PSEUDODIFFERENTIAL EXTENSION AND TODD CLASS

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Abstract

Let M be a closed manifold. Wodzicki shows that, in the stable range, the cyclic cohomology of the associative algebra of pseudodifferential symbols of order ≤ 0 is isomorphic to the homology of the cosphere bundle of M. In this article we develop a formalism which allows to calculate that, under this isomorphism, the Radul cocycle corresponds to the Poincaré dual of the Todd class. As an immediate corollary we obtain a purely algebraic proof of the Atiyah-Singer index theorem for elliptic pseudodifferential operators on closed manifolds.

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1 Introduction

Let M be a closed, not necessarily orientable, smooth manifold and denote by $\mathrm{CL}(M)$ the algebra of classical, one-step polyhomogeneous pseudodifferential operators on M. The space of smoothing operators $\mathrm{L}^{-\infty}(M)$ is a two-sided ideal in $\mathrm{CL}(M)$, and we call the quotient $\mathrm{CS}(M) = \mathrm{CL}(M)/\mathrm{L}^{-\infty}(M)$ the algebra of *formal symbols* on M. The multiplication on $\mathrm{CS}(M)$ is the usual $\star$-product of symbols. One thus gets an extension of associative algebras

$$0 \to \mathrm{L}^{-\infty}(M) \to \mathrm{CL}(M) \to \mathrm{CS}(M) \to 0 \ . \tag{1}$$

An "abstract index problem" then amounts to the computation of the corresponding excision map $HP^\bullet(\mathrm{L}^{-\infty}(M)) \to HP^{\bullet+1}(\mathrm{CS}(M))$ in periodic cyclic cohomology [9]. In even degree, $HP^0(\mathrm{L}^{-\infty}(M)) \cong \mathbb{C}$ is generated by the usual trace of smoothing operators, whereas in odd degree $HP^1(\mathrm{L}^{-\infty}(M)) \cong 0$. Using zeta-function renormalization, one shows (see for instance [10]) that the image of the trace under the excision map is represented by the following cyclic one-cocycle over $\mathrm{CS}(M)$,

$$c(a_0, a_1) = \int\!\!\!\!\!\!- \; a_0[\log q, a_1] \tag{2}$$

for any two formal symbols $a_0, a_1 \in \mathrm{CS}(M)$. Here the bar integral denotes the Wodzicki residue [12], which is a trace on $\mathrm{CS}(M)$, and $\log q$ is a log-polyhomogeneous symbol associated to a fixed positive elliptic symbol $q \in$

$\mathrm{CS}(M)$ of order one. Notice that the bilinear functional c was originally introduced by Radul in the context of Lie algebra cohomology [11]. A direct computation shows that c is in fact a cyclic one-cocycle over $\mathrm{CS}(M)$, and that its cyclic cohomology class does not depend on the choice of q. Hence the class $[c] \in HP^1(\mathrm{CS}(M))$ is completely canonical, in the sense that it only depends on M. On the other hand the cyclic cohomology of $\mathrm{CS}(M)$ is known [13], and corresponds to the ordinary homology (with complex coefficients) of a certain manifold. A natural question therefore is to identify the class $[c]$. In the present paper we give the answer for its image in the periodic cyclic cohomology of the subalgebra $\mathrm{CS}^0(M) \subset \mathrm{CS}(M)$, the formal symbols of order ≤ 0. The result is stated as follows. The leading symbol map gives rise to an algebra homomorphism $\lambda : \mathrm{CS}^0(M) \to C^\infty(S^*M)$, where S^*M is the cosphere bundle of M. This allows to pullback any homology class of S^*M to the periodic cyclic cohomology of the symbol algebra:

$$\lambda^* : H_\bullet(S^*M, \mathbb{C}) \to HP^\bullet(\mathrm{CS}^0(M)) \ . \qquad (3)$$

Wodzicki shows that λ^* is an *isomorphism*, provided that the natural locally convex topology of $\mathrm{CS}^0(M)$ is taken into account [13]. Our main result is the following theorem (6.8), which holds in the algebraic setting or the locally convex setting regardless to Wodzicki's isomorphism.

Theorem 1.1 *Let M be a closed manifold. The periodic cyclic cohomology class of $[c] \in HP^1(\mathrm{CS}^0(M))$ is*

$$[c] = \lambda^*\big([S^*M] \cap \pi^*\mathrm{Td}(T_{\mathbb{C}}M)\big) \ , \qquad (4)$$

where $\mathrm{Td}(T_{\mathbb{C}}M) \in H^\bullet(M, \mathbb{C})$ is the Todd class of the complexified tangent bundle, and $\pi : S^*M \to M$ is the cosphere bundle endowed with its canonical orientation and fundamental class $[S^*M] \in H_\bullet(S^*M)$.

We give a purely algebraic proof of this theorem. The central idea is to introduce the $\mathbb{Z}_2$-graded algebra $\mathrm{CL}(M, E)$ of pseudodifferential operators acting on differential forms, that is, on the sections of the exterior bundle $E = \Lambda T^*M$, and view the corresponding algebra of formal symbols $\mathrm{CS}(M, E)$ as a bimodule over itself. Using this bimodule structure we develop a formalism of abstract Dirac operators. This leads to the construction of cyclic cocycles for the subalgebra $\mathrm{CS}^0(M) \subset \mathrm{CS}(M, E)$. These cocycles are given by algebraic analogues of the JLO formula [6], and are all cohomologous in $HP^\bullet(\mathrm{CS}^0(M))$. By choosing genuine Dirac operators we obtain both sides of equality (4). Let us mention that the JLO formula in the right-hand-side provides a representative of the Todd class as a closed differential form over M

$$\mathrm{Td}(\mathrm{i}R/2\pi) = \det\left(\frac{\mathrm{i}R/2\pi}{e^{\mathrm{i}R/2\pi} - 1}\right) \qquad (5)$$

where R is the curvature two-form of an affine torsion-free connection on M. Hence our method gives an "explicit formula" for the class $[c]$. In the same way, we also prove that the cyclic cohomology class of the Wodzicki residue vanishes in $HP^0(\mathrm{CS}^0(M))$.

As an immediate corollary of Theorem 1.1 we obtain the Atiyah-Singer index formula for elliptic pseudodifferential operators [1]. If Q is an elliptic operator acting on the sections of a (trivially graded) vector bundle over M, its leading symbol is an invertible matrix g with entries in the commutative algebra $C^\infty(S^*M)$, hence it defines a class in the algebraic K-theory $K_1(C^\infty(S^*M))$. Its Chern character in $H^\bullet(S^*M,\mathbb{C})$ is represented by the closed differential form of odd degree

$$\mathrm{ch}(g) = \sum_{k\geq 0} \frac{k!}{(2k+1)!}\,\mathrm{tr}\Big(\frac{(g^{-1}dg)^{2k+1}}{(2\pi \mathrm{i})^{k+1}}\Big)\ . \qquad (6)$$

Corollary 1.2 (Index theorem) *Let Q be an elliptic pseudodifferential operator of order ≤ 0 acting on the sections of a trivially graded vector bundle over M, with leading symbol class $[g]\in K_1(C^\infty(S^*M))$. Then the Fredholm index of Q is the integer*

$$\mathrm{Ind}(Q) = \langle [S^*M], \pi^*\mathrm{Td}(T_{\mathbb{C}}M)\cup \mathrm{ch}([g])\rangle\ . \qquad (7)$$

This is a direct consequence of the fact that the class $[c]\in HP^1(\mathrm{CS}^0(M))$ of the residue cocycle is the image of the operator trace $\mathrm{Tr}:\mathrm{L}^{-\infty}(M)\to\mathbb{C}$ under the excision map of the fundamental extension

$$0\to \mathrm{L}^{-\infty}(M)\to \mathrm{CL}^0(M)\to \mathrm{CS}^0(M)\to 0\ . \qquad (8)$$

In fact (4) and the index formula are equivalent. Hence our method gives a new algebraic proof of the index theorem. This should however not be confused with what is usually called an *algebraic index theorem* ([8]). The latter calculates the cyclic cohomology class of the canonical trace on a (formal) deformation quantization of the algebra of smooth functions on a symplectic manifold, and relates it to the Todd class. In the special case of the symplectic manifold T^*M, one may take the algebra of smoothing operators $\mathrm{L}^{-\infty}(M)$ as a deformation quantization of the commutative algebra of functions over T^*M and obtain in this way the usual index theorem. This is *not* what we are doing here. In fact our approach is in some sense opposite, because instead of working with the operator ideal $\mathrm{L}^{-\infty}(M)\subset \mathrm{CL}^0(M)$ we directly deal with the quotient algebra of formal symbols $\mathrm{CS}^0(M)$. As a consequence, we drop the delicate analytic issues inherent to the highly non-local algebra $\mathrm{L}^{-\infty}(M)$ and its operator trace, and entirely transfer the index problem on the algebra $\mathrm{CS}^0(M)$ endowed with the residue cocycle (2). The computation is purely local because only a finite number of terms in the asymptotic expansion of symbols contribute to the index, which relates our approach to the Connes-Moscovici residue index formula [4]. For this reason our formalism is well-adapted (and in fact motivated by) the study of more general index problems appearing in non-commutative geometry [3], for which a genuine extension of algebras and the corresponding residue cocycle are available. This includes higher equivariant index theorems for non-isometric actions of non-compact groups, higher index theorems on Lie groupoids, and so on. These ideas will be developped elsewhere.

Here is a brief description of the paper. In section 2 we recall basic things about pseudodifferential operators. In section 3 we look at $\mathrm{CS}(M,E)$ as a bimodule over itself and introduce the relevant spaces of operators acting on it. In section 4 a canonical trace is defined by means of the Wodzicki residue.

Section 5 introduces generalized Dirac operators acting on $\mathrm{CS}(M,E)$. Theorem 1.1 is proved in section 6 by means of the algebraic JLO formula, and the index theorem is deduced in section 7.
All manifolds are supposed to be Hausdorff, paracompact, smooth and without boundary.

2 Pseudodifferential operators

Let M be a n-dimensional manifold. We denote by $C^\infty(M)$ (resp. $C^\infty_c(M)$) the space of smooth complex-valued (resp. compactly supported) functions over M. A linear map $A : C^\infty_c(M) \to C^\infty(M)$ is a pseudodifferential operator of order $m \in \mathbb{R}$ if for every coordinate chart $(x^1, \ldots, x^n)$ over an open subset $U \subset M$, there exists a smooth function $a \in C^\infty(U \times \mathbb{R}^n)$ such that

$$(A \cdot f)(x) = \frac{1}{(2\pi)^n} \int_{U\times\mathbb{R}^n} e^{\mathrm{i}p\cdot(x-y)} a(x,p)\, f(y)\, dy\, dp \tag{9}$$

for any $f \in C^\infty_c(U)$. We use the notation $\mathrm{i} = \sqrt{-1}$. For any multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, the symbol a has to satisfy the estimate

$$|\partial_x^\alpha \partial_p^\beta a(x,p)| \leq C_{\alpha,\beta}(1 + \|p\|)^{m-|\beta|} \tag{10}$$

for some constant $C_{\alpha,\beta}$, where $|\beta| = \beta_1 + \ldots + \beta_n$, $\partial_x = \frac{\partial}{\partial x}$ and $\partial_p = \frac{\partial}{\partial p}$ are the partial derivatives with respect to the variables $x = (x^1, \ldots, x^n)$ and $p = (p_1, \ldots, p_n)$, and $\|p\|$ is the euclidian norm of $p \in \mathbb{R}^n$. Note that (x,p) is the canonical coordinate system on the cotangent bundle $T^*U \cong U \times \mathbb{R}^n$. In addition, A is a *classical* (one-step polyhomogeneous) pseudodifferential operator of order m if its symbol in any coordinate chart has an asymptotic expansion as $\|p\| \to \infty$ of the form

$$a(x,p) \sim \sum_{j=0}^\infty a_{m-j}(x,p) \tag{11}$$

where the functions $a_{m-j} \in C^\infty(U \times \mathbb{R}^n)$ are homogeneous of degree $m - j$ with respect to the variable p. For any $m \in \mathbb{R}$, we denote by $\mathrm{CL}^m(M)$ the space of all classical pseudodifferential operators of order $\leq m$. One has $\mathrm{CL}^m(M) \subset \mathrm{CL}^{m'}(M)$ whenever $m \leq m'$. Define as usual the space of all classical pseudodifferential operators and the space of smoothing operators, respectively

$$\mathrm{CL}(M) = \bigcup_{m\in\mathbb{R}} \mathrm{CL}^m(M)\ , \qquad \mathrm{L}^{-\infty} = \bigcap_{m\in\mathbb{R}} \mathrm{CL}^m(M)\ . \tag{12}$$

Two operators in $\mathrm{CL}(M)$ are equal modulo smoothing operators if and only if their asymptotic expansions (11) agree in all coordinate charts. The space of *formal classical symbols* $\mathrm{CS}(M)$ is defined via the exact sequence

$$0 \to \mathrm{L}^{-\infty}(M) \to \mathrm{CL}(M) \to \mathrm{CS}(M) \to 0 \tag{13}$$

Thus, a formal symbol of order m corresponds to a formal series as the right-hand-side of (11) in any local chart, which fulfills complicated gluing formulas

under coordinate-change. $\mathrm{CS}(M)$ is of course the union, for all $m \in \mathbb{R}$, of the subspaces $\mathrm{CS}^m(M)$ of formal symbols of order $\leq m$. Recall that $\mathrm{CS}^m(M)$ is *complete*, in the sense that any formal series of homogeneous functions a_{m-j} is the formal symbol of some pseudodifferential operator. We denote by $\mathrm{PS}(M) \subset \mathrm{CS}(M)$ the subalgebra of formal symbols which are polynomial with respect to the variable p in any chart. $\mathrm{PS}(M)$ is isomorphic to the space of differential operators on M.
The composition of pseudodifferential operators is not always defined, unless these operators are properly supported. This happens in particluar when M is compact. In that case, $\mathrm{CL}(M)$ becomes a filtered associative algebra, i.e. $\mathrm{CL}^m(M) \cdot \mathrm{CL}^{m'}(M) \subset \mathrm{CL}^{m+m'}(M)$, and $\mathrm{L}^{-\infty}(M)$ is a two-sided ideal. Hence (13) is actually an exact sequence of associative algebras. The product of two formal symbols $a, b \in \mathrm{CS}(M)$ in a local chart is the $\star$-product

$$(ab)(x,p) = \sum_{|\alpha|=0}^{\infty} \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} \, \partial_p^\alpha a(x,p) \, \partial_x^\alpha b(x,p) \tag{14}$$

where the sum runs over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\alpha! = \alpha_1! \ldots \alpha_n!$. Notice that, in contrast with $\mathrm{CL}(M)$, the product in $\mathrm{CS}(M)$ is defined without any condition on the support (compact or not) of the symbols.

If E is a (possibly $\mathbb{Z}_2$-graded) complex vector bundle over M, the algebra of classical pseudodifferential operators $\mathrm{CL}(M,E)$ acting on the smooth sections of E is defined analogously. The only difference is that over a local chart which also trivialises E, the symbol becomes a function of (x,p) with values in the matrix algebra $M_k(\mathbb{C})$ where k is the rank of E. One has the exact sequence

$$0 \to \mathrm{L}^{-\infty}(M,E) \to \mathrm{CL}(M,E) \to \mathrm{CS}(M,E) \to 0 \ . \tag{15}$$

$\mathrm{PS}(M,E) \subset \mathrm{CS}(M,E)$ denotes the algebra of polynomial symbols with respect to p. It is isomorphic to the algebra of differential operators acting on the smooth sections of E. In the sequel we will essentially focus on the $\mathbb{Z}_2$-graded bundle $E = \Lambda T^*_{\mathbb{C}} M$, the exterior algebra of the complexified cotangent bundle of M. The smooth sections of E are the complex differential forms over M. Consider the (real) vector bundle $TM \oplus T^*M$ endowed with its canonical inner product. Then E is a spinor representation of the complexified Clifford algebra bundle $C(TM \oplus T^*M)$. In other words, the endomorphism bundle $\mathrm{End}(E)$ is canonically isomorphic to $C(TM \oplus T^*M)$. We use this identification in order to find a set of generators for the algebra $\mathrm{PS}(M,E)$ in a local coordinate system $(x^1, \ldots, x^n)$ over an open $U \subset M$. For each i we view x^i as the multiplication operator of a differential form by the function x^i, and $\mathrm{i}p_i$ as the Lie derivative of a differential form with respect to the vector field $\frac{\partial}{\partial x^i}$. For all indices i, j they fulfill the usual Canonical Commutation Relations

$$[x^i, x^j] = 0 \ , \qquad [x^i, p_j] = \mathrm{i}\delta^i_j \ , \qquad [p_i, p_j] = 0 \tag{16}$$

($\mathrm{i} = \sqrt{-1}$), and generate the even part of the algebra of differential operators $\mathrm{PS}(U,E)$. The odd generators are defined by the operators

$$\psi^i = \mu(dx^i) \ , \qquad \bar{\psi}_i = \iota(\partial_{x^i}) \ , \tag{17}$$

where μ is exterior multiplication by a differential form (on the left) and ι is interior multiplication by a vector field (on the left). These are odd sections of the endomorphism bundle $\mathrm{End}(E)$ over U, and their *graded* commutators (hence anticommutators) fulfill the Clifford relations (Canonical Anticommutation Relations)

$$[\psi^i, \psi^j] = 0 \ , \qquad [\psi^i, \bar\psi_j] = \delta^i_j \ , \qquad [\bar\psi_i, \bar\psi_j] = 0 \tag{18}$$

while the commutators between x, p on one hand and $\psi, \bar\psi$ on the other hand all vanish. The odd operators $\psi, \bar\psi$ generate a basis of sections for $\mathrm{End}(E)$ over U. Hence a differential operator $a \in \mathrm{PS}(M, E)$ is represented over U as a function $a(x, p, \psi, \bar\psi)$ which depends polynomially on the even variable p. Since the odd variables generate a finite-dimensional algebra, a is also a polynomial with respect to $\psi, \bar\psi$. In the same way, any symbol $a \in \mathrm{CS}(M, E)$ of order m is locally represented as a formal series, over $j \in \mathbb{N}$, of functions $a_{m-j}(x, p, \psi, \bar\psi)$ which are homogeneous of degree $m - j$ with respect to p and polynomial with respect to the odd variables $\psi, \bar\psi$.
Let us end this paragraph with the effect of a coordinate change (or local diffeomorphism) γ on the generators $(x, p, \psi, \bar\psi)$ of $\mathrm{CS}(U, E)$. If one puts $\gamma(x^i) = y^i$ for all i, then

$$\gamma(\psi^i) = \mu(dy^i) = \mu\Big(\frac{\partial y^i}{\partial x^j}\, dx^j\Big) = \frac{\partial y^i}{\partial x^j}\, \psi^j \tag{19}$$

where we use Einstein's convention of summation over repeated indices. In the same way

$$\gamma(\bar\psi_i) = \iota(\partial_{y^i}) = \iota\Big(\frac{\partial x^j}{\partial y^i}\, \partial_{x^j}\Big) = \frac{\partial x^j}{\partial y^i}\, \bar\psi_j \ . \tag{20}$$

Finally, the identification of $\mathrm{i}p_i$ with the Lie derivative $\iota(\partial_{x^i}) \circ d + d \circ \iota(\partial_{x^i}) = \bar\psi_i \circ d + d \circ \bar\psi_i$ yields

$$\gamma(p_i) = -\mathrm{i}(\gamma(\bar\psi_i) \circ d + d \circ \gamma(\bar\psi_i)) = \frac{\partial x^j}{\partial y^i}\, p_j - \mathrm{i}\, d\Big(\frac{\partial x^j}{\partial y^i}\Big)\bar\psi_j \ .$$

Since the exterior derivative is $d = \psi^k \partial_{x^k}$, and ψ^k commutes with functions of x, one has

$$\gamma(p_i) = \frac{\partial x^j}{\partial y^i}\, p_j - \mathrm{i}\, \frac{\partial}{\partial x^k}\Big(\frac{\partial x^j}{\partial y^i}\Big)\psi^k \bar\psi_j \ . \tag{21}$$

3 The bimodule of formal symbols

Let M be an n-dimensional manifold and consider the $\mathbb{Z}_2$-graded algebra of formal symbols $\mathrm{CS}(M, E)$ with $E = \Lambda T^*_{\mathbb{C}} M$. We view $\mathrm{CS}(M, E)$ as a left $\mathrm{CS}(M, E)$-module and right $\mathrm{PS}(M, E)$-module: the left action of a formal symbol $a \in \mathrm{CS}(M, E)$ and the right action of a polynomial symbol $b \in \mathrm{PS}(M, E)$ on a vector $\xi \in \mathrm{CS}(M, E)$ read

$$a_L \cdot \xi = a\xi \ , \qquad b_R \cdot \xi = \pm \xi b \ , \tag{22}$$

where the sign $\pm$ depends on the respective parities of b and ξ: it is $-$ if both b and ξ are odd, $+$ otherwise. The left action of a induces a representation of $\mathrm{CS}(M, E)$ in the algebra of linear endomorphisms $\mathrm{End}(\mathrm{CS}(M, E))$. The right action of b induces a representation of the opposite algebra $\mathrm{PS}(M,E)^{\mathrm{op}}$ in $\mathrm{End}(\mathrm{CS}(M,E))$. The left and right actions commute in the graded sense, whence an algebra homomorphism from the (graded) tensor product $\mathrm{CS}(M,E)\otimes \mathrm{PS}(M,E)^{\mathrm{op}}$ to $\mathrm{End}(\mathrm{CS}(M,E))$. This homomorphism is not injective. Its range defines a $\mathbb{Z}_2$-graded algebra

$$\mathscr{L}(M) = \mathrm{Im}\big(\mathrm{CS}(M,E)\otimes \mathrm{PS}(M,E)^{\mathrm{op}} \to \mathrm{End}(\mathrm{CS}(M,E))\big)\ . \tag{23}$$

