Classical limit for a system of non-linear random Schrödinger equations

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Abstract

This work is concerned with the semi-classical analysis of mixed state solutions to a Schrödinger-Poisson equation perturbed by a random potential with weak amplitude and fast oscillations in time and space. We show that the Wigner transform of the density matrix converges weakly and in probability to solutions to a Vlasov-Poisson-Boltzmann equation with a linear collision kernel. We obtain in addition that the local density is self-averaging. The proof brings together the martingale method for stochastic equations with compactness techniques for non-linear PDE in a semi-classical regime. It partly relies on the derivation of an energy estimate that is straightfoward in a deterministic setting but requires the use of a martingale formulation and well-chosen perturbed test functions in the random context.

1 Introduction

This paper investigates the semi-classical limit of a system of Schrödinger equations coupled via the Poisson equation (or equivalently the Quantum Liouville-Poisson system) perturbed by random heterogeneities with weak amplitude and fast oscillations. This system describes the quantum motion of a large number of electrons subject to the Coulomb interaction and random perturbations. Models of this sort are widely used for semiconductors modeling or nuclear physics, see for instance [24, 28]. More precisely,

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we are interested in the following random non-linear Liouville equation for the density operator ϱ^ε (i.e. a trace class, positive, hermitian operator) in three dimensions:

$$\begin{aligned} &i\varepsilon\partial_t \varrho^\varepsilon = \left[-\frac{\varepsilon^2}{2}\Delta + V^\varepsilon + U^\varepsilon, \varrho^\varepsilon\right], \qquad t>0, \qquad x\in\mathbb{R}^3, \\ &\varrho^\varepsilon(t=0) = \varrho_0^\varepsilon, \end{aligned} \tag{1}$$

where ϱ_0^ε is a given initial density operator, $[\cdot,\cdot]$ denotes the commutator between two operators ($[A,B]=AB-BA$), and ε is the rescaled Planck constant and is therefore small. With an abuse of notations, V^ε and U^ε denote multiplication operators by the corresponding potentials. Here, U^ε is the Poisson interaction potential that has the form

$$U^\varepsilon = \frac{1}{4\pi|x|} * n^\varepsilon, \tag{2}$$

where the symbol $*$ denotes convolution in the space variables and n^ε is the local density associated to ϱ^ε. Its expression is given by $n^\varepsilon(t,x)=\rho^\varepsilon(t,x,x)$, where $\rho^\varepsilon(t,x,y)$ is the integral kernel of ϱ^ε and is usually referred to as the density matrix. The real potential $V^\varepsilon \equiv V^\varepsilon(t,x)$ is random and accounts for some random perturbations that depend both in time and space. The weak fluctuations are assumed to be fast and to oscillate at the scale of the rescaled Planck constant ε, so that we set $V^\varepsilon(t,x)=\sqrt{\varepsilon}V\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)$, for some random potential V to be defined later on. Further motivations for such a choice will be given in the sequel.

The deterministic system (1) (i.e. when $V^\varepsilon\equiv 0$), or its equivalent formulation in terms of Schrödinger or Wigner equations, see (3) and (6) below, has an extensive history in the mathematical literature in many physical configurations, see e.g. [36, 10, 12, 29, 33, 27, 3, 15] for a few references. The semi-classical limit $\varepsilon\to 0$ of (1) when $V^\varepsilon\equiv 0$, relates the quantum motion of the particles to its classical counterpart. It is generally performed with the help of Wigner transforms [43], see its definition below, or by means of WKB expansions. The analysis via Wigner transforms leads to Vlasov type non-linear transport equations, and requires the non-linearity be smooth enough [33, 17], which is the case for the Poisson potential. See [25, 4, 2] for recent results on semi-classical limits for rough potentials. Non-linear Schrödinger (NLS) equations with power non-linearities [40] are excluded, but a semi-classical analysis using WKB techniques in some configurations is still possible [16].

On the other hand, the semi-classical limit of the random linear system (1) (i.e. when $U^\varepsilon\equiv 0$) is now well-established in several physical contexts: particles submitted to random impurities [39, 20, 21], or waves propagating in random media in the paraxial approximation, see the review paper [5]. In the latter situation, waves generally satisfy the important property of self-averaging, meaning the stochastic process given by the random Wigner function converges in probability to a deterministic quantity. Such property is paramount for imaging purposes, see [14, 9], and generated some interest in the analysis of the remaining stochasticity (corrector analysis) [8, 31]. In the appropriate scaling, the limiting Wigner transform is solution to a linear transport equation with a collision kernel depending on the two-point statistics of the random fluctuations. The nature of the collision process depends on the oscillation scales of the random fluctuations [5].

This work proposes to bring together these two types of semi-classical analysis and to consider a non-linear problem with randomness. While each of them has attracted a lot of interest separately, it seems there are relatively few works on semi-classical limits with both non-linearity and randomness. The perturbations of NLS equations by randomness have extensively been studied in the focusing case or in the context of Anderson localization, see for instance [11, 19, 1, 42], but the semi-classical limit was not investigated. Our analysis relies on the combination of the deterministic compactness method used by Lions and Paul in [33] with martingale and perturbed test functions techniques for stochastic equations [32, 37]. Under appropriate assumptions on the random potential and the initial condition, our main result is that the (unique) solution to (1) or the related Wigner equation (6), converges weakly and in probability to the unique solution to the Vlasov-Poisson-Boltzmann system (9) with a linear collision kernel. In addition, we obtain that the density n^ε, and not just the Wigner transform, is self-averaging, which is apparently a new observation, even in the linear context.

To be more specific, we recast the system (1)-(2) in the following form: if the spectral decomposition of the initial density operator ϱ_0^ε reads, denoting by $(\cdot,\cdot)_2$ the usual $L^2(\mathbb{R}^3)$ inner product,

$$\varrho_0^\varepsilon \varphi = \sum_{i\in\mathbb{N}} \rho_i^\varepsilon (\psi_{i,0}^\varepsilon, \varphi)_2 \psi_{i,0}^\varepsilon, \qquad \forall \varphi \in L^2(\mathbb{R}^3),$$

with $\rho_i^\varepsilon \geq 0$, $\forall i \geq 0$, $\sum_{i\in\mathbb{N}} \rho_i^\varepsilon < \infty$ (since ϱ_0^ε is positive and trace class), and where $(\rho_i^\varepsilon, \psi_{i,0}^\varepsilon)_{i\in\mathbb{N}}$ are the eigenvalues and eigenvectors of ϱ_0^ε, then $\varrho^\varepsilon(t)$ can be written as

$$\varrho^\varepsilon(t)\varphi = \sum_{i\in\mathbb{N}} \rho_i^\varepsilon (\psi_i^\varepsilon(t), \varphi)_2 \psi_i^\varepsilon(t), \qquad \forall \varphi \in L^2(\mathbb{R}^3),$$

where $(\psi_i^\varepsilon(t))_{i\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{R}^3)$ at all times t and solve the system of coupled Schrödinger equations:

$$\begin{aligned} & i\varepsilon\partial_t \psi_i^\varepsilon = -\frac{\varepsilon^2}{2}\Delta\psi_i^\varepsilon + V^\varepsilon \psi_i^\varepsilon + U^\varepsilon \psi_i^\varepsilon, \qquad t>0, \qquad x\in\mathbb{R}^3, \\ & \psi_i^\varepsilon(t=0) = \psi_{0,i}^\varepsilon. \end{aligned} \tag{3}$$

In addition, the integral kernel and the density have the formal expressions

$$\rho^\varepsilon(t,x,y) = \sum_{i\in\mathbb{N}} \rho_i^\varepsilon \psi_i^\varepsilon(t,x)(\psi_i^\varepsilon)^*(t,y) \qquad ; \qquad n^\varepsilon(t,x) = \sum_{i\in\mathbb{N}} \rho_i^\varepsilon |\psi_i^\varepsilon(t,x)|^2. \tag{4}$$

Above, $(\psi_i^\varepsilon)^*$ is the complex conjugate of ψ_i^ε. The semi-classical limit is performed with the Wigner function defined by

$$W^\varepsilon(t,x,k) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ik\cdot y}\rho^\varepsilon\left(t, x-\frac{\varepsilon}{2}y, x+\frac{\varepsilon}{2}y\right)dy. \tag{5}$$

The Wigner function is generally seen as a distribution function in the phase space, even though it is not always positive. See [33, 26] for the main properties of Wigner transforms, one of which being that all limits of Wigner transforms in a suitable sense are non-negative. Starting from (3), it can be seen that the Wigner function satisfies the following random Wigner-Poisson equation (WP):

$$\begin{aligned} & \partial_t W^\varepsilon + k\cdot\nabla_x W^\varepsilon = \mathcal{L}_1^\varepsilon W^\varepsilon + \mathcal{L}_2^\varepsilon W^\varepsilon, \qquad t>0, \qquad (x,k)\in\mathbb{R}^3\times\mathbb{R}^3, \\ & W^\varepsilon(t=0) = W_0^\varepsilon, \end{aligned} \tag{6}$$

where W_0^ε is the Wigner transform of the integral kernel of ϱ_0^ε, denoted by ρ_0^ε, and the operators $\mathcal{L}_1^\varepsilon$ and $\mathcal{L}_2^\varepsilon$ read, with, $j=1,2$:

$$\begin{aligned}(\mathcal{L}_j^\varepsilon W^\varepsilon)(t,x,k) &= \int_{\mathbb{R}^3} f_j^\varepsilon(t,x,k-\eta)W^\varepsilon(t,x,\eta)d\eta \\ f_1^\varepsilon(t,x,k) &= \frac{i}{\sqrt{\varepsilon}\pi^3}\left[\hat{V}\left(\frac{t}{\varepsilon},-2k\right)e^{-i2k\cdot x/\varepsilon}-\hat{V}\left(\frac{t}{\varepsilon},2k\right)e^{i2k\cdot x/\varepsilon}\right] \\ f_2^\varepsilon(t,x,k) &= \frac{i}{\varepsilon(2\pi)^3}\int_{\mathbb{R}^3} e^{-ik\cdot y}\left[U^\varepsilon(t,x-\frac{\varepsilon}{2}y)-U^\varepsilon(t,x+\frac{\varepsilon}{2}y)\right]dy.\end{aligned}$$

Above, $\hat{V}$ denotes the Fourier transform of V with the convention

$$\hat{V}(k)=\int_{\mathbb{R}^3} e^{-ik\cdot x}V(x)dx,$$

and the density n^ε is formally related to W^ε by

$$n^\varepsilon(t,x)=\int_{\mathbb{R}^3} W^\varepsilon(t,x,k)dk.$$

If W denotes the limit of the Wigner function, a simple formal expansion shows that the operator $\mathcal{L}_2^\varepsilon$ introduces the standard force term of the Vlasov-Poisson equation

$$\mathcal{L}_2^\varepsilon W^\varepsilon \to \nabla_x U\cdot\nabla_k W$$

where

$$U=\frac{1}{|x|}*n, \qquad n(t,x)=\int_{\mathbb{R}^3} W(t,x,k)dk. \tag{7}$$

On the other hand, the operator $\mathcal{L}_1^\varepsilon$ brings in the limit a linear collision term of the form [6]

$$\mathcal{L}W(k)=\int_{\mathbb{R}^3}\frac{dp}{(2\pi)^3}\sigma\left(\frac{|k|^2-|p|^2}{2},p-k\right)(W(p)-W(k)), \tag{8}$$

where σ is a smooth collision cross-section and is related to the Fourier transform in all variables of the two-point correlation function of the random fluctuations $\hat{R}$, see (12). The formal asymptotics therefore leads to the following Vlasov-Poisson-Boltzmann (VPB) equation with linear collision kernel

$$\begin{aligned}&\partial_t W + k\cdot\nabla_x W - \nabla_x U\cdot\nabla_k W = \mathcal{L}W, \qquad t>0, \qquad (x,k)\in\mathbb{R}^3\times\mathbb{R}^3, \\ &W(t=0)=W_0.\end{aligned} \tag{9}$$

