu \wedge dy^\nu \theta^i_{\mu\nu} \) with values in vertical vector fields, whose components read

\[
\theta^i_{\mu\nu} = \frac{\partial \omega^i_\nu}{\partial y^\mu} + \omega^i_{\mu\nu}.
\]

Now we promote \( d_H \) to a derivation on all vertical differential forms. The new local expression has to be modified as follows:

\[
d_H = dy^\mu \frac{\partial}{\partial y^\mu} + dy^\mu \left( i \omega^i_\mu \partial_i + \frac{\partial \omega^i_\mu}{\partial x^j} \psi^j \overline{\psi}_i \right) - \frac{1}{2} dy^\mu \wedge dy^\nu (\theta^i_{\mu\nu} \overline{\psi}_i).
\]

Note that \( i \omega^i_\mu \partial_i + \frac{\partial \omega^i_\mu}{\partial x^j} \psi^j \overline{\psi}_i \) is the Lie derivative of vertical differential forms with respect to the vector field \( \omega^i_\mu \partial_i \), while \( \theta^i_{\mu\nu} \overline{\psi}_i \) is the interior product by the vector field \( \theta^i_{\mu\nu} \). Both are smooth sections of the bundle \( \text{PS}^1(E \times_B M, V) \) of vertical polynomial symbols (i.e. differential operators) locally defined over \( U \). The action by commutator \( \tilde{d}_H = [d_H, \cdot] \) is an odd derivation on the space of sections \( C^\infty(E, \Lambda^* \mathcal{O} \otimes \mathrm{CS}(E \times_B M, V)) \), explicitly

\[
\tilde{d}_H = dy^\mu \frac{\partial}{\partial y^\mu} + dy^\mu \left( i \omega^i_\mu \partial_i + \frac{\partial \omega^i_\mu}{\partial x^j} \psi^j \overline{\psi}_i \right)_L - dy^\mu \left( i \omega^i_\mu \partial_i + \frac{\partial \omega^i_\mu}{\partial x^j} \psi^j \overline{\psi}_i \right)_R
\]

\[
-\frac{1}{2} dy^\mu \wedge dy^\nu (\theta^i_{\mu\nu} \overline{\psi}_i)_L + \frac{1}{2} dy^\mu \wedge dy^\nu (\theta^i_{\mu\nu} \overline{\psi}_i)_R.
\]

Except for the derivative term \( dy^\mu \frac{\partial}{\partial y^\mu} \), all other terms are local sections of \( \Lambda^* \mathcal{O} \otimes \mathcal{D}(E \times_B M) \) over \( U \). Hence using the formalism of superconnections we can add genuine global sections of this algebra bundle to \( \tilde{d}_H \). We consider the family of Dirac superconnections

\[
\mathcal{D} = i\epsilon(\tilde{d}_H + A) + D,
\]

where \( A \in C^\infty(E, \Lambda^1 \mathcal{O} \otimes \mathcal{D}(E \times_B M)) \), and \( D \in C^\infty(E, \mathcal{D}(E \times_B M)) \) is a generalized Dirac operator. Since the complementary part of \( D \) has an horizontal form degree \( \geq 1 \), a Duhamel expansion shows that the heat operator \( \exp(-D^2) \in C^\infty(E, \Lambda^* \mathcal{O} \otimes \mathcal{D}(E \times_B M)) \) is a trace-class section. Now observe that the bundles \( \mathcal{D}(E \times_B M) \), \( \mathcal{D}(E \times_B M) \), \( \mathcal{D}(E \times_B M) \), etc... over \( E \) are all \( \Gamma \)-bundles. In particular we can form the convolution algebra

\[
\mathcal{U} = C^\infty_p(E, r^* (\Lambda^* \mathcal{O} \otimes \mathcal{D}_c(E \times_B M))) \rtimes \Gamma.
\]

It naturally inherits \( \mathbb{Z}_2 \)-grading from \( \Lambda^* \mathcal{O} \) and \( \mathcal{D}_c(E \times_B M) \). \( \mathcal{U} \) is a bimodule over the algebra of sections \( C^\infty(E, \Lambda^* \mathcal{O} \otimes \mathcal{D}_c(E \times_B M)) \), and if the horizontal
distribution $H$ is $\Gamma$-invariant, the commutator $[\tilde{d}_H, \cdot]$ defines an odd derivation on $\mathcal{U}$. Following the general recipe we construct an extension of the convolution algebra. Define the vector space $\mathcal{V} = \bigoplus_{n \geq 1} \mathcal{V}_n$, where

$$\mathcal{V}_n = C^\infty_p(\Gamma^{(n)} \cap \mathcal{V}_n), r^n_1(\Lambda^*F^* \otimes \mathcal{D}, (E \times_B M)) \otimes \cdots \otimes r^n_1(\Lambda^\max \Lambda^*F^*)$$

As a vector space $\mathcal{V}_1$ is isomorphic to $\mathcal{U}$. A product $\mathcal{V}_{n_1} \otimes \mathcal{V}_{n_2} \rightarrow \mathcal{V}_{n_1+n_2}$ is defined as usual, as well as the multiplication map $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}$. Hence $\mathcal{V}$ is a $\mathbb{Z}_2$-graded extension of $\mathcal{U}$. The commutator $[\tilde{d}_H, \cdot]$ lifts uniquely to an odd derivation on $\mathcal{V}$. More generally the commutator with any Dirac superconnection yields an odd derivation, where $C^\infty(E, \Lambda^*F^* \otimes \mathcal{D}, (E \times_B M))$ acts by multipliers on $\mathcal{V}$ in the obvious way. Finally we denote by $\mathcal{W} = \text{Ker}(\mathcal{V} \otimes \mathcal{V})$ the kernel of the multiplication map, which is stable by $[\tilde{d}_H, \cdot]$, and by $\hat{\mathcal{V}}$ the $\mathcal{W}$-adic completion of $\mathcal{V}$.

Since the space of vertical $0$-forms on $E \times_B M$ is a direct summand in the space of all vertical forms, the algebra bundle of scalar symbols $\text{CS}(E \times_B M)$ sits naturally as an algebra subbundle of $\text{CS}(E \times_B M, V)$. The latter may further be identified with a subbundle of $\mathcal{D}(E \times_B M)$ through the left representation $L$. Hence one gets a canonical inclusion $\text{CS}_L(E \times_B M) \rightarrow \mathcal{D}_L(E \times_B M)$, which in turn induces an homomorphism of pro-algebras $\hat{\mathcal{D}} \rightarrow \hat{\mathcal{V}}$. Composing with the homomorphism $\hat{\sigma}' : \hat{T}_\mathcal{A} \rightarrow \hat{\mathcal{D}}$ constructed above one gets a representation

$$\sigma'' = L \circ \hat{\sigma}' : \hat{T}_\mathcal{A} \rightarrow \hat{\mathcal{D}} \hookrightarrow \hat{\mathcal{V}}.$$  \hspace{1cm} (101)$$

Choose as above an elliptic section $Q \in C^\infty(E, \Lambda^1(E \times_B M))$, with symbol $q \in C^\infty(E, \Lambda^1(E \times_B M))$. Extend $q$ to an elliptic symbol acting on vertical differential forms $\tilde{q} \in C^\infty(E, \Lambda^1(E \times_B M, V))$, requiring that the leading symbol of $\tilde{q}$ remains of scalar type. Let $\epsilon, \epsilon^2 = 0$ be the odd parameter introduced above. For any Dirac superconnection $D$, the new superconnection

$$\nabla = D + \epsilon \ln \tilde{q}$$

acting on the algebra $\hat{\mathcal{V}}[\epsilon]$ by graded commutators is a graded derivation. We use the homomorphism $\sigma''$ and the superconnection $\nabla$ to construct a cocycle $\chi^{\nabla}(\sigma'', \nabla) \in \text{Hom}_0(\Omega\hat{T}_\mathcal{A}, X(\mathcal{H}_{\text{top}}[\epsilon], d)_{[E]} )$ by a JLO-type formula. We first extend the graded trace (98) to an $X$-complex map

$$\text{Tr}_\chi : X(\hat{\mathcal{V}}) \rightarrow X(\mathcal{H}_{\text{top}}[\epsilon], d)_{[E]} ,$$

taking into account a rescaling factor of $(i\epsilon)^{-1}$ for each leafwise form degree in $\mathcal{H}$. Then on any $n$-form $\tilde{a}_0 \tilde{d}_1 \ldots \tilde{d}_n$ set

$$\chi^{\nabla}(\sigma'', \nabla)(\tilde{a}_0 \tilde{d}_1 \ldots \tilde{d}_n) = \sum_{i=0}^{n} (-)^{i(n-i)} \int_{\Delta_{n+1}} \text{Tr}_\chi (e^{-t_{n+1} \nabla^2}[\nabla, \sigma'']_i \ldots e^{-t_n \nabla^2} \sigma''_0 e^{-t_0 \nabla^2}[\nabla, \sigma']_0 \ldots e^{-t_i \nabla^2} dt$$

$$+ \sum_{i=1}^{n} \int_{\Delta_n} \text{Tr}_\chi (\sigma''_0 e^{-t_n \nabla^2}[\nabla, \sigma']_0 \ldots e^{-t_i \nabla^2} \sigma''_i e^{-t_i \nabla^2}[\nabla, \sigma''_0] \ldots e^{-t_i \nabla^2} dt)$$

where $\sigma''_i = \sigma''(\tilde{a}_i) \in \hat{\mathcal{V}}$ for all $i$. As above the $\epsilon$-component of $\chi'(\sigma'', \nabla)$