Thus $\mathscr{L}(M)$ is linearly generated by products $a_L b_R$ with $a \in \mathrm{CS}(M,E)$ and $b \in \mathrm{PS}(M,E)$. Let $(x^1,\dots,x^n)$ be a local coordinate system over an open subset $U\subset M$. The function x^i is a symbol (of order zero) in $\mathrm{PS}(U,E)$, so that x^i_L and x^i_R are elements of $\mathscr{L}(U)$. Moreover for any $\xi\in \mathrm{CS}(U,E)$ one has

$$(x^i_L - x^i_R)\cdot \xi = [x^i,\xi] = \mathrm{i}\frac{\partial \xi}{\partial p_i}\ , \tag{24}$$

whence $x^i_L - x^i_R = \mathrm{i}\frac{\partial}{\partial p_i}$. In the same way the conjugate coordinate p_i is a symbol (of order one) in $\mathrm{PS}(U,E)$, so p_{iL} and p_{iR} are elements of $\mathscr{L}(U)$, and for any $\xi\in\mathrm{CS}(U,E)$,

$$(p_{iL} - p_{iR})\cdot \xi = [p_i,\xi] = -\mathrm{i}\frac{\partial \xi}{\partial x^i}\ , \tag{25}$$

whence $p_{iL}-p_{iR} = -\mathrm{i}\frac{\partial}{\partial x^i}$. The situation is analogous for the odd coordinates ψ^i and $\bar\psi_i$, and one finds that $\psi^i_L - \psi^i_R$ is the partial derivative with respect to $\bar\psi_i$, while $\bar\psi_{iL} - \bar\psi_{iR}$ is the partial derivative with respect to ψ^i. If $b\in \mathrm{PS}^k(M,E)$ is a differential operator of order $k\in\mathbb{N}$, we can write, locally over U

$$b(x,p,\psi,\bar\psi) = \sum_{|\alpha|=0}^{k} b_\alpha(x,\psi,\bar\psi)\, p^\alpha = \sum_{|\alpha|=0}^{k}\sum_{|\eta|=0}^{n}\sum_{|\theta|=0}^{n} b_{\alpha,\eta,\theta}(x)\, p^\alpha\, \psi^\eta\, \bar\psi^\theta\ ,$$

where $b_{\alpha,\eta,\theta}$ are functions of the only variable x, and α,η,θ are multi-indices. Using formula (14) for the star-product, one finds

$$(b_{\alpha,\eta,\theta})_R\cdot\xi = \sum_{|\beta|=0}^{\infty}\frac{(-\mathrm{i})^{|\beta|}}{\beta!}\,(\partial_x^\beta b_{\alpha,\eta,\theta})_L\cdot \partial_p^\beta \xi$$

for any $\xi\in\mathrm{CS}(M,E)$. Since left and right actions commute, the operator b_R reads

$$b_R = \sum_{|\alpha|=0}^{k}\sum_{|\beta|=0}^{\infty}\sum_{|\eta|=0}^{n}\sum_{|\theta|=0}^{n}\frac{(-\mathrm{i})^{|\beta|}}{\beta!}\,(\partial_x^\beta b_{\alpha,\eta,\theta})_L\,(\psi^\eta\bar\psi^\theta)_R\, p_R^\alpha\,\partial_p^\beta$$

Using the identity $p_R = p_L + \mathrm{i}\partial_x$, one concludes that a generic element $a_L b_R \in \mathscr{L}(M)$ can be expressed, locally over a subset $U\subset M$, as a series

$$a_L b_R = \sum_{|\alpha|=0}^{k}\sum_{|\beta|=0}^{\infty}\sum_{|\eta|=0}^{n}\sum_{|\theta|=0}^{n}(s_{\alpha,\beta,\eta,\theta})_L\,(\psi^\eta\bar\psi^\theta)_R\,\partial_x^\alpha\,\partial_p^\beta\ , \tag{26}$$

for some coefficients $s_{\alpha,\beta,\eta,\theta}\in\mathrm{CS}(U,E)$ and finite $k\in\mathbb{N}$. Notice however that the converse is not true: a series (26) with arbitrary coefficients $s_{\alpha,\beta,\eta,\theta}$ does not necessarily come from an element of $\mathscr{L}(M)$.

Now let $\mathscr{S}(M) = \mathscr{L}(M)[[\varepsilon]]$ be the $\mathbb{Z}_2$-graded algebra of *formal power series* in the indeterminate ε, with coefficients in $\mathscr{L}(M)$. The generator ε has trivial grading. An element of $\mathscr{S}(M)$ is therefore an infinite sum $s = \sum_{k=0}^{\infty} s_k \varepsilon^k$ where each coefficient s_k is given by a series of the form (26) in any local chart. We can view $\mathscr{S}(M)$ as an algebra of linear operators acting on the space of formal series $\mathrm{CS}(M,E)[[\varepsilon]]$. This algebra is filtered by the subalgebras $\mathscr{S}_k(M) = \mathscr{S}(M)\varepsilon^k$, $\forall k \in \mathbb{N}$. For each $m \in \mathbb{R}$, we define a subspace $\mathscr{D}^m(M) \subset \mathscr{S}(M)$ as follows. An element $s = \sum s_k \varepsilon^k$ is in $\mathscr{D}^m(M)$ if and only if in any local chart over $U \subset M$,

$$s_k = \sum_{|\alpha|=0}^{k} \sum_{|\beta|=0}^{\infty} \sum_{|\eta|=0}^{n} \sum_{|\theta|=0}^{n} (s^m_{k,\alpha,\beta,\eta,\theta})_L \, (\psi^\eta \bar\psi^\theta)_R \, \partial_x^\alpha \, \partial_p^\beta \tag{27}$$

where $s^m_{k,\alpha,\beta,\eta,\theta} \in \mathrm{CS}(U,E)$ is a symbol of order $\leq m + (k + |\beta| - 3|\alpha|)/2$. Moreover we set $\mathscr{D}^m_k(M) = \mathscr{D}^m(M) \cap \mathscr{S}_k(M)$ for all $k \in \mathbb{N}$. Hence the subscript k counts the minimal power of ε appearing in a formal series. Observe that in local coordinates, the partial derivative ∂_x always appears with at least one power of ε. Here are some examples: $1 \in \mathscr{D}^0_0(M)$, $\mathrm{CS}^m(M,E)_L \subset \mathscr{D}^m_0(M)$, $\varepsilon \in \mathscr{D}^{-1/2}_1(M)$, $\varepsilon\partial_x \in \mathscr{D}^1_1(U)$, $\partial_p \in \mathscr{D}^{-1/2}_0(U)$, and $\varepsilon\partial_x\partial_p \in \mathscr{D}^{1/2}_1(U)$. One obviously has $\mathscr{D}^m(M) \subset \mathscr{D}^{m'}(M)$ whenever $m \leq m'$, and we set $\mathscr{D}(M) = \bigcup_{m\in\mathbb{R}} \mathscr{D}^m(M)$. The following lemma shows that $\mathscr{D}(M)$ is a subalgebra of $\mathscr{S}(M)$.

Lemma 3.1 *The inclusion $\mathscr{D}^m_k(M)\mathscr{D}^{m'}_{k'}(M) \subset \mathscr{D}^{m+m'}_{k+k'}(M)$ holds in all degrees $m, m' \in \mathbb{R}$ and $k, k' \in \mathbb{N}$. Hence $\mathscr{D}(M)$ is a unital, $\mathbb{Z}_2$-graded, bi-filtered algebra.*

Proof: Since ψ_R and $\bar\psi_R$ play no role in the filtration degrees, it suffices to show that, in a local coordinate system over U, the composition $s \circ s'$ of two operators

$$\begin{aligned} s &= \sum_{k=0}^{\infty} \sum_{|\alpha|=0}^{k} \sum_{|\beta|=0}^{\infty} \varepsilon^k (s^m_{k,\alpha,\beta})_L \, \partial_x^\alpha \partial_p^\beta \ \in \mathscr{D}^m(U) \ , \\ s' &= \sum_{k'=0}^{\infty} \sum_{|\alpha'|=0}^{k'} \sum_{|\beta'|=0}^{\infty} \varepsilon^{k'} (s^{m'}_{k',\alpha',\beta'})_L \, \partial_x^{\alpha'} \partial_p^{\beta'} \ \in \mathscr{D}^{m'}(U) \end{aligned}$$

is in $\mathscr{D}^{m+m'}(U)$. Note that the commutator $[\partial_p, \ ]$ on a symbol decreases the order by one, whereas the commutator $[\partial_x, \ ]$ leaves the order unaffected. Hence we can write the composition $\partial_x^\alpha \partial_p^\beta \circ (s^{m'}_{k',\alpha',\beta'})_L$ as a sum

$$\partial_x^\alpha \partial_p^\beta \circ (s^{m'}_{k',\alpha',\beta'})_L = \sum_{|\gamma|=0}^{|\alpha|} \sum_{|\delta|=0}^{|\beta|} (t^{m',\alpha,\beta}_{k',\alpha',\beta',\gamma,\delta})_L \partial_x^\gamma \partial_p^\delta$$

where $t^{m',\alpha,\beta}_{k',\alpha',\beta',\gamma,\delta}$ is a symbol of order $\leq m' - |\beta| + |\delta| + (k' + |\beta'| - 3|\alpha'|)/2$. Then

$$s \circ s' = \sum_{k,k',|\beta|,|\beta'|\geq 0} \sum_{|\alpha|=0}^{k} \sum_{|\alpha'|=0}^{k'} \sum_{|\gamma|=0}^{|\alpha|} \sum_{|\delta|=0}^{|\beta|} \varepsilon^{k+k'} \big(s^m_{k,\alpha,\beta} t^{m',\alpha,\beta}_{k',\alpha',\beta',\gamma,\delta}\big)_L \partial_x^{\gamma+\alpha'} \partial_p^{\delta+\beta'}$$

The symbol $s^m_{k,\alpha,\beta} t^{m',\alpha,\beta}_{k',\alpha',\beta',\gamma,\delta}$ has order $\leq m + m' - |\beta| + |\delta| + \frac{1}{2}(k + k' + |\beta| + |\beta'| - 3|\alpha| - 3|\alpha'|) = m + m' + \frac{1}{2}(k + k' + |\delta + \beta'| - 3|\gamma + \alpha'|) - \frac{3}{2}(|\alpha| - |\gamma|) - \frac{1}{2}(|\beta| - |\delta|)$.

For fixed indices $k, k', \alpha', \beta', \gamma, \delta$ this order is a strictly decreasing function of $|\alpha|$ and $|\beta|$. Moreover $\frac{3}{2}(|\alpha| - |\gamma|) \geq 0$ and $\frac{1}{2}(|\beta| - |\delta|) \geq 0$. Hence by completeness of the space of symbols, the series

$$u^{m+m'}_{k,k',\alpha',\beta',\gamma,\delta} = \sum_{|\alpha|=|\gamma|}^{k} \sum_{|\beta|=0}^{\infty} s^m_{k,\alpha,\beta} t^{m',\alpha,\beta}_{k',\alpha',\beta',\gamma,\delta}$$

converges to a symbol of order $\leq m + m' + \frac{1}{2}(k + k' + |\delta + \beta'| - 3|\gamma + \alpha'|)$. It follows that

$$s \circ s' = \sum_{k,k' \geq 0} \sum_{|\alpha'|=0}^{k'} \sum_{|\gamma|=0}^{k} \sum_{|\beta|=0}^{\infty} \sum_{|\delta|=0}^{|\beta|} \varepsilon^{k+k'} \big(u^{m+m'}_{k,k',\alpha',\beta',\gamma,\delta}\big)_L \partial_x^{\gamma+\alpha'} \partial_p^{\delta+\beta'}$$

is indeed an element of $\mathscr{D}^{m+m'}(U)$. This shows the inclusion $\mathscr{D}^m(M)\mathscr{D}^{m'}(M) \subset \mathscr{D}^{m+m'}(M)$. Furthermore $\mathscr{S}_k(M)\mathscr{S}_{k'}(M) \subset \mathscr{S}_{k+k'}(M)$ is obvious, one concludes that $\mathscr{D}^m_k(M)\mathscr{D}^{m'}_{k'}(M) \subset \mathscr{D}^{m+m'}_{k+k'}(M)$. ∎

Definition 3.2 *An operator* $\Delta \in \mathscr{D}^{1/2}_1(M)$ *of even parity is called a* generalized Laplacian *if in any coordinate system over an open set* $U \subset M$ *it reads*

$$\Delta \equiv \mathrm{i}\varepsilon \frac{\partial}{\partial x^i} \frac{\partial}{\partial p_i} \mod \mathscr{D}^0_1(U) \tag{28}$$

(summation over repeated indices).

Lemma 3.3 *A generalized Laplacian exists over any manifold* M.

Proof: It is actually a consequence of the existence of Dirac operators (section 5) but we can give a direct proof by looking at the behaviour of the canonical "flat" Laplacian $\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$ under a coordinate change $x^i \mapsto \gamma(x^i) = y^i$ over U. One has $\frac{\partial}{\partial x^i} = \mathrm{i}(p_{iL} - p_{iR})$ hence

$$\gamma\Big(\frac{\partial}{\partial x^i}\Big) = \mathrm{i}\gamma(p_i)_L - \mathrm{i}\gamma(p_i)_R \ .$$

Recall that $\gamma(p_i) = \frac{\partial x^j}{\partial y^i} p_j - \mathrm{i} \frac{\partial}{\partial x^k} \frac{\partial x^j}{\partial y^i} \psi^k \bar{\psi}_j$. The operators $(\frac{\partial}{\partial x^k} \frac{\partial x^j}{\partial y^i} \psi^k \bar{\psi}_j)_L$ and $(\frac{\partial}{\partial x^k} \frac{\partial x^j}{\partial y^i} \psi^k \bar{\psi}_j)_R$ belong to $\mathscr{D}^0(U)$, so that

$$\gamma\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\Big) \equiv -\varepsilon\Big(\frac{\partial x^j}{\partial y^i} p_j\Big)_L + \varepsilon\Big(\frac{\partial x^j}{\partial y^i} p_j\Big)_R \mod \mathscr{D}^0_1(U) \ .$$

Then we use the expansion

$$\varepsilon\Big(\frac{\partial x^j}{\partial y^i} p_j\Big)_R = \varepsilon \sum_{|\beta|=0}^{\infty} \frac{(-\mathrm{i})^{|\beta|}}{\beta!} \Big(\partial_x^\beta \frac{\partial x^j}{\partial y^i}\Big)_L (p_j)_R \partial_p^\beta \ .$$

Since $\varepsilon p_R \in \mathscr{D}^1_1(U)$ and $\partial_p \in \mathscr{D}^{-1/2}_0(U)$, the terms in the right-hand side belong to $\mathscr{D}^{1/2}_1(U)$ whenever $|\beta| \geq 1$. Thus we only retain the principal term $|\beta| = 0$:

$$\gamma\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\Big) \equiv \varepsilon\Big(\frac{\partial x^j}{\partial y^i}\Big)_L\big(-p_{jL} + p_{jR}\big) \equiv \mathrm{i}\varepsilon\Big(\frac{\partial x^j}{\partial y^i}\Big)_L \frac{\partial}{\partial x^j} \mod \mathscr{D}^{1/2}_1(U) \ .$$

We proceed in the same way with $\frac{\partial}{\partial p_i} = -\mathrm{i}x^i_L + \mathrm{i}x^i_R$:

$$\gamma\Big(\frac{\partial}{\partial p_i}\Big) = -\mathrm{i}\gamma(x^i)_L + \mathrm{i}\gamma(x^i)_R = -\mathrm{i}y^i_L + \mathrm{i}y^i_R \ .$$

We use the expansion

$$y^i_R = \sum_{|\beta|=0}^{\infty} \frac{(-\mathrm{i})^{|\beta|}}{\beta!} (\partial_x^\beta y^i)_L \partial_p^\beta \ .$$

Since $\partial_p^\beta \in \mathscr{D}_0^{-|\beta|/2}(U)$, we only retain the principal terms $|\beta|=1$ in the following sum:

$$\gamma\Big(\frac{\partial}{\partial p_i}\Big) = \mathrm{i} \sum_{|\beta|=1}^{\infty} \frac{(-\mathrm{i})^{|\beta|}}{\beta!} (\partial_x^\beta y^i)_L \partial_p^\beta \equiv \Big(\frac{\partial y^i}{\partial x^j}\Big)_L \frac{\partial}{\partial p_j} \mod \mathscr{D}_0^{-1}(U) \ .$$

Finally we can write

$$\begin{aligned}
\gamma\Big(\frac{\partial}{\partial p_i}\Big)\gamma\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\Big) \quad &\equiv \quad \Big(\Big(\frac{\partial y^i}{\partial x^j}\Big)_L \frac{\partial}{\partial p_j} \mod \mathscr{D}_0^{-1}\Big)\Big(\mathrm{i}\varepsilon\Big(\frac{\partial x^j}{\partial y^i}\Big)_L \frac{\partial}{\partial x^j} \mod \mathscr{D}_1^{1/2}\Big) \\
&\equiv \quad \mathrm{i}\varepsilon \frac{\partial}{\partial p_j}\frac{\partial}{\partial x^j} \mod \big(\mathscr{D}_0^{-1}\mathscr{D}_1^{1} + \mathscr{D}_0^{-1/2}\mathscr{D}_1^{1/2}\big)
\end{aligned}$$

This shows that the operator $\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$ is invariant modulo $\mathscr{D}_1^0(U)$ under coordinate change. Now let (c_I) be a partition of unity associated to an atlas (U_I, x_I) on M. Denoting by Δ_I the canonical flat Laplacian in local coordinates x_I, the sum

$$\Delta = \sum_I (c_I)_L \Delta_I$$

globally defines a generalized Laplacian on M. ∎

Observe that a generalized Laplacian Δ carries at least one power of ε, hence any formal power series of Δ is a well-defined element of $\mathscr{S}(M)$. For example, for any parameter $t \in \mathbb{R}$ the exponential

$$\exp(t\Delta) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\,\Delta^k \tag{29}$$

is an invertible element of $\mathscr{S}(M)$, with inverse $\exp(-t\Delta)$. However, these elements do not belong to $\mathscr{D}(M)$. We define an automorphism σ^t_Δ of the algebra $\mathscr{S}(M)$ as follows:

$$\sigma^t_\Delta(s) = \exp(t\Delta)\, s\, \exp(-t\Delta) \qquad \forall s \in \mathscr{S}(M) \ . \tag{30}$$

Clearly $\sigma^t_\Delta \circ \sigma^{t'}_\Delta = \sigma^{t+t'}_\Delta$ so the map $t \mapsto \sigma^t_\Delta$ defines a one-parameter group of automorphisms.

Lemma 3.4 *For any generalized Laplacian Δ, the automorphism group σ_Δ preserves the subalgebra $\mathscr{D}(M)$. More precisely one has $[\Delta, \mathscr{D}^m_k(M)] \subset \mathscr{D}^m_{k+1}(M)$ and $\sigma^t_\Delta(\mathscr{D}^m_k(M)) = \mathscr{D}^m_k(M)$ for all $m \in \mathbb{R}$, $k \in \mathbb{N}$ and $t \in \mathbb{R}$.*

Proof: In local coordinates $\Delta = \mathrm{i}\varepsilon \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} + r$ with $r \in \mathscr{D}^0_1$. Hence for any $s \in \mathscr{D}^m_k$

$$[\Delta, s] = \mathrm{i}\varepsilon(\partial_x s\, \partial_p + \partial_p s\, \partial_x + \partial_x \partial_p s) + [r, s] \ .$$

One has $\varepsilon\partial_x s \partial_p \in \mathscr{D}^{m-1}_{k+1}$, $\varepsilon\partial_p s \partial_x \in \mathscr{D}^m_{k+1}$, $\varepsilon\partial_x\partial_p s \in \mathscr{D}^{m-3/2}_{k+1}$, $rs \in \mathscr{D}^m_{k+1}$ and $sr \in \mathscr{D}^m_{k+1}$. Hence $[\Delta, \mathscr{D}^m_k(M)] \subset \mathscr{D}^m_{k+1}(M)$ as claimed.
Next we show $\exp(t\Delta)\mathscr{D}^m_k(M)\exp(-t\Delta) \subset \mathscr{D}^m_k(M)$ for all m, k. Replacing t by $-t$ then gives the inverse inclusion. For $s \in \mathscr{D}^m_k(M)$ consider the identity

$$\exp(t\Delta)\, s\, \exp(-t\Delta) = \sum_{l=0}^{\infty} \frac{t^l}{l!}\, s^{(l)}$$

where $s^{(l)}$ denotes the l-th power of the derivation $[\Delta,\ ]$ on s. By induction one has $s^{(l)} \in \mathscr{D}^m_{k+l}(M)$ for any $l \geq 0$ so that the infinite sum over l gives a well-defined element of $\mathscr{D}^m_k(M)$. ■

Lemma 3.5 *Let $\Delta + s$ be a perturbation of a generalized Laplacian Δ, with $s \in \mathscr{D}^0_1(M)$. Then the Duhamel formula holds in $\mathscr{S}(M)$:*

$$\exp(\Delta + s) = \sum_{k=0}^{\infty} \int_{\Delta_k} \exp(t_0\Delta)\, s\, \exp(t_1\Delta)\, s \ldots s\, \exp(t_k\Delta)\, dt \ , \tag{31}$$

where $\Delta_k = \{(t_0, \ldots, t_k) | \sum_{i=0}^k t_i = 1\}$ is the standard k-simplex and $dt = dt_0 \ldots dt_{k-1}$.

