A Nekrasov-Okounkov type formula for $\widetilde{C}$

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Motivations

GOAL: A **combinatorial expansion** of powers of Dedekind η function, involving partition **hook lengths**, by generalizing **Macdonald identities** in type $\widetilde{C}$.
APPLICATIONS: new hook length formulas, some relations between Macdonald identity in different types, number theoretic application...

Partitions and Dedekind η function

DEFINITION:
$$\eta(x) = x^{1/24}\prod_{k\geq 1}(1-x^k)$$

THEOREM (MACDONALD IDENTITY IN TYPE $\widetilde{A}$, 1972):
$$\eta(x)^{t^2-1} = c_0 \sum_{\mathrm{v}} x^{\|\mathrm{v}\|^2/2t}\prod_{i<j}(v_i - v_j),$$
where the sum is over t-tuples $\mathrm{v} := (v_0, \ldots, v_{t-1}) \in \mathbb{Z}^t$ such that $v_i \equiv i \bmod t$ and $v_0 + \cdots + v_{t-1} = 0$.

DEFINITION A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of the integer $n \geq 0$ is a finite non-increasing sequence of positive integers whose sum is n.

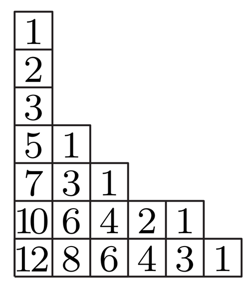


Figure: The Ferrers diagram of the partition $\lambda = (6,5,3,2,1,1,1)$

DEFINITION We define the principal hooks as the hooks coming from a box on the diagonal of the Ferrers diagram.

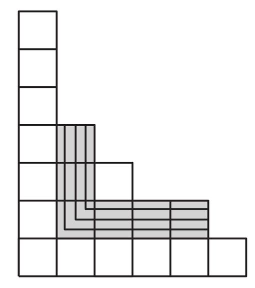


Figure: A principal hook of the partition $\lambda = (6,5,3,2,1,1,1)$

Nekrasov-Okounkov formula

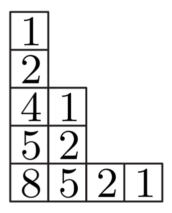
THEOREM (NEKRASOV-OKOUNKOV, 2006; HAN, 2009) For any complex number z, we have:
$$\prod_{k\geq 1}(1-x^k)^{z-1} = \sum_{\lambda\in\mathcal{P}} x^{|\lambda|}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{z}{h^2}\right)$$

QUESTION: what happens for the other affine types?
Macdonald identity in type $\widetilde{C}_t$, 1972:
$$\eta(X)^{2t^2+t} = c_1 \sum \prod_i v_i \prod_{i<j}(v_i^2 - v_j^2) X^{\|v\|^2/4(t+1)},$$
where the sum ranges over $(v_1, \ldots, v_t) \in \mathbb{Z}^t$ such that $v_i \equiv i \mod 2t+2$.

t-cores

Set t a positive integer.

- NOTATION: We denote by $\mathcal{H}(\lambda)$ the hook length multiset of the partition λ, and by $\mathcal{H}_t(\lambda)$ the multiset of hook length which are integral multiple of t.
- DEFINITION: λ is a t-core if there is no hook with length t in λ, or equivalently, if $\mathcal{H}_t(\lambda) = \emptyset$.
- EXAMPLE: $(4,2,2,1,1)$ is a 3-core.