$$\chi'(\sigma'', D, \ln \tilde{q}_L) = \frac{\partial}{\partial \epsilon} \chi(\sigma'', D + \epsilon \ln \tilde{q}_L)$$  \hspace{1cm} (103)
is a cocycle in $\text{Hom}(\hat{\Omega}\hat{T}\mathcal{A}, X(\mathcal{H}_{\text{top}}, d)_{|E})$, whose cohomology class does not depend on the choice of Dirac superconnection $D$ and elliptic symbol $\hat{q}$. Choosing genuine Dirac superconnections thus allows to build cohomologous cocycles. We now exploit this fact. Let $d_V$ be the vertical part of the de Rham operator on $E \times_\Gamma M$, acting on the vertical differential forms. One thus has $d_V \in C^\infty(E, \text{PS}(E \times_M B, V))$ and in local coordinates

$$d_V = i\psi \partial \psi.$$  

Of course $d_V$ is completely canonical and $d_V^2 = 0$. The choice of horizontal distribution $H$ allows to identify $V$ with a subbundle of $\Lambda^* T^*(E \times B M)$, and according to this identification the total de Rham differential on $E \times B M$ is exactly the sum $d_H + d_V$. Thus in particular $(d_H + d_V)^2 = d_H^2 + |d_H, d_V| = 0$. This can be explicitly checked in local coordinates using the formulas above. Next, taking the image of $d_V$ under the right representation $R$ yields a global section of $\mathcal{D}(E \times B M)$. In local coordinates one has

$$(i\psi \partial \psi)_R = -i(\psi \partial \psi)_L \left( \frac{\partial}{\partial x^i} - (i\psi)_L \right)$$

using the commutation relations. A Dirac operator $D$ is called de Rham-Dirac if it is exactly given by

$$D = i\varepsilon(\psi \partial \psi)_R \left( \frac{\partial}{\partial x^i} - (i\psi)_L \right) + (\overline{\psi})_R \left( \frac{\partial}{\partial p_i} + \ldots \right)$$

where the dots represent an expansion in higher powers of $\frac{\partial}{\partial p}$ ensuring that $D$ is globally defined. Since $d_V$ is completely canonical, the term proportional to $\psi_R$ is always $\Gamma$-invariant. Only the term proportional to $\overline{\psi}_R$ may not be $\Gamma$-invariant, however using the properness of the action we can always find a $\Gamma$-invariant de Rham-Dirac operator.

**Proposition 7.2** Let $D = i\varepsilon \hat{d}_H + D$ be a Dirac superconnection constructed from a $\Gamma$-invariant horizontal distribution $H$ and a $\Gamma$-invariant de Rham-Dirac operator $D$. Let $q, \hat{q}$ be $\Gamma$-invariant elliptic symbols of scalar type. Then one has the equality of cocycles in $\text{Hom}(\hat{\Omega}\hat{T}\mathcal{A}, X(\mathcal{H}_{\text{top}}, d)_{|E})$

$$\chi_{\text{Tr}}(\sigma''_\star, D, \ln \hat{q}_L) = \chi_{\text{Res}}(\sigma'_\star, d_H, \ln q) \quad (104)$$

**Proof:** Let $\Pi \in C^\infty(E, \text{PS}(E \times_B V))$ be the natural projection operator of vertical differential forms onto their 0-degree component (scalar functions). Write $D = -i\varepsilon(d_V)_R + \nabla$ where $\nabla$ contains all the terms proportional to $\overline{\psi}_R$. Since left and right representations commute one has $[\hat{d}_H, \sigma''_\star(\hat{a})] = ([d_H, \sigma'_\star(\hat{a})]\Pi)_L$ and $[D, \sigma''_\star(\hat{a})] = [\nabla, \sigma''_\star(\hat{a})]$ for any $\hat{a} \in \hat{T}A$. Thus

$$[D, \sigma''_\star(\hat{a})] = ([d_H, \sigma'_\star(\hat{a})]\Pi)_L + [\nabla, \sigma''_\star(\hat{a})].$$

On the other hand $D^2 = -\varepsilon^2(\hat{d}_H)^2 + i\varepsilon[\hat{d}_H, D] + D^2$, with $(\hat{d}_H)^2 = \theta_L - \theta_R$ and $i\varepsilon[\hat{d}_H, D] = \varepsilon^2([d_H, d_V])_R + i\varepsilon[d_H, \nabla] = -\varepsilon^2 \theta_R + i\varepsilon[\hat{d}_H, \nabla]$. Hence the Laplacian reads

$$-D^2 = \varepsilon^2 \theta_L - D^2 - i\varepsilon[\hat{d}_H, \nabla]$$

At this point we can apply verbatim the proof of [22] Proposition 6.4 (and 6.7), showing that the terms $[\nabla, \sigma''_\star(\hat{a})]$ and $[d_H, \nabla]$ do not contribute, and that the

$$\chi_{\text{Tr}}(\sigma''_\star, D, \ln \hat{q}_L) = \chi_{\text{Res}}(\sigma'_\star, d_H, \ln q) \quad (104)$$


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effect of the graded trace $\operatorname{Tr}_s$ is to reduce the JLO cocycle $\chi(\sigma', D, \ln q)$ to the JLO cocycle $\chi(\sigma, d_{\mathcal{H}}, \ln q)$.

Choose a torsion-free affine connection on the manifold $E \times_B M$ and take its restriction $\nabla^T$ to the vertical tangent bundle $E \times_B T_z M$. In local coordinates we can write $\nabla^T = dx^a \partial_a + dy^\mu \partial_\mu + dx^i \partial_i$, and because we are interested to differentiate in the directions of the leaves ($z$ fixed) we only retain the covariant derivatives $\nabla^T_\mu$ and $\nabla^T_{ij}$ in the horizontal and vertical directions respectively. Their effect on vertical vector fields is expressed in terms of the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ij}$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$ of the affine connection:

$$\nabla^T_\mu \left( \frac{\partial}{\partial x^j} \right) = \Gamma^k_{\mu j} \frac{\partial}{\partial x^k}, \quad \nabla^T_{ij} \left( \frac{\partial}{\partial x^j} \right) = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

In particular we denote $\nabla^T_\mu \omega^k = \frac{\partial \omega^k}{\partial x^\mu} + \Gamma^k_{ij} \omega^j$ the covariant derivative of the 1-form $\omega^k = \omega^k_i dy^\mu$ defining the horizontal distribution $H$. In local coordinates the curvature of the affine connection acts on vertical vector fields by

$$(\nabla^T)^2 \left( \frac{\partial}{\partial x^j} \right) = R^k_{ij} \frac{\partial}{\partial x^k},$$

where the coefficients $R^k_{ij}$ are leafwise 2-forms over $E \times_B M$. We decompose them into purely horizontal, mixed and purely vertical components

$$R^k_{ij} = \frac{1}{2} R^k_{ilm} dy^\mu \wedge dy^\nu + R^k_{ij} d^\mu \wedge dx^j + \frac{1}{2} R^k_{ij} dx^i \wedge dx^j,$$

where the components of the curvature tensor are expressed as usual via the Christoffel symbols

$$R^k_{ij} = \left[ \nabla^T_\mu, \nabla^T_\nu \right]_l^k = \frac{\partial \Gamma^k_{il}}{\partial y^\nu} - \frac{\partial \Gamma^k_{il}}{\partial y^\nu} + \Gamma^k_{im} \Gamma^m_{ij} - \Gamma^k_{im} \Gamma^m_{il},$$

$$R^k_{ij} = \left[ \nabla^T_\mu, \nabla^T_\nu \right]_l^k = \frac{\partial \Gamma^k_{ij}}{\partial y^\nu} - \frac{\partial \Gamma^k_{ij}}{\partial y^\nu} + \Gamma^k_{jm} \Gamma^m_{ij} - \Gamma^k_{jm} \Gamma^m_{ij},$$

$$R^k_{ij} = \left[ \nabla^T_\mu, \nabla^T_\nu \right]_l^k = \frac{\partial \Gamma^k_{il}}{\partial x^j} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ jm} \Gamma^m_{ij} - \Gamma^k_{ jm} \Gamma^m_{il}.$$

**Definition 7.3** A Dirac superconnection $D = i\varepsilon (\tilde{a}_H + A) + D$ is called affiliated to the affine connection $\nabla^T$ and the horizontal distribution $H$ if in local coordinates

$$A = (\nabla^T_\mu \omega^k + \Gamma^k_{ij} dy^\mu)_L \left( (p_k)_L \frac{\partial}{\partial p_j} + (\psi^\mu)_L + \ldots \right) + (\psi^i)_R \left( \frac{\partial}{\partial x^i} + (\Gamma^k_{ij} \bar{\psi}^\mu)_L + \ldots \right)_L$$

$$D = i\varepsilon (\psi^i)_R \left( \frac{\partial}{\partial x^i} + (\Gamma^k_{ij} \bar{\psi}^\mu)_L + \ldots \right) + \left( \psi^i \right)_R \left( \frac{\partial}{\partial p_i} + \ldots \right)_L$$

where the dots denote an expansion in higher powers of $\frac{\partial}{\partial q}$.