Proof: Since the exponential of a generalized Laplacian is defined by its formal power series, the identity (31) which holds at a formal level makes sense in $\mathscr{S}(M)$. Indeed $s \in \mathscr{D}^0_k$ carries at least one power of ε, so that the product $\exp(t_0\Delta)s\exp(t_1\Delta)s\ldots s\exp(t_k\Delta)$ is in $\mathscr{S}_k(M)$, and its expansion in powers of ε has polynomial coefficients with respect to $(t_0, \ldots, t_k)$. Hence the integral over the simplex Δ_k gives a well-defined element of $\mathscr{S}_k(M)$, and the infinite sum over k converges in $\mathscr{S}(M)$. ■

Notice that the Duhamel formula can be rewritten by means of the automorphism group σ_Δ as follows:

$$\exp(\Delta + s) = \sum_{k=0}^{\infty} \int_{\Delta_k} \sigma^{t_0}_\Delta(s)\sigma^{t_0+t_1}_\Delta(s)\ldots\sigma^{t_0+\ldots+t_{k-1}}_\Delta(s)\, \exp(\Delta)\, dt \tag{32}$$

Fix a generalized Laplacian Δ and consider the following vector subspace of $\mathscr{S}(M)$:

$$\mathscr{T}(M) = \mathscr{D}(M)\exp\Delta \tag{33}$$

Proposition 3.6 *$\mathscr{T}(M)$ is a $\mathscr{D}(M)$-bimodule and does not depend on the choice of generalized Laplacian. We call $\mathscr{T}(M)$ the bimodule of* trace-class operators.

Proof: $\mathscr{T}(M)$ is clearly a left $\mathscr{D}(M)$-module. Moreover by Lemma 3.4, one has $\mathscr{D}(M)\exp(\Delta)\mathscr{D}(M) = \mathscr{D}(M)\sigma^1_\Delta(\mathscr{D}(M))\exp\Delta = \mathscr{D}(M)\exp\Delta$ hence $\mathscr{T}(M)$ is a right $\mathscr{D}(M)$-module. Further on, if Δ and Δ' are two Laplacians, then

$\Delta' = \Delta + s$ with $s \in \mathscr{D}_1^0(M)$. We know that $\sigma_\Delta^t(s) \in \mathscr{D}_1^0(M)$ for any $t \in \mathbb{R}$, so the series

$$S = \sum_{k=0}^{\infty} \int_{\Delta_k} \sigma_\Delta^{t_0}(s)\sigma_\Delta^{t_0+t_1}(s)\ldots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(s)\,dt$$

converges in $\mathscr{D}^0(M)$. Hence $\exp\Delta' = S\exp\Delta$ by the Duhamel formula. In the same way $\exp\Delta = S'\exp\Delta'$. Therefore $\mathscr{D}(M)\exp\Delta' = \mathscr{D}(M)\exp\Delta$, and $\mathscr{T}(M)$ does not depend on the choice of generalized Laplacian. ∎

$\mathscr{T}(M)$ is not a subalgebra of $\mathscr{S}(M)$. For example the product $\exp(\Delta)\exp(\Delta) = \exp(2\Delta)$ does not belong to the space of trace-class operators.

4 Canonical trace

Let M be a closed manifold. The Wodzicki residue ([12]) is a canonical trace on the algebra of classical pseudodifferential operators $\mathrm{CL}(M)$. It is in fact the unique trace (up to a numerical factor) on $\mathrm{CL}(M)$ when the manifold has dimension $n > 1$. The Wodzicki residue vanishes on $\mathrm{CL}^m(M)$ whenever $m < -n$, hence vanishes on the ideal of smoothing operators $\mathrm{L}^{-\infty}(M)$, so that it is really a trace on the algebra of formal symbols $\mathrm{CS}(M)$. Wodzicki gives a concrete formula for the residue of a symbol $a \in \mathrm{CS}^m(M)$ in terms of its expansion $a(x,p) = \sum_j a_{m-j}(x,p)$ in a local system of canonical coordinates over an open subset $U \subset M$. Let $\omega = dp_i \wedge dx^i$ be the symplectic two-form on the cotangent bundle $T^*U \subset T^*M$ (summation over repeated indices). Then T^*U is canonically oriented by the volume form $\omega^n/n! = dp_1 \wedge dx^1 \ldots dp_n \wedge dx^n$. The cosphere bundle S^*U inherits this orientation. The Wodzicki residue of a symbol $a(x,p)$ with compact x-support over U is the integral of a $(2n-1)$-form

$$⨍ a = \frac{1}{(2\pi)^n} \int_{S^*U} \iota(L) \cdot \Big(a_{-n}(x,p)\, \frac{\omega^n}{n!} \Big) \ , \tag{34}$$

where a_{-n} is the degree $-n$ component of the symbol and $L = p_i \frac{\partial}{\partial p_i}$ is the fundamental vector field on T^*U . We can write

$$\iota(L) \cdot \frac{\omega^n}{n!} = (\iota(L)\cdot\omega) \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{\eta \wedge \omega^{n-1}}{(n-1)!}$$

where $\eta = p_i dx^i$ is the canonical one-form on T^*U. It is non-trivial to check that the Wodzicki residue is a trace and does not depend on the choice of coordinate system. Hence such expressions can be patched together using a partition of unity, allowing to define the residue of a symbol a with arbitrary support. If E is a ($\mathbb{Z}_2$-graded) complex vector bundle over M, one defines analogously the Wodzicki residue as a (graded) trace on the algebra $\mathrm{CS}(M,E)$: at each point (x,p) the symbol $a_{-n}(x,p)$ is now a endomorphism acting on the fibre E_x, so (34) has to be modified according to

$$⨍ a = \frac{1}{(2\pi)^n} \int_{S^*U} \iota(L) \cdot \Big(\mathrm{tr}_s\big(a_{-n}(x,p)\big)\, \frac{\omega^n}{n!} \Big) \ , \tag{35}$$

where tr_s is the (graded) trace of endomorphisms. We focus on $E = \Lambda T^*_{\mathbb{C}}M$. In a local coordinate system over U we know that a basis of sections of $\mathrm{End}(E)$ is provided by all products of ψ^i or $\bar{\psi}_j$, $i,j=1,\ldots,n$ among themselves, taking the Clifford relations (18) into account. A symbol $a \in \mathrm{CS}(U,E)$ may thus be decomposed into a finite sum over multi-indices $\eta = (\eta_1,\ldots,\eta_n)$, $\theta = (\theta_1,\ldots,\theta_n)$,

$$a(x,p,\psi,\bar{\psi}) = \sum_{\eta,\theta} a^{\eta,\theta}(x,p)\, \psi^\eta \bar{\psi}^\theta \ , \tag{36}$$

where the coefficients $a^{\eta,\theta}$ are functions of x and p only. It is easy to see that the graded trace of endomorphisms, which acts on polynomials $\psi^\eta\bar{\psi}^\theta$, vanishes whenever $(|\eta|,|\theta|) \neq (n,n)$ and is normalized as follows on the polynomial of highest weight:

$$\mathrm{tr}_s(\psi^1 \ldots \psi^n \bar{\psi}_n \ldots \bar{\psi}_1) = (-1)^n \ . \tag{37}$$

An equivalent normalization is $\mathrm{tr}_s(\Pi) = 1$ where $\Pi = \bar{\psi}_1\psi^1 \ldots \bar{\psi}_n\psi^n$ is the projection operator from the space of differential forms $\Omega^*(U)$ to the subspace of scalar functions $\Omega^0(U)$.

In section 3 we introduced the algebra $\mathscr{S}(M)$ acting on the space of formal power series $\mathrm{CS}(M,E)[[\varepsilon]]$, its subalgebra $\mathscr{D}(M) \subset \mathscr{S}(M)$, and the $\mathscr{D}(M)$-bimodule of trace-class operators $\mathscr{T}(M) \subset \mathscr{S}(M)$. By means of the Wodzicki residue, our goal now is to construct a graded trace on $\mathscr{T}(M)$, that is, a linear map $\mathscr{T}(M) \to \mathbb{C}$ vanishing on the subspace of graded commutators $[\mathscr{D}(M), \mathscr{T}(M)]$. We start by doing this locally on an open subset $U \subset M$. Choose a coordinate system (x,p) over U and fix the canonical "flat" Laplacian $\Delta = \mathrm{i}\varepsilon \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$. For all multi-indices α and β set

$$\langle \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle = \partial_x^\alpha \partial_p^\beta \cdot \exp\Big(\frac{\mathrm{i}}{\varepsilon}(p_i - q_i)(x^i - y^i)\Big)\Big|_{\substack{x=y\\p=q}} \tag{38}$$

For example one has

$$\langle \exp\Delta \rangle = 1 \ , \quad \langle \partial_{x^i} \exp\Delta\rangle = 0 = \langle \partial_{p_j}\exp\Delta\rangle \ , \quad \langle \partial_{x^i}\partial_{p_j}\exp\Delta\rangle = \frac{\mathrm{i}}{\varepsilon}\,\delta_i^j$$

and more generally with a polynomial $\partial_x^\alpha\partial_p^\beta$ the formula involves all possible contractions between ∂_x and ∂_p. In particular

$$\langle \partial_{x^i}\partial_{x^j}\partial_{p_k}\partial_{p_l}\exp\Delta\rangle = \Big(\frac{\mathrm{i}}{\varepsilon}\Big)^2 (\delta_i^k\delta_j^l + \delta_i^l\delta_j^k) \ .$$

Notice that $\langle \partial_x^\alpha\partial_p^\beta \exp\Delta\rangle$ vanishes unless $|\alpha| = |\beta|$. We define similarly a contraction map for the polynomials in the odd variables $\psi_R, \bar{\psi}_R$. If $(\psi^\eta\bar{\psi}^\theta)_R$ is a generic product with multi-indices η,θ set

$$\langle(\psi^\eta\bar{\psi}^\theta)_R\rangle = (-1)^n \mathrm{tr}_s(\psi^\eta\bar{\psi}^\theta) \ . \tag{39}$$

Hence from the normalization (37) holds $\langle \psi^1\ldots\psi^n\bar{\psi}_n\ldots\bar{\psi}_1\rangle = 1$, and the contraction vanishes on polynomials of lower degree. The even and odd contractions assemble in a linear map

$$\langle\!\langle\ \rangle\!\rangle : \mathscr{T}(U) \to \mathrm{CS}(U,E)[[\varepsilon]] \tag{40}$$

defined as follows. Let $s = \sum_{k=0}^{\infty} s_k \varepsilon^k$ belong to $\mathscr{D}^m(U)$, so that $s \exp\Delta$ is a generic element of $\mathscr{T}(U)$. We can write, for all components $s_k \in \mathscr{L}(U)$,

$$s_k = \sum_{|\alpha|=0}^{k} \sum_{|\beta|=0}^{\infty} \sum_{|\eta|=0}^{n} \sum_{|\theta|=0}^{n} (s_{k,\alpha,\beta,\eta,\theta})_L \, (\psi^\eta \bar{\psi}^\theta)_R \, \partial_x^\alpha \, \partial_p^\beta$$

with $s_{k,\alpha,\beta,\eta,\theta} \in \mathrm{CS}(U,E)$ a symbol of order $\leq m + (k + |\beta| - 3|\alpha|)/2$. Set

$$\langle\!\langle s_k \exp\Delta \rangle\!\rangle = \sum_{|\alpha|=0}^{k} \sum_{|\beta|=0}^{\infty} \sum_{|\eta|=0}^{n} \sum_{|\theta|=0}^{n} s_{k,\alpha,\beta,\eta,\theta} \, \langle(\psi^\eta \bar{\psi}^\theta)_R\rangle \langle \partial_x^\alpha \, \partial_p^\beta \exp\Delta\rangle \ .$$

Observe that the sum over α is finite, as is the sum over β because of the contractions $\langle \partial_x^\alpha \, \partial_p^\beta \exp\Delta\rangle$. Hence $\langle\!\langle s_k \exp\Delta \rangle\!\rangle$ is a polynomial of degree at most k in the indeterminate ε^{-1}, with coefficients in $\mathrm{CS}(U,E)$. Consequently $\langle\!\langle s_k \exp\Delta \rangle\!\rangle \varepsilon^k$ is a polynomial in ε of degree at most k, with coefficients in $\mathrm{CS}(U,E)$. However it is not at all obvious that the sum

$$\langle\!\langle s \exp\Delta \rangle\!\rangle = \sum_{k=0}^{\infty} \langle\!\langle s_k \exp\Delta \rangle\!\rangle \varepsilon^k \tag{41}$$

makes sense even in the space of formal series $\mathrm{CS}(U,E)[[\varepsilon]]$. The completeness of the space of symbols is an essential ingredient of the following lemma.

Lemma 4.1 *$\langle\!\langle s \exp\Delta \rangle\!\rangle$ is a well-defined element of $\mathrm{CS}(U,E)[[\varepsilon]]$ for any $s \in \mathscr{D}(U)$.*

Proof: Let $s \in \mathscr{D}^m(U)$. For each power $l \in \mathbb{N}$, we have to show that the coefficient of ε^l in the formal series

$$\langle\!\langle s \exp\Delta \rangle\!\rangle = \sum_{k=0}^{\infty} \sum_{|\alpha|=0}^{k} \sum_{|\beta|=0}^{\infty} \sum_{|\eta|=0}^{n} \sum_{|\theta|=0}^{n} s_{k,\alpha,\beta,\eta,\theta} \, \langle(\psi^\eta \bar{\psi}^\theta)_R\rangle \langle \partial_x^\alpha \, \partial_p^\beta \exp\Delta\rangle \varepsilon^k$$

is a well-defined element of $\mathrm{CS}(U,E)$. The contraction $\langle \partial_x^\alpha \, \partial_p^\beta \exp\Delta\rangle$ forces $|\beta| = |\alpha|$, hence the symbol $s_{k,\alpha,\beta,\eta,\theta}$ has order $\leq m + k/2 - |\alpha|$. Moreover $\langle \partial_x^\alpha \, \partial_p^\beta \exp\Delta\rangle$ brings a factor $\varepsilon^{-|\alpha|}$. It follows that for fixed $l \in \mathbb{N}$, the coefficient of ε^l in the above series is proportional to

$$a_l = \sum_{k=0}^{\infty} \sum_{|\alpha|=k-l} a_{k,\alpha}$$

where $a_{k,\alpha}$ is a symbol of order $\leq m + k/2 - |\alpha| = m + l - k/2$. Since m and l are fixed, the order of $a_{k,\alpha}$ is a strictly decreasing function of k, hence a_l converges in $\mathrm{CS}(U,E)$. ∎

Let $\mathscr{D}_c(U) \subset \mathscr{D}(U)$ and $\mathscr{T}_c(U) \subset \mathscr{T}(U)$ be the subspaces of operators with compact x-support on U. Any element of $\mathscr{T}_c(U)$ reads $s \exp\Delta$ for some $s \in \mathscr{D}_c(U)$, and $\mathscr{T}_c(U)$ is a $\mathscr{D}(U)$-bimodule.

Lemma 4.2 *Let* $\langle\!\langle s \exp \Delta \rangle\!\rangle[n] \in \mathrm{CS}(U,E)$ *be the coefficient of* ε^n*,* $n = \dim M$*, in the formal series* $\langle\!\langle s \exp \Delta \rangle\!\rangle$*. The linear map* $\mathrm{Tr}_s^U : \mathscr{T}_c(U) \to \mathbb{C}$ *defined by*

$$\mathrm{Tr}_s^U(s \exp \Delta) = ⨍ \langle\!\langle s \exp \Delta \rangle\!\rangle[n] \ , \qquad \forall \ s \in \mathscr{D}_c(U) \ , \tag{42}$$

is a graded trace on the space of compactly-supported trace-class operators viewed as a $\mathscr{D}(U)$*-bimodule.*

Proof: In fact we will show that the map $\mathscr{T}_c(U) \to \mathbb{C}[[\varepsilon]]$ defined by

$$s \exp \Delta \mapsto ⨍ \langle\!\langle s \exp \Delta \rangle\!\rangle$$

is a graded trace. Selecting the coefficient of ε^n then yields Tr_s^U. By linearity it is sufficient to check the trace property on operators $s \in \mathscr{D}(U)$ which depend *polynomially* on ε and the partial derivatives ∂_x and ∂_p. So let $s = s_k \varepsilon^k$,

$$s_k = a_L (\psi^\eta \bar{\psi}^\theta)_R \partial_x^\alpha \partial_p^\beta$$

be such an operator, for some $a \in \mathrm{CS}(U,E)$ and multi-indices $\alpha, \beta, \eta, \theta$. It is enough to show that

$$⨍ \langle\!\langle (s \exp \Delta) s' \rangle\!\rangle = \pm ⨍ \langle\!\langle s' s \exp \Delta \rangle\!\rangle \tag{43}$$

in the following cases: $s' = b_L$ for a symbol $b \in \mathrm{CS}(U,E)$, or $s' = \partial_x$, ∂_p, ψ_R, $\bar{\psi}_R$. The sign must be $-$ if s and s' are both odd, $+$ otherwise. Since the contraction map involves the supertrace on the Clifford algebra generated by $\psi_R, \bar{\psi}_R$, (43) is obvious when $s' = \psi_R$ or $\bar{\psi}_R$. Then for $s' = \frac{\partial}{\partial x^i}$ one has

$$⨍ \langle\!\langle [s', s \exp \Delta] \rangle\!\rangle = ⨍ \frac{\partial a}{\partial x^i} \langle (\psi^\eta \bar{\psi}^\theta)_R \rangle \langle \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle \varepsilon^k$$

The Wodzicki residue vanishes on the derivative $\partial a / \partial x^i$, hence (43) is verified. The case $s' = \frac{\partial}{\partial p_i}$ is similar. It remains to deal with the case $s' = b_L$ for a symbol b. If $F(\partial_x, \partial_p)$ is any formal power series with respect to the variables $X = \partial_x$ and $P = \partial_p$, one has the identity

$$F(\partial_x, \partial_p) \circ b_L = \sum_{|\gamma|=0}^{\infty} \sum_{|\delta|=0}^{\infty} \frac{1}{\gamma! \delta!} (\partial_x^\gamma \partial_p^\delta b)_L \, \partial_X^\gamma \partial_P^\delta F(\partial_x, \partial_p) \ .$$

Applying this to the series $F(\partial_x, \partial_p) = \partial_x^\alpha \partial_p^\beta \exp \Delta$ one gets

$$\langle (a_L \partial_x^\alpha \partial_p^\beta \exp \Delta) b_L \rangle = \sum_{|\gamma|=0}^{\infty} \sum_{|\delta|=0}^{\infty} \frac{1}{\gamma! \delta!} \langle (a \partial_x^\gamma \partial_p^\delta b)_L \, \partial_X^\gamma \partial_P^\delta (\partial_x^\alpha \partial_p^\beta \exp \Delta) \rangle \ .$$

But the contraction map vanishes on a derivative $\partial_X F(\partial_x, \partial_p)$ or $\partial_P F(\partial_x, \partial_p)$. This only selects the terms $|\gamma| = |\delta| = 0$:

$$\langle (a_L \partial_x^\alpha \partial_p^\beta \exp \Delta) b_L \rangle = \langle (ab)_L \, \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle = ab \, \langle \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle \ .$$

Finally, the Wodzicki residue is a trace on the algebra of (compactly supported) symbols, hence

$$-\!\!\!\!\!\!\int \langle (a_L \partial_x^\alpha \partial_p^\beta \exp \Delta) b_L \rangle = -\!\!\!\!\!\!\int ba \, \langle \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle = -\!\!\!\!\!\!\int \langle b_L a_L \partial_x^\alpha \partial_p^\beta \exp \Delta \rangle \ .$$

This shows that (43) is verified for $s' = b_L$ as well. ∎

Proposition 4.3 *The map* Tr_s^U *does not depend on the choice of coordinate system* (x,p) *over* T^*U*. Hence using a partition of unity relative to an open covering of* M*, such maps can be patched together, giving rise to a* canonical *graded trace*

$$\mathrm{Tr}_s : \mathscr{T}(M) \to \mathbb{C} \tag{44}$$

on the $\mathscr{D}(M)$*-bimodule of trace-class operators.*

Proof: First observe that under an *affine* change of coordinates $x^i \mapsto y^i$, $p_i \mapsto \frac{\partial x^j}{\partial y^i} p_j$, the flat laplacian $\Delta = \mathrm{i}\varepsilon \frac{\partial}{\partial x^i} \frac{\partial}{\partial p_i}$ is invariant, as well as Eq. (38). It follows that the contraction map $\mathscr{T}(U) \to \mathrm{CS}(U,E)[[\varepsilon]]$ is equivariant under affine transformations. Since the Wodzicki residue is also invariant, it follows that the trace Tr_s^U is invariant under affine transformations.
Now let γ be any (smooth) change of coordinates. By linearity it is enough to show that, if $t \in \mathscr{T}(U)$ has support in an arbitrary small neighborhood of a point $x_0 \in U$, then $\mathrm{Tr}_s^U(t) = \mathrm{Tr}_s^U(\gamma(t))$. After composition with an appropriate affine transformation, we can even suppose that γ leaves the point x_0 and its tangent space $T_{x_0}U$ fixed. Then there exists a small neighborhood V of x_0 such that the restriction of γ to the domain V is a diffeomorphism homotopic to identity. Hence we only need to show that the trace Tr_s^U is invariant under infinitesimal transformations induced by vector fields on U. This follows from the fact that such transformations are given by commutators. Indeed let $X = X^i \frac{\partial}{\partial x^i} \in \mathrm{Vect}(U)$ be any smooth vector field and consider the following symbol $L_X \in \mathrm{PS}(U,E)$:

$$L_X = \mathrm{i} X^j p_j + \frac{\partial X^j}{\partial x^k} \psi^k \bar{\psi}_j \ .$$

As an operator on the smooth sections of E (that is, the differential forms over U), L_X corresponds to the Lie derivative along X. One easily checks that its induced action on the generators of the algebra $\mathrm{CS}(U,E)$ reads

$$\begin{aligned} [L_X, x^i] &= X^i \ , \qquad [L_X, p_i] = -\frac{\partial X^j}{\partial x^i} p_j + \mathrm{i} \frac{\partial^2 X^j}{\partial x^i \partial x^k} \psi^k \bar{\psi}_j \ , \\ [L_X, \psi^i] &= \frac{\partial X^i}{\partial x^k} \psi^k \ , \qquad [L_X, \bar{\psi}_i] = -\frac{\partial X^j}{\partial x^i} \bar{\psi}_j \ , \end{aligned}$$

which are the correct transformation laws. Further on, the induced action on the algebra $\mathscr{S}(U)$ is given by the commutator with $(L_X)_L + (L_X)_R \in \mathscr{L}(U)$. Restricting this action to the subspace of trace-class operators $\mathscr{T}(U)$ shows that the trace Tr_s^U vanishes on Lie derivatives. ∎