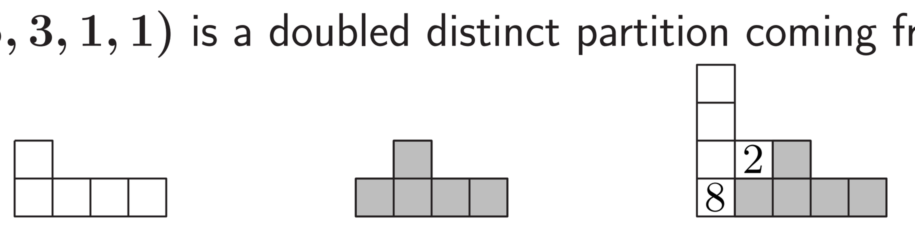


- PREVIOUS WORK:
  Nakayama (1940): introduction and conjectures in representation theory
  Garvan-Kim-Stanton (1990): generating function, proof of Ramanujan's congruences
  Ono (1994): positivity of the number of t-cores
  Anderson (2002), Olsson-Stanton (2007): simultaneous s- and t-core
  Han (2009): expansion of η in terms of hooks

Doubled distinct partitions

- DEFINITION: Define the doubled distinct partitions as follows:
  - start with a partition μ^0 with distinct parts
  - for all i, shift by i to the right the $i^t h$ row of μ^0
  - add μ_i^0 boxes to the i^{th} column.

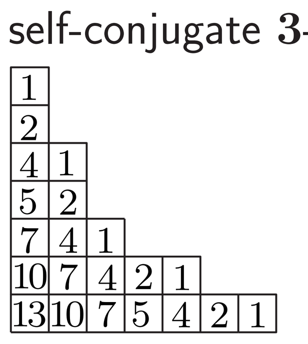
  The resulting partition μ is a doubled distinct partition.
- EXAMPLE: $\mu = (5,3,1,1)$ is a doubled distinct partition coming from $\mu^0 = (3,1)$.



- NOTATIONS: We denote by DD the set of doubled distinct partitions and by $DD_{(t)}$ its subset of t-cores.
- REMARK: A doubled distinct partition is uniquely determined by its principal hook lengths.

Self-conjugate partitions

- NOTATION: We denote by SC the set of self-conjugate partions and by $SC_{(t)}$ its subset of t-cores.
- EXAMPLE: $(7,5,3,2,2,1,1)$ is a self-conjugate 3-core.



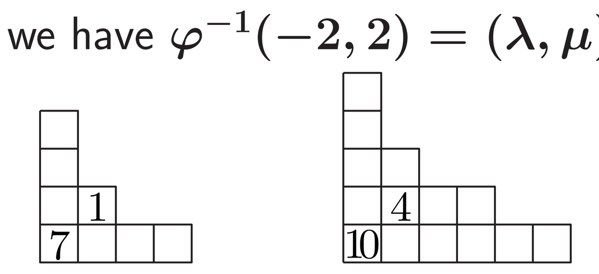
A bijection between pairs of $t+1$-cores and vectors of integers

- DEFINITION: If $(\lambda,\mu) \in SC_{(t+1)} \times DD_{(t+1)}$, let Δ be the set of principal hook lengths of λ and μ, and for $i \in \{1,\ldots,t\}$ we define
$$\Delta_i := Max\left(\{h \in \Delta, h \equiv \pm i - t - 1 \bmod 2t+2\} \cup \{i-t-1\}\right).$$
- THEOREM: There is a bijection φ between $SC_{(t+1)} \times DD_{(t+1)}$ and $\mathbb{Z}^t$ such that $\varphi(\lambda,\mu) := \mathrm{n} = (n_1,\ldots,n_t)$ satisfies:
$$|\lambda| + |\mu| = \sum (t+1)n_i^2 + in_i$$
$$t+1+\Delta_i = ((2t+2)n_i + i)\sigma_i, \text{ with } \sigma_i = \begin{cases} 1 \text{ if } n_i \geq 0 \\ -1 \text{ if } n_i < 0\end{cases}$$

Description of the bijection φ^{-1}

We give a recursive description of the reverse bijection φ^{-1}.