The global existence of such operators $A$ and $D$ on $E \times_B M$ is proved as usual by gluing together local operators by means of a partition of unity. From now on...
on let us abusively denote by \( \pi \) the submersion \( E \times_B S^*_\pi M \to E \), and by \( \pi_* \) its tangent map. The preimage \( \pi^{-1}_*F \) of the subbundle \( F \subset TE \) under the tangent map defines a foliation on \( E \times_B S^*_\pi M \). Pulling back the vertical tangent bundle and its affine connection from \( E \times_B M \) to \( E \times_B S^*_\pi M \), we can view the above curvature \( R \) as a leafwise 2-form on \( E \times_B S^*_\pi M \) with values in the endomorphisms of the vertical tangent bundle. By Chern-Weil theory, the Todd class of the complexified vertical tangent bundle \( \operatorname{Td}((E \times_B T^*_\pi M) \otimes \mathbb{C}) \) is represented by the closed leafwise differential form

\[
\operatorname{Td}(iR/2\pi) = \det \left( \frac{iR/2\pi}{e^{iR/2\pi} - 1} \right) \in C^\infty(E \times_B S^*_\pi M, \Lambda^4 \pi^{-1}_* F), \tag{105}
\]

which is a polynomial in the Pontryagin classes. Next, the submersion \( \pi : E \times_B S^*_\pi M \to E \) being \( \Gamma \)-equivariant, it extends to a morphism of Lie groupoids \( \pi : (E \times_B S^*_\pi M) \times \Gamma \to \Gamma \). Its tangent map \( \pi_* \) is a morphism of the corresponding tangent groupoids. Hence the preimage of the flat connection \( \Phi \subset \Gamma \),

\[
\pi^{-1}_*(\Phi) \subset T((E \times_B S^*_\pi M) \times \Gamma), \tag{106}
\]

is a flat connection on the groupoid \( (E \times_B S^*_\pi M) \times \Gamma \), and the latter acts on leafwise differential forms over \( E \times_B S^*_\pi M \). If we start from a \( \Gamma \)-equivariant affine connection \( \nabla^T \), which is always possible by the properness of the action, then the Todd form \( \operatorname{Td}(iR/2\pi) \in C^\infty(E \times_B S^*_\pi M, \Lambda^4 \pi^{-1}_* F) \) is invariant, and a classical homotopy argument shows that its class in the cohomology of invariant leafwise differential forms is independent of the choice of connection. Also note that one can further assume \( \nabla^T \) to be a \( \Gamma \)-equivariant metric connection, in the sense that it preserves an invariant scalar product on the vertical tangent bundle. Now define the convolution algebra

\[
\mathcal{X} = C^\infty_c(E \times_B S^*_\pi M, \Lambda^\bullet \pi^{-1}_* F) \times \Gamma. \tag{107}
\]

It comes equipped with the grading of leafwise differential forms, and the leafwise de Rham operator \( d \) which squares to zero. Hence \( (\mathcal{X}, d) \) is a DG algebra. As usual we build an extension \( \mathcal{Y} = \bigoplus_{n \geq 1} \mathcal{Y}_n \) with

\[
\mathcal{Y}_n = C^\infty_c((E \times_B S^*_\pi M) \times \Gamma)^{(n)}, \quad r^*_1 \Lambda^\bullet \pi^{-1}_* F^* \otimes r^*_1 [\Lambda^{\text{max}} A^\Gamma | \otimes \ldots \otimes r^*_n [\Lambda^{\text{max}} A^\Gamma])
\]

where for notational simplicity we identify the fiber of the algebroid \( \Lambda \mathcal{G} \) with that of \( A((E \times_B S^*_\pi M) \times \Gamma) \). Then \( \mathcal{Y}_1 = \mathcal{X} \) as a vector space, there is a product \( \mathcal{Y}_n \times \mathcal{Y}_{n_2} \to \mathcal{Y}_{n_1 + n_2} \) and a multiplication map \( \mathcal{Y} \to \mathcal{X} \). Moreover the differential \( d \) on \( \mathcal{X} \) extends uniquely to a differential on \( \mathcal{Y} \). We let \( \mathcal{Z} = \ker(\mathcal{Y} \to \mathcal{X}) \) be the kernel of the multiplication map which is a DG ideal in \( \mathcal{Y} \), and denote by \( \mathcal{Y} \) the \( \mathcal{E} \)-adic completion of \( \mathcal{Y} \). The DG pro-algebra \( (\mathcal{Y}, d) \) is the analogue, for the groupoid \( (E \times_B S^*_\pi M) \times \Gamma \) with connection \( \pi^{-1}_*(\Phi) \), of the DG pro-algebra \( (\mathcal{Y}, d) \) introduced in section 6 for the groupoid \( \Gamma \) with connection \( \Phi \). In order to apply Lemma 6.7 in this context, we need to build an homomorphism from \( T_\mathcal{Y} \) to \( \mathcal{Y} \). To this end, observe that the cut-off function previously chosen on the submersion \( \nu : N \to B \) yields, by pullback to the cosphere bundle, a cut-off function on the submersion \( N \times_B S^*_\pi M \to S^*_\pi M \), whence an homomorphism of algebras

\[
\mathcal{A} = C^\infty_c(S^*_\pi M) \times G \to C^\infty_c(N \times_B S^*_\pi M) \times \nu^* G
\]
realizing the Morita equivalence between the corresponding groupoids. Then as before the \( \Gamma \)-equivariant map \( \eta : E \to N \) induces a pullback homomorphism 
\[
C^\infty_c(N \times_B S^*_\varepsilon M) \times \nu^*G \to C^\infty_b(E \times_B S^*_\varepsilon M) \times \Gamma \subset \mathcal{X}.
\]
The resulting homomorphism \( \rho : \mathcal{A} \to \mathcal{X} \) extends as usual to an homomorphism of pro-algebras
\[
\rho_* : \hat{T}\mathcal{A} \to \hat{T}\mathcal{X}
\]  
(108)
Remark that \( \rho : \mathcal{A} \to \mathcal{X} \) could as well be obtained by composition of the linear map \( \sigma : \mathcal{A} \to \mathcal{E} \) with the homomorphism \( \mu : \mathcal{E} \to \mathcal{E}^0 \) (where the range only contains formal symbols of order \( \leq 0 \)), followed by the projection homomorphism \( \mathcal{E}^0 \to \mathcal{X} \) onto leading symbols (i.e. scalar functions over the cosphere bundle). Hence (108) is the composition of the homomorphism \( \sigma_* : \hat{T}\mathcal{A} \to \hat{T}\mathcal{E} \) with the projection \( \hat{T}\mathcal{E} \to \hat{T}\mathcal{X} \) onto leading symbols. Lemma 6.7 yields a chain map \( \chi(\rho_*,d) \in \text{Hom}(\tilde{\Omega}\tilde{T}\mathcal{A},X(\mathcal{X},d)) \), defined on any \( n \)-form \( \tilde{a}_0d\tilde{a}_1 \cdots d\tilde{a}_n \in \Omega^n\hat{T}\mathcal{A} \) by
\[
\chi(\rho_*,d)(\tilde{a}_0d\tilde{a}_1 \cdots d\tilde{a}_n) = 
\frac{1}{(n+1)!} \sum_{i=0}^n (-1)^{i(n-i)} d\rho_*(\tilde{a}_{i+1}) \cdots d\rho_*(\tilde{a}_n) \rho_*(\tilde{a}_0) d\rho_*(\tilde{a}_1) \cdots d\rho_*(\tilde{a}_i)
+ \frac{1}{n!} \sum_{i=1}^n i(\rho_*(\tilde{a}_0) d\rho_*(\tilde{a}_1) \cdots d\rho_*(\tilde{a}_i)) \cdot d\rho_*(\tilde{a}_i) \cdots d\rho_*(\tilde{a}_n) 
\]
Also note that the wedge product by a closed, invariant, leafwise differential form on \( E \times_B S^*_\varepsilon M \) defines in an obvious way an endomorphism of the localized complex \( X(\mathcal{X}_{\text{top}},d)[E \times_B S^*_\varepsilon M] \). Moreover, integration of differential forms along the fibers of the submersion \( E \times_B S^*_\varepsilon M \to E \) gives rise to a chain map of total complexes
\[
\int_{S^*_\varepsilon M} : X(\mathcal{X}_{\text{top}},d)[E \times_B S^*_\varepsilon M] \to X(\mathcal{X}_{\text{top}},d)[E] .
\]

**Proposition 7.4** Let \( D = i\varepsilon(\tilde{d}_H + A) + D \) be a Dirac superconnection affiliated to a \( \Gamma \)-equivariant metric connection \( \nabla^T \) on the vertical tangent bundle and a \( \Gamma \)-invariant horizontal distribution \( H \). Let \( \tilde{q} \) be a \( \Gamma \)-invariant elliptic symbol of scalar type. Then one has the equality of cocycles in \( \text{Hom}(\tilde{\Omega}\tilde{T}\mathcal{A},X(\mathcal{X}_{\text{top}},d)[E]) \)
\[
\chi^{Tr_c}(\sigma''_*,D,\ln \tilde{q}_L) = \int_{S^*_\varepsilon M} Td(\varepsilon R/2\pi) \wedge \chi(\rho_*,d) ,
\]  
(109)
where \( R \) is the curvature 2-form of \( \nabla^T \).