We end this section with a useful formula in local coordinates (x,p) over $U \subset M$. Let $R = (R^i_j)$ be an $n \times n$ matrix with entries in $\mathbb{C}[[\varepsilon]]$. We suppose that R has no term of degree zero with respect to ε. Hence the Todd series of R and its determinant are well-defined as formal power series in $M_n(\mathbb{C}[[\varepsilon]])$ and $\mathbb{C}[[\varepsilon]]$ respectively:

$$\frac{R}{e^R-1}=1-\frac{1}{2}R+\frac{1}{12}R^2+\dots\ , \qquad \mathrm{Td}(R)=\det\left(\frac{R}{e^R-1}\right)\ . \tag{45}$$

We consider the operator $s=p_{iL}R^i_j\partial_{p_j}=p_L\cdot R\cdot\partial_p$ as a formal perturbation of the flat Laplacian $\Delta=\mathrm{i}\varepsilon\partial_x\cdot\partial_p$. Note however that $\Delta+s$ is not a generalized Laplacian. Then by the Duhamel formula,

$$\exp(\Delta+p_L\cdot R\cdot\partial_p)=\sum_{k=0}^{\infty}\int_{\Delta_k}\big(\sigma_\Delta^{t_0}(s)\sigma_\Delta^{t_0+t_1}(s)\dots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(s)\,\exp\Delta\big)dt$$

is a well-defined element of $\mathscr{T}(U)$. There is an explicit formula computing the contraction of this series with an arbitrary polynomial in the derivatives ∂_x and ∂_p:

Lemma 4.4 *For any multi-indices α and β holds*

$$\langle\partial_x^\alpha\partial_p^\beta\exp(\Delta+p_L\cdot R\cdot\partial_p)\rangle=\mathrm{Td}(R)\,s(R,p) \tag{46}$$

where the symbol $s(R,p)\in\mathrm{CS}(U,E)[[\varepsilon]]$ is a polynomial in p:

$$s(R,p)=\partial_x^\alpha\partial_p^\beta\exp\left(\frac{\mathrm{i}}{\varepsilon}q\cdot R\cdot(x-y)+\frac{\mathrm{i}}{\varepsilon}(p-q)\cdot\frac{R}{1-e^{-R}}\cdot(x-y)\right)\Big|_{\substack{x=y\\p=q}}$$

Proof: The operator $\exp(\Delta+p_L\cdot R\cdot\partial_p)\exp(-\Delta)$ can be expanded as a formal power series in R, whose coefficients depend polynomially on p_L and the partial derivatives ∂_x,∂_p. Thus one has

$$\langle\partial_x^\alpha\partial_p^\beta\exp(\Delta+p_L\cdot R\cdot\partial_p)\rangle=\partial_x^\alpha\partial_p^\beta H_\varepsilon(R,x,y,p,q)\Big|_{\substack{x=y\\p=q}}$$

where $H_\varepsilon(R,x,y,p,q)=\exp(\Delta+p\cdot R\cdot\partial_p)\exp(-\Delta)\big(\exp\big(\frac{\mathrm{i}}{\varepsilon}(p-q)\cdot(x-y)\big)\big)$. We introduce a deformation parameter $t\in[0,1]$ and replace R by tR. The function $H_\varepsilon(tR,x,y,p,q)$ is viewed as a formal power series in t. For $t=0$ it reduces to

$$H_\varepsilon(0,x,y,p,q)=\exp\left(\frac{\mathrm{i}}{\varepsilon}(p-q)\cdot(x-y)\right)\ .$$

We are going to show that $H_\varepsilon(tR,x,y,p,q)$ fulfills a differential equation of first order with respect to t. One has

$$\frac{\partial}{\partial t}\exp(\Delta+p\cdot tR\cdot\partial_p)=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(p\cdot R\cdot\partial_p)^{(n)}\exp(\Delta+p\cdot tR\cdot\partial_p)$$

where the superscript $^{(n)}$ denotes the derivation $X\mapsto[\Delta+p\cdot tR\cdot\partial_p,X]$ applied n times. Hence $(p\cdot R\cdot\partial_p)^{(1)}=[\Delta,p\cdot R\cdot\partial_p]=\mathrm{i}\varepsilon\partial_x\cdot R\cdot\partial_p=\mathrm{i}\varepsilon R^i_j\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_j}$. Furthermore

$$(p\cdot R\cdot\partial_p)^{(n)}=[p\cdot tR\cdot\partial_p,(p\cdot R\cdot\partial_p)^{(n-1)}]=\mathrm{i}\varepsilon(-t)^{n-1}\partial_x\cdot R^n\cdot\partial_p$$

for all $n \geq 2$. Hence we can write

$$
\begin{aligned}
&\frac{\partial}{\partial t} \exp(\Delta + p \cdot tR \cdot \partial_p) \\
&\qquad = \Big(p \cdot R \cdot \partial_p - \mathrm{i}\varepsilon \sum_{n=1}^{\infty} \partial_x \cdot \frac{(-tR)^n}{t(n+1)!} \cdot \partial_p\Big) \exp(\Delta + p \cdot tR \cdot \partial_p) \\
&\qquad = \Big(p \cdot R \cdot \partial_p + t^{-1}\Delta + \mathrm{i}\varepsilon\, \partial_x \cdot \frac{e^{-tR} - 1}{t^2 R} \cdot \partial_p\Big) \exp(\Delta + p \cdot tR \cdot \partial_p)
\end{aligned}
$$

It is important to note that, by construction, the t-expansion of the differential operator $p \cdot R \cdot \partial_p + t^{-1}\Delta + \mathrm{i}\varepsilon\, \partial_x \cdot \frac{e^{-tR}-1}{t^2 R} \cdot \partial_p$ only involves non-negative powers of t. Hence the function H_ε is a solution of the differential equation

$$
\Big(- \frac{\partial}{\partial t} + p \cdot R \cdot \partial_p + t^{-1}\Delta + \mathrm{i}\varepsilon\, \partial_x \cdot \frac{e^{-tR} - 1}{t^2 R} \cdot \partial_p\Big)\, H_\varepsilon(tR, x, y, p, q) = 0
$$

and is uniquely specified, as a formal power series in t, by its value at $t = 0$. A routine computation shows that the Ansatz

$$
H_\varepsilon(tR, x, y, p, q) = \mathrm{Td}(tR) \exp\Big(\frac{\mathrm{i}}{\varepsilon} q \cdot tR \cdot (x - y) + \frac{\mathrm{i}}{\varepsilon}(p - q) \cdot \frac{tR}{1 - e^{-tR}} \cdot (x - y)\Big)
$$

is this unique solution. ■

Let us apply this lemma in some particular cases. One has

$$
\begin{aligned}
\langle \exp(\Delta + p_L \cdot R \cdot \partial_p)\rangle &= \mathrm{Td}(R) \qquad (47)\\
\Big\langle \frac{\partial}{\partial x^i} \exp(\Delta + p_L \cdot R \cdot \partial_p)\Big\rangle &= \frac{\mathrm{i}}{\varepsilon}\, \mathrm{Td}(R)\, (p \cdot R)_i
\end{aligned}
$$

Observe that the right-hand-side of the second equation contains no negative power of ε because R brings at least one factor ε. One thus gets the identity

$$
\Big\langle \Big(\mathrm{i}\varepsilon \frac{\partial}{\partial x^i} + (p_L \cdot R)_i\Big) \exp(\Delta + p_L \cdot R \cdot \partial_p)\Big\rangle = 0 \ . \qquad (48)
$$

More generally for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$:

$$
\big\langle \big(\mathrm{i}\varepsilon\partial_x + p_L \cdot R\big)^\alpha \exp(\Delta + p_L \cdot R \cdot \partial_p)\big\rangle = 0 \ . \qquad (49)
$$

5 Dirac operators

Let M be an n-dimensional manifold and $E = \Lambda T^*_{\mathbb{C}} M$. The space of smooth sections of E is isomorphic to the space $\Omega^*(M)$ of complex differential forms over M. The exterior multiplication of $\Omega^*(M)$ on the sections of E (from the left) gives rise to an homomorphism of algebras

$$
\mu : \Omega^*(M) \to \mathrm{PS}^0(M, E) \ . \qquad (50)
$$

Remark that the algebra $\mathrm{PS}^0(M, E)$ of differential operators of order zero is isomorphic to the algebra of smooth sections of the endomorphism bundle $\mathrm{End}(E)$. The map μ is injective. In a local coordinate system $(x^1, \dots, x^n)$ over $U \subset M$, the image of a k-form $\alpha = \alpha_{i_1\ldots i_k}(x)dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is the endomorphism $\mu(\alpha) = \alpha_{i_1\ldots i_k}(x)\psi^{i_1}\ldots\psi^{i_k}$. Also, the operation of interior multiplication by vector fields on the sections of E gives rise to an injective linear map

$$\iota : \mathrm{Vect}(M) \to \mathrm{PS}^0(M,E) \ . \tag{51}$$

In local coordinates the image of a vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ is the endomorphism $\iota(X) = X^i(x)\bar\psi_i$. In the sequel we consider $\Omega^*(M)$ and $\mathrm{Vect}(M)$ as subspaces of the algebra of differential operators $\mathrm{PS}(M,E)$. Finally, we introduce another subspace $\mathrm{SPS}^1(M,E) \subset \mathrm{PS}^1(M,E)$ of differential operators, characterized by their expression in any local coordinate system over U as follows:

$$a \in \mathrm{SPS}^1(U,E) \Leftrightarrow a(x,p) = a^i(x)p_i + a^i_j(x)\psi^j\bar\psi_i + b(x) \tag{52}$$

where $a_i, a^i_j, b \in \Omega^0(U)$ are scalar functions. $\mathrm{SPS}^1(M)$ is the space of differential operators of order one, even parity, and *scalar* leading symbol. This definition is coordinate-independent, because under a coordinate change $x^i \mapsto y^i$ one has $p_i \mapsto \frac{\partial x^j}{\partial y^i}p_j - \mathrm{i}\frac{\partial}{\partial x^k}\frac{\partial x^j}{\partial y^i}\psi^k\bar\psi_j$, $\psi^i \mapsto \frac{\partial y^i}{\partial x^j}\psi^j$ and $\bar\psi_i \mapsto \frac{\partial x^j}{\partial y^i}\bar\psi_j$. Moreover, one easily checks that the commutator $[\mathrm{SPS}^1(M,E), \mathrm{SPS}^1(M,E)]$ is again in $\mathrm{SPS}^1(M,E)$. Thus $\mathrm{SPS}^1(M,E)$ is a Lie algebra.
As before we denote $\mathscr{L}(M)$ the subalgebra of linear operators on the vector space $\mathrm{CS}(M,E)$, generated by left multiplication by $\mathrm{CS}(M,E)$ and right multiplication by $\mathrm{PS}(M,E)$. In other words $\mathscr{L}(M) = \mathrm{CS}(M,E)_L\mathrm{PS}(M,E)_R$. From the discussion above we can also form various subspaces of $\mathscr{L}(M)$, for instance $\mathrm{SPS}^1(M,E)_L\Omega^1(M)_R$ or $\Omega^0(M)_L\mathrm{Vect}(M)_R$. The latter operators are easy to characterize in local coordinates:

$$\begin{aligned} s \in \mathrm{SPS}^1(U,E)_L\Omega^1(U)_R \quad &\Leftrightarrow \quad s = \sum_{|\alpha|=0}^{\infty} (s^k_{\alpha i}p_k + s^k_{\alpha ij}\psi^j\bar\psi_k + s_{\alpha i})_L\psi^i_R\partial^\alpha_p \\ r \in \Omega^0(U)_L\mathrm{Vect}(U)_R \quad &\Leftrightarrow \quad r = \sum_{|\alpha|=0}^{\infty} (r^i_\alpha)_L\bar\psi_{iR}\partial^\alpha_p \end{aligned} \tag{53}$$

for some scalar functions $s^k_{\alpha i}, s^k_{\alpha ij}, s_{\alpha i}, r^i_\alpha \in \Omega^0(U)$. From these expressions it is clear that $\mathrm{SPS}^1(M,E)_L\Omega^1(M)_R \subset \mathscr{D}^1_0(M)$ and $\Omega^0(M)_L\mathrm{Vect}(M)_R \subset \mathscr{D}^0_0(M)$. We are now ready to define Dirac operators as particular elements of $\mathscr{D}(M)$.

Definition 5.1 *Suppose that $\nabla \in \mathscr{L}(M)$ and $\overline\nabla \in \mathscr{L}(M)$ are odd operators on* $\mathrm{CS}(M,E)$ *such that, in any local coordinate system over $U \subset M$,*

$$\nabla = \psi^i_R\frac{\partial}{\partial x^i} + s \ , \qquad \overline\nabla = \bar\psi_{iR}\frac{\partial}{\partial p_i} + r \ , \tag{54}$$

with $s \in \mathrm{SPS}^1(U,E)_L\Omega^1(U)_R$ and $r \in \Omega^0(U)_L\mathrm{Vect}(U)_R \cap \mathscr{D}^{-1}_0(U)$. The sum

$$D = \mathrm{i}\varepsilon\nabla + \overline\nabla \ \in \ \mathscr{D}^1_1(M) + \mathscr{D}^{-1/2}_0(M) \tag{55}$$

is called a generalized Dirac operator *on M.*

Hence the operator $\mathrm{i}\varepsilon\nabla$ is locally the sum of its leading part $\mathrm{i}\varepsilon\psi^i_R\frac{\partial}{\partial x^i} \in \mathscr{D}^1_1(U)$ and a perturbation term $\mathrm{i}\varepsilon s \in \mathscr{D}^{1/2}_1(U)$. Similarly $\overline\nabla$ is locally the sum of its leading part $\bar\psi_{iR}\frac{\partial}{\partial p_i}\in\mathscr{D}_0^{-1/2}(U)$ and a perturbation

$$r=\sum_{|\alpha|=2}^{\infty}(r^i_\alpha)_L\bar\psi_{iR}\partial_p^\alpha\ \in\mathscr{D}_0^{-1}(U)\ .$$

In fact $\bar\psi_{iR}\frac{\partial}{\partial p_i}=\mathrm{i}\bar\psi_{iR}(x^i_R-x^i_L)$ so that $\overline\nabla\in\Omega^0(M)_L\mathrm{Vect}(M)_R\cap\mathscr{D}_0^{-1/2}(M)$.

The existence of ∇ and $\overline\nabla$ as global operators on M is not a priori obvious. In order to understand the above definition, we first examine the behaviour of $\psi^i_R\frac{\partial}{\partial x^i}$ under a coordinate change $x^i\mapsto\gamma(x^i)=y^i$. One has

$$\gamma\Big(\psi^i_R\frac{\partial}{\partial x^i}\Big)=\gamma(\psi^i_R)\gamma(\mathrm{i}p_{iL}-\mathrm{i}p_{iR})=\mathrm{i}\gamma(\psi^i)_R\gamma(p_i)_L-\mathrm{i}\gamma(p_i\psi^i)_R\ .$$

$\mathrm{i}p_i\psi^i$ is the symbol of the exterior derivative d on differential forms (see Example 5.3), hence it is ivariant under coordinate change. This can also be checked by direct computation, with $\gamma(p_i)=\frac{\partial x^j}{\partial y^i}p_j-\mathrm{i}\frac{\partial}{\partial x^k}\frac{\partial x^j}{\partial y^i}\psi^k\bar\psi_j$ and $\gamma(\psi^i)=\frac{\partial y^i}{\partial x^l}\psi^l$. One thus has

$$\gamma\Big(\psi^i_R\frac{\partial}{\partial x^i}\Big)=\mathrm{i}\Big(\frac{\partial y^i}{\partial x^l}\psi^l\Big)_R\Big(\frac{\partial x^j}{\partial y^i}p_j-\mathrm{i}\frac{\partial}{\partial x^k}\frac{\partial x^j}{\partial y^i}\psi^k\bar\psi_j\Big)_L-\mathrm{i}(p_i\psi^i)_R\ .$$

Write $-\mathrm{i}(p_i\psi^i)_R=\psi^i_R\frac{\partial}{\partial x^i}-\mathrm{i}p_{iL}\psi^i_R$ and use the commutation of left and right actions:

$$\gamma\Big(\psi^i_R\frac{\partial}{\partial x^i}\Big)=\psi^i_R\frac{\partial}{\partial x^i}-\mathrm{i}p_{iL}\psi^i_R+\Big(\mathrm{i}\frac{\partial x^j}{\partial y^i}p_j+\frac{\partial}{\partial x^k}\frac{\partial x^j}{\partial y^i}\psi^k\bar\psi_j\Big)_L\Big(\frac{\partial y^i}{\partial x^l}\psi^l\Big)_R\ .\quad(56)$$

The right-hand side reads $\psi^i_R\frac{\partial}{\partial x^i}+s$ with $s\in\mathrm{SPS}^1(U,E)_L\Omega^1(U)_R$. We can choose a partition of unity (c_I) relative to an atlas (U_I,x_I) of M and define a global operator ∇ by:

$$\nabla=\sum_I(c_I)_L(\psi^i_I)_R\frac{\partial}{\partial x^i_I}\ .\quad(57)$$

Then (56) shows that ∇ has the required form in any local coordinate system. In order to build a global operator $\overline\nabla$, we proceed analogously and examine how the local operator $\bar\psi_{iR}\frac{\partial}{\partial p_i}$ transforms under coordinate change. One has

$$\gamma\Big(\bar\psi_{iR}\frac{\partial}{\partial p_i}\Big)=\gamma(\bar\psi_{iR})\gamma(\mathrm{i}(x^i_R-x^i_L))=\mathrm{i}\Big(\frac{\partial x^j}{\partial y^i}\bar\psi_j\Big)_R(y^i_R-y^i_L)\ .$$

This still belongs to the subspace $\Omega^0(U)_L\mathrm{Vect}(U)_R\cap\mathscr{D}_0^{-1/2}(U)$. Then use the expansion $y^i_R=\sum_{|\alpha|=0}^\infty\frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}(\partial_x^\alpha y^i)_L\partial_p^\alpha$:

$$\gamma\Big(\bar\psi_{iR}\frac{\partial}{\partial p_i}\Big)=\mathrm{i}\Big(\frac{\partial x^j}{\partial y^i}\bar\psi_j\Big)_R\sum_{|\alpha|=1}^{\infty}\frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}(\partial_x^\alpha y^i)_L\partial_p^\alpha\ .$$

For $|\alpha|\geq 2$ the terms of the series belong to $\mathscr{D}_0^{-1}(U)$. Keeping only the first term ($|\alpha|=1$) one gets

$$\gamma\Big(\bar\psi_{iR}\frac{\partial}{\partial p_i}\Big)\equiv\Big(\frac{\partial x^j}{\partial y^i}\bar\psi_j\Big)_R\Big(\frac{\partial y^i}{\partial x^k}\Big)_L\frac{\partial}{\partial p_k}\ \mathrm{mod}\ \mathscr{D}_0^{-1}(U)\ .$$

But $\left(\frac{\partial x^j}{\partial y^i}\right)_R = \sum_{|\alpha|=0}^\infty \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}\left(\partial_x^\alpha \frac{\partial x^j}{\partial y^i}\right)_L \partial_p^\alpha$ equals $\left(\frac{\partial x^j}{\partial y^i}\right)_L$ modulo $\mathscr{D}_0^{-1/2}(U)$, so

$$\gamma\Big(\bar\psi_{iR}\frac{\partial}{\partial p_i}\Big) \equiv \bar\psi_{jR}\Big(\frac{\partial x^j}{\partial y^i}\frac{\partial y^i}{\partial x^k}\Big)_L\frac{\partial}{\partial p_k} \equiv \bar\psi_{jR}\frac{\partial}{\partial p_j} \mod \mathscr{D}_0^{-1}(U)\ .$$

Hence we have $\gamma(\bar\psi_{iR}\frac{\partial}{\partial p_i}) = \bar\psi_{iR}\frac{\partial}{\partial p_i} + r$ with $r \in \Omega^0(U)_L \mathrm{Vect}(U)_R \cap \mathscr{D}_0^{-1}(U)$. As before we obtain a global operator $\overline\nabla$ on M by gluing local pieces $\bar\psi_{iR}\frac{\partial}{\partial p_i}$ together by means of a partition of unity. This shows that generalized Dirac operators always exist.

Proposition 5.2 *Let D be a generalized Dirac operator on M. Then $-D^2$ is a generalized Laplacian (Definition 3.2).*

Proof: $D = \mathrm{i}\varepsilon\nabla + \overline\nabla$ so $D^2 = \overline\nabla^2 + \mathrm{i}\varepsilon[\nabla,\overline\nabla] - \varepsilon^2\nabla^2$. Since $\overline\nabla \in \Omega^0(M)_L\mathrm{Vect}(M)_R$, one has $\overline\nabla^2 = 0$ (indeed $\Omega^0(M)$ is a commutative algebra, and vector fields anticommute in $\mathrm{PS}^0(M,E)$). Then choose a local coordinate system and write $\nabla = \psi^i_R\frac{\partial}{\partial x^i} + s$ with $s \in \mathrm{SPS}^1_L\Omega^1_R$. One has $(\psi^i_R\frac{\partial}{\partial x^i})^2 = 0$ hence

$$\nabla^2 = \Big(\psi^i_R\frac{\partial}{\partial x^i} + s\Big)^2 = [\psi^i_R\frac{\partial}{\partial x^i}, s] + s^2\ .$$

Recall that $\psi^i_R\frac{\partial}{\partial x^i}$ and s are odd, so the commutator is taken in the graded sense. Let us decompose s as a sum of generic elements a_Lb_R, with $a \in \mathrm{SPS}^1$ (even) and $b \in \Omega^1$ (odd). Then ψ^i and b commute in the graded sense so

$$[\psi^i_R\frac{\partial}{\partial x^i}, a_Lb_R] = \psi^i_R[\frac{\partial}{\partial x^i}, a_Lb_R] = \Big(\frac{\partial a}{\partial x^i}\Big)_L\psi^i_R b_R + a_L\psi^i_R\Big(\frac{\partial b}{\partial x^i}\Big)_R\ .$$

Hence $[\psi^i_R\frac{\partial}{\partial x^i}, s] \in \mathrm{SPS}^1_L\Omega^2_R$. Then writing $s = \sum_I a^I_L b^I_R$, one has

$$s^2 = \sum_{I,J} a^I_L a^J_L b^I_R b^J_R = -\sum_{I,J}(a^Ia^J)_L(b^Jb^I)_R = -\frac12\sum_{I,J}[a^I,a^J]_L(b^Jb^I)_R$$

because b^I and b^J are anticommuting one-forms. SPS^1 is a Lie algebra, hence $[a^I,a^J] \in \mathrm{SPS}^1$ and $s^2 \in \mathrm{SPS}^1_L\Omega^2_R$. This shows that $\nabla^2 \in \mathrm{SPS}^1_L\Omega^2_R \subset \mathscr{D}^1_0$ and $-\varepsilon^2\nabla^2 \in \mathscr{D}^0_2$.