Above, W_0 is the limit of the initial Wigner function. The system (9) is deterministic, and this is a consequence of the self-averaging properties of W^ε and n^ε in the limit $\varepsilon\to 0$. We will show in this paper that the limit can be performed rigously in the appropriate setting. A first assumption is that the random potential is Markovian with bounded generator, which allows us to treat the process $(V^\varepsilon, W^\varepsilon)$ as jointly Markov and to make use of the martingale formulation along with the perturbed test functions method in a relatively simple manner. A second one is the traditional and crucial mixed state hypothesis which provides uniform in ε L^2 bounds for the initial Wigner function. We will also assume that the initial total energy is bounded uniformly in ε. This property is paramount for the treatment of the non-linearity as it shows that the wave function is ε-oscillatory [33]. A consequence of this fact is that the density n^ε associated with the Wigner function W^ε, converges to the density n associated with W. This is not an obvious result since it requires some control of W^ε for large k, and such control is provided by the energy estimate. Notice that showing that the total energy remains uniformly bounded at all times is by no means straightforward in this random context. It is in the deterministic case when $V^\varepsilon \equiv 0$, since the conservation of the energy follows formally by a multiplication of (3) by $\partial_t \psi_i^*$, by integrating by parts, taking the real part and summing over i. When V^ε is not zero, this procedure brings either a term of the form

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t\int_{\mathbb{R}^3}(\partial_t V)(\frac{s}{\varepsilon},\frac{x}{\varepsilon})n^\varepsilon(s,x)dsdx \qquad \text{or} \qquad \frac{1}{\sqrt{\varepsilon}}\int_0^t\int_{\mathbb{R}^3}(\nabla_x V)(\frac{s}{\varepsilon},\frac{x}{\varepsilon})\cdot J^\varepsilon(s,x)dsdx,$$

where J^ε is the density current (see (47)), and these terms have a wrong homogeneity in ε. An important part of the proof is therefore to obtain some bounds on the total energy. We were actually able to show that the total energy is uniformly bounded in average, but have no information on the random energy itself. For this, we use the martingale formulation along with well-chosen perturbed test functions.

The rest of the paper is organized as follows: in section 2, we present the several hypotheses required to prove the result, we describe the construction of the random potential, recall some standard existence results for the WP and VPB systems, and finally state our main result. In section 3, we introduce the martingale formulation, give an outline of the proof, derive the energy estimate and obtain some tightness property. In section 4, we pass to the limit in the martingale formulation and prove our main theorem.

Notations. For a function $f \equiv f(x,k)$, the partial Fourier transforms w.r.t. x and k are denoted by

$$\mathcal{F}_{x\to\xi}f(\xi,k)=\int_{\mathbb{R}^3}e^{-ix\cdot\xi}f(x,k)dx \qquad ; \qquad \mathcal{F}_{k\to y}f(x,y)=\int_{\mathbb{R}^3}e^{-ik\cdot y}f(x,k)dk.$$

We use the classical notation for the continuous functions $\mathcal{C}^0(\mathbb{R}^3\times\mathbb{R}^3)$, for the infinitely differentiable functions with compact support $\mathcal{C}_c^\infty(\mathbb{R}^3\times\mathbb{R}^3)$, for the Schwartz class $\mathcal{S}(\mathbb{R}^3\times\mathbb{R}^3)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^3\times\mathbb{R}^3)$. We denote by $\mathcal{O}$ the class of $\mathcal{C}^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ functions, slowly increasing in the variable k and bounded in variable x, i.e. the set of $\mathcal{C}^\infty$ functions with all derivatives growing at most polynomially in k and bounded in x. An example of function in this class that we will use for the derivation of the energy estimate is $|k|^2$. When there is no possible confusion, L^p denotes the usual L^p spaces $L^p(\mathbb{R}^3)$ or $L^p(\mathbb{R}^3\times\mathbb{R}^3)$, $p\in[0,\infty]$. We also introduce the multi-index notation

$$\partial_x^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}}, \qquad \alpha_i\geq 0, \qquad \alpha=(\alpha_1,\alpha_2,\alpha_3), \quad |\alpha|=\alpha_1+\alpha_2+\alpha_3.$$

Depending on the regularity on f,g, $\langle f,g\rangle$ denotes both the integral

$$\int_{\mathbb{R}_x^3\times\mathbb{R}_k^3} f^*(x,k)g(x,k)dxdk$$

or the duality product $\mathcal{S}' - \mathcal{S}$. Throughout the paper, C denotes a generic constant independent of ε and T a finite time in $(0,\infty)$ independent of ε.

2 Main result

2.1 Settings

The random potential. We use a setting close to the one of [6, 7], see also [22] for generalities about Markov processes. Namely, we construct a potential $V(t,x)$ that is a stationary ergodic mean-zero Markov process in t, whose Fourier transform in space is a measure with bounded total variation. We actually use the bounded total variation property in the proof of the result and comment on this fact in Remark 4. The potential is essentially obtained by performing a discrete Fourier transform of appropriate random coefficients and is constructed as follows. Let $\mathcal{V}$ be the set of measures on $\mathbb{R}^3$ with bounded total variation with support in a ball $B_L = \{p \in \mathbb{R}^3, |p| \leq L\}$:

$$\mathcal{V} = \left\{ \hat{V} : \int_{\mathbb{R}^3} |d\hat{V}| < \infty, \quad \text{supp}\hat{V} \subset B_L, \quad \hat{V}(p) = \hat{V}^*(-p) \right\},$$

where $|d\hat{V}|$ denotes the total variation of the measure $d\hat{V}$. The last property in the definition insures that the Fourier transform of $d\hat{V}$ is real. Let B be the unit ball of $\mathbb{C}$ and $\mathbb{Z}^3_+ = \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}$. For $m = (m_1, m_2, m_3) \in \mathbb{Z}^3_+$, consider a collection of stationary ergodic independent Markov processes $\hat{v}(t) = (\hat{v}_m(t))_{m \in \mathbb{Z}^3_+}$, each with the same invariant measure π_0, such that for $t \in [0,T]$: $\hat{v}_m(t) \in B$ and

$$\mathbb{E}_{\pi_0}\{\hat{v}_m(t)\} = 0, \qquad \mathbb{E}_{\pi_0}\{(\hat{v}_m(t))^2\} = 0, \qquad \mathbb{E}_{\pi_0}\{|\hat{v}_m(t)|^2\} = R_0,$$

where $R_0 \in (0,1)$. Such construction is achieved for instance by setting $\hat{v}_m(t) = |\hat{v}_m(t)| e^{2i\pi\phi_m}$ for appropriate random variables $|\hat{v}_m(t)|$ and ϕ_m. The processes $\hat{v}_m(t)$ are assumed to be right-continuous with left limits jump processes with generator Q bounded on $L^\infty(B)$, and given by, for any $g \in L^\infty(B)$:

$$Qg(\hat{v}) = \int_B g(\hat{u}) d\pi_0(\hat{u}) - g(\hat{v}), \qquad \int_B d\pi_0(\hat{u}) = 1. \tag{10}$$

Note that $Q\hat{v}_m(t) = -\hat{v}_m(t)$ since $\hat{v}_m$ is mean-zero for all m. This means that $\hat{v}_m$ is an eigenfunction of Q associated with the eigenvalue -1, and therefore $e^{sQ}\hat{v}_m(t) = e^{-s}\hat{v}_m(t)$. Using this, the particular form of the generator yields the following correlation function, for $s > 0$:

$$\begin{aligned}
\mathbb{E}_{\pi_0}\{\hat{v}_m(t+s)\hat{v}^*_m(t)\} &:= R_0(s), \\
&= \mathbb{E}_{\pi_0}\{\hat{v}^* \mathbb{E}_{\pi_0}\{\hat{v}_m(t+s) | \hat{v}_m(t) = \hat{v}\}\}, \\
&= \mathbb{E}_{\pi_0}\{\hat{v}^* e^{sQ} \hat{v}\} = \mathbb{E}_{\pi_0}\{\hat{v}^* e^{-s} \hat{v}\}, \\
&= R_0 e^{-s}.
\end{aligned}$$

For all $s \in \mathbb{R}$, we find $R_0(s) = R_0 e^{-|s|}$. We extend $\hat{v}_m$ to all indices m in $\mathbb{Z}^3$ by symmetry by setting $\hat{v}_m = \hat{v}^*_{-m}$. The measure-valued Fourier transform $d\tilde{V}(t,p)$ of $V(t,x)$ with respect to x is finally defined by

$$d\tilde{V}[\hat{v}](t,p) = \sum_{m\in\mathbb{Z}^3} \sqrt{\hat{R}_1(p_m)}\hat{v}_m(t)\delta(p-p_m).$$

Above, δ denotes the Dirac measure, $(p_m)_{m\in\mathbb{Z}^3}$ is a set of points in $\mathbb{R}^3$ symmetric with respect to the origin (and thus satisfying $p_{(\sigma_1 m_1,\sigma_2 m_2,\sigma_3 m_3)} = \sigma_1\sigma_2\sigma_3 p_{(m_1,m_1,m_1)}$, for $\sigma_1,\sigma_2,\sigma_3=\pm 1$). Moreover, $\hat{R}_1$ a non-negative function in $\mathcal{C}_c^\infty(\mathbb{R}^3)$ with support included in B_L such that $\hat{R}_1(p)=\hat{R}_1(-p)$. We verify that almost surely, $d\tilde{V}\in L^\infty((0,T),\mathcal{V})$. The random potential $V(t,x)$ is then given by

$$V(t,x)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\tilde{V}(t,p)e^{ix\cdot p}.$$

We introduce the product measure $\pi=\pi_0^{\otimes\mathbb{Z}^3_+}$ on $(\mathcal{B},\sigma(B))$, where $\mathcal{B}=B^{\otimes\mathbb{Z}^3_+}$ and $\sigma(\mathcal{B})$ is the Borel $\sigma-$algebra on $\mathcal{B}$. We then have

$$\mathbb{E}_\pi\{d\tilde{V}(t,p)d\tilde{V}(t+s,q)\} = (2\pi)^3\delta(p+q)R_0(s)\sum_{m\in\mathbb{Z}^3}\hat{R}_1(p_m)\delta(p-p_m), \qquad (11)$$

so that the correlation function of the random potential is

$$\mathbb{E}_\pi\{V(t,y)V(t+s,y+x)\} = R(s,x) = (2\pi)^{-3}R_0(s)\sum_{m\in\mathbb{Z}^3}\hat{R}_1(p_m)e^{ix\cdot p_m}. \qquad (12)$$