**Proof:** This is a direct generalization of [22] Theorem 6.5, stating that for Dirac operators affiliated to metric connections the JLO cocycle \( \chi^{Tr_c}(\sigma''_*,D,\ln \tilde{q}_L) \) only involves the leading symbols of its arguments, so has a simple expression as an integral of ordinary differential forms over the cosphere bundle. Indeed one has
\[
[D,\sigma''_*(\tilde{a})] = i\varepsilon dy^\mu \left( \frac{\partial \sigma''_*(\tilde{a})}{\partial y^\mu} \Pi + \Gamma^{k}_{\mu j} p_k \frac{\partial \sigma''_*(\tilde{a})}{\partial p_j} \Pi - \Pi \psi_k \nabla^T_k \omega^k \sigma''_*(\tilde{a}) \right)_L
+ i\varepsilon(\psi' + \omega') R \left( \frac{\partial \sigma''_*(\tilde{a})}{\partial x^i} \Pi + \Gamma^{k}_{ij} p_k \frac{\partial \sigma''_*(\tilde{a})}{\partial p_j} \Pi - \Pi \psi_k (\nabla^T_k \omega^k + \Gamma^{k}_{ij} dy^m) \sigma''_*(\tilde{a}) \right)_L
+ (\psi_i)_R \left( \frac{\partial \sigma''_*(\tilde{a})}{\partial p_i} \Pi \right)_L + \ldots
\]
for any $\tilde{a} \in \tilde{T}\mathcal{A}$, where the dots denote expansions in higher powers of $\partial/\partial p$. On the other hand the Laplacian reads

\[-D^2 = i\epsilon \left( \frac{\partial}{\partial \psi^i} \frac{\partial}{\partial \psi^i} + (\Gamma^k_{ij})_L((\psi^i + \omega^i)\nabla_k)_R \frac{\partial}{\partial \psi^i} + (\Gamma^k_{ij})_L dy^i((\nabla_k)_R + \ldots)\right) + \frac{\epsilon^2}{2} dy^i \wedge dy^j \left( (R^k_{ij})_L \frac{\partial}{\partial \psi^k} + (R^k_{ij})_L ((\psi^j + \omega^j)_L - (\psi^j + \omega^j)_R) + \frac{\epsilon^2}{2} dy^i \wedge (\psi^j + \omega^j)_R \left( (R^k_{ij})_L \frac{\partial}{\partial \psi^k} + (R^k_{ij})_L ((\psi^j + \omega^j)_L + \ldots) \right) \right) \]

The identities $\nabla^T \Pi = \Pi \nabla^T \psi^i$ and $\nabla^T \Pi = 0 = \Pi \nabla^T$ for all indices $k, l$, together with the fact that $\nabla^T$ is a metric connection ($R^k_{ijk} = R^k_{kij} = 0$), allows to show as in [22] Theorem 6.5 that the terms involving $\nabla_k$ do not contribute. The rest of the proof follows exactly the lines of [22] Theorems 6.5 and 6.8.

Now remark that the bicomplex of equivariant leafwise differential forms $C^\bullet((E \times_B S^*_\mathcal{A} M) \times \Gamma, \Lambda^\bullet \pi^\perp F)$ is naturally a module (for the wedge product) over the algebra of closed invariant leafwise differential forms. In particular, if $c \in C^\bullet(\Gamma, \Lambda^\bullet F)$ is a normalized total cocycle, we can take its pullback $\pi^*(c)$ under the projection $\pi : E \times_B S^*_\mathcal{A} M \to E$, and the product

$\text{Td}(iR/2\pi) \wedge \pi^*(c) \in C^\bullet((E \times_B S^*_\mathcal{A} M) \times \Gamma, \Lambda^\bullet \pi^\perp F)$

is a normalized total cocycle. Clearly the latter descends to a cup-product on the corresponding cohomologies. Collecting all the preceding results we obtain

**Theorem 7.5** Let $G \rightrightarrows B$ be a Lie groupoid acting on a surjective submersion $\pi : M \to B$. The excision map localized at units

$\pi^1_G : HP^\bullet(C^\infty_c(B) \rtimes G)_R \to HP^\bullet(C^\infty_c(S^*_\mathcal{A} M \rtimes G))_{S^*_\mathcal{A} M}$

sends the cyclic cohomology class of a proper geometric cocycle $(N, E, \Phi, c)$ to the cyclic cohomology class

$\pi^1_G([N, E, \Phi, c]) = [N \times_B S^*_\mathcal{A} M, E \times_B S^*_\mathcal{A} M, \pi^\perp(\Phi), \text{Td}(T\pi M \otimes \mathcal{C}) \wedge \pi^*(c)]$

where $T\pi(M \otimes \mathcal{C})$ is the Todd class of the complexified vertical tangent bundle in the invariant leafwise cohomology of $E \times_B S^*_\mathcal{A} M$.

**Proof:** By Proposition 7.1 the class $\pi^1_G([N, E, \Phi, c])$ is represented by the cocycle $\lambda(c) \circ \chi^\text{res}(\sigma, dH, \ln q) \circ \gamma$ in Hom($X(T\mathcal{A}, \mathcal{C})$. Then Proposition 7.2 implies that this cocycle is cohomologous to $\lambda(c) \circ \chi^\text{Tr}(\sigma, D, \ln \tilde{q}_L) \circ \gamma$ for any choice of Dirac superconnection $D$. By Proposition 7.4 it is also cohomologous to $\lambda(\text{Td}(T\pi M \otimes \mathcal{C}) \wedge \pi^*(c)) \circ \chi(\rho, d) \circ \gamma$, which is precisely the construction of the class $[N \times_B S^*_\mathcal{A} M, E \times_B S^*_\mathcal{A} M, \pi^\perp(\Phi), \text{Td}(T\pi M \otimes \mathcal{C}) \wedge \pi^*(c)]$.  

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Combining Theorems 5.6 and 7.5 yields a commutative diagram computing explicitly the excision map of the fundamental pseudodifferential extension on the range of proper geometric cocycles localized at units:

\[
\begin{array}{cccc}
HP^\bullet(C^\infty_c(B, \text{CL}^{-1}_e(M)) \rtimes G) & \xrightarrow{E^\tau} & HP^\bullet+1(C^\infty_c(S^*_\pi M) \rtimes G) \\
\downarrow & & \downarrow \\
HP^\bullet_{\text{top}}(C^\infty_c(B) \rtimes G) & \xrightarrow{\tau} & HP^\bullet_{\text{top}}+1(C^\infty_c(S^*_\pi M) \rtimes G)_{[\pi^*_G, M]}
\end{array}
\] (111)

This together with the adjointness theorem in K-theory [17, 21] gives the following

**Corollary 7.6** Let \( P \in M_\infty(\text{CL}^0_e(M) \rtimes G)^+ \) be an elliptic operator and let \((N, E, \Phi, c)\) be a proper geometric cocycle localized at units for \(G\). The K-theoretical index \( \text{Ind}([P]) \in K_0(\text{CL}^{-1}_e(M) \rtimes G) \) evaluated on the cyclic cohomology class \( \tau_{[N, E, \Phi, c]} \) is

\[
\langle \tau_{[N, E, \Phi, c]} \rangle, \text{Ind}([P]) = \\
\langle [N \times_B S^*_\pi M, E \times_B S^*_\pi M, \pi^{-1}(\Phi), \text{Td}(\pi_M \otimes C) \wedge \pi^*(c)] \rangle, [P]\rangle
\]

where \([P] \in K_1(C^\infty_e(S^*_\pi M) \rtimes G)\) is the leading symbol class of \(P\).

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### 8 Foliated dynamical systems

Let \((V, \mathcal{F})\) be a compact foliated manifold without boundary. We recall [5] that its *holonomy groupoid* \(H\) is a Lie groupoid with \(V\) as set of units, and the arrows are equivalence classes of leafwise paths \(\gamma\) from a source point \(x = s(\gamma)\) to a range point \(y = r(\gamma)\) belonging to the same leaf, with the following relation: two leafwise paths \(\gamma\) and \(\gamma'\) from \(x\) to \(y\) are equivalent if and only if they induce the same diffeomorphism (holonomy) from a transverse neighborhood of \(x\) to a transverse neighborhood of \(y\). The composition of arrows is the concatenation product of paths. It is well-known that the holonomy groupoid can be reduced to a Morita equivalent *étale* groupoid upon a choice of complete transversal. Indeed, let \(B \to V\) be an immersion of a closed manifold everywhere transverse to the leaves and intersecting each leaf at least once. The subgroupoid

\[
H_B = \{\gamma \in H \mid s(\gamma) \in B \text{ and } r(\gamma) \in B\}
\] (112)

is étale, i.e. the range and source maps from \(H_B\) to \(B\) are local diffeomorphisms, and is Morita equivalent to the holonomy groupoid. \(H_B\) naturally acts on the submersion \(s : M \to B\), corresponding to the restriction of the source map \(s : H \to V\) to the submanifold

\[
M = \{x \in H \mid s(x) \in B\}.
\] (113)

The right action of \(H_B\) on \(M\) is given by composition of arrows in \(H\): for any \(x \in M\) and \(\gamma \in H_B\) such that \(s(x) = r(\gamma) \in B\), the composite \(x \cdot \gamma\) is indeed in \(M\). The range map \(r : M \to V, x \mapsto r(x)\) is a covering, mapping the fibers of the submersion \(M\) to the leaves of \(V\).

Now suppose in addition the foliation \((V, \mathcal{F})\) endowed with a transverse flow
of \( \mathbb{R} \). Hence there is a one-parameter group of diffeomorphisms \( \phi_t, t \in \mathbb{R} \) on \( V \), such that \( \phi_t \) maps leaves to leaves, and the vector field \( \Phi \) generating the flow is nowhere tangent to the leaves (thus in particular does not vanish). Since \( M \to V \) is a covering, the generator of the flow \( \phi \) can be lifted in a unique way to a vector field also denoted \( \Phi \) on the submersion \( M \). Since \( \phi \) preserves the leaves, \( \Phi \) descends to a vector field \( \overline{\Phi} \) on the base \( B \). By hypothesis \( B \) is a closed manifold, hence \( \overline{\Phi} \) can be integrated to a flow \( \Phi \) on \( B \), and consequently \( \Phi \) also generates a flow \( \phi \) on \( M \), mapping fibers to fibers. By construction the range map \( r: M \to V \) is \( \mathbb{R} \)-equivariant:

\[
r(\phi_t(x)) = \phi_t(r(x)) \quad \forall \ x \in M , \ t \in \mathbb{R} .
\] (114)

The vector field \( \overline{\Phi} \) on \( B \) is obtained as follows: at each point \( b \) of the submanifold \( B \subset V \), the tangent space \( T_bV \) decomposes canonically as the direct sum of \( T_bB \) and the tangent space to the leaf. \( \Phi \) projects accordingly to a vector field on \( B \) precisely corresponding to the generator \( \Phi \). The flow \( \Phi \) also lifts to a flow on the groupoid \( H_B \) because the range and source maps \( H_B \Rightarrow B \) are étale. This results in an action of \( \mathbb{R} \) on the groupoid \( H_B \) by homomorphisms, i.e.