- $\forall i, n_i = 0 \longrightarrow \lambda = \mu = \emptyset$
- $\exists i, n_i = 1$, λ or μ, (depending on the parity of $t+1+i$) contains a principal hook of length $t+1+i$,
- $\exists i, n_i = -1$, λ or μ contains a principal hook of length $t+1-i$,
- If $n_i > 0$, $\varphi^{-1}(n_1,\ldots,n_i+1,\ldots,n_t) = \varphi^{-1}(n_1,\ldots,n_i,\ldots,n_t)+$ a principal hook of length $(t+1)(2n_i-1)+i$
- if $n_i < 0$, $\varphi^{-1}(n_1,\ldots,n_i-1,\ldots,n_t) = \varphi^{-1}(n_1,\ldots,n_i,\ldots,n_t)+$ a principal hook of length $(t+1)(-2n_i-1)-i)$
- EXAMPLE: for $t+1=3$, we have $\varphi^{-1}(-2,2) = (\lambda,\mu)$ with



A Nekrasov-Okounkov type formula for $\widetilde{C}$

- THEOREM For any complex number t, the following expansion holds:
$$\prod_{k\geq 1}(1-x^k)^{2t^2+t} = \sum_{\lambda\in DD}\delta_\lambda x^{|\lambda|/2}\prod_{h\in\mathcal{H}(\lambda)}\left(1-\frac{2t+2}{h\,\varepsilon_h}\right),$$
$$\delta_\lambda = \begin{cases} 1 & \text{if the Durfee size of } \lambda \text{ is odd} \\ -1 & \text{otherwise}\end{cases} \qquad \varepsilon_h = \begin{cases} 1 & \text{if the box corresponding to } h \text{ is above the principal diagonal} \\ -1 & \text{otherwise}\end{cases}$$
- IDEAS OF THE PROOF:
  - Start from the Macdonald formula in type $\widetilde{C}_t$ for t integer
  - Write $v_i = (2t+2)n_i + i$ and apply the bijection φ
  - Transform the resulting sum over $SC_{(t+1)} \times DD_{(t+1)}$
  - By polynomiality, what we get is true for any complex z
  - Use a bijection between $SC \times DD$ and DD to conclude

Applications

THEOREM: We have a symplectic analogue of the hook length formula.
$$\sum_{\substack{\lambda\in DD\\|\lambda|=2n}}\prod_{h\in\mathcal{H}(\lambda)}\frac{1}{h} = \frac{1}{2^n n!}$$

THEOREM: (refinement of a result of Kostant) Let k be a positive integer and s be a real number such that $s > k-1$. Then $(-1)^k f_k(2s^2+s) > 0$, where
$$\prod_{n\geq 1}(1-x^n)^s = \sum_{k\geq 0} f_k(s)x^k$$

THEOREM: The following families of formulas are all generalized by our Theorem:
(i) the Macdonald formula in type $\widetilde{C}_t$;
(ii) the Macdonald formula in type $\widetilde{B}_t$: where the sum ranges over t-tuples $\mathrm{v} := (v_1,\ldots,v_t) \in \mathbb{Z}^t$ such that $v_i \equiv 2i-1 \bmod 4t-2$ and $v_1 + \cdots + v_t = t^2 \bmod 8t-4$;
(iii) the Macdonald formula in type $\widetilde{BC}_t$: where the sum ranges over t-tuples $\mathrm{v} := (v_1,\ldots,v_t) \in \mathbb{Z}^t$ such that $v_i \equiv 2i-1 \bmod 4t+2$.

By using the Littlewood decomposition, we prove:
THEOREM: Let $t = 2t'+1$ be an odd positive integer, and let y and z be two complex numbers. The following expansion holds:
$$\sum_{\lambda\in DD}\delta_\lambda x^{|\lambda|/2}\prod_{h\in\mathcal{H}_t(\lambda)}\left(y - \frac{yt(2z+2)}{\varepsilon_h\, h}\right) = \prod_{k\geq 1}(1-x^k)(1-x^{kt})^{t'-1} \times \left(1-x^{kt}y^{2k}\right)^{(2z+1)(zt+3(t-1)/2)}$$