The latter infinite sums are actually finite since $\hat{R}_1$ has a bounded support. The limiting Boltzmann equation collision term defined in (8) involves the Fourier transform in all variables of $R(s,x)$, namely, for a test function $\varphi\in\mathcal{S}(\mathbb{R}^3)$,

$$\begin{aligned}\mathcal{L}\varphi(k) &= \int_{\mathbb{R}^3}\frac{dp}{(2\pi)^3}\hat{R}\left(\frac{|k|^2-|p|^2}{2},p-k\right)(\varphi(p)-\varphi(k)),\\ &= \frac{1}{(2\pi)^3}\sum_{m\in\mathbb{Z}^3}\hat{R}_0\left(\frac{|k|^2-|p_m|^2}{2}\right)\hat{R}_1\left(p_m-k\right)(\varphi(p_m)-\varphi(k)),\end{aligned}$$

and is therefore a discrete operator with our construction. If one is fine with a discrete collision operator, one can leave the definition of the random potential here. We actually decide to deal with continuous collision operators and interpret the latter sum as the Riemann sum of the corresponding integral. This can be done either by introducing a small parameter independent of ε, and taking a second limit after the limit $\varepsilon\to 0$, or by allowing the small parameter to depend on ε and perform both limits at once. We choose the second option and define a parameter $h_\varepsilon = o(\varepsilon)$, and modify the Fourier transform of the potential as

$$d\tilde{V}[\hat{v}](t,p) = h_\varepsilon^3\sum_{m\in\mathbb{Z}^3}\sqrt{\hat{R}_1(p_m)}\hat{v}_m(t)\delta(p-p^\varepsilon_m), \qquad (13)$$

where $p^\varepsilon_m = h_\varepsilon m$ and for notational simplicity, we decided to omit the dependence of $d\tilde{V}$ on ε. This leads to

$$V(t,x)=\frac{h_\varepsilon^3}{(2\pi)^3}\sum_{m\in\mathbb{Z}^3}\sqrt{\hat{R}_1(p^\varepsilon_m)}\hat{v}_m(t)e^{ix\cdot p^\varepsilon_m}.$$

Since $\hat{R}_1$ has a bounded support, we then have, for k fixed here:

$$\frac{h_\varepsilon^3}{(2\pi)^3} \sum_{m\in\mathbb{Z}^3} \hat{R}_0 \left(\frac{|k|^2 - |p_m^\varepsilon|^2}{2} \right) \hat{R}_1 \left(p_m^\varepsilon - k \right) \left(\varphi(p_m^\varepsilon) - \varphi(k) \right)$$
$$= \int_{\mathbb{R}^3} \frac{dp}{(2\pi)^3} \hat{R}_0 \left(\frac{|k|^2 - |p|^2}{2} \right) \hat{R}_1 \left(p - k \right) \left(\varphi(p) - \varphi(k) \right) + \mathcal{O}(h_\varepsilon).$$

The following bounds also hold π almost surely, for some constant C independent of ε:

$$\sup_{\alpha\in\mathbb{N}^3} \|\partial_x^\alpha V\|_{L^\infty((0,T)\times\mathbb{R}^3)} + \left\| \int_{\mathbb{R}^3} |d\tilde{V}(t,p)| \right\|_{L^\infty(0,T)} \leq C h_\varepsilon^3 \sum_{m\in\mathbb{Z}^3} \sqrt{\hat{R}_1(p_m)} \leq C \left\| \sqrt{\hat{R}_1} \right\|_{L^1} \leq C. \tag{14}$$

This concludes our construction of the random potential.

Existence and regularity for the Wigner equation. When $V^\varepsilon \equiv 0$, it is proven in [3] for the Liouville-Poisson system (1)-(2) with ε fixed, under the hypotheses on the initial condition

$$\operatorname{Tr} \varrho_0^\varepsilon < \infty \qquad \text{and} \qquad \operatorname{Tr} \sqrt{-\Delta} \varrho_0^\varepsilon \sqrt{-\Delta} < \infty,$$

that there exists a unique solution $\varrho^\varepsilon(t)$, trace class, hermitian and positive for all times $t \in [0,T]$ and such that

$$\operatorname{Tr} \varrho_0^\varepsilon = \operatorname{Tr} \varrho^\varepsilon(t) \qquad \text{and} \qquad \operatorname{Tr} \sqrt{-\Delta} \varrho^\varepsilon(t) \sqrt{-\Delta} < \infty.$$

Above, Tr denotes the trace of an operator. Same type of results are obtained in [29, 15] for the Wigner-Poisson system (6)-(2) with more regular initial conditions. Using the regularity of the random potential (14), it is possible to adapt the proofs of [3, 29, 15] to show that the Wigner equation (6) admits a unique solution such that $W^\varepsilon \in \mathcal{C}^0([0,T], L^2)$ π almost surely. This solution admits in addition the conserved quantities

$$\|W^\varepsilon(t)\|_{L^2} = \|W_0^\varepsilon\|_{L^2}, \qquad \|n^\varepsilon(t)\|_{L^1} = \|n_0^\varepsilon\|_{L^1}, \qquad \forall t \in [0,T], \tag{15}$$

with the notation $n_0^\varepsilon = n^\varepsilon(t=0)$. As mentioned in [15], it is possible to use bootstrapping arguments to obtain a better regularity, so that if W_0^ε belong to $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ for instance (which holds if $\rho_0^\varepsilon \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$), then it can be shown that $W^\varepsilon(t) \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$, for all $t \in [0,T]$ and almost surely. This is the regularity we will assume in the sequel for simplicity, even though such strength is not required in the proofs. This regularity theory holds for ε fixed, but does not hold in the limit. The kinetic energy is defined by

$$\mathcal{E}_{\text{kin}}^\varepsilon(t) = \frac{1}{2} \int_{\mathbb{R}^6_{xk}} |k|^2 W^\varepsilon(t) dx dk \tag{16}$$

which in terms of ϱ^ε or ψ_i^ε reads

$$2\mathcal{E}_{\text{kin}}^\varepsilon(t) = \varepsilon^2 \operatorname{Tr} \sqrt{-\Delta} \varrho^\varepsilon(t) \sqrt{-\Delta} = \sum_{i\in\mathbb{N}} \rho_i^\varepsilon \|\varepsilon \nabla \psi_i^\varepsilon(t)\|_{L^2}^2. \tag{17}$$

The total energy, kinetic plus potential is given by

$$\mathcal{E}^\varepsilon(t) = \mathcal{E}^\varepsilon_{\text{kin}}(t) + \frac{1}{2}\|\nabla U^\varepsilon(t)\|^2_{L^2}, \tag{18}$$

with the notation $\mathcal{E}^\varepsilon_0 = \mathcal{E}^\varepsilon(t=0)$. The assumptions we make on the initial condition are therefore:

H: W^ε_0 is deterministic and the Wigner transform of an integral kernel $\rho^\varepsilon_0 \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ with the uniform estimates

$$\|n^\varepsilon_0\|_{L^1} + \|W^\varepsilon_0\|_{L^2} + \mathcal{E}^\varepsilon_0 \leq C, \tag{19}$$

where the constant C does not depend on ε. This ensures that almost surely $W^\varepsilon \in \mathcal{C}^0([0,T], \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3))$ with the conservations (15). Notice the fact that W^ε_0 being uniformly bounded in L^2 implies

$$\sum_{i\in\mathbb{N}} (\rho^\varepsilon_i)^2 = \text{Tr}(\varrho^\varepsilon_0)^2 = \|\rho^\varepsilon_0\|^2_{L^2} = \varepsilon^3 2^{-3} \|W^\varepsilon_0\|^2_{L^2} \leq C\varepsilon^3. \tag{20}$$

H': W^ε_0 converges weakly in L^2 to a Lipschitz function $W_0 \in L^1 \cap L^\infty$ which verifies

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^m W_0(x,k)dxdk < \infty, \qquad \text{for some } m > 6.$$

The regularity assumed in **H'** insures [34] that the Vlasov-Poisson system in $\mathbb{R}^3$ has a unique solution $f \in \mathcal{C}^0([0,T], L^p(\mathbb{R}^3\times\mathbb{R}^3)) \cap L^\infty((0,T), L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$, for all $p \in [0,\infty)$, and such that

$$\forall m' < m, \qquad \sup_{t\in[0,T]} \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^{m'} f(t,x,k)dxdk < \infty. \tag{21}$$

The existence of solutions only requires weaker assumptions on the initial condition. The hypotheses made above allow to show [34] that the density $n(t)$ belongs to $L^\infty(\mathbb{R}^3)$, which is known [35] be a sufficient condition for uniqueness. The existence and uniqueness theory for the Vlasov-Poisson-Boltzmann system (9) is essentially the same. The asymptotic limit of (6) provides the existence of a solution of (9) in $L^1 \cap L^2$ that verifies

$$\text{ess}\sup_{t\in[0,T]} \int_{\mathbb{R}^3\times\mathbb{R}^3} |k|^2 W(t,x,k)dxdk < \infty.$$

Assumption **H'** then gives as in [34] the $L^\infty(\mathbb{R}^3)$ bound for the density and consequently the uniqueness of a solution $W \in \mathcal{C}^0([0,T], L^p(\mathbb{R}^3\times\mathbb{R}^3)) \cap L^\infty((0,T), L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$, for all $p \in [0,\infty)$ and satisfying estimate (21).

2.2 Main theorem

We will prove the following theorem:

Theorem 1 *Under Assumptions **H** and **H'**, the solution W^ε to the random Wigner-Poisson equation* (6) *converges weakly in L^2 and in probability to the unique solution W to the Vlasov-Poisson-Boltzmann equation* (9) *W, i.e., uniformly on compact intervals, for all $\lambda_0 \in L^2$:*

$$\lim_{\varepsilon\to 0} \mathbb{P}^\varepsilon\left(|\langle W^\varepsilon(t), \lambda_0\rangle - \langle W(t), \lambda_0\rangle| > \delta\right) = 0, \qquad \forall \delta > 0.$$

Moreover, the density is self-averaging: uniformly on compact intervals, for all $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^3)$:

$$\lim_{\varepsilon\to 0} \mathbb{P}^\varepsilon\left(|(n^\varepsilon(t), \lambda_0)_2 - (n(t), \lambda_0)_2| > \delta\right) = 0, \qquad \forall \delta > 0.$$

In the theorem, $\mathbb{P}^\varepsilon$ refers to the measure defined on the space $\mathcal{C}^0([0,T], L^2(\mathbb{R}^3 \times \mathbb{R}^3))$ generated by the Cauchy problem (6), see section 3, and $(f,g)_2 = \int_{\mathbb{R}^3} f^* g dx$. The cross-section of the collision operator in the Boltzmann equation is $\sigma(u,p) = \hat{R}_0(u)\hat{R}_1(p)$, where $\hat{R}_0$ and $\hat{R}_1$ are defined in section 2.1. Theorem 1 shows that not only the Wigner transform W^ε is self-averaging, but also the density n^ε. As explained in the introduction, the latter is a consequence of the fact that the total energy is bounded in average independently of ε at all times. Such property was apparently not known. It is independent of the non-linearity and also holds for linear problems.