\[
\Phi_t(\gamma, \delta) = \Phi_t(\gamma) \Phi_t(\delta) \quad \text{for all } \gamma, \delta \in H_B \text{ and } t \in \mathbb{R}.
\]

We form a new Lie groupoid by taking the crossed product

\[
G = H_B \rtimes \mathbb{R} .
\] (115)

The arrows of \( G \) are pairs \( (\gamma, t) \in H_B \times \mathbb{R} \), with composition law \( (\gamma, t)(\delta, u) = (\gamma \Phi_t(\gamma, \delta), t+u) \). Combining the respective actions of \( B \) and \( \mathbb{R} \) on the submersion \( s: M \to B \) yields the following action of the crossed product: \( x \cdot (\gamma, t) = \phi_t(x \cdot \gamma) = \phi_t(x) \cdot \Phi_t(\gamma) \). To summarize we are left with two Morita equivalent groupoids, namely \( V \rtimes \mathbb{R} \) and \( M \rtimes G \). This equivalence can be realized at the level of convolution algebras via a homomorphism

\[
\rho: C_c^\infty(V \rtimes \mathbb{R}) \to C_c^\infty(M \rtimes G)
\] (116)

constructed as follows. The range map \( r: M \to V \) is a covering of a compact manifold, hence choosing and partition of unity relative to a local trivialization of the covering one can build a “cut-off” function \( c \in C_c^\infty(M) \) with the property

\[
\sum_{v \in r^{-1}(v)} c(x)^2 = 1 \quad \text{for all } v \in V .
\]

Then \( \rho \) sends \( f \in C_c^\infty(V \rtimes \mathbb{R}) \) to the function

\[
\rho(f)(x, (\gamma, t)) = c(x) f(r(x), t) c(\phi_t(x \cdot \gamma))
\] (117)

for any \( x \in M, \gamma \in H_B \) and \( t \in \mathbb{R} \). Note that \( \phi_t(x \cdot \gamma) \) is the action of \( (\gamma, t) \in G \) on \( x \). One checks that \( \rho \) is a homomorphism of algebras, and that the resulting map in \( K \)-theory \( \rho_0: K_0(C_c^\infty(V \rtimes \mathbb{R})) \to K_0(C_c^\infty(M \rtimes G)) \) is independent of any choice of cut-off function.

Now let \( E = E_+ \oplus E_- \) be a \( \mathbb{Z}_2 \)-graded, \( \mathbb{R} \)-equivariant vector bundle over \( V \). We consider an \( \mathbb{R} \)-invariant leafwise differential elliptic operator \( D \) of order one and odd degree acting on the sections of \( E \). Hence according to the \( \mathbb{Z}_2 \)-grading \( D \) splits as the sum of two differential operators of order one acting only in the directions of the leaves \( D_+: C_c^\infty(V, E_+) \to C_c^\infty(V, E_+) \) and \( D_-: C_c^\infty(V, E_-) \to C_c^\infty(V, E_-) \). This may be rewritten in the usual matrix form

\[
D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} .
\] (118)
We make the further assumption that $D$ is formally self-adjoint with respect to some metric on $V$ and hermitean structure on $E$. Note that the condition of $\mathbb{R}$-invariance for $D$ imposes strong restrictions on the flow on $V$. The vector bundle $E$ defines a vector bundle (still denoted by $E$) on the manifold $M$ by pullback with respect to the range map $r$. Since $D$ is an $\mathbb{R}$-invariant leafwise differential operator on $V$, it pullbacks accordingly to a $G$-invariant fiberwise differential operator on the submersion $s : M \to B$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle, endowed with the trivial action of $G$. We view the product $M' = S^1 \times M$ as a submersion over the base manifold $B$, endowed with its $G$-action, and consider the differential operator

$$Q = \begin{pmatrix} \partial_x & D_- \\ D_+ & -\partial_x \end{pmatrix}$$

(119)

where $\partial_x = \frac{\partial}{\partial x}$ denotes partial differentiation with respect to the circle variable $x \in [0, 1]$. Then $Q$ is a $G$-invariant fiberwise differential operator on $M'$, and is elliptic because $Q^2 = D^2 + (\partial_x)^2$. As pseudodifferential operators, the modulus $|Q|$ and its parametrix $[Q]^{-1}$ are well-defined only modulo the ideal of smoothing operators. Hence the “sign” of $Q$ and the “spectral projection” onto the 1-eigenspace

$$F = \text{sign}(Q) = \frac{Q}{|Q|}, \quad P = \frac{1 + F}{2}$$

(120)

are represented by fiberwise pseudodifferential operators of order zero on the submersion $M' \to B$, and modulo smoothing operators $F$ and $P$ are $G$-invariant and fulfill the identities $F^2 = 1$ and $P^2 = P$. Let $[c] \in K_0(C^\infty_*(V \times \mathbb{R}))$ be a $K$-theory class represented by an idempotent matrix $c$. For simplicity we assume $c \in C^\infty_*(V \times \mathbb{R})$. The suspension of the idempotent $\rho(c) \in C^\infty_*(M \times G)$ is the invertible element

$$u = 1 + \rho(c)(\beta - 1) \in C^\infty_*(M' \times G)^+,$$

(121)

where the function $\beta \in C^\infty(S^1)$, $\beta(x) = \exp(2\pi i x)$ is the Bott generator of the circle. The algebra $C^\infty_*(M')$ acting by pointwise multiplication on the space of sections $C^\infty_*(M', E)$ is naturally represented in the algebra of sections of the bundle of pseudodifferential operators of order zero $\text{Cl}^0_*(M', E)$. This representation is $G$-equivariant, hence extends to a representation of $C^\infty_*(M') \rtimes G$ in the convolution algebra $\mathcal{E} = C^\infty_*(B, \text{Cl}^0_*(M', E)) \rtimes G$. Accordingly $u$ is represented by an invertible element $U$ in the unitalization of $\mathcal{E}$. The Toeplitz operator

$$T = PUP + (1 - P) \in \mathcal{E}^+$$

(122)

is uniquely defined modulo smoothing operators. One has $T \equiv 1 + P(U - 1)$ modulo the ideal $\mathcal{B} \subset \mathcal{E}$ of order $-1$ pseudodifferential operators, and the inverse of $T$ modulo this ideal is represented by the Toeplitz operator $PU^{-1}P + (1 - P) \equiv 1 + P(U^{-1} - 1)$. The quotient $\mathcal{E}/\mathcal{B}$ is isomorphic to the convolution algebra of leading symbols $\mathcal{A} = C^\infty_*(B, \text{LS}^0_*(M', E)) \rtimes G$. Note that the leading symbol of $T$ is uniquely defined and reads

$$\sigma_T = 1 + \sigma_P(u - 1) \in \mathcal{A}^+,$$

(123)

where the leading symbol $\sigma_P$ of $P$ is a $G$-invariant idempotent in the algebra of sections of $\text{LS}^0_*(M', E)$. Hence the equality $\sigma_P u = u \sigma_P$ holds in $\mathcal{A}$ and the
inverse of $\sigma_T$ is exactly $1 + \sigma_T(u^{-1} - 1)$. It is easy to see that the $K$-theory class $[\sigma_T] \in K_1(\mathcal{A})$ only depends on the $K$-theory class $[e] \in K_0(C_\infty^\infty(V \times \mathbb{R}))$ of the idempotent $e$.

**Definition 8.1** Let $(V, \mathcal{F})$ be a compact foliated manifold endowed with a transverse flow of $\mathbb{R}$, and $D$ be an $\mathbb{R}$-equivariant leafwise elliptic differential operator of order one and odd degree acting on the sections of a $\mathbb{Z}_2$-graded equivariant vector bundle over $V$. Then for any class $[e] \in K_0(C_\infty^\infty(V \times \mathbb{R}))$, we define the index of $D$ with coefficients in $[e]$ as the $K$-theory class

$$\text{Ind}(D, [e]) = \text{Ind}_E([\sigma_T]) \in K_0(\mathcal{D}),$$

where $T$ is the Toeplitz operator constructed above and $\text{Ind}_E$ is the index map of the extension $(E) : 0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{A} \to 0$.

From now on we specialize to a *codimension one* foliation $(V, \mathcal{F})$ endowed with a transverse flow $\phi$. The connected components of the closed transversal $B$ are all diffeomorphic to the circle. Hence, the induced flow $\bar{\sigma}$ on a connected component $B_p$ acts by rotation with a given period $p \in \mathbb{R}$. Choosing a basepoint we canonically get a parametrization of $B_p$ by the variable $b \in \mathbb{R}/\mathbb{Z}$, and the flow is simply $\bar{\sigma}(b) \equiv b + t \mod p$. In the same way, we consider the range $r(\gamma) \in B_p$ and the source $s(\gamma) \in B_p$ of any arrow $\gamma \in H_B$ as numbers respectively in $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$, and denote by $r^*db$, $s^*db$ the measures on $H_B$ pulled back from the Lebesgue measure by the étale maps $r$, $s$ respectively. Note that $db$ provides a canonical holonomy-invariant transverse measure on $(V, \mathcal{F})$ because the flow $\phi$ preserves the leaves of the foliation. In particular $r^*db = s^*db$ on $H_B$. Remark also that because the transverse Lebesgue measure is holonomy-invariant, the holonomy of a closed leafwise path is always the identity. As a consequence, the holonomy groupoid $H$ and its reduced groupoid $H_B$ have no automorphisms except the units (that is, $r(\gamma) = s(\gamma)$ implies $\gamma$ is a unit).