Finally we compute the graded commutator $[\nabla,\overline\nabla]$. Write $\overline\nabla = \bar\psi_{iR}\frac{\partial}{\partial p_i} + r$ with $r \in \Omega^0_L\mathrm{Vect}_R \cap \mathscr{D}_0^{-1}$. One has $[\nabla, \bar\psi_{iR}\frac{\partial}{\partial p_i}] = [\psi^j_R\frac{\partial}{\partial x^j}, \bar\psi_{iR}\frac{\partial}{\partial p_i}] + [s, \bar\psi_{iR}\frac{\partial}{\partial p_i}]$, where

$$[\psi^j_R\frac{\partial}{\partial x^j}, \bar\psi_{iR}\frac{\partial}{\partial p_i}] = [\psi^j_R, \bar\psi_{iR}]\frac{\partial}{\partial x^j}\frac{\partial}{\partial p_i} = -[\bar\psi_i,\psi^j]_R\frac{\partial}{\partial x^j}\frac{\partial}{\partial p_i} = -\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$$

As before decompose s as a sum of generic elements $a_Lb_R \in \mathrm{SPS}^1_L\Omega^1_R$. Since $b = \sum_i b_i(x)\psi^i \in \Omega^1$ does not depend on p,

$$[a_Lb_R, \bar\psi_{iR}\frac{\partial}{\partial p_i}] = a_L[b_R,\bar\psi_{iR}]\frac{\partial}{\partial p_i} + \bar\psi_{iR}\Big(\frac{\partial a}{\partial p_i}\Big)_L b_R = -a_Lb_{iR}\frac{\partial}{\partial p_i} - \Big(\frac{\partial a}{\partial p_i}\Big)_L(b\bar\psi_i)_R$$

One has $a_Lb_{iR} \in \mathrm{SPS}^1_L\Omega^0_R$, hence $a_Lb_{iR}\frac{\partial}{\partial p_i} \in \mathrm{SPS}^1_L\Omega^0_R \cap \mathscr{D}_0^{1/2}$. Moreover $\left(\frac{\partial a}{\partial p_i}\right)_L(b\bar\psi_i)_R \in \Omega^0_L\mathrm{PS}^0_R \subset \mathscr{D}^0_0$. Then $[\mathrm{i}\varepsilon\nabla, r] \in [\mathscr{D}^1_1, \mathscr{D}_0^{-1}] \subset \mathscr{D}^0_1$ so that finally $\mathrm{i}\varepsilon[\nabla,\overline\nabla] \equiv -\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} \mod \mathscr{D}^0_1$. In conclusion

$$D^2 \equiv -\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} \mod (\mathscr{D}^0_1 + \mathscr{D}^0_2) \equiv -\mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} \mod \mathscr{D}^0_1$$

which shows that $-D^2$ is a generalized Laplacian. ■

Example 5.3 Let d be the exterior derivative of differential forms over M. Hence $d \in \mathrm{PS}(M,E)$ is a differential operator of order one. Its right multiplication on $\mathrm{CS}(M,E)$ defines an element of odd degree $d_R \in \mathscr{L}(M)$. In a local coordinate system over U one has $d = \mathrm{i}p_i\psi^i$, hence

$$d_R = \mathrm{i}(p_i\psi^i)_R = \mathrm{i}\psi^i_R p_{iR} = -\psi^i_R \frac{\partial}{\partial x^i} + \mathrm{i}p_{iL}\psi^i_R \tag{58}$$

with $p_{iL}\psi^i_R \in \mathrm{SPS}^1(U,E)_L\Omega^1(U)_R$. This shows that $\nabla = -d_R$ is a possible choice. Adding any $\overline{\nabla}$, the generalized Dirac operator $D = -\mathrm{i}\varepsilon d_R + \overline{\nabla}$ thus obtained will be called a *de Rham-Dirac* operator on M. Note that $\nabla = -d_R$ is completely canonical, only the $\overline{\nabla}$ part requires some choice.

Proposition 5.4 *Let $D = -\mathrm{i}\varepsilon d_R + \overline{\nabla}$ be a de Rham-Dirac operator on M. In a local coordinate system over an open set $U \subset M$, the associated generalized Laplacian reads*

$$\begin{aligned} -D^2 \;=\;& \mathrm{i}\varepsilon\Big(\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} + \sum_{|\alpha|=2}^{\infty}(a^i_\alpha)_L\frac{\partial}{\partial x^i}\Big(\frac{\partial}{\partial p}\Big)^\alpha\Big) \\ &+\varepsilon\Big(p_{iL}\frac{\partial}{\partial p_i} + \sum_{|\alpha|=2}^{\infty}(a^i_\alpha p_i)_L\Big(\frac{\partial}{\partial p}\Big)^\alpha\Big) \\ &+\varepsilon\Big((\psi^i\bar\psi_i)_R + \sum_{|\alpha|=1}^{\infty}(b^i_{\alpha j})_L(\psi^j\bar\psi_i)_R\Big(\frac{\partial}{\partial p}\Big)^\alpha\Big) \end{aligned} \tag{59}$$

where $a^i_\alpha, b^i_{\alpha j} \in \Omega^0(U)$ are scalar functions.

Proof: Since $d^2 = 0$ and $\overline{\nabla}^2 = 0$ one has $-D^2 = \mathrm{i}\varepsilon[d_R, \overline{\nabla}]$. In a local coordinate system one can write $\overline{\nabla} \equiv \bar\psi_{iR}\frac{\partial}{\partial p_i} \mod \Omega^0_L\mathrm{Vect}_R \cap \mathscr{D}_0^{-1}$. Let us calculate

$$[d_R, \bar\psi_{iR}\frac{\partial}{\partial p_i}] = \mathrm{i}[(p_j\psi^j)_R, \bar\psi_{iR}\frac{\partial}{\partial p_i}] = -\mathrm{i}[\bar\psi_i, p_j\psi^j]_R\frac{\partial}{\partial p_i} - \mathrm{i}\bar\psi_{iR}[(p_j\psi^j)_R, \frac{\partial}{\partial p_i}]\ .$$

One has $[\bar\psi_i, p_j\psi^j]_R = (p_j[\bar\psi_i, \psi^j])_R = p_{iR}$ and $\bar\psi_{iR}[(p_j\psi^j)_R, \frac{\partial}{\partial p_i}] = -\bar\psi_{iR}\psi^i_R = (\psi^i\bar\psi_i)_R$, so that

$$[d_R, \bar\psi_{iR}\frac{\partial}{\partial p_i}] = -\mathrm{i}p_{iR}\frac{\partial}{\partial p_i} - \mathrm{i}(\psi^i\bar\psi_i)_R = \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} - \mathrm{i}p_{iL}\frac{\partial}{\partial p_i} - \mathrm{i}(\psi^i\bar\psi_i)_R\ .$$

This gives the three principal terms in (59). The other terms, which are perturbations, come from the commutator of d_R with $\Omega^0_L\mathrm{Vect}_R \cap \mathscr{D}_0^{-1}$. Indeed, a generic element in $\Omega^0_L\mathrm{Vect}_R \cap \mathscr{D}_0^{-1}$ can be expanded as $\sum_{|\alpha|=2}^{\infty}(a^i_\alpha)_L\bar\psi_{iR}\partial^\alpha_p$, with $a^i_\alpha \in \Omega^0$. One has

$$[d_R, (a^i_\alpha)_L\bar\psi_{iR}\partial^\alpha_p] = (a^i_\alpha)_L\big([d_R, \bar\psi_{iR}]\partial^\alpha_p - \bar\psi_{iR}[d_R, \partial^\alpha_p]\big)$$

where $[d_R, \bar\psi_{iR}] = \mathrm{i}[(p_j\psi^j)_R, \bar\psi_{iR}] = -\mathrm{i}[\bar\psi_i, \psi^j]_R p_{jR} = -\mathrm{i}p_{iR} = \frac{\partial}{\partial x^i} - \mathrm{i}p_{iL}$. Moreover $[d_R, \partial_p^\alpha] = \mathrm{i}[(p_j\psi^j)_R, \partial_p^\alpha] = \mathrm{i}\psi_R^j[p_{jR}, \partial_p^\alpha]$ is a sum of terms proportional to $\psi_R^j\partial_p^\beta$ for all multi-indices β such that $|\beta| = |\alpha| - 1$. Hence we can write

$$[d_R, (a_\alpha^i)_L\bar\psi_{iR}\partial_p^\alpha] = (a_\alpha^i)_L\frac{\partial}{\partial x^i}\partial_p^\alpha - \mathrm{i}(a_\alpha^i p_i)_L\partial_p^\alpha - \mathrm{i}\sum_{|\beta|=|\alpha|-1}(b_{\beta j}^i)_L(\psi^j\bar\psi_i)_R\partial_p^\beta$$

where the terms of the right-hand-side contribute to the first, second and third line of (59) respectively. ∎

Remark 5.5 For any generalized Dirac operator $D = \mathrm{i}\varepsilon\nabla + \overline{\nabla}$, we can write

$$\nabla = -d_R + s \quad \text{with} \quad s \in \mathrm{SPS}^1(M,E)_L\Omega^1(M)_R \tag{60}$$

globally on M. This property completely characterizes the class of operators ∇ without reference to any local coordinate system.

Example 5.6 We now give another important example of generalized Dirac operator related to a choice of torsion-free affine connection Γ on M. Such a connection is characterized in any local coordinate system over $U \subset M$ by its Christoffel symbols $\Gamma_{ij}^k(x)$, for $i,j,k = 1,\ldots,n$, which are symmetric with respect to the lower indices ij. Under a coordinate transformation $x^i \mapsto \gamma(x^i) = y^i$ the Christoffel symbols change according to

$$\Gamma_{ij}^k(x) \mapsto {}^\gamma\Gamma_{ij}^k(x) = \frac{\partial x^k}{\partial y^l}\frac{\partial^2 y^l}{\partial x^i\partial x^j} + \frac{\partial x^k}{\partial y^l}\frac{\partial y^p}{\partial x^i}\frac{\partial y^q}{\partial x^j}\Gamma_{pq}^l(y)\ . \tag{61}$$

In the given coordinate system we define a "covariant derivative" operator acting on $\mathrm{CS}(U,E)$:

$$\nabla_i^\Gamma = \frac{\partial}{\partial x^i} + \big(\Gamma_{ij}^k(x)\big)_L\Big(p_{kL}\frac{\partial}{\partial p_j} + (\bar\psi_k\psi^j)_L - (\bar\psi_k\psi^j)_R\Big)\ . \tag{62}$$

Note that it is not quite a derivation on the algebra $\mathrm{CS}(U,E)$, because x and p do not commute, however its action on the generators $x, p, \psi, \bar\psi$ is what we expect from a covariant derivative:

$$\nabla_i^\Gamma(x^k) = \delta_i^k\ ,\quad \nabla_i^\Gamma(p_j) = \Gamma_{ij}^k p_k\ ,\quad \nabla_i^\Gamma(\psi^k) = -\Gamma_{ij}^k\psi^j\ ,\quad \nabla_i^\Gamma = \Gamma_{ij}^k\bar\psi_k\ .$$

We say that a generalized Dirac operator $D = \mathrm{i}\varepsilon\nabla + \overline{\nabla}$ is *affiliated to the connection* Γ is in any coordinate system one has

$$\nabla = \psi_R^i\nabla_i^\Gamma + s\ , \tag{63}$$

where the remainder s has an expansion of the form

$$s = \psi_R^i\Big(\sum_{|\alpha|=2}^\infty (s_{\alpha i}^k p_k)_L\partial_p^\alpha + \sum_{|\alpha|=1}^\infty (s_{\alpha ij}^k\bar\psi_k\psi^j + s_{\alpha i})_L\partial_p^\alpha\Big) \tag{64}$$

for some scalar functions $s_{\alpha i}^k, s_{\alpha ij}^k, s_{\alpha i} \in \Omega^0(U)$. Observe that s belongs to $\mathrm{SPS}^1(U,E)_L\Omega^1(U)_R \cap \mathscr{D}_0^0(U)$. In order to check that this definition makes sense, one has to inspect the transformation law of $\psi^i_R \nabla^\Gamma_i$ under a coordinate change γ. Using the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ one has

$$\psi^i_R \nabla^\Gamma_i = \psi^i_R \frac{\partial}{\partial x^i} + \psi^i_R \big(\Gamma^k_{ij}(x) p_k\big)_L \frac{\partial}{\partial p_j} + \psi^i_R \big(\Gamma^k_{ij}(x) \bar\psi_k \psi^j\big)_L \ .$$

We already know that $\gamma(\psi^i_R \frac{\partial}{\partial x^i}) \equiv \psi^i_R \frac{\partial}{\partial x^i} \mod \mathrm{SPS}^1(U,E)_L \Omega^1(U)_R$, but a closer examination of Equation (56)

$$\gamma\Big(\psi^i_R \frac{\partial}{\partial x^i}\Big) = \psi^i_R \frac{\partial}{\partial x^i} - \mathrm{i} p_{iL} \psi^i_R + \Big(\mathrm{i}\frac{\partial x^k}{\partial y^l} p_k + \frac{\partial}{\partial x^q}\frac{\partial x^k}{\partial y^l}\psi^q \bar\psi_k\Big)_L \Big(\frac{\partial y^l}{\partial x^i}\psi^i\Big)_R$$

gives, by means of the expansion $\big(\frac{\partial y^l}{\partial x^i}\psi^i\big)_R = \sum_{|\alpha|=0}^\infty \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} \partial^\alpha_x \big(\frac{\partial y^l}{\partial x^i}\big)_L \psi^i_R \partial^\alpha_p$,

$$\begin{aligned}
\gamma\Big(\psi^i_R \frac{\partial}{\partial x^i}\Big) &= \psi^i_R \frac{\partial}{\partial x^i} + \Big(\frac{\partial x^k}{\partial y^l} p_k \frac{\partial^2 y^l}{\partial x^i \partial x^j}\Big)_L \psi^i_R \frac{\partial}{\partial p_j} + \Big(\frac{\partial x^k}{\partial y^l}\frac{\partial^2 y^l}{\partial x^i \partial x^j}\bar\psi_k \psi^j\Big)_L \psi^i_R \\
&\quad + \sum_{|\alpha|=2}^\infty \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}\Big(\mathrm{i}\frac{\partial x^k}{\partial y^l} p_k \partial^\alpha_x \frac{\partial y^l}{\partial x^i}\Big)_L \psi^i_R \partial^\alpha_p \\
&\quad + \sum_{|\alpha|=1}^\infty \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}\Big(\frac{\partial}{\partial x^q}\frac{\partial x^k}{\partial y^l}\psi^q \bar\psi_k \partial^\alpha_x \frac{\partial y^l}{\partial x^i}\Big)_L \psi^i_R \partial^\alpha_p \ .
\end{aligned}$$

We used the identities $-\mathrm{i} p_i + \mathrm{i}\frac{\partial x^k}{\partial y^l} p_k \frac{\partial y^l}{\partial x^i} = \frac{\partial x^k}{\partial y^l}\frac{\partial^2 y^l}{\partial x^k \partial x^i}$ and $\psi^j \bar\psi_k = \delta^j_k - \bar\psi_k \psi^j$ in order to simplify the first line. Since commutators with p are proportional to derivations with respect to x, the above expression reads

$$\gamma\Big(\psi^i_R \frac{\partial}{\partial x^i}\Big) = \psi^i_R \frac{\partial}{\partial x^i} + \psi^i_R \Big(\frac{\partial x^k}{\partial y^l}\frac{\partial^2 y^l}{\partial x^i \partial x^j}\Big)_L \Big(p_{kL}\frac{\partial}{\partial p_j} + (\bar\psi_k \psi^j)_L\Big) + s' \ ,$$

where the remainder s' has an expansion of the form (64). In the same way, one can show that

$$\begin{aligned}
&\gamma\Big(\psi^i_R \big(\Gamma^k_{ij}(x)\big)_L \Big(p_{kL}\frac{\partial}{\partial p_j} + (\bar\psi_k \psi^j)_L\Big)\Big) = \\
&\qquad \psi^i_R \Big(\frac{\partial x^k}{\partial y^l}\frac{\partial y^p}{\partial x^i}\frac{\partial y^q}{\partial x^j}\Gamma^l_{pq}(y)\Big)_L \Big(p_{kL}\frac{\partial}{\partial p_j} + (\bar\psi_k \psi^j)_L\Big) + s''
\end{aligned}$$

with a remainder s'' of the form (64). Hence $\gamma(\psi^i_R \nabla^\Gamma_i) = \psi^i_R \nabla^{\gamma\Gamma}_i + s$, and using a partition of unity we can build a global operator ∇ on M with the wanted property. The following proposition, which is an analogue of the Lichnerowicz formula, relates the square of the corresponding Dirac operator to the curvature tensor of the connection Γ, whose components in local coordinates are

$$R^k_{lij} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il} \ . \tag{65}$$

Proposition 5.7 *Let D be a Dirac operator affiliated to a torsion-free affine connection Γ on M. In a local coordinate system over an open set $U \subset M$, the associated generalized Laplacian reads*

$$\begin{aligned}
-D^2 &= \mathrm{i}\varepsilon\Big(\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} + (\Gamma^k_{ij})_L (\psi^i \bar\psi_k)_R \frac{\partial}{\partial p_j} + u + v\Big) \\
&\quad + \varepsilon^2 \Big(\frac{1}{2}(\psi^i \psi^j)_R (R^k_{lij})_L \Big(p_{kL}\frac{\partial}{\partial p_l} + (\bar\psi_k \psi^l)_L\Big) + w\Big)
\end{aligned} \tag{66}$$

where R^k_{lij} are the components of the curvature tensor, and

$$\begin{aligned} u &= \sum_{|\alpha|=2}^{\infty} \Big((u_{\alpha i})_L \frac{\partial}{\partial x^i} + (u^k_\alpha p_k)_L + (u^k_{\alpha i})_L (\psi^i \bar\psi_k)_R + (u_\alpha)_L \Big) \partial^\alpha_p \\ v &= \sum_{|\alpha|=1}^{\infty} (v^k_{\alpha i} \bar\psi_k \psi^i)_L \partial^\alpha_p \\ w &= (\psi^i \psi^j)_R \Big(\sum_{|\alpha|=2}^{\infty} (w^k_{\alpha ij} p_k)_L \partial^\alpha_p + \sum_{|\alpha|=1}^{\infty} (w^k_{\alpha lij} \bar\psi_k \psi^l + w_{\alpha ij})_L \partial^\alpha_p \Big) \end{aligned}$$

where $u_{\alpha i}, u^k_\alpha, u^k_{\alpha i}, u_\alpha, v^k_{\alpha i}, w^k_{\alpha ij}, w^k_{\alpha lij}, w_{\alpha ij} \ni \Omega^0(U)$ are scalar functions.