Theorem 1 can easily be adapted to yield different collisions kernel than the one in (9). For instance, if the random potential oscillates faster in time than the scale ε^{-1}, i.e. V^ε has the form $V^\varepsilon(t,x) = V(\frac{x}{\varepsilon^{1+\gamma}}, \frac{x}{\varepsilon})$, for some $\gamma > 0$, then the limiting kernel is given by

$$\mathcal{L}W(k) = \int_{\mathbb{R}^3} \frac{dp}{(2\pi)^2} \sigma(0, p-k)(W(p) - W(k)).$$

On the contrary, if now V^ε has the form $V^\varepsilon(t,x) = V(\frac{x}{\varepsilon^{1-\gamma}}, \frac{x}{\varepsilon})$, for some $\gamma > 0$ small enough, then the collision kernel becomes conservative and reads

$$\mathcal{L}W(k) = \int_{\mathbb{R}^3} \frac{dp}{(2\pi)^2} \sigma_0(p-k)(W(p) - W(k))\delta\left(\frac{|k|^2 - |p|^2}{2}\right), \qquad \sigma_0(p) = \int_{\mathbb{R}} \sigma(u,p) du.$$

Other transport regimes, such as Fokker-Planck-Poisson equations, can be obtained depending on the scale of the spatial fluctuations of the random potential, we refer to [5] for the corresponding scalings.

The Markov property of the random potential is essential in our analysis. It is known in the linear case that such assumption can be relaxed to a random potential independent of time by making use of diagrammatic expansions of the solution of the Schrödinger equation [21, 39]. Such strategy only provides the convergence of the expectation of W^ε and not the convergence in probability. In our non-linear setting, it is not clear this technique can still be applied as the non-linearity should break down the diagrammatic expansion procedure. We nevertheless expect the result to hold true for other types of random potentials with fast oscillations, as for instance the ones constructed by Fannjiang in [23] that are not Markovian but satisfy some subgaussian and mixing properties. The mixed-state assumption is also crucial, both for the treatment of the non-linearity and the randomness.

Other types of non-linearities can also be considered provided they are sufficiently regularizing, for instance non-linear potentials of the form $U^\varepsilon = (-\Delta)^{-s} n^\varepsilon$ for $s \leq 1$ positive large enough.

3 Martingale formulation

3.1 Outline of the proof

The proof combines the perturbed test functions method for martingale problems [32, 37] with compactness methods for the Wigner-Poisson problem [33]. First of all, the Cauchy problem (6) generates a measure $\mathbb{P}^\varepsilon$ on the space $\mathcal{C}^0([0,T], L^2(\mathbb{R}^3\times\mathbb{R}^3))$ supported inside the ball $X = \{W \in L^2,\ \|W\|_{L^2} \le C\}$, where the constant C is as in (19). The set X is the state space for the random process $W^\varepsilon(t)$. The trajectories are actually smoother (in $\mathcal{C}^0([0,T], S(\mathbb{R}^3\times\mathbb{R}^3))$ because of the regularity assumptions on W_0^ε), but that of the limiting measure are not because only the L^2 norm of W^ε is uniformly bounded. Let us introduce the notation $\hat{v}^\varepsilon(t) = (\hat{v}_m(\frac{t}{\varepsilon}))_{m\in\mathbb{Z}^3_+} \in \mathcal{B}$ for all $t \ge 0$, where the coefficients $\hat{v}_m$ and $\mathcal{B}$ are defined in section 2.1. The fact that $\hat{v}^\varepsilon(t)$ is Markov allows us to treat the process $(\hat{v}^\varepsilon(t), W^\varepsilon(t))$ as jointly Markov and to obtain the corresponding generator. The process takes values in $\mathcal{B}\times X$ and we denote by $\tilde{\mathbb{P}}^\varepsilon$ the corresponding measure on $\mathcal{B}\times X$. The first step of the proof is to write the martingale problem for $\tilde{\mathbb{P}}^\varepsilon$. For this, since $\hat{v}^\varepsilon(t)$ appears in the Schrödinger equation only through the potential, or equivalently its spatial Fourier transform $\tilde{V}^\varepsilon(t,p) := \tilde{V}[\hat{v}^\varepsilon](\frac{t}{\varepsilon}, p) \in \mathcal{V}$ for $t \in [0,T]$ defined in (13), we do not need to use general functions of the form $F \equiv F(\hat{v}^\varepsilon, W^\varepsilon)$, but only functions of the form $F \equiv F(\tilde{V}[v^\varepsilon], W^\varepsilon)$, for F smooth. The conditional expectation with respect to $\tilde{\mathbb{P}}^\varepsilon$ is given by

$$\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}_{W,\hat{v},t}\left\{F(\tilde{V}[\hat{v}^\varepsilon], W^\varepsilon)\right\}(\tau) = \mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\left\{F(\tilde{V}[\hat{v}^\varepsilon](\tau), W^\varepsilon(\tau))|\ W^\varepsilon(t) = W,\ \hat{v}^\varepsilon(t) = \hat{v}\right\}, \quad \tau \ge t,$$

and the generator $\mathcal{A}^\varepsilon$ is defined by

$$\frac{d}{ds}\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}_{W,\hat{v},t}\left\{F(\tilde{V}[\hat{v}^\varepsilon], W^\varepsilon)\right\}(t+s)\bigg|_{s=0} = \mathcal{A}^\varepsilon F(\tilde{V}[\hat{v}], W). \tag{22}$$

We then have the property

$$M_t^\varepsilon := F(\tilde{V}[\hat{v}^\varepsilon](t), W^\varepsilon(t)) - \int_0^t \mathcal{A}^\varepsilon F(\tilde{V}[\hat{v}^\varepsilon](s), W^\varepsilon(s))ds \qquad \text{is a } \tilde{\mathbb{P}}^\varepsilon \text{ martingale.} \tag{23}$$

Let now λ be a regular function in $L^\infty(\mathcal{V}, \mathcal{O})$ and let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be another smooth function and define

$$f_\varepsilon(t) = \phi(\langle W^\varepsilon(t), \lambda\rangle), \qquad f'_\varepsilon(t) = \phi'(\langle W^\varepsilon(t), \lambda\rangle), \qquad f''_\varepsilon(t) = \phi''(\langle W^\varepsilon(t), \lambda\rangle).$$

Only the choices $\phi(u) = u$ and $\phi(u) = u^2$ will be relevant to us in the sequel. Note that $\langle W^\varepsilon(t), \lambda\rangle$ makes sense since $W^\varepsilon(t) \in \mathcal{S}$ and $\lambda(\cdot,\cdot,\hat{V}) \in \mathcal{O}$ for $\hat{V} \in \mathcal{V}$. We will use test functions of the class $\mathcal{O}$ only for the derivation of the energy estimate, we will use otherwise $\mathcal{C}_c^\infty$ functions for the perturbed test function method or passing to the limit. In order to find the form of the generator $\mathcal{A}^\varepsilon$, consider first functions of $\tilde{V}^\varepsilon[\hat{v}^\varepsilon]$ only. Since $\hat{R}_1$ has a bounded support included in B_L, see section 2.1, $\tilde{V}[\hat{v}^\varepsilon]$ actually depends on a finite number of coefficients $\hat{v}_m$. They are such that $|m|h_\varepsilon \le L$, where $|m| = \sqrt{m_1^2 + m_2^2 + m_3^2}$, and we denote by N^ε the set of corresponding indices. Let us introduce the notation $g(\hat{v}^\varepsilon) = F(\tilde{V}[\hat{v}^\varepsilon])$, where $F \in \mathcal{C}_c^\infty(\mathcal{V})$. Recalling that π denotes the invariant measure of the process $\hat{v}(t)$, we then have

$$\frac{d}{ds}\mathbb{E}_\pi \left\{ g(\hat{v}^\varepsilon(t+s)) \middle| \, \hat{v}^\varepsilon(t) = \hat{v} \right\} \bigg|_{s=0} = \frac{1}{\varepsilon} Q_\varepsilon g(\hat{v}), \tag{24}$$

where

$$Q_\varepsilon g(\hat{v}) = \sum_{m \in N^\varepsilon} Q_m g(\hat{v}) := \sum_{m \in N^\varepsilon} \left[\int_B g(\hat{v}) d\pi_0(\hat{v}_m) - g(\hat{v}) \right].$$

The first term of the r.h.s above corresponds to averaging with respect to the $m-$th component $\hat{v}_m$ of $\hat{v} = (\hat{v}_m)_{m \in \mathbb{Z}^3_+}$. From (6), (22) and (24), we obtain the following weak form of the generator

$$\mathcal{A}^\varepsilon f_\varepsilon(t) \;\; = \;\; f'_\varepsilon(t) \left\langle W, \left(\frac{1}{\varepsilon} Q_\varepsilon + k \cdot \nabla_x + \frac{1}{\sqrt{\varepsilon}} \mathcal{K}[\hat{v}, \frac{x}{\varepsilon}] + \mathcal{L}_2^\varepsilon \right) \lambda \right\rangle \tag{25}$$

where we have introduced the notation ($\eta \equiv \frac{x}{\varepsilon}$ denotes the fast variable for simplicity)

$$\mathcal{K}[\hat{v}, \eta]\psi(x, \eta, k, \hat{v}) = \frac{1}{i} \int_{\mathbb{R}^3} \frac{d\tilde{V}[\hat{v}](t,p)}{(2\pi)^3} e^{ip\cdot\eta} \left[\psi(x, \eta, k - \frac{p}{2}) - \psi(x, \eta, k + \frac{p}{2}) \right]. \tag{26}$$

The operator $\mathcal{K}[\hat{v}, \eta]$ is simply a reformulation of $\mathcal{L}_1^\varepsilon$ that emphasizes the dependence on $(\hat{v}, \eta)$. Above, we have used that both $\mathcal{L}_1^\varepsilon$ and $\mathcal{L}_2^\varepsilon$ are self-adjoint for $\langle \cdot, \cdot \rangle$. The goal is then to pass to the limit in the martingale formulation (23) to obtain the weak formulation of the Vlasov-Poisson-Boltzmann equation. This requires two types of results: the tightness of $\mathbb{P}^\varepsilon$ for convergence to a limiting measure $\mathbb{P}$, this question is addressed in section 3.5; and regularity estimates, that allow both to prove the tightness and to pass to the limit in the non-linear term. The key ingredient is the energy estimate obtained in section 3.4 with the help of the perturbed test functions method. This in turn provides improved estimates on the density using Lieb-Thirring inequalities as in [33]. The limiting process is then shown to be a martingale with null quadratic variation that solves in the distribution sense the VPB equation. The VPB system admitting only one solution under assumption **H'**, our limit is this unique solution and deterministic.

The rest of the section is structured as follows: in section 3.2, we introduce some perturbed test functions that will be used all along the paper; in section 3.3, we recall basic estimates for the WP problem and in section 3.4 we prove the energy estimate.