**Proposition 8.2** The convolution algebra $C_\infty^\infty(G)$ acts on the space of functions $C_\infty^\infty(B)$ by smoothing operators. In this representation, the operator trace of an element $f \in C_\infty^\infty(G)$ reads

$$\text{Tr}(f) = \sum_{B_p} \sum_{n \in \mathbb{Z}} \int_{H_B} f(\gamma, np + r(\gamma) - s(\gamma)) r^*db(\gamma)$$

where the sum runs over all connected components $B_p$ of the flow $\bar{\sigma}$ on the transversal, $p$ is the period of $B_p$ and $db$ is the pullback of the Lebesgue measure on the groupoid $H_B$. In particular the distribution kernel of $\text{Tr}$ is a measure on $G$ whose support is the manifold of automorphisms in the groupoid $G$.

**Proof:** We define the action of $f \in C_\infty^\infty(G)$ on a test function $\xi \in C_\infty^\infty(B)$ as follows: at any point $b \in B$,

$$(f \cdot \xi)(b) = \sum_{\gamma \in H_B^b} \int_{-\infty}^{\infty} f(\gamma, t)\xi(\bar{\sigma}(s(\gamma))) dt$$

where $H_B^b$ is the set of arrows $\gamma \in H_B$ such that $r(\gamma) = b$. The sum is finite and the integral converges because $f$ is of compact support on $G$. One readily
verifies that this defines a representation of the convolution algebra $C^\infty_c(G)$ on $C^\infty(B)$, and that the distributional kernel of the operator $f$ on the manifold $B \times B$ is

$$k_f(b, b') = \sum_{\gamma \in H_B^b} \int_{-\infty}^{+\infty} f(\gamma, t) \delta(\phi_t(s(\gamma)) - b') \, dt$$

where $\delta$ is the Dirac measure. If $s(\gamma)$ is in a connected component of period $p$ for the flow $\phi$, one has $\phi_t(s(\gamma)) \equiv s(\gamma) + t \mod p$ and the integral selects all values $t = np + b' - s(\gamma)$, $n \in \mathbb{Z}$. Therefore

$$k_f(b, b') = \sum_{n \in \mathbb{Z}} \sum_{\gamma \in H_B^b} f(\gamma, np + b' - s(\gamma))$$

(where $p$ is the period of the connected component of $s(\gamma)$) is a smooth function. Hence $C^\infty_c(G)$ acts by smoothing operators. The trace $\text{Tr}(f)$ is given by the integral, with respect to the measure $db$, of the kernel restricted to the diagonal $k_f(b, b)$. In this case $b = b' = r(\gamma)$ and $s(\gamma)$ belong to the same connected component, say $B_p$. This leads to

$$\text{Tr}(f) = \sum_{b_p} \sum_{n \in \mathbb{Z}} \sum_{\gamma \in H^b_{np}} f(\gamma, np + b - s(\gamma)) \, db$$

$$= \sum_{b_p} \sum_{n \in \mathbb{Z}} \int_{H_B^{b_p}} f(\gamma, np + r(\gamma) - s(\gamma)) \, r^* db(\gamma)$$

as claimed. If $b = r(\gamma) \in B_p$ and $t = np + r(\gamma) - s(\gamma)$ for some $n \in \mathbb{Z}$, one has $\phi_t(s(\gamma)) \equiv r(\gamma) \mod p$ which is equivalent to $b \cdot (\gamma, t) = b$, that is $(\gamma, t) \in G$ is an automorphism. \qed

The operator trace on $C^\infty_c(G)$ is therefore a cyclic zero-cocycle localized at the isotropic set in $G$, and is of order $k = 0$ because its distribution kernel is a measure. We may split it into several parts. Let us first consider the linear functional $\text{Tr}_0 : C^\infty_c(G) \to \mathbb{C}$ whose support is localized at the set of units $B \subset G$. It amounts to sum only over the arrows $\gamma = r(\gamma) = s(\gamma) \in B$ and the integer $n = 0$ in the right-hand-side of (125):

$$\text{Tr}_0(f) = \int_B f(b, 0) \, db \ . \quad (126)$$

One easily checks that $\text{Tr}_0$ is a trace. By construction it is localized at the submanifold of units in $G$, and its distribution kernel corresponds to the (holonomy invariant) transverse measure $db$. By virtue of section 4, there is an associated cyclic cocycle $\tau_0$ over the algebra $\mathcal{B}$.

**Definition 8.3** Let $\text{Tr}_0 : C^\infty_c(G) \to \mathbb{C}$ be the trace localized at units given by the holonomy invariant transverse measure on $(V, \mathcal{F})$ induced by the flow $\phi$, and $\tau_0$ be the associated cyclic cocycle over $\mathcal{B}$. We define the Connes-Euler characteristics of the $\mathbb{R}$-invariant operator $D$ with coefficients in a K-theory class $[e] \in K_0(C^\infty_c(V \times \mathbb{R}))$ as the complex number

$$\chi(D, [e]) = \langle \tau_0, \text{Ind}(D, [e]) \rangle \quad (127)$$

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This number is the analogue, in the $\mathbb{R}$-equivariant context, of the index defined by Connes for longitudinal elliptic operators on a foliation endowed with a holonomy invariant transverse measure [5]. Now let $\tau$ be the cyclic cocycle over $\mathcal{B}$ corresponding to the full operator trace $\text{Tr}$. Our aim is to evaluate $\tau$ on the $K$-theory class $\text{Ind}(D, [e]) \in K_0(\mathcal{B})$. As we shall see, the complementary part $(\tau, \text{Ind}(D, [e])) - \chi(D, [e])$ is related to the periodic orbits of the flow $\phi$ on $(V, \mathcal{F})$. If $\Pi \subset V$ is a periodic orbit with period $p_\Pi$, a choice of base-point canonically provides a parametrization of the orbit by a variable $v \in [0, p_\Pi]$, with $\phi(v) \equiv v + t$ mod $p_\Pi$. The flow at $t = p_\Pi$ is called the return map. It defines an endomorphism $h_\Pi'(v)$ on the tangent space to the leaf at any point $v \in \Pi$, together with an even-degree endomorphism $j_\Pi(v)$ on the fiber of the $\mathbb{Z}_2$-graded vector bundle $E$ over $v$. We shall suppose that $h_\Pi'$ is non-degenerate, in the sense that $\det(1 - h_\Pi'(v)) \neq 0$ for all $v \in \Pi$. This implies in particular that the periodic orbits of the flow are isolated.

**Proposition 8.4** Let $\Pi \subset V$ be a periodic orbit of the flow $\phi$, and $v \in \Pi$. We denote by $h_\Pi'(v)$ the action of the return map on the leafwise tangent space, and by $\text{tr}_n(j_\Pi(v))$ the supertrace of the return map on the $\mathbb{Z}_2$-graded vector bundle $E$ at $v$. Suppose that $h_\Pi'$ is non-degenerate. Then the linear functional $\Theta_\Pi : C_c^\infty(V \times \mathbb{R}) \to \mathbb{C}$

$$\Theta_\Pi(f) = \sum_{n \in \mathbb{Z}} \int_\Pi \frac{\text{tr}_n(j_\Pi(v)^n)}{\det(1 - h_\Pi'(v)^n)} f(v, np_\Pi) \, dv$$

with $p_\Pi \in \mathbb{R}$ the period of the orbit, is a trace. Actually the function $v \mapsto \text{tr}_n(j_\Pi(v)^n)/\det(1 - h_\Pi'(v)^n)$ is constant along the orbit.

**Proof:** If $v$ and $v'$ are two distinguished points in the orbit $\Pi$, the endomorphisms $j_\Pi(v)$ and $j_\Pi(v')$ are conjugate, hence have the same supertrace. Similarly $\det(1 - h_\Pi'(v))$ and $\det(1 - h_\Pi'(v'))$ are equal. The function $v \mapsto \text{tr}_n(j_\Pi(v)^n)/\det(1 - h_\Pi'(v)^n)$ is therefore constant along the orbit, and can be pushed out of the integration over $v$. In fact for any fixed $n \in \mathbb{Z}^*$, the integral

$$\Theta_n(f) = \int_\Pi f(v, np_\Pi) \, dv$$

defines a trace on $C_c^\infty(V \times \mathbb{R})$. Indeed for two functions $f, g$ on $\Pi \times \mathbb{R}$ one has

$$\Theta_n(fg) = \int_\Pi \int_{-\infty}^{\infty} f(v, np_\Pi - t)g(v - t, t) \, dt \, dv = \int_\Pi \int_{-\infty}^{\infty} f(v, t)g(v + t, np_\Pi - t) \, dt \, dv = \int_\Pi \int_{-\infty}^{\infty} f(v - t, t)g(v, np_\Pi - t) \, dt \, dv = \Theta_n(gf)$$

where we used repeatedly the fact that the functions $f$ and $g$ are $p_\Pi$-periodic in the variable $v$.

**Theorem 8.5** Let $(V, \mathcal{F})$ be a codimension one compact foliated manifold endowed with a transverse flow, $E$ a $\mathbb{Z}_2$-graded $\mathbb{R}$-equivariant vector bundle over
At the level of leading symbols, one has $R$ normal to the circle, i.e. $h R$ symbol of the cocycle class $\tau$. Assume that the periodic orbits of the flow are non-degenerate. Then for any $V$ the canonical coordinates on the cotangent bundle of $\Pi$, and a complementary part of the Connes-Euler characteristics, the sum runs over the periodic orbits $\Pi$ of the flow, and $\Theta_{\Pi}$ is the canonical trace (128) on $C_\infty^c(V \rtimes \mathbb{R})$.