Proof: Since $\overline{\nabla}^2 = 0$ one has $-D^2 = -\mathrm{i}\varepsilon[\nabla, \overline{\nabla}] + \varepsilon^2 \nabla^2$. In a local coordinate system $\nabla = \psi^i_R \nabla^\Gamma_i + s$ and $\overline{\nabla} = \bar\psi_{kR} \frac{\partial}{\partial p_k} + r$ with

$$\begin{aligned} s &= \psi^i_R \Big(\sum_{|\alpha|=2}^{\infty} (s^k_{\alpha i} p_k)_L \partial^\alpha_p + \sum_{|\alpha|=1}^{\infty} (s^k_{\alpha ij} \bar\psi_k \psi^j + s_{\alpha i})_L \partial^\alpha_p \Big) \\ r &= \sum_{|\alpha|=2}^{\infty} (r^i_\alpha)_L \bar\psi_{iR} \partial^\alpha_p \ . \end{aligned}$$

Hence $[\nabla, \overline{\nabla}] = [\psi^i_R \nabla^\Gamma_i, \bar\psi_{kR} \frac{\partial}{\partial p_k}] + [\psi^i_R \nabla^\Gamma_i, r] + [s, \bar\psi_{kR} \frac{\partial}{\partial p_k}] + [s, r]$. We compute each commutator of the right hand side separately. Firstly,

$$[\psi^i_R \nabla^\Gamma_i, \bar\psi_{kR} \frac{\partial}{\partial p_k}] =$$
$$[\psi^i_R \frac{\partial}{\partial x^i}, \bar\psi_{kR} \frac{\partial}{\partial p_k}] + [\big(\Gamma^l_{ij} p_l\big)_L \psi^i_R \frac{\partial}{\partial p_j}, \bar\psi_{kR} \frac{\partial}{\partial p_k}] + [\big(\Gamma^l_{ij} \bar\psi_l \psi^j\big)_L \psi^i_R, \bar\psi_{kR} \frac{\partial}{\partial p_k}] \ .$$

One has

$$[\psi^i_R \frac{\partial}{\partial x^i}, \bar\psi_{kR} \frac{\partial}{\partial p_k}] = [\psi^i_R, \bar\psi_{kR}] \frac{\partial}{\partial x^i} \frac{\partial}{\partial p_k} = -[\bar\psi_k, \psi^i]_R \frac{\partial}{\partial x^i} \frac{\partial}{\partial p_k} = -\frac{\partial}{\partial x^i} \frac{\partial}{\partial p_i} \ .$$

Then

$$\begin{aligned} &[\big(\Gamma^l_{ij} p_l\big)_L \psi^i_R \frac{\partial}{\partial p_j}, \bar\psi_{kR} \frac{\partial}{\partial p_k}] \\ &= \big(\Gamma^l_{ij} p_l\big)_L [\psi^i_R, \bar\psi_{kR}] \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_j} - \bar\psi_{kR} \big(\Gamma^l_{ij}\big)_L [p_{lL}, \frac{\partial}{\partial p_k}] \psi^i_R \frac{\partial}{\partial p_j} \\ &= -\big(\Gamma^l_{ij} p_l\big)_L \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} - \big(\Gamma^k_{ij}\big)_L (\psi^i \bar\psi_k)_R \frac{\partial}{\partial p_j} \end{aligned}$$

and

$$[\big(\Gamma^l_{ij} \bar\psi_l \psi^j\big)_L \psi^i_R, \bar\psi_{kR} \frac{\partial}{\partial p_k}] = -\big(\Gamma^l_{ij} \bar\psi_l \psi^j\big)_L \frac{\partial}{\partial p_i}$$

so that

$$\begin{aligned} -\mathrm{i}\varepsilon[\psi^i_R \nabla^\Gamma_i, \bar\psi_{kR} \frac{\partial}{\partial p_k}] &= \mathrm{i}\varepsilon \frac{\partial}{\partial x^i} \frac{\partial}{\partial p_i} + \mathrm{i}\varepsilon (\Gamma^k_{ij})_L (\psi^i \bar\psi_k)_R \frac{\partial}{\partial p_j} \\ &\quad + \mathrm{i}\varepsilon \big(\Gamma^l_{ij} p_l\big)_L \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} + \mathrm{i}\varepsilon \big(\Gamma^l_{ij} \bar\psi_l \psi^j\big)_L \frac{\partial}{\partial p_i} \ . \end{aligned}$$

The first and second term appear in the first line of (66), while the third and fourth terms contribute to u and v respectively. We continue with the commutator $[\psi^i_R \nabla^\Gamma_i, r]$:

$$[\psi^i_R \frac{\partial}{\partial x^i}, r] = \sum_{|\alpha|=2}^{\infty} [\psi^i_R \frac{\partial}{\partial x^i}, (r^j_\alpha)_L \bar{\psi}_{jR}] \partial^\alpha_p = \sum_{|\alpha|=2}^{\infty} \Big(\Big(\frac{\partial r^j_\alpha}{\partial x^i} \Big)_L \psi^i_R \bar{\psi}_{jR} - (r^i_\alpha)_L \frac{\partial}{\partial x^i} \Big) \partial^\alpha_p$$

and

$$\begin{aligned} [(\Gamma^l_{ij} p_l)_L \psi^i_R \frac{\partial}{\partial p_j}, r] &= \sum_{|\alpha|=2}^{\infty} [(\Gamma^l_{ij} p_l)_L \psi^i_R, (r^j_\alpha)_L \bar{\psi}_{jR} \partial^\alpha_p] \frac{\partial}{\partial p_j} \\ &= \sum_{|\alpha|=3}^{\infty} (a^k_\alpha p_k)_L \partial^\alpha_p + \sum_{|\alpha|=2}^{\infty} \big((a^k_{\alpha i})_L (\psi^i \bar{\psi}_k)_R + (a_\alpha)_L \big) \partial^\alpha_p \end{aligned}$$

for some scalar functions $a^k_\alpha, a^k_{\alpha i}, a_\alpha$, and

$$[(\Gamma^l_{ij} \bar{\psi}_l \psi^j)_L \psi^i_R, r] = -\sum_{|\alpha|=2}^{\infty} \big(\Gamma^l_{ij} \bar{\psi}_l \psi^j r^i_\alpha \big)_L \partial^\alpha_p \ .$$

Hence $[\psi^i_R \nabla^\Gamma_i, r]$ can be absorbed inside $u + v$. Further on, we have

$$\begin{aligned} [\bar{\psi}_{jR} \frac{\partial}{\partial p_j}, s] &= \sum_{|\alpha|=2}^{\infty} [\bar{\psi}_{jR} \frac{\partial}{\partial p_j}, (s^k_{\alpha i} p_k)_L \psi^i_R] \partial^\alpha_p \\ &+ \sum_{|\alpha|=1}^{\infty} (s^k_{\alpha il} \bar{\psi}_k \psi^l)_L [\bar{\psi}_{jR} \frac{\partial}{\partial p_j}, \psi^i_R] \partial^\alpha_p + \sum_{|\alpha|=1}^{\infty} (s_{\alpha i})_L [\bar{\psi}_{jR} \frac{\partial}{\partial p_j}, \psi^i_R] \partial^\alpha_p \\ &= \sum_{|\alpha|=2}^{\infty} \Big((s^j_{\alpha i})_L \bar{\psi}_{jR} \psi^i_R - (s^k_{\alpha i} p_k)_L \frac{\partial}{\partial p_i} \Big) \partial^\alpha_p \\ &\quad - \sum_{|\alpha|=1}^{\infty} (s^k_{\alpha il} \bar{\psi}_k \psi^l)_L \frac{\partial}{\partial p_i} \partial^\alpha_p - \sum_{|\alpha|=1}^{\infty} (s_{\alpha i})_L \frac{\partial}{\partial p_i} \partial^\alpha_p \end{aligned}$$

The first and third series of the right-hand-side can be absorbed inside u, whereas the second series counts for v. Instead of computing the commutator $[s, r]$ explicitly, we only need to remark that $s \in \mathrm{SPS}^1_L \Omega^1_R \cap \mathscr{D}^0_0$ and $r \in \Omega^0_L \mathrm{Vect}_R \cap \mathscr{D}^{-1}_0$. Then

$$\begin{aligned} [\mathrm{SPS}^1_L \Omega^1_R, \Omega^0_L \mathrm{Vect}_R] &\subset [\mathrm{SPS}^1_L, \Omega^0_L] \mathrm{PS}^0_R + \mathrm{SPS}^1_L [\Omega^1_R, \mathrm{Vect}_R] \\ &\subset \Omega^0_L \mathrm{PS}^0_R + \mathrm{SPS}^1_L \Omega^0_R \end{aligned}$$

It follows that $[s, r] \in (\Omega^0_L \mathrm{PS}^0_R + \mathrm{SPS}^1_L \Omega^0_R) \cap \mathscr{D}^{-1}_0$ can be absorbed inside $u + v$. Now we look at

$$\nabla^2 = (\psi^i_R \nabla^\Gamma_i + s)^2 = (\psi^i_R \nabla^\Gamma_i)^2 + [\psi^i_R \nabla^\Gamma_i, s] + s^2 \ .$$

A routine computation gives

$$[\nabla^\Gamma_i, \nabla^\Gamma_j] = (R^k_{lij})_L \Big(p_{kL} \frac{\partial}{\partial p_l} + (\bar{\psi}_k \psi^l)_L - (\bar{\psi}_k \psi^l)_R \Big) \ .$$

Consequently, the Bianchi identity $(R^k_{lij})_L(\psi^l\psi^i\psi^j)_R = 0$ implies

$$(\psi^i_R\nabla^\Gamma_i)^2 = \frac{1}{2}\,\psi^i_R\psi^j_R[\nabla^\Gamma_i,\nabla^\Gamma_j] = \frac{1}{2}\,(\psi^i\psi^j)_R(R^k_{lij})_L\Big(p_{kL}\frac{\partial}{\partial p_l} + (\bar\psi_k\psi^l)_L\Big)\ .$$

This the leading term in the second line of (66). Then we have

$$[\psi^l_R\frac{\partial}{\partial x^l}, s] = (\psi^l\psi^i)_R\Big(\sum_{|\alpha|=2}^{\infty}\Big(\frac{\partial s^k_{\alpha i}}{\partial x^l}p_k\Big)_L\partial^\alpha_p + \sum_{|\alpha|=1}^{\infty}\Big(\frac{\partial s^k_{\alpha ij}}{\partial x^l}\bar\psi_k\psi^j + \frac{\partial s_{\alpha i}}{\partial x^l}\Big)_L\partial^\alpha_p\Big)$$

and

$$[\big(\Gamma^l_{ij}p_l\big)_L\psi^i_R\frac{\partial}{\partial p_j}, s] = (\psi^i\psi^j)_R\Big(\sum_{|\alpha|=2}^{\infty}(b^k_{\alpha ij}p_k)_L\partial^\alpha_p + \sum_{|\alpha|=1}^{\infty}(b^k_{\alpha lij}\bar\psi_k\psi^l + b_{\alpha ij})_L\partial^\alpha_p\Big)$$

for some scalar functions $b^k_{\alpha ij}, b^k_{\alpha lij}, b_{\alpha ij}$, and

$$[\big(\Gamma^l_{ij}\bar\psi_l\psi^j\big)_L\psi^i_R, s] = (\psi^i\psi^j)_R\sum_{|\alpha|=1}^{\infty}(c^k_{\alpha lij}\bar\psi_k\psi^l)_L\partial^\alpha_p$$

for some other scalar functions $c^k_{\alpha lij}$. Hence $[\psi^i_R\nabla^\Gamma_i, s]$ can be absorbed inside w. Finally one easily checks that s^2 is also of the form w. ∎

6 Algebraic JLO formula

We first recall Connes' definition of periodic cyclic cohomology [2]. Let $\mathscr{A}$ be a trivially-graded associative $\mathbb{C}$-algebra. Form the unitalized algebra $\mathscr{A}^+ = \mathscr{A}\oplus\mathbb{C}$, even if $\mathscr{A}$ already has a unit. For any $k\in\mathbb{N}^*$ denote by $CC^k(\mathscr{A})$ the space of $(k+1)$-linear maps $\mathscr{A}^+\times\mathscr{A}^{\times k}\to\mathbb{C}$, and $CC^0(\mathscr{A})$ the space of linear maps $\mathscr{A}\to\mathbb{C}$. The Hochschild operator $b: CC^k(\mathscr{A})\to CC^{k+1}(\mathscr{A})$ is defined on a k-cochain $\varphi_k\in CC^k(\mathscr{A})$ by

$$\begin{aligned} b\varphi_k(a_0,\ldots,a_{k+1}) &= \sum_{i=0}^{k}(-1)^i\varphi_k(a_0,\ldots,a_ia_{i+1},\ldots,a_{k+1}) \\ &\quad +(-1)^{k+1}\varphi_k(a_{k+1}a_0,\ldots,a_k) \end{aligned} \qquad (67)$$

for any $a_0\in\mathscr{A}^+$ and $a_1,\ldots,a_k\in\mathscr{A}$. The Connes operator $B: CC^k(\mathscr{A})\to CC^{k-1}(\mathscr{A})$ reads

$$B\varphi_k(a_0,\ldots,a_{k-1}) = \sum_{i=0}^{k-1}(-1)^{i(k-i)}\varphi_k(a_i,\ldots,a_{k-1},a_0,\ldots,a_{i-1})\ . \qquad (68)$$

One checks $b^2 = B^2 = bB + Bb = 0$. The direct sum $CP^\bullet(\mathscr{A}) = \sum_{k=0}^\infty CC^k(\mathscr{A})$ endowed with the boundary operator $b+B$ is therefore a $\mathbb{Z}_2$-graded complex. The cohomology $HP^\bullet(\mathscr{A})$, of this complex is the periodic cyclic cohomology of $\mathscr{A}$. Thus, an even periodic cyclic cocycle over $\mathscr{A}$ is a *finite* collection $\varphi = (\varphi_0,\varphi_2,\ldots,\varphi_{2n})$ of homogeneous cochains such that

$$b\varphi_k + B\varphi_{k+2} = 0 \quad \text{for } 0\le k<2n\ , \quad b\varphi_{2n} = 0\ . \qquad (69)$$

An odd periodic cyclic cocycle is a finite collection $\varphi = (\varphi_1, \varphi_3, \dots, \varphi_{2n+1})$ verifying analogous relations.

Example 6.1 (Connes [2]) If M is a compact manifold, any homology class $[C_k] \in H_k(M, \mathbb{C})$ represented by a k-dimensional closed de Rham current C_k gives rise to a periodic cyclic cohomology class over the commutative algebra $C^\infty(M)$ by setting

$$\varphi_k(a_0, \dots, a_k) = \frac{c_k}{k!} \langle C_k, a_0 da_1 \dots da_k \rangle \ , \quad \forall a_i \in C^\infty(M) \ , \tag{70}$$

where c_k is a normalization factor depending on the parity of k. We choose $c_{2k} = 1/(2\pi \mathrm{i})^k$ and $c_{2k+1} = 1/(2\pi \mathrm{i})^{k+1}$ for compatibility with the usual normalization of characteristic classes in de Rham cohomology. Then one checks $b\varphi_k = 0 = B\varphi_k$ so that $[\varphi_k] \in HP^{k \bmod 2}(C^\infty(M))$ is represented by a homogeneous cochain of degree k. One thus gets a linear map

$$H_\bullet(M, \mathbb{C}) \to HP^\bullet(C^\infty(M)) \tag{71}$$

for any compact manifold. In fact, Connes shows that this is an *isomorphism* [2], provided that cyclic cohomology is defined through continuous cochains with respect to the natural locally convex topology of $C^\infty(M)$. Since we are not concerned with analytical issues in this paper, the fact that (71) is an isomorphism will be irrelevant for us.

Example 6.2 Consider the non-commutative algebra $\mathrm{CS}^0(M)$ of formal symbols of order ≤ 0 on a closed manifold M. The leading symbol gives rise to an algebra homomorphism $\lambda : \mathrm{CS}^0(M) \to C^\infty(S^*M)$ to the commutative algebra of functions over the cosphere bundle S^*M. Since cyclic cohomology pullbacks under homomorphisms, one gets, modulo composition with (71), a canonical map

$$\lambda^* : H_\bullet(S^*M) \to HP^\bullet(\mathrm{CS}^0(M)) \ . \tag{72}$$

In fact, Wodzicki shows that this is an *isomorphism* [13], provided the natural locally convex topology of $\mathrm{CS}^0(M))$ is taken into account. Again, we will not use the fact that λ^* is an isomorphism.

Now fix a closed n-dimensional manifold M. We will construct some cyclic cocycles over the algebra $\mathrm{CS}^0(M)$ using Dirac operators as defined in section 5.1. By construction $\mathrm{CL}^0(M)$ is an algebra of operators on the space $C^\infty(M)$. We can view $\mathrm{CL}^0(M)$ as an algebra of operators on the space of sections of the vector bundle $E = \Lambda T^*_{\mathbb{C}}M$: indeed its action on the zero-forms $C^\infty(M) = \Omega^0(M)$ can be extended by zero on $\Omega^k(M)$, $\forall k \geq 1$. Therefore one has a canonical homomorphism of $\mathrm{CL}^0(M)$ into the even part of the $\mathbb{Z}_2$-graded algebra $\mathrm{CL}^0(M, E)$. It descends to an homomorphism $\pi : \mathrm{CS}^0(M) \to \mathrm{CS}^0(M, E)$. In a local coordinate system we can write

$$\pi(a)(x, p, \psi, \bar\psi) = a(x, p)\Pi \qquad \forall a \in \mathrm{CS}^0(M) \ , \tag{73}$$

where $\Pi = \bar\psi_1 \psi^1 \dots \bar\psi_n \psi^n$ is the Clifford section corresponding to the projection operator from $\Omega^*(M)$ onto $\Omega^0(M)$. Then we can compose π with the left representation of $\mathrm{CS}^0(M, E)$ as endomorphisms on the vector space $\mathrm{CS}(M, E)$. This yields an injective homomorphism of algebras

$$\rho : \mathrm{CS}^0(M) \hookrightarrow \mathscr{D}^0_0(M) \ , \quad \rho(a) = (a\Pi)_L \quad \forall a \in \mathrm{CS}^0(M) \ . \tag{74}$$

We are now ready to introduce the following algebraic version of the JLO cocycle [6]. It involves the graded trace on the algebra of trace-class operators $\mathscr{T}(M)$ introduced in section 4.

Proposition 6.3 *Let $D = \mathrm{i}\varepsilon\nabla + \overline{\nabla} \in \mathscr{D}^1(M)$ be a generalized Dirac operator. The homogeneous cochains over the algebra* $\mathrm{CS}^0(M)$

$$\varphi_k^D(a_0,\ldots,a_k) = \int_{\Delta_k} \mathrm{Tr}_s\big(\rho(a_0)e^{-t_0D^2}[D,\rho(a_1)]e^{-t_1D^2}\ldots[D,\rho(a_k)]e^{-t_kD^2}\big)dt \tag{75}$$

defined for all $k \in 2\mathbb{N}$, are the components of an even periodic cyclic cocycle φ^D and vanish whenever $k > 2n$, $n = \dim M$. Moreover, the periodic cyclic cohomology class $[\varphi^D] \in HP^0(\mathrm{CS}^0(M))$ does not depend on D.

Proof: The graded trace of a trace-class operator $s \in \mathscr{T}(M)$ vanishes if the Clifford part of s is not of heighest weight, that is, if s is not proportional to the product $(\psi^1\ldots\psi^n\bar\psi_1\ldots\bar\psi_n)_L(\psi^1\ldots\psi^n\bar\psi_1\ldots\bar\psi_n)_R$ in local coordinates. Hence in the computation of φ_k^D, we should only retain the terms which bring at least n powers of ψ_L (resp. of ψ_R) and exactly the same powers of $\bar\psi_L$ (resp. of $\bar\psi_R$), because we have to take into account the possible lowering of powers coming from commutators $[\psi^i,\bar\psi_j] = \delta^i_j$. All other combinations of $\psi_L,\bar\psi_L,\psi_R,\bar\psi_R$ will vanish under the graded trace. In fact the right sector $\psi_R,\bar\psi_R$ will be our main interest. One has

$$[D,\rho(a)] = \mathrm{i}\varepsilon[\nabla,(a\Pi)_L] + [\overline{\nabla},(a\Pi)_L]\ .$$

The first term brings a factor $\varepsilon\psi_R$, whereas the second term brings a factor $\bar\psi_R$. We define the *pseudodifferential order* of an operator according to the following rule: a_L is of order m for any symbol $a \in \mathrm{CS}^m(M,E)$, the operators $\psi_R,\bar\psi_R,\varepsilon$ are of order 0, while ∂_p is of order -1 and ∂_x of order $+1$. From these rules one sees that the operator $\mathrm{i}\varepsilon[\nabla,(a\Pi)_L]$ has order ≤ 0, and $[\overline{\nabla},(a\Pi)_L]$ has order ≤ -1. In the same way we inspect the generalized Laplacian

$$-D^2 = -\mathrm{i}\varepsilon[\nabla,\overline{\nabla}] + \varepsilon^2\nabla^2\ .$$

From the proof of Proposition 5.2 we know that $\nabla^2 \in \mathrm{SPS}^1_L\Omega^2_R$, hence $\varepsilon^2\nabla^2$ has pseudodifferential order ≤ 1 and brings a factor $\varepsilon^2\psi_R\psi_R$. Similarly one has $-\mathrm{i}\varepsilon[\nabla,\overline{\nabla}] = \Delta + u$ where $\Delta = \mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$ is the flat Laplacian in local coordinates. u has order ≤ 0 and its right sector is proportional to either $\varepsilon\psi_R\bar\psi_R$ or 1. We treat $-D^2$ as a perturbation of the flat Laplacian. A Duhamel expansion of the exponentials $\exp(-t_iD^2)$ appearing in the cochain φ^D leads to the computation of terms like

$$\mathrm{Tr}_s\big(\rho(a_0)\exp(t_0\Delta)\,X_1\,\exp(t_1\Delta)\ldots X_k\,\exp(t_k\Delta)\big) = \\ ⨏ \langle\!\langle(a_0\Pi)_L\sigma_\Delta^{t_0}(X_1)\ldots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(X_k)\,\exp\Delta\rangle\!\rangle[n]$$

where $X_i = \varepsilon^2\nabla^2$, or $X_i = u$, or $X_i = \mathrm{i}\varepsilon[\nabla,(a_j\Pi)_L]$, or $X_i = [\overline{\nabla},(a_j\Pi)_L]$ for some $a_j \in \mathrm{CS}^0(M)$. In order to achieve an exact balance between the powers of ψ_R and $\bar\psi_R$, we see that the number $\bar l$ of factors $[\overline{\nabla},(a_j\Pi)_L]$ should equal $l + 2m$, where l is the number of factors $\mathrm{i}\varepsilon[\nabla,(a_j\Pi)_L]$ and m the number of factors $\varepsilon^2\nabla^2$. The pseudodifferential order of each X_i is not modified by the action of the modular group σ_Δ because

$$[\Delta, X_i] = \mathrm{i}\varepsilon\Big(\frac{\partial X_i}{\partial x^j}\frac{\partial}{\partial p_j} + \frac{\partial X_i}{\partial p_j}\frac{\partial}{\partial x^j} + \frac{\partial^2 X_i}{\partial x^j \partial p_j}\Big) \ .$$