3.2 Perturbed test functions

We will need the following two functions in our analysis : let first $\lambda_0 \in \mathcal{O}$, $\hat{v} \in \mathcal{B}$ and consider the equation for λ_1:

$$k \cdot \nabla_\eta \lambda_1 + Q_\varepsilon \lambda_1 = -\mathcal{K}[\hat{v}, \eta]\lambda_0. \tag{27}$$

The function λ_1 depends on the fast variable $\eta = \frac{x}{\varepsilon}$, and on the slow variable x if λ_0 does. Solving (27) amounts to solving the following standard Poisson problem for $g \in L^\infty(\mathcal{B})$:

$$Q_\varepsilon f = g.$$

The kernel of Q_ε is given by functions satisfying

$$f(\hat{v}) = \frac{1}{|N_\varepsilon|} \sum_{m \in N^\varepsilon} Q_m f(\hat{v}),$$

which is reduced to constants functions with respect to the variables $\hat{v}_m$ for $m \in N^\varepsilon$. If π_{N^ε} denotes the marginal $\pi_{N^\varepsilon} = \int_{\mathbb{Z}^3_+ \setminus N_\varepsilon} \pi$, the Poisson equation is uniquely solvable according to the Fredholm alternative provided g is orthogonal to the kernel of Q_ε, that is $\mathbb{E}_{\pi_{N^\varepsilon}}\{g(\hat{v})\}$=0. In such case, the solution verifies $f \in L^\infty(\mathcal{B})$ with $\mathbb{E}_{\pi_{N^\varepsilon}}\{f(\hat{v})\} = 0$ and reads

$$f(\hat{v}) = Q_\varepsilon^{-1} g(\hat{v}) = -\int_0^\infty dr e^{rQ_\varepsilon} g(\hat{v}). \tag{28}$$

Going back to (27), the equation is uniquely solvable since

$$\mathbb{E}_{\pi_{N^\varepsilon}}\{\mathcal{K}[\hat{v},\eta]\lambda_0\} = \mathbb{E}_\pi\{\mathcal{K}[\hat{v},\eta]\lambda_0\} = 0$$

and (28) yields after an inverse Fourier transform

$$\begin{aligned} \lambda_1(x,\eta,k,\hat{v}) &= \frac{1}{i}\int_0^\infty dr e^{rQ_\varepsilon} g(r,x,\eta,k,\hat{v}), \\ g(r,x,\eta,k,\hat{v}) &= \int_{\mathbb{R}^3} \frac{d\tilde{V}[\hat{v}](p)}{(2\pi)^3} e^{irk\cdot p + i\eta\cdot p} \left[\lambda_0(x, k-\frac{p}{2}) - \lambda(x_0, k+\frac{p}{2})\right]. \end{aligned}$$

The expression can be simplified with the observation that g can be written as

$$g \equiv \sum_{m \in N^\varepsilon} \alpha_m \hat{v}_m,$$

for some coefficients α_m that depend on (r,x,η,k). We have consequently $Q_\varepsilon g = -g$ (see section 2.1) and therefore $e^{rQ_\varepsilon} g = e^{-r} g$. The function λ_1 is then reduced to

$$\lambda_1(x,\eta,k,\hat{v}) = \frac{1}{i}\int_0^\infty dr e^{-r} \int_{\mathbb{R}^3} \frac{d\tilde{V}[\hat{v}](p)}{(2\pi)^3} e^{irk\cdot p + i\eta\cdot p} \left[\lambda_0(x, k-\frac{p}{2}) - \lambda(x_0, k+\frac{p}{2})\right]. \tag{29}$$

Since $\lambda_0 \in \mathcal{O}$, it is not difficult using (29) and (14) to see that $\lambda_1 \in L^\infty(\mathcal{B}, \mathcal{O})$. A simple calculation involving (11) and (29) shows in addition that

$$\mathbb{E}_\pi\{\mathcal{K}[\hat{v},\eta]\lambda_1\} = \mathcal{L}_\varepsilon \lambda_0, \tag{30}$$

where $\mathcal{L}_\varepsilon$ is given by

$$\mathcal{L}_\varepsilon \lambda_0(x,k) = \frac{h_\varepsilon^3}{(2\pi)^3} \sum_{m \in \mathbb{Z}^3} \hat{R}_0\left(\frac{|k|^2 - |p_m^\varepsilon|^2}{2}\right) \hat{R}_1(p_m^\varepsilon - k)\,(\lambda_0(x, p_m^\varepsilon) - \lambda_0(x,k)). \tag{31}$$

See section (2.1) for the definitions of h_ε and p_m^ε. We will also need the second order corrector λ_2 given by the solution to

$$k \cdot \nabla_\eta \lambda_2 + Q_\varepsilon \lambda_2 = \mathcal{L}_\varepsilon \lambda_0 - \mathcal{K}[\hat{v},\eta]\lambda_1.$$

Thanks to (30), the equation is uniquely solvable and the solution reads

$$\lambda_2(x,\eta,k,\hat{v}) = -\int_0^\infty dr e^{rQ_\varepsilon}\left[(\mathcal{L}_\varepsilon \lambda_0)(x,k) - (\mathcal{K}[\hat{v},\eta+rk]\lambda_1)(x,\eta+rk,k,\hat{v})\right]. \tag{32}$$

Some estimates on λ_1 and λ_2 will be given in Lemmas 3.1 and 4.1. We have for instance that λ_1 and λ_2 are bounded uniformly in ε if λ_0 belongs to L^2.

3.3 Standard estimates for the Wigner-Poisson problem

In addition to the conservation of the L^2 norm of the Wigner transform and the L^1 norm of the density stated in (15), some other uniform estimates can be deduced from the Wigner equation (6). A crucial one is the energy estimate that we derive in section 3.4. As already mendioned, such energy estimate is straightforward in a deterministic setting (provided the exterior potential is slowly variable in terms of ε), while it requires an additional effort in the random case due to the fast oscillations both in time and space of the random potential. Owing such energy estimate, it is then possible to deduce improved bounds on the density n^ε and the potential U^ε as in [33]. Regarding the density, the use of Lieb-Thirring inequalities yields in [33, 3] the estimate

$$\|n^\varepsilon(t)\|_{L^q} \leq C_q \left(\mathrm{Tr}(\varrho^\varepsilon(t))^2\right)^{\theta/2} (\mathcal{E}^\varepsilon_{\mathrm{kin}}(t))^{1-\theta} \varepsilon^{2\theta-2}, \qquad \theta \in [0,1], \tag{33}$$

with

$$\theta = \frac{3}{2q} - \frac{1}{2} \qquad \text{and} \qquad \frac{7}{5} \leq q \leq 3.$$

Setting $q = \frac{7}{5}$ gives

$$\|n^\varepsilon(t)\|_{L^{7/5}} \leq C \left(\mathrm{Tr}(\varrho^\varepsilon(t))^2\right)^{\frac{2}{7}} (\mathcal{E}^\varepsilon_{\mathrm{kin}}(t))^{\frac{3}{7}} \varepsilon^{-\frac{6}{7}}.$$

According to (20), $\mathrm{Tr}(\varrho^\varepsilon(t))^2 \leq C\varepsilon^3$, so that the different ε compensate exactly and one finds for all $t \in [0,T]$ and almost surely:

$$\|n^\varepsilon(t)\|_{L^{7/5}} \leq C(\mathcal{E}^\varepsilon_{\mathrm{kin}}(t))^{\frac{3}{7}}. \tag{34}$$

Regarding the Poisson potential U^ε, the Hardy-Littlewood-Sobolev inequality [38] yields, for $n = 0, 1$

$$\|\partial^n_{x_i} U^\varepsilon\|_{L^p} \leq C\|n^\varepsilon\|_{L^r}, \qquad \frac{1}{r} = \frac{2-n}{3} + \frac{1}{p}, \qquad 1 < r, p < \infty. \tag{35}$$

3.4 Energy estimate

We prove in the section the following proposition, where we recall that $\mathcal{E}^\varepsilon$ is the total energy defined in (18):

Proposition 2 *We have the energy estimate*

$$\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon(t)\} \leq C. \tag{36}$$

Our strategy consists in using the martingale property of M^ε_t along with the perturbed test functions method with test functions $\phi(u) = u$ and $\lambda_0(x,k) = |k|^2 \in \mathcal{O}$. For $\hat{v}^\varepsilon(t) = \hat{v}(\frac{t}{\varepsilon}) \in \mathcal{B}$, we use the shorthand $\tilde{V}^\varepsilon(t,p) = \tilde{V}[\hat{v}^\varepsilon](\frac{t}{\varepsilon}, p) \in \mathcal{V}$ for $t \in [0,T]$, where $\tilde{V}$ is defined in (13). Using expression (29), we define first

$$\lambda^\varepsilon(x,k,\tilde{V}^\varepsilon(t)) = \lambda_0(x,k) + \sqrt{\varepsilon}\lambda_1(x, \frac{x}{\varepsilon}, k, \tilde{V}^\varepsilon(t)) \in L^\infty(\mathcal{V}, \mathcal{O}), \tag{37}$$

where λ_1 is defined in (29) (we identity here $\tilde{V}^\varepsilon$ with $\hat{v}^\varepsilon$). Since M_t^ε is a $\tilde{\mathbb{P}}^\varepsilon$ martingale, we have the relation

$$\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\{\langle W^\varepsilon(t), \lambda^\varepsilon(x,k,\tilde{V}^\varepsilon(t))\rangle\} = \mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\{\langle W^\varepsilon(0), \lambda^\varepsilon(x,k,\tilde{V}^\varepsilon(0))\rangle\} + \mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\int_0^t \mathcal{A}^\varepsilon f_\varepsilon(s)ds. \quad (38)$$

We will then show that

$$\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\{\langle W^\varepsilon(t), \lambda^\varepsilon(x,k,\tilde{V}^\varepsilon(t))\rangle\} = \mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon_{\text{kin}}(t)\} + \mathcal{O}(\sqrt{\varepsilon}) \quad (39)$$

and

$$\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\int_0^t \mathcal{A}^\varepsilon f_\varepsilon(s)ds = \frac{1}{2}\|\nabla U^\varepsilon(0)\|^2_{L^2} - \frac{1}{2}\mathbb{E}^{\mathbb{P}^\varepsilon}\|\nabla U^\varepsilon(t)\|^2_{L^2} + \int_0^t g^\varepsilon(s)ds, \quad (40)$$

where the term $|g^\varepsilon(s)|$ can be controlled by $1+\mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon(s)\}$. Together with (38)–(39), we obtain an estimate of the form

$$0 \leq \mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon(t)\} \leq \mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon(0)\} + C\int_0^t \mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon(s)\}ds + \mathcal{O}(\sqrt{\varepsilon}),$$

which yields the desired result thanks to the Gronwall Lemma. We give below the complete proof of Proposition 2. An important point is to show that g^ε is sublinear with respect to $\mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{E}^\varepsilon(s)\}$. This is not clear at first sight since g^ε involves $\langle W^\varepsilon, \mathcal{L}_2^\varepsilon(\lambda_0+\sqrt{\varepsilon}\lambda_1)\rangle$, whose homogeneity is like $(W^\varepsilon)^2$. The term $\langle W^\varepsilon, \mathcal{L}_2^\varepsilon\lambda_0\rangle$ directly gives the potential energy. For the second one $\sqrt{\varepsilon}\langle W^\varepsilon, \mathcal{L}_2^\varepsilon\lambda_1\rangle$, we make an explicit use of the extra $\sqrt{\varepsilon}$ which allows us to obtain a control in terms of $\|n^\varepsilon\|^{\alpha_1}_{L^{7/5}}$, $\|\nabla U^\varepsilon\|^{\alpha_2}_{L^2}$ and $\|U^\varepsilon\|^{\alpha_3}_{L^{14}}$ for α_1, α_2 and α_3 small enough. The Lieb-Thirring inequality (33) then provides a sublinear estimate for g^ε in terms of the energy. Without using the extra $\sqrt{\varepsilon}$, the powers α_1, α_2 and α_3 are too large to obtain a sublinear estimate.