Proof: We use zeta-function renormalization by means of the complex powers of the operator $|Q|$. Since $\text{Tr}$ is a trace on $C_\infty^c(G)$ of order zero and localized at the submanifold of automorphisms, Corollary 5.7 implies

$$\langle [\tau], \text{Ind}(D, [e]) \rangle = C \circ \text{Res}(T^{-1}[\ln |Q|, T])$$

with $T^{-1}$ the inverse of $T$ modulo smoothing operators, and $C$ the distribution kernel of $\text{Tr}$ which is a measure on $G$. Note that $T^{-1}[\ln |Q|, T]$ is a pseudodifferential operator of order $-1$ because $|Q|$ is $G$-invariant modulo smoothing operators. We know that $C$ splits as a sum of a part $C_0$ whose support is localized at units, and a complementary part $C_p$. By definition of the Connes-Euler characteristics,

$$\langle [\tau], \text{Ind}(D, [e]) \rangle - \chi(D, [e]) = C_p \circ \text{Res}(T^{-1}[\ln |Q|, T]) .$$

The support of $C_p$ is the set of arrows $(\gamma, t) \in G$ which are not units and such that $r(\gamma) = \tilde{\phi}(s(\gamma))$ in $B$. Such $(\gamma, t)$ acts on the fiber $M_{r(\gamma)}$ of the submersion $s : M \to B$ by a diffeomorphism $h : x \mapsto \phi_t(x \cdot \gamma)$. Since $(\gamma, t)$ is not a unit, one sees that $x \in M_{r(\gamma)}$ is a fixed point of $h$ if and only if its image $r(x) \in V$ is contained in a periodic orbit of the flow. By hypothesis the periodic orbits are isolated, so the fixed points for $h$ is discrete subset of $M_{r(\gamma)}$. Extending the action of $h$ on the fiber $M'_{r(\gamma)} = S^1 \times M_{r(\gamma)}$ in a trivial way, the fixed point set becomes a discrete union of circles. In this simple situation, Proposition 5.3 reduces to

$$\text{Res}_{z=0} \text{Tr}(RU_h|Q|^{-z}) = \sum_{s' \subset M'_{r(\gamma)} \text{ fixed}} \int_{S^1 \times S^1} [\sigma_{R}]^{-1}(x, \xi; 0, 0) \frac{\xi dx}{\det(1 - h')} 2\pi$$

for any pseudodifferential operator $R$ of order $-1$ on $M'_{r(\gamma)}$. Here $(x, \xi)$ are the canonical coordinates on the cotangent bundle of $S^1$, $[\sigma_R]^{-1}$ is the leading symbol of $R$, and $h'$ is the differential of the diffeomorphism $h$ in the direction normal to the circle, i.e. $h'$ is an endomorphism of the tangent space to $M_{r(\gamma)}$ at the fixed points. We apply this formula to the operator $R = T^{-1}[\ln |Q|, T]$. At the level of leading symbols, one has

$$\sigma_{T^{-1}[\ln |Q|, T]}(x, \xi; 0, 0) \sim \sigma_{T}^{-1} \frac{\partial \sigma_T}{\partial x}(x, \xi; 0, 0)$$

where $\sigma_T = 1 + \sigma_P(u - 1)$ is the leading symbol of $T$, $u = 1 + \rho(e)(\beta - 1)$, and $\sigma_P$ is the leading symbol of the spectral projection $P$. In matrix notation adapted to the $\mathbb{Z}_2$-grading of the vector bundle $E$,

$$\sigma_P(x, \xi; 0, 0) = \begin{pmatrix} \theta(\xi) & 0 \\ 0 & \theta(-\xi) \end{pmatrix} ,$$

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where $\theta : \mathbb{R} \to \{0, 1\}$ is the Heavyside function. The dependency of $\sigma_T$ upon the parameter $x$ comes only from the Bott element $\beta \in C^\infty(S^1)$. The cosphere bundle $S^* S^1$ being just a sum of two copies of the circle, the computation of the residue is straightforward. It is a sum over the (isolated) fixed points of $h$ in $M_r(\gamma)$:

$$\text{Res}_{z=0} \text{Tr}(T^{-1} [\ln |Q|, T] (\gamma, t) U_h |Q|^{-z}) = \sum_{y \in M_r(\gamma) \text{ fixed}} \frac{\text{tr}_x (\rho(e)(y, (\gamma, t)))}{|\det(1 - h'(y))|}$$

Since $y \in M$ is fixed by $(\gamma, t) \in G$, one has $\rho(e)(y, (\gamma, t)) = c(y)^2 e(r(y), t)$ where $c \in C^\infty_c(M)$ is the cut-off function used in the construction of the homomorphism $\rho$, and $r(y) \in V$ is the projection of $y$. Since the étale groupoid $H_B$ acts without fixed points on $B$, integrating with respect to the measure $C_p$ on $G$ amounts to integrate over all the periodic orbits $\Pi \subset V$ of the flow. A straightforward computation yields

$$C_p \circ \text{Res}(T^{-1} [\ln |Q|, T]) = \sum_{\Pi} \sum_{n \in \mathbb{Z}} \int_\Pi \frac{\text{tr}_x(j_\Pi(v)^n)}{|\det(1 - h'_\Pi(v)^n)|} e(v, np_\Pi) \, dv$$

where $h'_\Pi(v)$ and $j_\Pi(v)$ are the actions of the return map respectively on the tangent space to the leaf and on the fiber of $E$ at $v \in \Pi$.

**Corollary 8.6** If the Connes-Euler characteristics $\chi(D, [e])$ is not an integer, then the flow $\phi$ has periodic orbits.

**Proof:** Indeed the cyclic cocycle $\tau$ over $\mathcal{B}$ comes from the operator trace on $C^\infty_c(G)$, so the pairing $\langle [\tau], \text{Ind}(D, [e]) \rangle$ is always an integer. The sum over periodic orbits cannot vanish if the Connes-Euler characteristic does not belong to $\mathbb{Z}$.

We end this section with a brief discussion of the more general situation of a flow $\phi$ compatible with the foliation $(V, \mathcal{F})$, i.e. mapping leaves to leaves, but also admitting fixed points. Remark that a leaf containing a fixed point is necessarily preserved by $\phi_t$ for all $t \in \mathbb{R}$: in this case the vector field $\Phi$ generating the flow is tangent to the leaf. We shall suppose that the flow $\phi$ is *non-degenerate* in the following sense:

i) At any fixed point $v \in V$, the tangent map $T_v \phi_t$ on the tangent space $T_v V$ has no singular value equal to 1 for $t \neq 0$;

ii) Each periodic orbit $\Pi \subset V$ is transverse to the leaves, and the return map induced by the flow on the tangent space to the leaf $T_v \Lambda$ at any $v \in \Pi$ has no singular value equal to 1.

iii) If $v \in V$ belongs to a leaf $\Lambda$ preserved by the flow, the tangent map $T_v \phi_t$ acting on the transverse space $T_v V / T_v \Lambda$ has no singular value equal to 1 for $t \neq 0$;

iv) The holonomy of a closed path contained in a leaf $\Lambda$ preserved by the flow is always the identity.
Conditions i) and ii) immediately imply that the fixed points and the periodic orbits of the flow are isolated. In particular, a fixed point cannot belong to a dense leaf. If $B \subset V$ is a complete closed transversal, the induced flow $\bar{\phi}$ on $B$ may also have fixed points, consisting in the intersection of $B$ with all the leaves preserved by $\phi$. Conditions i) and ii) also imply that the fixed points of $\bar{\phi}$ are isolated in $B$. Moreover, by condition iii) the tangent map $T_b \bar{\phi}_t$ acting on the tangent space $T_b B$ at a fixed point $b \in B$ is not the identity if $t \neq 0$; thus $T_b \bar{\phi}_t$ acts by the multiplication operator $e^{\kappa_b t}$ where $\kappa_b \neq 0$ is the exponent of the flow at $b$. We add condition iv) because, unlike in the previous situation of a transverse flow, the holonomy of a closed path is not automatically the identity if the leaf is preserved by the flow. Now the closed manifold $B$ can be partitioned into the orbits of the flow $\bar{\phi}$, which are of three different types:

- The periodic orbits, denoted $B_p$, all diffeomorphic to a circle. Any such orbit of period $p$ is naturally parametrized by $b \in \mathbb{R}/p\mathbb{Z}$ once a base-point is chosen, and the flow is $\bar{\phi}_t(b) \equiv b + t \mod p$ for all $t$. We denote $db$ the Lebesgue measure on $B_p$;
- The infinite orbits, denoted $B_\infty$, all diffeomorphic to $\mathbb{R}$. They are also naturally parametrized by $b \in \mathbb{R}$ once a base-point is chosen, with $\bar{\phi}_t(b) = b + t$ for all $t$. We denote $db$ the Lebesgue measure on $B_\infty$;
- The fixed points $\bar{\phi}_t(b) = b$ for all $t$, each of them connecting two infinite orbits.

The reduced holonomy groupoid $H_B$ still carries an action of $\mathbb{R}$ by homomorphisms, so we can form the crossed product $H_B \rtimes \mathbb{R}$ as before. If $B_p$, a periodic orbit, we denote $H_{B_p}$ the subgroupoid of $H_B$ consisting in arrows $\gamma$ with source and range contained in $B_p$. Similarly with the infinite orbits. As before, the measure $db$ on periodic and infinite orbits is holonomy-invariant. However, compared with the previous situation of a transverse flow, the main difficulty with the presence of fixed points is that the convolution algebra $C^\infty_c(G)$ does no longer act by smoothing operators on the space $C^\infty(B)$. In order to define a trace we are forced to restrict to the subalgebra

$$C^\infty_c(G^*_+) = \{ f \in C^\infty_c(G) \mid \text{supp}(f) \subset H_B \times \mathbb{R}^*_+ \} \quad (130)$$

where $\mathbb{R}^*_+ \subset \mathbb{R}$ is the abelian monoid (with addition) of real numbers $> 0$. One easily checks that the convolution product on $C^\infty_c(G)$ preserves the subspace $C^\infty_c(G^*_+)$. The submanifold $G^*_+ = H_B \times \mathbb{R}^*_+$ of $G$ has a partially defined composition law coming from the composition of arrows in $G$; however $G^*_+$ is not a subgroupoid. We nevertheless regard $C^\infty_c(G^*_+)$ as the convolution algebra of the “crossed-product” $G^*_+ = H_B \rtimes \mathbb{R}^*_+$.