The contractions $\langle \partial_x^\alpha \partial_p^\beta \exp\Delta\rangle$ also preserve the pseudodifferential order (∂_x and ∂_p are simultaneously contracted). It follows that the pseudodifferential order of the symbol $\langle\!\langle \rho(a_0)\sigma_\Delta^{t_0}(X_1)\dots\sigma_\Delta^{t_0+\dots+t_{k-1}}(X_k)\,\exp\Delta\rangle\!\rangle[n]$ is $\leq -\bar{l}+m = -l-m$, and its Wodzicki residue vanishes unless $-l-m \geq -n$ ($n = \dim M$). The latter condition implies $l \leq n-m$ and $\bar{l} \leq n+m$, so $l+\bar{l} \leq 2n$. This means that φ_k^D vanishes whenever it involves more than $2n$ commutators $[D,\rho(a)]$, that is, whenever $k > 2n$.
Hence φ^D is a cochain in the periodic complex $CP^\bullet(\mathrm{CS}^0(M))$. The cocycle identity $b\varphi_k^D + B\varphi_{k+2}^D = 0$ then follows from well-known algebraic manipulations which we do not need to reproduce here, see [6]. Finally observe that given two operators D_0 and D_1 the linear homotopy

$$D = tD_1 + (1-t)D_0 \ , \qquad t \in [0,1] \ ,$$

is a Dirac operator for all t. It is again a classical result that the cocycles φ^{D_0} and φ^{D_1} are related by a transgression formula of JLO type (see for instance [5]). One shows as above that the transgressed cochain, in our case, lies in the periodic complex. Hence the periodic cyclic cohomology class of φ^D does not depend on D. ■

Proposition 6.4 *Let $D = -\mathrm{i}\varepsilon d_R + \overline{\nabla}$ be a de Rham-Dirac operator. Then φ_0^D is the Wodzicki residue on $\mathrm{CS}^0(M)$, while the other components φ_k^D vanish for $k > 0$. Hence $[\varphi^D]$ is the periodic cyclic cohomology class of the Wodzicki residue.*

Proof: Let us first look at the commutator $[D,\rho(a)]$. Since $\rho(a) = (a\Pi)_L$ belongs to the left sector, it commutes with d_R, so that

$$[D,\rho(a)] = [\overline{\nabla}, (a\Pi)_L] \ .$$

By definition $\overline{\nabla} \in \Omega^0(M)_L\mathrm{Vect}(M)_R$ is proportional to $\bar{\psi}_R$ and not to ψ_R. Thus $[D,\rho(a)]$ brings a factor $\bar{\psi}_R$. On the other hand, the generalized Laplacian $-D^2$ is given by Formula (59), and brings either $(\psi\bar{\psi})_R$ or 1 in the right sector. This means that whenever some commutators $[D,\rho(a)]$ appear, the graded trace must vanish because the $\bar{\psi}_R$'s cannot be balanced with the same amount of ψ_R's. Hence $\varphi_k^D = 0$ whenever $k > 0$, and the only remaining component is

$$\varphi_0^D(a) = \mathrm{Tr}_s(\rho(a)\exp(-D^2)) = \mathrm{Tr}_s((a\Pi)_L\exp(-D^2)) \ .$$

We work in local coordinates (x,p) over $U \subset M$ and suppose that the symbol a has x-support contained in U (the general case follows by linearity). Write $-D^2 = \Delta + s$, where $\Delta = \mathrm{i}\varepsilon\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$ is the canonical flat Laplacian, and the remainder s is given by Equation (59):

$$s = \varepsilon\Big(p_{iL}\frac{\partial}{\partial p_i} + (\psi^i\bar{\psi}_i)_R + \sum_{|\alpha|=2}^{\infty}\Big(\mathrm{i}(a_\alpha^i)_L\frac{\partial}{\partial x^i} + (a_\alpha^i p_i)_L\Big)\partial_p^\alpha + \sum_{|\alpha|=1}^{\infty}(b_{\alpha j}^i)_L(\psi^j\bar{\psi}_i)_R\partial_p^\alpha\Big)$$

for some scalar functions $a^i_\alpha, b^i_{\alpha j} \in \Omega^0(U)$. Our goal is to show that the series over the multi-index α do not contribute to φ_0^D. We use a Duhamel expansion for $\exp(-D^2)$:

$$\varphi_0^D(a) = \sum_{k=0}^{\infty} \int_{\Delta_k} \mathrm{Tr}_s\big((a\Pi)_L \sigma_\Delta^{t_0}(s)\sigma_\Delta^{t_0+t_1}(s)\ldots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(s)\,\exp\Delta\big)dt$$

Now rewrite the product $\sigma_\Delta^{t_0}(s)\ldots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(s)$ by moving all the derivation operators ∂_x and ∂_p to the right, in front of $\exp\Delta$. The graded trace would vanish if the resulting powers of ∂_x and ∂_p are not exactly equal, because it involves the contractions $\langle\partial_x\partial_p\exp\Delta\rangle$. We remark that all the terms in s except $(\psi^i\bar\psi_i)_R$ bring a power of ∂_p strictky higher than the power of ∂_x. However, a ∂_p can be absorbed by commutation with p_L when it moves to the right, and a ∂_x can appear from $\sigma^t_\Delta(p_L) = p_L + t[\Delta, p_L] = p_L + \mathrm{i}\varepsilon t\partial_x$. A rapid inspection shows that an exact balance between ∂_x and ∂_p cannot occur if either $(\mathrm{i}(a^i_\alpha)_L\frac{\partial}{\partial x^i} + (a^i_\alpha p_i)_L)\partial_p^\alpha$ with $|\alpha| \geq 2$, or $(b^i_{\alpha j})_L(\psi^j\bar\psi_i)_R\partial_p^\alpha$ with $|\alpha| \geq 1$ appears. Thus we can keep the only relevant part $\varepsilon(p_{iL}\frac{\partial}{\partial p_i} + (\psi^i\bar\psi_i)_R)$ of s in the product $\sigma_\Delta^{t_0}(s)\ldots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(s)$, and write

$$\varphi_0^D(a) = \mathrm{Tr}_s\Big((a\Pi)_L\exp\big(\Delta + \varepsilon p_L\cdot\partial_p + \varepsilon(\psi^i\bar\psi_i)_R\big)\Big)\ .$$

$(\psi^i\bar\psi_i)_R$ commutes with $\Delta + \varepsilon p_L\cdot\partial_p$, hence the exponential splits as the product of $\exp(\varepsilon(\psi^i\bar\psi_i)_R)$ and $\exp(\Delta + \varepsilon p_L\cdot\partial_p)$. Expanding $\exp(\varepsilon(\psi^i\bar\psi_i)_R)$ in powers of ε, only the term of order n survives because it involves the product of all ψ_R's and $\bar\psi_R$'s, and the higher powers of ε are ignored by the graded trace. One finds

$$\begin{aligned}\varphi_0^D(a) &= \mathrm{Tr}_s\Big((a\Pi)_L\varepsilon^n(\psi^1\bar\psi_1\ldots\psi^n\bar\psi_n)_R\exp\big(\Delta + \varepsilon p_L\cdot\partial_p\big)\Big)\\ &= ⨍ \mathrm{tr}_s(a\Pi)\big\langle\!\big\langle\varepsilon^n(\psi^1\bar\psi_1\ldots\psi^n\bar\psi_n)_R\exp\big(\Delta + \varepsilon p_L\cdot\partial_p\big)\big\rangle\!\big\rangle[n]\ .\end{aligned}$$

By definition of the graded trace on the Clifford algebra, $\mathrm{tr}_s(a\Pi) = a$ and $\langle(\psi^1\bar\psi_1\ldots\psi^n\bar\psi_n)_R\rangle = (-1)^n\mathrm{tr}_s(\psi^1\bar\psi_1\ldots\psi^n\bar\psi_n) = 1$ so that

$$\varphi_0^D(a) = ⨍ a\,\big\langle\exp\big(\Delta + \varepsilon p_L\cdot\partial_p\big)\big\rangle[0]\ .$$

Then we apply Lemma 4.4 to the matrix $R = \varepsilon\mathrm{Id}$. This yields the formal power series in ε

$$\langle\exp\big(\Delta + \varepsilon p_L\cdot\partial_p\big)\exp(-\Delta)\rangle = \mathrm{Td}(\varepsilon\mathrm{Id}) = \Big(\frac{\varepsilon}{e^\varepsilon - 1}\Big)^n\ ,$$

whose coefficient of degree zero is $\mathrm{Td}(\varepsilon\mathrm{Id})[0] = 1$. Therefore $\varphi_0^D(a)$ is the Wodzicki residue as claimed. ∎

Theorem 6.5 *The periodic cyclic cohomology class of the Wodzicki residue vanishes in $HP^0(\mathrm{CS}^0(M))$ for any closed manifold M.*

Proof: Let Γ be the Levi-Civita connection associated to a given Riemannian metric on M, and let $D = \mathrm{i}\varepsilon\nabla + \overline{\nabla}$ be a generalized Dirac operator affiliated

to Γ. We will show that all the components of the corresponding cocycle φ^D vanish. The theorem is then a consequence of Propositions 6.3 and 6.4.
In a local coordinate system ∇ is expressed in terms of the Christoffel symbols Γ^k_{ij} of the connection:

$$\nabla = \psi^i_R \frac{\partial}{\partial x^i} + \big(\Gamma^k_{ij}(x)p_k\big)_L \psi^i_R \frac{\partial}{\partial p_j} + \big(\Gamma^k_{ij}(x)\bar\psi_k \psi^j\big)_L \psi^i_R + s\ .$$

The remainder s can be expanded in power series of the partial derivative ∂_p,

$$s = \sum_{|\alpha|=2}^{\infty} (s^k_{\alpha i} p_k)_L \psi^i_R \partial^\alpha_p + \sum_{|\alpha|=1}^{\infty} (s^k_{\alpha ij}\bar\psi_k\psi^j + s_{\alpha i})_L \psi^i_R \partial^\alpha_p$$

where $s^k_{\alpha i}, s^k_{\alpha ij}$ and $s_{\alpha i}$ are scalar functions of x. As in the proof of Proposition 6.3 we look at the pseudodifferential order of these operators. The leading part $\psi^i_R \frac{\partial}{\partial x^i}$ of ∇ has order $+1$, the two sub-leading terms have order ≤ 0, while the remainder s has order ≤ -1. We calculate, for any $a \in \mathrm{CS}^0(M)$,

$$[\mathrm{i}\varepsilon\nabla, \rho(a)] = [\mathrm{i}\varepsilon\nabla, (a\Pi)_L] = \mathrm{i}\varepsilon\Big(\frac{\partial a}{\partial x^i}\Pi\Big)_L \psi^i_R + \mathrm{i}\varepsilon\Big(\Gamma^k_{ij}(x)p_k \frac{\partial a}{\partial p_j}\Pi\Big)_L \psi^i_R + \ldots$$

We only write the terms of order 0, and ignore the dots of order -1. In the same way

$$\overline{\nabla} = \bar\psi_{iR} \frac{\partial}{\partial p_i} + r$$

has a leading term of order -1, and the remainder r of order -2 can be expanded as $\sum_{|\alpha|=2}^{\infty} (r^i_\alpha)_L \bar\psi_{iR} \partial^\alpha_p$ for some scalar functions r^i_α. Hence

$$[\overline{\nabla}, \rho(a)] = [\overline{\nabla}, (a\Pi)_L] = \Big(\frac{\partial a}{\partial p_i}\Pi\Big)_L \bar\psi_{iR} + \ldots$$

is of order -1 and we ignore the dots of order -2. On the other hand, the generalized Laplacian $-D^2$ is given by (66). Keeping only the leading terms we write

$$\begin{aligned} -D^2 \quad &= \quad \mathrm{i}\varepsilon\Big(\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i} + (\Gamma^k_{ij})_L (\psi^i\bar\psi_k)_R \frac{\partial}{\partial p_j}\Big) \\ & \quad + \frac{\varepsilon^2}{2}(\psi^i\psi^j)_R (R^k_{lij})_L \Big(p_{kL}\frac{\partial}{\partial p_l} + (\bar\psi_k\psi^l)_L\Big) + \ldots\ , \end{aligned}$$

where the dots have the form of the leading terms but involve higher powers of the partial derivative ∂_p (hence have strictly lower order). We proceed as in the proof of Proposition 6.3 and consider $-D^2 = \Delta + u$ as a perturbation of the flat Laplacian $\Delta = \mathrm{i}\varepsilon \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i}$. A Duhamel expansion of the exponentials $\exp(-t_i D^2)$ appearing in the cochain φ^D leads to the computation of terms like

$$\begin{aligned} &\mathrm{Tr}_s\big(\rho(a_0)\exp(t_0\Delta)\, X_1\, \exp(t_1\Delta)\ldots X_k\, \exp(t_k\Delta)\big) = \\ &\qquad ⨍ \langle\!\langle \rho(a_0)\sigma^{t_0}_\Delta(X_1)\ldots\sigma^{t_0+\ldots+t_{k-1}}_\Delta(X_k)\, \exp\Delta\rangle\!\rangle[n] \end{aligned}$$

where $X_i = u$ or $X_i = [D, \rho(a_j)]$ for some $a_j \in \mathrm{CS}^0(M)$. In particular X_i has pseudodifferential order ≤ 0, and this order is not modified by the action of the modular group σ_Δ because

$$[\Delta, X_i] = \mathrm{i}\varepsilon\Big(\frac{\partial X_i}{\partial x^j}\frac{\partial}{\partial p_j} + \frac{\partial X_i}{\partial p_j}\frac{\partial}{\partial x^j} + \frac{\partial^2 X_i}{\partial x^j \partial p_j}\Big) \ .$$

Now observe that in the above expressions for $-D^2$ and $[D, \rho(a)]$, a factor $\varepsilon\psi_R$ always appears together with a pseudodifferential order ≤ 0, whereas a factor $\bar\psi_R$ always appears together with a pseudodifferential order ≤ -1. The contraction map on the odd variables $\psi_R, \bar\psi_R$ selects the only part of $\sigma_\Delta^{t_0}(X_1)\ldots\sigma_\Delta^{t_0+\ldots+t_{k-1}}(X_k)$ containing the product $(\psi^1\ldots\psi^n\bar\psi_n\ldots\bar\psi_1)_R$. This part has order $\leq -n$. Moreover, the dots in the above expressions for $-D^2$ and $[D, \rho(a)]$ contribute to an order $< -n$. A crucial consequence is that we only need to keep the leading terms of all quantities and ignore the dots because the Wodzicki residue vanishes on symbols of order $< -n$ (recall that the contractions $\langle \partial_x^\alpha \partial_p^\beta \exp\Delta\rangle$ do not affect the pseudodifferential order). Another crucial consequence is that all the derivatives $\partial X_i/\partial x^j$ appearing in the action of the modular group can be neglected, because these terms also contribute to an overall order $< -n$. Hence *all functions of the variable x behave like constants.* This drastically simplifies the computation of φ^D. One has

$$\varphi_k^D(a_0,\ldots,a_k) = \int_{\Delta_k} \mathrm{Tr}_s\big(\rho(a_0)\sigma_{-D^2}^{t_0}([D,\rho(a_1)])\ldots\sigma_{-D^2}^{t_0+\ldots+t_{k-1}}([D,\rho(a_k)])\exp(-D^2)\big)dt$$

If we localize the supports of the symbols a_i around a point $x_0 \in U$ and choose a coordinate system in which $\Gamma_{ij}^k(x_0) = 0$, we can write

$$\begin{aligned}
[D, \rho(a)] &\simeq \mathrm{i}\varepsilon\Big(\frac{\partial a}{\partial x^i}\Pi\Big)_L \psi_R^i + \Big(\frac{\partial a}{\partial p_i}\Pi\Big)_L \bar\psi_{iR} \ , \\
-D^2 &\simeq \Delta + \frac{\varepsilon^2}{2}(\psi^i\psi^j)_R (R_{lij}^k)_L \Big(p_{kL}\frac{\partial}{\partial p_l} + (\bar\psi_k\psi^l)_L\Big)
\end{aligned}$$

because we only keep the leading terms and ignore the x-derivatives of Γ_{ij}^k, hence $\Gamma_{ij}^k \simeq \Gamma_{ij}^k(x_0) = 0$. For notational simplicity set $R_l^k = \frac{\varepsilon^2}{2}(R_{lij}^k)_L(\psi^i\psi^j)_R$ and recall that it behaves like a constant with respect to x. The generator of the modular group σ_{-D^2} is the commutator with $-D^2$. Its iterated actions on $X = [D, \rho(a)]$ read

$$\begin{aligned}
-[D^2, X] &\simeq [\Delta + R_l^k p_{kL}\frac{\partial}{\partial p_l}, X] \simeq \frac{\partial X}{\partial p_i}\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^i} + R_i^k p_{kL}\Big) \\
[D^2, [D^2, X]] &\simeq \frac{\partial^2 X}{\partial p_i \partial p_j}\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^i} + R_i^k p_{kL}\Big)\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^j} + R_j^l p_{lL}\Big) \\
&\quad + R_i^j \frac{\partial X}{\partial p_i}\Big(\mathrm{i}\varepsilon\frac{\partial}{\partial x^j} + R_j^l p_{lL}\Big)
\end{aligned}$$

Observe that the term $(R_{lij}^k\bar\psi_k\psi^l)_L$ multiplied by $\rho(a) = (a\Pi)_L$ vanishes, because $R_{lij}^k\bar\psi_k\psi^l\Pi = R_{lij}^k(\delta_k^l - \psi^l\bar\psi_k)\Pi = R_{kij}^k\Pi$, and since Γ is by hypothesis a

Riemannian connection, $R^k_{kij} = 0$. More generally

$$\begin{aligned} \sigma^t_{-D^2}(X) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \underbrace{[D^2, \ldots [D^2}_{k}, X] \ldots] \\ &\simeq X + \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{|\alpha|=1}^{k} P_\alpha(X)(\mathrm{i}\varepsilon\partial_x + p_L \cdot R)^\alpha \end{aligned}$$

where α is a multi-index and $P_\alpha(X)$ is a linear combination of the partial p-derivatives of X. Since we drop the x-derivatives, the operator $(\mathrm{i}\varepsilon\partial_x + p_L \cdot R)^\alpha$ commutes with all operators under the graded trace so it can be moved to the right in front of $\exp(-D^2)$. Moreover $\rho(a)$ brings a factor Π_L and we know that $R^k_{lij}\bar{\psi}_k\psi^l\Pi = 0$, so we may replace everywhere $-D^2$ by $\Delta + p_L \cdot R \cdot \partial_p$. Then identities (49) lead to

$$\begin{aligned} &\mathrm{Tr}_s\big(\rho(a_0)\sigma^{t_0}_{-D^2}([D,\rho(a_1)]) \ldots \sigma^{t_0+\ldots+t_{k-1}}_{-D^2}([D,\rho(a_k)]) \exp(\Delta + p_L \cdot R \cdot \partial_p)\big) \\ &\qquad = ⨍ \big\langle\!\big\langle \rho(a_0)[D,\rho(a_1)] \ldots [D,\rho(a_k)] \exp(\Delta + p_L \cdot R \cdot \partial_p) \big\rangle\!\big\rangle[n] \end{aligned}$$

The integral over $(t_0, \ldots, t_k) \in \Delta_k$ simply brings a factor $1/k!$. Lemma 4.4 applied to the matrix then gives

$$\varphi^D_k(a_0, \ldots, a_k) = \frac{1}{k!} ⨍ \big\langle\!\big\langle \rho(a_0)[D,\rho(a_1)] \ldots [D,\rho(a_k)] \mathrm{Td}(R) \exp \Delta \big\rangle\!\big\rangle[n] \ .$$

We have to select the coefficient of ε^n in this expression. ε always comes with a factor ψ_R and the graded trace on the Clifford algebra selects the only polynomial $(\psi^1 \ldots \psi^n \bar{\psi}_n \ldots \bar{\psi}_1)_R$, hence the variables ψ_R and $\bar{\psi}_R$ behave as if they anticommute. We make the identification with differential forms $\varepsilon\psi^i_R \leftrightarrow dx^i$ and $\bar{\psi}_{iR} \leftrightarrow dp_i - \Gamma^k_{ij} p_k dx^j$ over T^*U, which is consistent with the action of a coordinate change. Locally in our coordinate system one has $\Gamma^k_{ij} \simeq 0$ so that

$$[D,\rho(a)] \leftrightarrow \Big(\mathrm{i}\frac{\partial a}{\partial x^i} dx^i + \frac{\partial a}{\partial p_i} dp_i\Big)\Pi \ , \quad \frac{\varepsilon^2}{2} R^k_{lij}(\psi^i\psi^j)_R \leftrightarrow \frac{1}{2} R^k_{lij} dx^i \wedge dx^j = R^k_l \ .$$

To be more precise, if we multiply the bracket by the volume form of the cotangent bundle $\omega^n/n! = dp_1 \wedge dx^1 \ldots dp_n \wedge dx^n$, and compare it to the normalization condition $\langle(\bar{\psi}_1\psi^1 \ldots \bar{\psi}_n\psi^n)_R\rangle = (-1)^n$, one finds the equality of $2n$-forms over T^*U (the subscript $_{\mathrm{vol}}$ denotes the top-component of a differential form)

$$\begin{aligned} &\big\langle\!\big\langle \rho(a_0)[D,\rho(a_1)] \ldots [D,\rho(a_k)] \mathrm{Td}(R) \exp \Delta \big\rangle\!\big\rangle[n] \, \frac{\omega^n}{n!} = \\ &\qquad (-1)^n \mathrm{i}^{k-n} \big(a_0 da_1 \ldots da_k \mathrm{Td}(R)\Pi\big)_{\mathrm{vol}} + \text{ terms of order } < -n \end{aligned}$$

The first term of the right-hand-side is a scalar symbol of order $\leq -n$, times the volume form. We claim that this symbol in fact has order $< -n$. Indeed the product $a_0 da_1 \ldots da_k$ brings n partial derivatives with respect to the variables $(p_1, \ldots p_n)$. Writing its leading symbol in polar coordinates $(\|p\|, \theta_1, \ldots, \theta_{n-1})$, one sees that is is proportional to $\|p\|^{1-n}$ times a partial derivative $\frac{\partial a}{\partial \|p\|}$. The latter has order ≤ -2. Hence the Wodzicki residue vanishes. ■