Proof. [Proposition 2] We need first an adequate expression for the first corrector λ_1. Fix as a start some $\hat{V} \in \mathcal{V}$. Plugging $\lambda_0(x,k) = |k|^2$ into (29) yields

$$\lambda_1(\eta,k,\hat{V}) = 2i\int_0^\infty dr e^{-r}\int_{\mathbb{R}^3}\frac{d\hat{V}(p)}{(2\pi)^3}e^{irk\cdot p+i\eta\cdot p}(k\cdot p) \in L^\infty(\mathcal{V},\mathcal{O}).$$

The latter expression is not convenient for estimating λ_1 as it involves $k\cdot p$. It can be simplified by performing an integration by part in r. This leads to

$$\lambda_1(\eta,k) = -2V(\eta) + 2\int_0^\infty dr e^{-r}V(\eta+rk)$$

and its partial Fourier transforms in $\mathcal{S}'$ read

$$\mathcal{F}_{\eta\to\xi}\lambda_1(\xi,k) = -2d\hat{V}(\xi) + 2\int_0^\infty dr e^{-r}e^{irk\cdot\xi}d\hat{V}(\xi) \quad (41)$$

$$\mathcal{F}_{k\to y}\lambda_1(\eta,y) = -2(2\pi)^3\delta(y)V(\eta) + 2\int_0^\infty dr e^{-r}\int_{\mathbb{R}^3}d\hat{V}(\xi)e^{i\eta\cdot\xi}\delta(y-r\xi). \quad (42)$$

The expressions (41) and (42) are understood in the distribution sense and need to be integrated against a test function to make sense. For instance, for all $\psi \in \mathcal{S}$:

$$\langle\psi, \mathcal{F}_{k\to y}\lambda_1\rangle = -2(2\pi)^3\int_{\mathbb{R}^3}d\eta\psi^*(\eta,0)V(\eta) + 2\int_0^\infty dr e^{-r}\int_{\mathbb{R}^6}d\eta d\hat{V}(\xi)e^{i\eta\cdot\xi}\psi^*(\eta,r\xi).$$

We are ready now to use the martingale formulation. For the λ^ε given in (37), let us define

$$G_{\lambda^\varepsilon}(t) = \langle W^\varepsilon(t), \lambda^\varepsilon(x,k,\tilde{V}^\varepsilon(t)\rangle - \int_0^t \mathcal{A}^\varepsilon f_\varepsilon(s)ds,$$

which is a $\tilde{\mathbb{P}}^\varepsilon$ martingale. Using the definition of λ_1, (25) and (27), we find

$$\begin{aligned} G_{\lambda_\varepsilon}(t) &= \langle W^\varepsilon(t), \lambda^\varepsilon\rangle - \int_0^t ds \left\langle W^\varepsilon(s), \left(\mathcal{K}[\hat{v}^\varepsilon(s), \frac{x}{\varepsilon}] + \sqrt{\varepsilon}\mathcal{L}_2^\varepsilon\right)\lambda_1\right\rangle - \int_0^t ds \left\langle W^\varepsilon(s), \mathcal{L}_2^\varepsilon\lambda_0\right\rangle, \\ &:= \langle W^\varepsilon(t), \lambda^\varepsilon\rangle - A_1^\varepsilon - A_2^\varepsilon - A_3^\varepsilon, \end{aligned}$$

with obvious notations. We treat each term separately. For simplicity, we denote by $\hat{V} \equiv \tilde{V}^\varepsilon(t) \equiv \tilde{V}[\hat{v}^\varepsilon(t)] \in \mathcal{V}$ which satisfies the uniform bound (14).

The term A_1^ε. It is not difficult to see that $\mathcal{K}[\hat{v}^\varepsilon, \frac{x}{\varepsilon}]\lambda_1 \in L^\infty(\mathcal{B},\mathcal{O})$, so that its partial Fourier transforms in x and k are well-defined in $\mathcal{S}'$. Besides, since $W^\varepsilon(t) \in \mathcal{S}$, we have

$$\left\langle W^\varepsilon, \mathcal{K}[\hat{v}^\varepsilon, \frac{x}{\varepsilon}]\lambda_1\right\rangle = \frac{1}{(2\pi)^3}\langle \mathcal{F}_{x\to\xi}W^\varepsilon, \mathcal{F}_{x\to\xi}\mathcal{K}[\hat{v}^\varepsilon, \frac{x}{\varepsilon}]\lambda_1\rangle$$

and

$$(\mathcal{F}_{x\to\xi}\mathcal{K}[\hat{v}^\varepsilon, \frac{x}{\varepsilon}]\lambda_1)(\xi,k) = \frac{1}{i}\sum_{\sigma_1=\pm1}\sigma_1\int_{\mathbb{R}^3}\frac{d\hat{V}(p)}{(2\pi)^3}\mathcal{F}_{x\to\xi}\lambda_1(\xi-\frac{p}{\varepsilon}, k-\sigma_1\frac{p}{2}) \in L^\infty(\mathcal{V},\mathcal{S}').$$

Using (41), it comes

$$\sum_{\sigma_1=\pm1}\sigma_1\mathcal{F}_{x\to\xi}\lambda_1(\xi-\frac{p}{\varepsilon}, k-\sigma_1\frac{p}{2}) = 2\varepsilon^3\sum_{\sigma_1=\pm1}\sigma_1\int_0^\infty e^{-r}d\hat{V}(\varepsilon\xi-p)e^{ir(k-\sigma_1\frac{p}{2})\cdot(\varepsilon\xi-p)}$$

so that

$$\begin{aligned} &\left\langle W^\varepsilon(s), \mathcal{K}[\hat{v}^\varepsilon, \frac{x}{\varepsilon}]\lambda_1\right\rangle \\ &= \frac{2}{i(2\pi)^6}\sum_{\sigma_1=\pm1}\sigma_1\int_{\mathbb{R}^3_p}d\hat{V}(p)\int_0^\infty dre^{-r}\int_{\mathbb{R}^6_{\xi k}}d\hat{V}(\xi)dke^{irk\cdot\xi}(\mathcal{F}_{x\to\xi}W^\varepsilon)^*(\xi+\frac{p}{\varepsilon}, k+\sigma_1\frac{p}{2}). \end{aligned}$$

Using (14), the latter expression can estimated by:

$$\begin{aligned} &\left|\left\langle W^\varepsilon(s), \mathcal{K}[\hat{v}^\varepsilon, \frac{x}{\varepsilon}]\lambda_1\right\rangle\right| \\ &\le C\int_{\mathbb{R}^3}|d\hat{V}(u)|\sup_{p\in\mathbb{R}^3}\left|\int_0^\infty dre^{-r}\int_{\mathbb{R}^6_{\xi k}}d\hat{V}(\xi)dke^{irk\cdot\xi}(\mathcal{F}_{x\to\xi}W^\varepsilon)^*(\xi+\sigma_1\frac{p}{\varepsilon}, k+\sigma_1\frac{p}{2})\right| \\ &\le C\int_0^\infty dre^{-r}\sup_{p\in\mathbb{R}^3}\int_{\mathbb{R}^6_{\xi k}}|d\hat{V}(\xi)|dk|\mathcal{F}_{x\to\xi}W^\varepsilon(\xi+\sigma_1\frac{p}{\varepsilon}, k+\sigma_1\frac{p}{2})| \\ &\le C\int_{\mathbb{R}^3}|d\hat{V}(\xi)|\sup_{u\in\mathbb{R}^3}\int_{\mathbb{R}^3_k}dk|\mathcal{F}_{x\to\xi}W^\varepsilon(u,k)| \\ &\le C\sup_{u\in\mathbb{R}^3}\int_{\mathbb{R}^3_k}dk|\mathcal{F}_{x\to\xi}W^\varepsilon(u,k)|. \end{aligned}$$

In order to control the last term, we notice that following the definition of the Wigner transform and (4):

$$\mathcal{F}_{x\to\xi}W^\varepsilon(\xi,k) = \frac{1}{(2\pi\varepsilon)^3}\sum_{i\in\mathbb{N}}\rho_i^\varepsilon\mathcal{F}\psi_i^\varepsilon\left(\frac{k}{\varepsilon}+\frac{\xi}{2}\right)(\mathcal{F}\psi_i^\varepsilon)^*\left(\frac{k}{\varepsilon}-\frac{\xi}{2}\right),$$

which yields, together with the conservation of the L^1 norm of the density (15)

$$\sup_{\xi\in\mathbb{R}^3}\int_{\mathbb{R}^3}dk|\mathcal{F}_{x\to\xi}W^\varepsilon(s,\xi,k)| \le C\sum_{i\in\mathbb{N}}\rho_i^\varepsilon\|\mathcal{F}\psi_i^\varepsilon\|_{L^2}^2 \le C\sum_{i\in\mathbb{N}}\rho_i^\varepsilon = C\|n_0^\varepsilon\|_{L^1}\le C. \tag{43}$$

To summarize, we therefore have proven that:

$$\sup_{s\in[0,T]}\left|\left\langle W^\varepsilon(s),\mathcal{K}[\tilde{V}^\varepsilon(s),\frac{x}{\varepsilon}]\lambda_1\right\rangle\right| \le C, \qquad \tilde{\mathbb{P}}^\varepsilon \text{ almost surely.} \tag{44}$$

The term A_3^ε. The Fourier transform of $\mathcal{L}_2^\varepsilon\lambda_0$ in $\mathcal{S}'$ with respect to k reads

$$\mathcal{F}_{k\to y}\mathcal{L}_2^\varepsilon\lambda_0(x,y) = \frac{i}{\varepsilon}(U^\varepsilon(x+\frac{\varepsilon}{2}y)-U^\varepsilon(x-\frac{\varepsilon}{2}y))\mathcal{F}_{k\to y}\lambda_0(y), \tag{45}$$

with in the distribution sense

$$\mathcal{F}_{k\to y}\lambda_0(y) = \int_{\mathbb{R}^3}e^{-ik\cdot y}|k|^2dk = -(2\pi)^3\Delta_y\delta(y).$$

Since

$$\mathcal{F}_{k\to y}W^\varepsilon(x,y) = \sum_{i\in\mathbb{N}}\rho_i^\varepsilon\psi_i^\varepsilon\left(x-\frac{\varepsilon}{2}y\right)(\psi_i^\varepsilon)^*\left(x+\frac{\varepsilon}{2}y\right), \tag{46}$$

this implies that

$$\begin{aligned}\langle W^\varepsilon(s),\mathcal{L}_2^\varepsilon\lambda_0\rangle &= -\frac{i}{\varepsilon}\int_{\mathbb{R}^3}dx\left[\Delta_y\left((\mathcal{F}_{k\to y}W^\varepsilon)^*(s,x,y)(U^\varepsilon(x+\frac{\varepsilon}{2}y)-U^\varepsilon(x-\frac{\varepsilon}{2}y))\right)\right]\Big|_{y=0}\\ &= -i\int_{\mathbb{R}^3}dx\nabla_y(\mathcal{F}_{k\to y}W^\varepsilon)^*(s,x,y)|_{y=0}\cdot\nabla_xU^\varepsilon(x)\\ &= -\int_{\mathbb{R}^3}dxJ^\varepsilon(s,x)\cdot\nabla_xU^\varepsilon(x).\end{aligned}$$