**Proposition 8.7** The algebra $C^\infty_c(G^*_+)$ acts on the space $C^\infty(B)$ by distributional kernels which may be singular on the diagonal. When the flow $\phi$ is non-degenerate, the operator trace on smoothing kernels canonically extends to

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a trace on $C^\infty_c(G^*_+) \times B$ by

$$\text{Tr}(f) = \sum_{B_p} \sum_{n \in \mathbb{Z}} \int_{H_{n_p}} f(\gamma, np + r(\gamma) - s(\gamma)) r^* db(\gamma)$$

$$+ \sum_{B_\infty} \int_{H_{\infty}} f(\gamma, r(\gamma) - s(\gamma)) r^* db(\gamma)$$

$$+ \sum_b \int_0^\infty \frac{f(b, t)}{|1 - e^{\kappa t}|} dt \tag{131}$$

where the sums run over all periodic orbits $B_p$, infinite orbits $B_\infty$ and fixed points $b$ of the flow $\tilde{\phi}$ on the transversal; $p > 0$ is the period of the orbit $B_p$ and $\kappa_b \neq 0$ is the exponent of $\tilde{\phi}$ at the fixed point $b$.

Proof: As in the proof of Proposition 8.2, the distributional kernel of an element $f \in C^\infty_c(G^*_+) \times B$ acting on $C^\infty(B)$ is

$$k_f(b, b') = \sum_{\gamma \in H_b^\infty} \int_0^\infty f(\gamma, t) \delta(\tilde{\phi}_t(s(\gamma)) - b') dt \, .$$

Its restriction to the diagonal is still a smooth function except at the fixed points of the flow $\tilde{\phi}$ where it is proportional to a Dirac measure. Its integral over $B$ thus extends the operator trace of smoothing kernels. We split $B$ into three parts corresponding to the different types of connected components of the flow. The first term of (131) associated to the periodic components is calculated exactly as in 8.2, and similarly for the second term associated to the infinite components. At a fixed point, the equality $\tilde{\phi}_t(s(\gamma)) = r(\gamma)$ implies $s(\gamma) = r(\gamma)$, hence $\gamma$ is necessarily a unit by hypothesis iv). The kernel $k_f(b, b)$ for $b$ in the vicinity of a fixed point thus reads

$$k_f(b, b) = \int_0^\infty f(b, t) \delta(\tilde{\phi}_t(b) - b) dt \, .$$

Taking the fixed point as the origin of a local coordinate system, the equality $\tilde{\phi}_t(b) \sim b e^{\kappa t}$ holds at first order in the variable $b$, where $\kappa$ is the exponent of $\tilde{\phi}$ at the fixed point. Thus

$$k_f(b, b) = \delta(b) \int_0^\infty \frac{f(b, t)}{|1 - e^{\kappa t}|} dt \, ,$$

and integrating over $b$ yields the contribution of the fixed point to the trace, whence the third term of (131).

Let $C^\infty_c(V \times \mathbb{R}^*_+)$ be the space of smooth compactly supported functions on $V \times \mathbb{R}$ with support contained in the open subset $V \times \mathbb{R}^*_+$. It is a subalgebra of the convolution algebra $C^\infty_c(V \times \mathbb{R})$. Let $D$ be an $\mathbb{R}$-equivariant leafwise elliptic differential operator $D$ of order one and odd degree acting on the sections of a $\mathbb{Z}_2$-graded equivariant vector bundle $E$ over $V$, and $c \in C^\infty_c(V \times \mathbb{R}^*_+)$ be an idempotent. The index $\text{Ind}(D, [c])$ defined in 8.1 is then a $K$-theory class of the subalgebra $R^*_+ = C_\infty_c(B, C_{\mathbb{C}, 1}^{\infty}(M', E)) \times G^*_+$ of $R$. As before we denote $\tau$ the cyclic cocycle over $R^*_+$ induced by the trace $\text{Tr} : C^\infty_c(G^*_+) \to \mathbb{C}$. The pairing $\langle [\tau], \text{Ind}(D, [c]) \rangle$ is a well-defined complex number. Its explicit computation involves, besides the periodic orbits of the flow, also the fixed points.
Proposition 8.8 Let \( v \in V \) be a fixed point of the non-degenerate flow \( \phi \). We denote by \( \kappa_v \) the generator of the flow on the tangent space \( T_v V \), and by \( j_v \) the generator of the flow on the fiber \( E_v \). Then the linear functional \( W_v : C^\infty_c(V \times \mathbb{R}_+^*) \to \mathbb{C} \)

\[
W_v(f) = \int_0^\infty \frac{\text{tr}_{\mathbb{R}}(e^{jt})}{|\det(1 - e^{jt})|} f(v, t) \, dt
\]

is a trace.

Proof: Since \( \phi_t(v) = v \), \( \forall t > 0 \), one has for any functions \( f, g \in C^\infty_c(V \times \mathbb{R}_+^*) \)

\[
(fg)(v, t) = \int_{-\infty}^{+\infty} f(v, u)g(v, t - u) \, du = (gf)(v, t)
\]

so that \( W_v \) is obviously a trace. \( \Box \)

Since the support of the index \( \text{Ind}(D, [e]) \in K_0(\mathcal{R}_+^\times) \) does not meet the part \( t \leq 0 \) of the groupoid \( \mathcal{G} \), the Connes-Euler characteristic does not appear in the computation of \( \langle [\tau], \text{Ind}(D, [e]) \rangle \). One is left with the periodic orbits and the fixed points only. The method of proof of Theorem 8.5 leads at once to the following result.

Theorem 8.9 Let \( (V, \mathcal{F}) \) be a codimension one compact foliated manifold endowed with an \( \mathcal{F} \)-compatible flow, \( E \) a \( \mathbb{Z}_2 \)-graded \( \mathbb{R} \)-equivariant vector bundle over \( V \), and \( D \) an odd \( \mathbb{R} \)-invariant leafwise elliptic differential operator of order one. Assume that the periodic orbits and the fixed points of the flow are non-degenerate. Then for any class \( [e] \in K_0(C^\infty_c(\mathbb{R}_+^*)) \), the pairing of the index \( \text{Ind}(D, [e]) \) with the cyclic cocycle \( \tau \) induced by the operator trace is

\[
\langle [\tau], \text{Ind}(D, [e]) \rangle = \sum_{\Pi} \Theta_\Pi(e) + \sum_{v \text{ fixed}} W_v(e),
\]

where the sums runs over the periodic orbits \( \Pi \) and the fixed points \( v \) of the flow, \( \Theta_\Pi \) and \( W_v \) are the canonical traces (128) and (132) on \( C^\infty_c(V \times \mathbb{R}_+^*) \).

Proof: As in the proof of 8.5 we let \( C \) be the measure on \( G_+^* \) corresponding to the trace \( \text{Tr} : C^\infty_c(G_+^*) \to \mathbb{C} \). Then

\[
\langle [\tau], \text{Ind}(D, [e]) \rangle = C \circ \text{Res}(T^{-1}[\ln |Q|, T])
\]

According to (131), the trace can be decomposed into three parts respectively associated to periodic orbits, infinite orbits and fixed points of the flow \( \phi \) on the transversal \( B \). Hence \( C \) can be accordingly decomposed into three measures \( C_p, C_\infty \) and \( C_f \) over \( G_+^* \). The contributions of \( C_p \) and \( C_\infty \) to the pairing \( \langle [\tau], \text{Ind}(D, [e]) \rangle \) are computed exactly as in 8.5 and yield the sum over the periodic orbits \( \Pi \) of the flow \( \phi \) on \( V \). Concerning the measure \( C_f \), whose support is localized at the arrows \( (b, t) \in G_+^* \) with \( b \in B \) a fixed point of \( \phi \) and \( t \in \mathbb{R}_+^* \), we arrive at

\[
C_f \circ \text{Res}(T^{-1}[\ln |Q|, T]) = \sum_{b \in B \text{ fixed}} \sum_{y \in \mathcal{M}_b \text{ fixed}} \int_0^\infty \frac{\text{tr}_{\mathbb{R}}(\rho(e)(y, (b, t)))}{|1 - e^{\rho(t)}| \det(1 - h_i(y))} \, dt
\]
where $h'_t$ is the action of the flow $\phi_t$ on the tangent space to the fiber $M_b$ at the fixed point $y$, and $\kappa_t$ is the exponent of the flow $\phi_t$ at the fixed point $b$. One has $(1 - e^{\kappa_t}) \det(1 - h'_t(y)) = \det(1 - T\phi_t(y))$ with $T\phi_t(y)$ the flow on the tangent space $T_yM$. The latter is of the form $T\phi_t(y) = e^{\kappa_y t}$ with $\kappa_y$ the generator of the tangent flow at $y$. To end the computation, we remark that $\rho(e)(y, (b, t)) = e(y)^2 e(r(y), t)$ where $e \in C^\infty_c(M)$ is the cut-off function and $r(y) \in V$. Since $\sum_{y \in \pi^{-1}(v)} e(y)^2 = 1$ for any fixed point $v$ in $V$ of the flow $\phi$, one gets

$$C_f \circ \text{Res}(T^{-1}[\ln |Q|, T]) = \sum_{v \text{ fixed}} \int_0^\infty \frac{\text{tr}_x(e^{j_y t})}{|\det(1 - e^{\kappa_y t})|} e(v, t) \, dt$$

where $j_y$ is the generator of the flow on the fiber $E_v$ and $\kappa_v$ is the generator of the flow on the tangent space $T_vV$. 

References