We now deal with the Radul cocycle. Let $q \in \mathrm{CS}^1(M)$ be a symbol of order one, with positive and invertible leading symbol. The logarithm $\log q$ is no longer classical, but belongs to the larger class of log-polyhomogeneous symbols: its asymptotic expansion in a local coordinate system (x,p) reads

$$(\log q)(x,p) = \log \|p\| + q_0'(x,p) \tag{76}$$

where $q_0' \in \mathrm{CS}^0(M)$ is a classical symbol of order ≤ 0. It is easy to check that the commutator (for the $\star$-product) of $\log q$ with any classical symbol $a \in \mathrm{CS}^m(M)$ is in $\mathrm{CS}^{m-1}(M)$. In fact $[\log q, a]$ has an expansion

$$[\log q, a] = \sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k-1} a^{(k)} q^{-k} \tag{77}$$

where $a^{(k)} \in \mathrm{CS}^m(M)$ denotes the k-th power of the derivation $[q,\ ]$ on a. Thus $[\log q,\ ]$ is an outer derivation on the algebra of classical symbols $\mathrm{CS}(M)$. The Radul cocycle [11] is the bilinear map $c : \mathrm{CS}(M) \times \mathrm{CS}(M) \to \mathbb{C}$ defined by means of the Wodzicki residue

$$c(a_0, a_1) = {\textstyle -\!\!\!\!\!\!\int} a_0 [\log q, a_1] \ , \quad \forall a_i \in \mathrm{CS}(M) \ . \tag{78}$$

The expansion (77) shows that the Wodzicki residue vanishes on commutators $[\log q, a]$ for any classical symbol a. Hence the Wodzicki residue is trace on $\mathrm{CS}(M)$ which is closed with respect to the derivation $[\log q,\ ]$. Elementary algebraic manipulations show the antisymetry property $c(a_0,a_1) = -c(a_1,a_0)$. Moreover the Hochschild coboundary of c is

$$bc(a_0,a_1,a_2) = c(a_0a_1,a_2) - c(a_0,a_1a_2) + c(a_2a_0,a_1) = 0$$

for all $a_i \in \mathrm{CS}(M)$. Thus c is a cyclic one-cocycle. Originally c was introduced as a two-cocycle over the Lie algebra $\mathrm{CS}(M)$, with commutator as Lie bracket, but the cyclic cocycle property is actually stronger. From now on we view c as a cyclic one-cocycle over the subalgebra $\mathrm{CS}^0(M) \subset \mathrm{CS}(M)$ of symbols of order ≤ 0.
Then we extend the commutator $[\log q,\ ]$ to a derivation on the algebra $\mathscr{L}(M) \subset \mathrm{End}(\mathrm{CS}(M,E))$ as follows. Recall that $\mathscr{L}(M)$ is generated by left multiplications a_L for all symbols $a \in \mathrm{CS}(M,E)$, and right multiplications b_R for all polynomial symbols $b \in \mathrm{PS}(M,E)$. Then extend $q \in \mathrm{CS}^1(M)$ to an elliptic positive symbol $\tilde{q} \in \mathrm{CS}^1(M,E)$ of scalar type and set

$$\delta(a_L b_R) = ([\log \tilde{q}, a])_L b_R \qquad \forall\ a \in \mathrm{CS}(M,E)\ ,\ b \in \mathrm{PS}(M,E)\ . \tag{79}$$

Since the left representation $a \mapsto a_L$ is faithful, $\delta : \mathscr{L}(M) \to \mathscr{L}(M)$ is well-defined. It is clearly a derivation. In an obvious fashion we extend it to a derivation, still denoted δ, on the algebra of formal power series $\mathscr{S}(M) = \mathscr{L}(M)[[\varepsilon]]$ by setting $\delta\varepsilon = 0$. It has good properties with respect to the subspaces $\mathscr{D}_k^m(M)$. Indeed in a local coordinate system over $U \subset M$, one has

$$\begin{aligned} \delta(\partial_p) &= \mathrm{i}\delta(x_R - x_L) = -\mathrm{i}([\log \tilde{q}, x])_L \in \mathrm{CS}^{-1}(U,E)_L \\ \delta(\partial_x) &= \mathrm{i}\delta(p_L - p_R) = \mathrm{i}([\log \tilde{q}, p])_L \in \mathrm{CS}^{0}(U,E)_L \end{aligned} \tag{80}$$

and also $\delta(\mathrm{CS}^m(M,E)_L) \subset \mathrm{CS}^{m-1}(M,E)_L$ for all $m \in \mathbb{R}$. This shows that $\delta(\mathscr{D}^m_k(M)) \subset \mathscr{D}^{m-1/2}_k(M)$ for all $m \in \mathbb{R}$ and $k \in \mathbb{N}$. If $\Delta \in \mathscr{D}^{1/2}_1(M)$ is a generalized Laplacian, one has

$$\delta \exp(\Delta) = \int_0^1 e^{t\Delta}\, \delta\Delta\, e^{(1-t)\Delta}\, dt = \int_0^1 \sigma^t_\Delta(\delta\Delta) \exp(\Delta)\, dt$$

hence δ restricts to a derivation on the $\mathscr{D}(M)$-bimodule $\mathscr{T}(M)$ of trace-class operators. The analogue of expansion (77) for δ shows that the graded trace $\mathrm{Tr}_s : \mathscr{T}(M) \to \mathbb{C}$ is δ-closed.

Proposition 6.6 *Let $D \in \mathscr{D}^1(M)$ be a generalized Dirac operator and δ the derivation associated to an elliptic positive symbol $\tilde{q} \in \mathrm{CS}^1(M,E)$. The homogeneous cochains over the algebra $\mathrm{CS}^0(M)$*

$$\begin{aligned}\varphi^{D,\delta}_k(a_0,\dots,a_k) = & \qquad\qquad (81)\\ \sum_{i=1}^{k}(-1)^{i+1}\int_{\Delta_k} & \mathrm{Tr}_s\big(\rho(a_0)e^{-t_0D^2}[D,\rho(a_1)]e^{-t_1D^2}\dots\delta\rho(a_i)e^{-t_iD^2}\dots[D,\rho(a_k)]e^{-t_kD^2}\big)dt\\ +\sum_{i=1}^{k+1}(-1)^{i}\int_{\Delta_{k+1}} & \mathrm{Tr}_s\big(\rho(a_0)e^{-t_0D^2}[D,\rho(a_1)]e^{-t_1D^2}\dots\delta De^{-t_iD^2}\dots[D,\rho(a_k)]e^{-t_kD^2}\big)dt\end{aligned}$$

defined for all $k \in 2\mathbb{N}+1$, are the components of an odd periodic cyclic cocycle $\varphi^{D,\delta}$ and vanish whenever $k > 2n+1$, $n = \dim M$. Moreover, the periodic cyclic cohomology class $[\varphi^{D,\delta}] \in HP^1(\mathrm{CS}^0(M))$ does not depend on D nor $\tilde{q}$.

Proof: Analogous to Proposition 6.3. Details are left to the reader. ■

Proposition 6.7 *Let $D = -\mathrm{i}\varepsilon d_R + \overline{\nabla}$ be a de Rham-Dirac operator. Then the first component $\varphi^{D,\delta}_1 = c$ is the Radul cocycle on $\mathrm{CS}^0(M)$, while the other components $\varphi^{D,\delta}_k$ vanish for $k > 1$. Hence $[\varphi^{D,\delta}]$ is the periodic cyclic cohomology class of $[c]$.*

Proof: We proceed as in Proposition 6.4. The commutator $[D,\rho(a)]$ only brings $\bar\psi_R$ which cannot be balanced by ψ_R, hence $\varphi^{D,\delta}_k$ vanishes whenever $k > 1$. The only non-zero component is

$$\varphi^{D,\delta}_1(a_0,a_1) = \int_0^1 \mathrm{Tr}_s\big(\rho(a_0)e^{-tD^2}\delta\rho(a_1)e^{(t-1)D^2}\big)dt\ .$$

Observe that

$$\frac{d}{dt}\mathrm{Tr}_s\big(\rho(a_0)e^{-tD^2}\delta\rho(a_1)e^{(t-1)D^2}\big) = -\mathrm{Tr}_s\big(\rho(a_0)e^{-tD^2}[D^2,\delta\rho(a_1)]e^{(t-1)D^2}\big)\ .$$

The identity $[D^2,\delta\rho(a_1)] = D[D,\delta\rho(a_1)] + [D,\delta\rho(a_1)]D$ and the graded trace property yield

$$-\mathrm{Tr}_s\big(\rho(a_0)e^{-tD^2}[D^2,\delta\rho(a_1)]e^{(t-1)D^2}\big) = \mathrm{Tr}_s\big([D,\rho(a_0)]e^{-tD^2}[D,\delta\rho(a_1)]e^{(t-1)D^2}\big)$$

This quantity vanishes because the commutators $[D, \rho(a)]$ are proportional to $\bar{\psi}_R$. Hence $\mathrm{Tr}_s\big(\rho(a_0)e^{-tD^2}\delta\rho(a_1)e^{(t-1)D^2}\big)$ does not depend on t and we can rewrite the integral $\varphi_1^{D,\delta}$ in terms of its integrand at $t=0$:

$$\varphi_1^{D,\delta}(a_0,a_1) = \mathrm{Tr}_s\big(\rho(a_0)\delta\rho(a_1)e^{-D^2}\big) = \mathrm{Tr}_s\big((a_0[\log q, a_1]\Pi)_L e^{-D^2}\big) \ .$$

The computation is now completely analogous to Proposition 6.4 and one finds

$$\varphi_1^{D,\delta}(a_0,a_1) = ⨍ a_0[\log q, a_1]$$

as claimed. ∎

Choose an affine torsion-free connection on the tangent bundle TM, and let $R \in \Omega^2(M, \mathrm{End}(TM))$ be its curvature two-form. The Todd class of the complexified tangent bundle $\mathrm{Td}(T_{\mathbb{C}}M) \in H^\bullet(M,\mathbb{C})$ is the cohomology class of even degree represented by the closed differential form

$$\mathrm{Td}(\mathrm{i}R/2\pi) = \det\left(\frac{\mathrm{i}R/2\pi}{e^{\mathrm{i}R/2\pi}-1}\right) \ , \tag{82}$$

where the determinant acts on the sections of the endomorphism bundle of $T_{\mathbb{C}}M$.

Theorem 6.8 *Let M be a closed manifold. The periodic cyclic cohomology class of $[c] \in HP^1(\mathrm{CS}^0(M))$ is*

$$[c] = \lambda^*\big([S^*M] \cap \pi^*\mathrm{Td}(T_{\mathbb{C}}M)\big) \ , \tag{83}$$

where λ^* is the pullback (72) induced by the leading symbol homomorphism, $\mathrm{Td}(T_{\mathbb{C}}M) \in H^\bullet(M,\mathbb{C})$ is the Todd class of the complexified tangent bundle, and $\pi : S^*M \to M$ is the cosphere bundle endowed with its canonical orientation and fundamental class $[S^*M] \in H_\bullet(S^*M)$.

Proof: We apply verbatim the proof of Theorem 6.5. We can replace the commutator $[D,\rho(a)]$ by $\mathrm{i}\varepsilon(\frac{\partial a}{\partial x^i}\Pi)_L\psi^i_R + (\frac{\partial a}{\partial p_i}\Pi)_L\bar{\psi}_{iR}$ in a local coordinate system, and consider $R^k_l = \frac{\varepsilon^2}{2}(R^k_{lij})_L(\psi^i\psi^j)_R$ as independent of x. Then

$$\begin{aligned}\varphi_k^{D,\delta}(a_0,\ldots,a_k) = \\ &\sum_{i=1}^{k}\frac{(-1)^{i+1}}{k!}⨍ \big\langle\!\big\langle \rho(a_0)[D,\rho(a_1)]\ldots\delta\rho(a_i)\ldots[D,\rho(a_k)]\mathrm{Td}(R)\exp\Delta\big\rangle\!\big\rangle[n] \\ &+\sum_{i=1}^{k+1}\frac{(-1)^{i}}{(k+1)!}⨍ \big\langle\!\big\langle \rho(a_0)[D,\rho(a_1)]\ldots\delta D\ldots[D,\rho(a_k)]\mathrm{Td}(R)\exp\Delta\big\rangle\!\big\rangle[n]\end{aligned}$$

The first term of the right-hand-side vanishes. Indeed, the bracket selects the polynomial $(\bar{\psi}_1\ldots\bar{\psi}_n)_R$ which brings n derivatives with respect to p, and $\delta\rho(a) = ([\log q, a]\Pi)_L$ is of order -1. Hence the symbol under the Wodzicki residue has order $< -n$ and disappears. We are left with the second term involving δD. Recall that $(\log q)(x,p) = \log\|p\| + q_0'(x,p)$ where q_0' is a classical symbol of order ≤ 0. At leading order one has

$$\begin{aligned}\delta D &= -\mathrm{i}\varepsilon\Big(\frac{\partial \log q}{\partial x^i}\Big)_L\psi^i_R - \Big(\frac{\partial \log q}{\partial p_i}\Big)_L\bar{\psi}_{iR} + \ldots \\ &= -\mathrm{i}\varepsilon\Big(\frac{\partial q_0'}{\partial x^i}\Big)_L\psi^i_R - \Big(\frac{\partial q_0'}{\partial p_i} + \frac{p^i}{\|p\|^2}\Big)_L\bar{\psi}_{iR} + \ldots\end{aligned}$$

where $p^i = \delta^{ij} p_j$. The leading term proportional to ψ_R (resp. $\bar{\psi}_R$) is of order ≤ 0 (resp. ≤ -1), and the dots proportional to ψ_R (resp. $\bar{\psi}_R$) are of order < 0 (resp. < -1). The bracket under the residue is expressed by means of differential forms:

$$\langle\!\langle \rho(a_0)[D,\rho(a_1)]\ldots\delta D\ldots[D,\rho(a_k)]\mathrm{Td}(R)\exp\Delta\rangle\!\rangle[n]\,\frac{\omega^n}{n!} =$$
$$-(-1)^n \mathrm{i}^{k+1-n}\Big(a_0 da_1\ldots\Big(dq_0' + \frac{p^i dp_i}{\|p\|^2}\Big)\ldots da_k \mathrm{Td}(R)\Pi\Big)_{\mathrm{vol}} + \ldots$$

The leading part is a symbol of order $\leq -n$, while the dots of order $< -n$ are killed by the Wodzicki residue. One shows as in the proof of 6.5 that the term $a_0 da_1 \ldots dq_0' \ldots da_k$ is also killed. Hence the only remaining term is proportional to $p^i dp_i/\|p\|^2$. At leading order we can view $a_0, \ldots, a_k$ as scalar functions over the cosphere bundle. Since $\mathrm{tr}_s(\Pi) = 1$ the residue becomes the integral of a $(2n-1)$-form (remark that it is globally defined)

$$\begin{aligned}\varphi_k^{D,\delta}(a_0,\ldots,a_k) &= \frac{(-1)^n \mathrm{i}^{k+1-n}}{(2\pi)^n k!}\int_{S^*M} \iota(L)\cdot\Big(\frac{p^i dp_i}{\|p\|^2}\wedge a_0 da_1\ldots da_k \mathrm{Td}(R)\Big)\\ &= \frac{\mathrm{i}^{k+n+1}}{(2\pi)^n k!}\int_{S^*M} a_0 da_1 \ldots da_k \mathrm{Td}(R)\end{aligned}$$

where $L = p_i \frac{\partial}{\partial p_i}$ is the fundamental vector field on T^*M. The dimension of S^*M equals $2n-1$ and the parity of the cochain is actually odd, so one gets

$$\varphi_{2k+1}^{D,\delta}(a_0,\ldots,a_{2k+1}) = \frac{1}{(2\pi \mathrm{i})^{k+1}(2k+1)!}\int_{S^*M} a_0 da_1 \ldots da_{2k+1}\mathrm{Td}(\mathrm{i}R/2\pi)$$

for any $k \in \mathbb{N}$. This is precisely the pullback, under the morphism λ, of the degree $2k+1$ component of the de Rham cycle $[S^*M] \cap \mathrm{Td}(\mathrm{i}R/2\pi)$. ∎

7 Atiyah-Singer index theorem

An immediate corollary of Theorem 6.8 is the Atiyah-Singer index theorem, which computes the index of an elliptic pseudodifferential operator on a closed manifold M, in terms of local data. We consider the algebra $\mathrm{CL}^0(M)$ of scalar pseudodifferential operators of order ≤ 0 as an extension of the algebra $\mathrm{CS}^0(M)$ of formal symbols, with kernel the algebra of smoothing operators:

$$(E): \quad 0 \to \mathrm{L}^{-\infty}(M) \to \mathrm{CL}^0(M) \to \mathrm{CS}^0(M) \to 0\ . \qquad (84)$$

An operator $Q \in \mathrm{CL}^0(M)$ is elliptic if and only if its leading symbol is invertible, or equivalently, if its formal symbol is invertible in $\mathrm{CS}^0(M)$. Thus Q has a parametrix $P \in \mathrm{CL}^0(M)$ which is an inverse modulo smoothing operators, that is $PQ-1$ and $QP-1$ are in $\mathrm{L}^{-\infty}(M)$. The obstruction of perturbing Q to an exactly invertible operator in $\mathrm{CL}^0(M)$ is measured by the *index map* of the extension (E) in algebraic K-theory

$$\mathrm{Ind}_E : K_1(\mathrm{CS}^0(M)) \to K_0(\mathrm{L}^{-\infty}(M)) \cong \mathbb{Z} \qquad (85)$$

cf. [7]. Indeed the formal symbol of Q is invertible hence defines a class $[Q]$ in the algebraic K-theory group $K_1(\mathrm{CS}^0(M))$, and its image under the map (85) coincides with the Fredholm index of Q as a bounded operator on $L^2(M)$. In [10] we presented a general procedure allowing to compute local index formulas associated to extensions. In the simple case of pseudodifferential operators on a closed manifold, the calculation of the index reduces to the Radul cocycle evaluated on Q and its parametrix (more precisely, on their formal symbols):

$$\mathrm{Ind}_E([Q]) = c(P,Q) \ . \tag{86}$$

In terms of Connes' pairing between K-theory and cyclic cohomology [2], the above formula is precisely the pairing of $[Q] \in K_1(\mathrm{CS}^0(M))$ with the cyclic cohomology class $[c] \in HP^1(\mathrm{CS}^0(M))$. Since by Theorem 6.8, this class is a pullback under the leading symbol map $\lambda : \mathrm{CS}^0(M) \to C^\infty(S^*M)$, we are able to express the index of Q in terms of its leading symbol which is an invertible function $g \in C^\infty(S^*M)$. This is not surprising because the algebra $\mathrm{CS}^0(M)$ is a pro-nilpotent extension of $C^\infty(S^*M)$, and this implies an isomorphism of the algebraic K-theory groups $K_1(\mathrm{CS}^0(M)) \cong K_1(C^\infty(S^*M))$. In fact one has a diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{L}^{-\infty}(M) & \longrightarrow & \mathrm{CL}^0(M) & \longrightarrow & \mathrm{CS}^0(M) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{CL}^{-1}(M) & \longrightarrow & \mathrm{CL}^0(M) & \longrightarrow & C^\infty(S^*M) & \longrightarrow & 0 \end{array}$$

The vertical arrows are isomorphisms both at the K-theoretic and periodic cyclic cohomology levels. Thus the index map of (E) should really be viewed as a map

$$\mathrm{Ind}_E : K_1(C^\infty(S^*M)) \to \mathbb{Z} \ , \tag{87}$$

sending the leading symbol class $[g] \in K_1(C^\infty(S^*M))$ to the Fredholm index of Q. Of course, everything extends to pseudodifferential operators acting on the sections of a (trivially graded) complex vector bundle over M, the leading symbols being matrix-valued functions over S^*M. In order to state the index formula we need to recall that any class $[g] \in K_1(C^\infty(S^*M))$, represented by an invertible matrix-valued function g, has a Chern character in the cohomology $H^\bullet(S^*M, \mathbb{C})$ of odd degree represented by the closed differential form

$$\mathrm{ch}(g) = \sum_{k\geq 0} \frac{k!}{(2k+1)!} \, \mathrm{tr}\Big(\frac{(g^{-1}dg)^{2k+1}}{(2\pi \mathrm{i})^{k+1}} \Big) \ . \tag{88}$$

Corollary 7.1 (Index theorem) *Let Q be an elliptic pseudodifferential operator of order ≤ 0 acting on the sections of a trivially graded vector bundle over M, with leading symbol class $[g] \in K_1(C^\infty(S^*M))$. Then the Fredholm index of Q is the integer*

$$\mathrm{Ind}(Q) = \langle [S^*M], \pi^*\mathrm{Td}(T_{\mathbb{C}}M) \cup \mathrm{ch}([g]) \rangle \ . \tag{89}$$

Proof: If $\varphi = (\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$ is an odd $(b+B)$-cocycle over $\mathrm{CS}^0(M)$, its pairing with the K-theory class $[Q] \in K_1(\mathrm{CS}^0(M))$ reads ([3])

$$\langle [\varphi], [Q] \rangle = \sum_{k\geq 0} (-1)^k \, k! \, (\varphi_{2k+1} \otimes \mathrm{tr})(P,Q,\ldots,P,Q)$$

where, strictly speaking, Q and its parametrix P should be replaced by their formal symbols. If φ is the pullback of an odd homology class $[C] \in H_\bullet(S^*M, \mathbb{C})$ under the leading symbol map λ, the above formula factors through the leading symbols $g = \lambda(Q)$ and $g^{-1} = \lambda(P)$. Using the identity $dg^{-1} = -g^{-1}dgg^{-1}$ one gets

$$\langle [\varphi], [Q] \rangle = \sum_{k \geq 0} \frac{(-1)^k k!}{(2\pi \mathrm{i})^{k+1}(2k+1)!} \langle C_{2k+1}, \mathrm{tr}(g^{-1}dg(dg^{-1}dg)^k) \rangle = \langle [C], \mathrm{ch}([g]) \rangle \ .$$

Applying this formula to the periodic cyclic cohomology class of c given by Theorem 6.8 gives the desired formula for $\mathrm{Ind}(Q) = \langle [c], [Q] \rangle$. ■

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