Above, the current J^ε is defined by

$$J^\varepsilon(s,x) = \frac{\varepsilon}{2}\Im\sum_{i\in\mathbb{N}}\rho_i^\varepsilon(\psi_i^\varepsilon)^*\nabla\psi_i^\varepsilon\,(s,x)\,(s,x)\,, \tag{47}$$

where $\Im$ denotes the imaginary part. The classical identity

$$\partial_t n^\varepsilon + \nabla_x\cdot J^\varepsilon = 0,$$

together with the fact that $-\Delta U^\varepsilon = n^\varepsilon$ finally yield

$$\int_0^t\langle W^\varepsilon(s),\mathcal{L}_2^\varepsilon\lambda_0\rangle\,ds = \frac{1}{2}\|\nabla U^\varepsilon(0)\|_{L^2}^2 - \frac{1}{2}\|\nabla U^\varepsilon(t)\|_{L^2}^2. \tag{48}$$

The term A_2^ε. Let

$$\delta U^\varepsilon(x,\varepsilon y) := U^\varepsilon(x+\frac{\varepsilon}{2}y) - U^\varepsilon(x-\frac{\varepsilon}{2}y). \tag{49}$$

Then, using (45) and (42), we have in the distribution sense

$$\mathcal{F}_{k\to y}\mathcal{L}_2^\varepsilon\lambda_1(x,y) = \frac{2i}{\varepsilon}\delta U^\varepsilon(x,\varepsilon y)\left[\int_0^\infty e^{-r}\int_{\mathbb{R}^3} d\hat{V}(\xi)e^{ix\cdot\xi/\varepsilon}\,\delta(y-r\xi) - (2\pi)^3\delta(y)V(\frac{x}{\varepsilon})\right].$$

This implies that

$$\langle W^\varepsilon(s), \mathcal{L}_2^\varepsilon\lambda_1\rangle = \frac{2i}{(2\pi)^3\varepsilon}\int_0^\infty dre^{-r}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} d\hat{V}(\xi)dxe^{ix\cdot\xi/\varepsilon}(\mathcal{F}_{k\to y}W^\varepsilon)^*(s,x,r\xi)\delta U^\varepsilon(x,\varepsilon r\xi).$$

From (46) and the Cauchy-Schwarz inequality, we deduce that

$$|\mathcal{F}_{k\to y}W^\varepsilon(x,y)| \leq \sqrt{n^\varepsilon(x-\frac{\varepsilon}{2}y)}\sqrt{n^\varepsilon(x+\frac{\varepsilon}{2}y)}$$

which yields, together with the Hölder inequality

$$\begin{aligned} |\langle W^\varepsilon(s), \mathcal{L}_2^\varepsilon\lambda_1\rangle| &\leq \frac{C}{\varepsilon}\int_0^\infty dre^{-r}\int_{\mathbb{R}^3}|d\hat{V}(\xi)|\int_{\mathbb{R}^3} dx|\delta U^\varepsilon(x,\varepsilon r\xi)| \\ &\qquad\times\sqrt{n^\varepsilon(x-\frac{\varepsilon}{2}r\xi)}\sqrt{n^\varepsilon(x+\frac{\varepsilon}{2}r\xi)} \\ &\leq \frac{C}{\varepsilon}\|n^\varepsilon\|_{L^{\frac{7}{5}}}\int_0^\infty dre^{-r}\int_{\mathbb{R}^3}|d\hat{V}(\xi)|\|\delta U^\varepsilon(\cdot,\varepsilon r\xi)\|_{L^{\frac{7}{2}}}. \end{aligned} \tag{50}$$

We already know from (34) that

$$\|n^\varepsilon\|_{L^{\frac{7}{5}}} \leq C\mathcal{E}_{\text{kin}}^{\frac{3}{7}}$$

so it only remains to treat δU^ε. Standard interpolation estimates along with classical results for differential quotients show that

$$\|\delta U^\varepsilon(\cdot,\varepsilon r\xi)\|_{L^{\frac{7}{2}}} \leq \sqrt{2}\|\delta U^\varepsilon(\cdot,\varepsilon r\xi)\|_{L^2}^{\frac{1}{2}}\|U^\varepsilon\|_{L^{14}}^{\frac{1}{2}} \leq \sqrt{2\varepsilon r|\xi|}\|\nabla U^\varepsilon\|_{L^2}^{\frac{1}{2}}\|U^\varepsilon\|_{L^{14}}^{\frac{1}{2}}.$$

This is where we take advantage of the extra $\sqrt{\varepsilon}$ factor related to the test function λ_1. This allows us to control $\|\delta U^\varepsilon(\cdot,\varepsilon r\xi)\|_{L^{\frac{7}{2}}}$ by $\|\nabla U^\varepsilon\|_{L^2}^{\frac{1}{2}}$ and not by a larger power of ∇U^ε in some L^p norm to compensate exactly the ε^{-1} factor. Using estimate (35) with $p=14$ and $n=0$, together with estimate (33) with $\theta = \frac{51}{84}$, we find

$$\begin{aligned} \|U^\varepsilon\|_{L^{14}} &\leq C\|n^\varepsilon\|_{L^{\frac{42}{31}}} \\ &\leq C\left(\text{Tr}(\varrho^\varepsilon)^2\right)^{\theta/2}(\mathcal{E}_{\text{kin}}^\varepsilon)^{1-\theta}\varepsilon^{2\theta-2} \\ &\leq C\varepsilon^{\frac{7}{2}\theta-2}(\mathcal{E}_{\text{kin}}^\varepsilon)^{1-\theta}. \end{aligned}$$

Above, $\frac{7}{2}\theta - 2 > 0$. Going back to (50), it comes, using the Young inequality, the fact that $\hat{V}$ has a bounded support and that $\frac{3}{7}+\frac{1}{2}(1-\theta) < \frac{3}{4}$

$$\begin{aligned} |\sqrt{\varepsilon}\langle W^\varepsilon, \mathcal{L}_2^\varepsilon\lambda_1\rangle| &\leq C\varepsilon^{\frac{7}{4}\theta-1}\mathcal{E}_{\text{kin}}^{\frac{3}{7}+\frac{1}{2}(1-\theta)}\|\nabla U^\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\leq C(1+\mathcal{E}_{\text{kin}}^\varepsilon + \|\nabla U^\varepsilon\|_{L^2}^2) \\ &\leq C + C\mathcal{E}^\varepsilon. \end{aligned} \tag{51}$$

End of the proof. We have, using (41)

$$\begin{aligned}\langle W^\varepsilon, \lambda_1 \rangle &= \frac{1}{(2\pi)^3} \langle \mathcal{F}_{x\to\xi} W^\varepsilon, \mathcal{F}_{x\to\xi}\lambda_1 \rangle \\ &= -\frac{2}{(2\pi)^3} \int_{\mathbb{R}^6_{\xi,k}} d\hat{V}(\xi) dk (\mathcal{F}_{x\to\xi} W^\varepsilon)^*(\xi,k) \\ &\quad + \frac{2}{(2\pi)^3} \int_0^\infty dr e^{-r} \int_{\mathbb{R}^6_{\xi,k}} d\hat{V}(\xi) dk e^{irk\cdot\xi} (\mathcal{F}_{x\to\xi} W^\varepsilon)^*(\xi,k).\end{aligned}$$

This implies, together with (43)

$$|\, \langle W^\varepsilon, \lambda_1 \rangle \,| \leq C \int_{\mathbb{R}^3} |d\hat{V}| \sup_{u\in\mathbb{R}^3} \int_{\mathbb{R}^3_k} dk |\mathcal{F}_{x\to\xi} W^\varepsilon(u,k)| \leq C.$$

We therefore have just proved (39). Gathering (44)-(48)-(51), we obtain (40), which, as announced previously at the beginning of this section, leads to the desired estimate after the use of the Gronwall Lemma in (38). □

Remark 3 *Proposition 2 with* (34) *yields*

$$\mathbb{E}^{\mathbb{P}^\varepsilon} \| n^\varepsilon(t) \|^{\frac{7}{3}}_{L^{\frac{7}{5}}} \leq C, \qquad \forall t \in [0,T]. \tag{52}$$

We extensively used in the proof of Proposition 2 that W^ε is the Wigner transform of a density operator. It is not clear such an energy estimate holds for general solutions to the Wigner equation with an initial condition which is not the Wigner transform of an initial density operator.

Remark 4 *We want to comment here on our form of the random potential and how it is used for Proposition 2. The Markov assumption allows us to solve the equation for the corrector λ_1* (27) *and can be seen a regularization of the same equation with $Q_\varepsilon \equiv 0$. As already mentioned, such an hypothesis can be replaced by some other provided λ_1 is well-defined. The fact that $\hat{v}(t)$ is a jump process with the generator* (10) *provides us with simple, pointwise estimates (with respect to $d\hat{V}$) for λ_1. Such assumption could be relaxed to more general bounded generators with additional technicalities. More essential is the fact that $d\hat{V}$ is a measure with bounded total variation. This allows us to estimate the terms A_1^ε and A_3^ε uniformly with respect to ε. In particular, we can control stochastic integrals of the form*

$$I = \int_{\mathbb{R}^6_{\xi k}} d\hat{V}(\xi) dk \mathcal{F}_{x\to\xi} W^\varepsilon(\xi,k)$$

directly by

$$|I| \leq \int_{\mathbb{R}^3} |d\hat{V}(\xi)| \sup_{u\in\mathbb{R}^3} \int_{\mathbb{R}^3_k} dk |\mathcal{F}_{x\to\xi} W^\varepsilon(u,k)| \leq C \|n_0^\varepsilon\|_{L^1} \leq C.$$

It is not clear to us how to relax the bounded total variation constraint, in particular for estimating the non-linear term A_3^ε. A possibility could be to make use of Malliavin calculus and treat the stochastic integral as a divergence operator. This would require to derive some estimates for the Malliavin derivative of W^ε.

3.5 Tightness

We prove in this section that the family of measures $\mathbb{P}^\varepsilon$ is tight in the space $D([0,T],X)$ of X-valued right continuous processes with left limits endowed with the Skohorod topology [13]. Here X is the state space for W^ε and is defined in section 3. Since the measure $\mathbb{P}^\varepsilon$ is actually defined on $\mathcal{C}^0([0,T],X)$ and the Skohorod topology relativized to continuous functions coincides with the uniform topology [13], we also obtain tightness in $\mathcal{C}^0([0,T],X)$. Moreover, since the space X is a metric space, compact for the weak topology, the tightness of the family $\{W^\varepsilon, \varepsilon \in (0,1)\}$ is equivalent to the tightness of the family $\{\langle W^\varepsilon, \lambda_0\rangle, \varepsilon \in (0,1)\}$, for all $\lambda_0 \in L^2$, and by density for all $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^6)$. We use the following criterion of tightness of [32], Chapter 3, Theorem 4. We need to prove that

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}^\varepsilon\{\sup_{t\in[0,T]} |\langle W^\varepsilon(t), \lambda_0\rangle| \geq N\} = 0, \qquad \forall T < \infty \tag{53}$$

and that, for each $\phi \in \mathcal{C}^\infty(\mathbb{R})$, there exists a sequence $g_\varepsilon \in L^\infty((0,T))$ such that for each $T < \infty$, the family $\{\mathcal{A}^\varepsilon g_\varepsilon(t),\, \varepsilon \in (0,1),\, t \in (0,T)\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}^\varepsilon\{\sup_{t\in[0,T]} |\phi(\langle W^\varepsilon(t), \lambda_0\rangle) - g_\varepsilon(t)| \geq \delta\} = 0, \qquad \forall \delta > 0. \tag{54}$$

Item (53) is immediate because of the conservation of the L^2 norm of W^ε. Let $\hat{V} \equiv \tilde{V}^\varepsilon(t) \equiv \tilde{V}[\hat{v}^\varepsilon(t)] \in \mathcal{V}$ which satisfies the uniform bound (14). For (54), we introduce the corrector

$$f_\varepsilon^1(t) = \sqrt{\varepsilon} f'_\varepsilon(t) \langle W^\varepsilon(t), \lambda_1\rangle \tag{55}$$

with $f_\varepsilon^0(t) = \phi(\langle W^\varepsilon(t), \lambda_0\rangle)$, $f'_\varepsilon(t) = \phi'(\langle W^\varepsilon(t), \lambda_0\rangle)$ and $\lambda_1 \equiv \lambda_1(x, \frac{x}{\varepsilon}, k, \tilde{V}^\varepsilon(t))$ is defined in (29). Decomposing the gradient ∇_x into

$$\nabla_x \to \nabla_x + \frac{1}{\varepsilon}\nabla_\eta, \qquad \eta = \frac{x}{\varepsilon},$$

we find

$$\begin{aligned} \mathcal{A}^\varepsilon f_\varepsilon^1(t) &= \sqrt{\varepsilon} f'_\varepsilon(t) \left\langle W^\varepsilon, \left(\frac{1}{\varepsilon} Q_\varepsilon + \frac{1}{\varepsilon} k\cdot\nabla_\eta + k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon, \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_1\right\rangle \\ &\quad + \sqrt{\varepsilon}\langle W^\varepsilon, \lambda_1\rangle f''_\varepsilon(t) \left\langle W^\varepsilon, \left(k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon, \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_0\right\rangle. \end{aligned} \tag{56}$$

Hence

$$\mathcal{A}^\varepsilon(f_\varepsilon^0(t) + f_\varepsilon^1(t)) = \mathcal{A}_1^\varepsilon + \mathcal{A}_2^\varepsilon$$

where

$$\begin{aligned} \mathcal{A}_1^\varepsilon &= f'_\varepsilon(t) \langle W^\varepsilon, (k\cdot\nabla_x + \mathcal{L}_2^\varepsilon)\lambda_0\rangle \\ &\quad + \sqrt{\varepsilon}\langle W^\varepsilon, \lambda_1\rangle f''_\varepsilon(t) \left\langle W^\varepsilon, \left(k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon, \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_0\right\rangle \\ \mathcal{A}_2^\varepsilon &= \sqrt{\varepsilon} f'_\varepsilon(t) \left\langle W^\varepsilon, \left(k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon, \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_1\right\rangle. \end{aligned}$$

We set $g_\varepsilon = f_\varepsilon^0 + f_\varepsilon^1$ for criterion (54), so that we need estimates for $\mathcal{A}_1^\varepsilon$ and $\mathcal{A}_2^\varepsilon$ in order to prove the uniform integrability. To this end, we use the following series of Lemmas. The first one is given without proof and is a simple adaptation of the one proved in [7]:

Lemma 3.1 *There exists a constant C independent of ε such that, for all $\lambda_0 \in \mathcal{S}$ and all $\hat{V} \in \mathcal{V}$ satisfying estimate* (14)*:*

$$\|\lambda_1\|_{L^\infty(\mathcal{V},L^2)} + \|k \cdot \nabla_x \lambda_1\|_{L^\infty(\mathcal{V},L^2)} + \|\mathcal{K}[\hat{V}, \frac{x}{\varepsilon}]\|_{\mathcal{L}(L^2)} \leq C.$$

In the Lemma, the differentiation is meant with respect to the slow variable x and $\mathcal{L}(L^2)$ is the space of bounded operators on L^2. We will also need the following two Lemmas.

Lemma 3.2 *Let $\lambda \in \mathcal{S}$. Then, we have the estimate:*

$$|\langle W^\varepsilon, \mathcal{L}_2^\varepsilon \lambda \rangle| \leq C\|W^\varepsilon\|_{L^2}\|\nabla U^\varepsilon\|_{L^2} \left(\int_{\mathbb{R}^3} dy|y|^2 \sup_{x\in\mathbb{R}^3} |\mathcal{F}_{k\to y}\lambda(x,y)|^2\right)^{\frac{1}{2}}.$$

Proof. Using notation (49), we have

$$\langle W^\varepsilon, \mathcal{L}_2^\varepsilon \lambda \rangle = \frac{i}{(2\pi)^3\varepsilon}\int_{\mathbb{R}^6_{xy}} dxdy (\mathcal{F}_{k\to y}W^\varepsilon(x,y))^* \mathcal{F}_{k\to y}\lambda(x,y)\delta U^\varepsilon(x,\varepsilon y)$$

so that

$$\begin{aligned} |\langle W^\varepsilon, \mathcal{L}_2^\varepsilon \lambda \rangle| &\leq \frac{1}{(2\pi)^3\varepsilon}\|W^\varepsilon\|_{L^2}\left(\int_{\mathbb{R}^6_{xy}} dxdy|\mathcal{F}_{k\to y}\lambda(x,y)\delta U^\varepsilon(x,\varepsilon y)|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{(2\pi)^3}\|W^\varepsilon\|_{L^2}\|\nabla U^\varepsilon\|_{L^2}\left(\int_{\mathbb{R}^3} dy|y|^2 \sup_{x\in\mathbb{R}^3}|\mathcal{F}_{k\to y}\lambda(x,y)|^2\right)^{\frac{1}{2}}. \end{aligned}$$

□

Lemma 3.3 *There exists a constant C independent of ε such that, for all $\lambda_0 \in \mathcal{S}$ and all $\hat{V} \in \mathcal{V}$ satisfying estimate* (14)*:*

$$\int_{\mathbb{R}^3} dy|y|^2 \sup_{x\in\mathbb{R}^3} |\mathcal{F}_{k\to y}\lambda_1(x, \frac{x}{\varepsilon}, y, \hat{V})|^2 \leq C.$$

The proof of the last Lemma is postponed to the Appendix.

Gathering Lemmas (3.1)-(3.2)-(3.3), we conclude that there exists a constant C independent of ε such that

$$|\mathcal{A}_1^\varepsilon| \leq C\|W^\varepsilon\|_{L^2}(1 + \|\nabla U^\varepsilon\|_{L^2} + \|W^\varepsilon\|_{L^2}\|\nabla U^\varepsilon\|_{L^2}) \leq C + C\|\nabla U^\varepsilon\|_{L^2}.$$

Using the energy estimate of Proposition 2, this yields

$$\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}^\varepsilon}\{|\mathcal{A}_1^\varepsilon|^2\} \leq C,$$

which shows that $\mathcal{A}_1^\varepsilon$ is uniformly integrable. Proceeding analogously, we deduce in addition that $\mathcal{A}_2^\varepsilon$ is uniformly integrable. The property (54) follows from the fact that

$$|f_\varepsilon^1(t)| \leq \sqrt{\varepsilon}\|\phi'\|_{L^\infty}\|W_0^\varepsilon\|_{L^2}\|\lambda_1\|_{L^2} \leq C\sqrt{\varepsilon}.$$

The proof of tightness is now complete.

4 Passing to the limit

In this section, we pass to the limit in the martingale formulation and prove Theorem 1. We use again the notation $\hat{V} \equiv \tilde{V}^\varepsilon(t) \equiv \tilde{V}[\hat{v}^\varepsilon(t)] \in \mathcal{V}$. Recall first that for smooth functions F,

$$M_t^\varepsilon = F(\tilde{V}^\varepsilon(t), W^\varepsilon(t)) - \int_0^t \mathcal{A}^\varepsilon F(\tilde{V}^\varepsilon(s), W^\varepsilon(s))ds \qquad \text{is a } \tilde{\mathbb{P}}^\varepsilon \text{ martingale,}$$

where $\mathcal{A}^\varepsilon$ is defined in (25). The martingale property of M_t^ε implies that for any sequence $0 < t_1 < t_2 < \cdots < t_n \leq t$, any bounded continuous fonction h, we have, for all $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^6)$ and $s > 0$:

$$\mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon} \left\{ h\left(\langle W^\varepsilon(t_1), \lambda_0\rangle, \cdots, \langle W^\varepsilon(t_n), \lambda_0\rangle\right) [M_{t+s}^\varepsilon - M_t^\varepsilon] \right\} = 0. \tag{57}$$

We have shown in section 3.5 that the family of processes $\{W^\varepsilon, \varepsilon \in (0,1)\}$ is tight in the space $\mathcal{C}^0([0,T], X)$, so that there exists a subsequence, still denoted by W^ε, that converges in law to a process $W \in \mathcal{C}^0([0,T], X)$. Notice that since the limits of Wigner transforms are non-negative [33], W is non-negative. We will pass to the limit in (57) and show that the limit of M_t^ε is also a martingale, and that the corresponding process for $F \equiv \phi(\langle W, \lambda_0\rangle)$ and $\phi(u) = u$ is a martingale with null quadratic variation. This then allows to identify W with the unique solution to the VPB system (9). The passage to the limit is done first by introducing different correctors. In addition to f_ε^1 defined in (55), we set, using the same notations as section 3.5, with λ_1 and λ_2 be the test functions defined in (29)-(32) :

$$f_\varepsilon^2(t) = \varepsilon f_\varepsilon'(t)\langle W^\varepsilon, \lambda_2\rangle \qquad ; \qquad f_\varepsilon^3(t) = \frac{\varepsilon}{2} f_\varepsilon''(t)(\langle W^\varepsilon, \lambda_1\rangle)^2.$$

Then, for $\eta = \frac{x}{\varepsilon}$,

$$\begin{aligned}
\mathcal{A}^\varepsilon f_\varepsilon^2(t) &= \varepsilon f_\varepsilon'(t) \left\langle W^\varepsilon, \left(\frac{1}{\varepsilon}Q + \frac{1}{\varepsilon}k\cdot\nabla_\eta + k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon(t), \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_2 \right\rangle \\
&\quad + \varepsilon\langle W^\varepsilon(t), \lambda_2\rangle f_\varepsilon''(t) \left\langle W^\varepsilon, \left(k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon(t), \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_0 \right\rangle \\
\mathcal{A}^\varepsilon f_\varepsilon^3(t) &= \varepsilon f_\varepsilon''(t)\langle W^\varepsilon, \lambda_1\rangle \left\langle W^\varepsilon, \left(\frac{1}{\varepsilon}Q + \frac{1}{\varepsilon}k\cdot\nabla_\eta + k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon(t), \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_1 \right\rangle \\
&\quad + \varepsilon(\langle W^\varepsilon, \lambda_1\rangle)^2 f_\varepsilon'''(t) \left\langle W^\varepsilon, \left(k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{v}^\varepsilon(t), \eta] + \mathcal{L}_2^\varepsilon\right)\lambda_0 \right\rangle
\end{aligned}$$

and therefore, together with (56)

$$\begin{aligned}
\mathcal{A}^\varepsilon(f_\varepsilon^0(t) + f_\varepsilon^1(t) + f_\varepsilon^2(t) + f_\varepsilon^3(t)) &= f_\varepsilon'(t)\left\langle W^\varepsilon(t), (k\cdot\nabla_x + \mathcal{L}_2^\varepsilon + \mathcal{L}_\varepsilon)\lambda_0\right\rangle + \sqrt{\varepsilon}R_\varepsilon(t) \\
&:= \mathcal{A}_L^\varepsilon(t) + \mathcal{A}_{NL}^\varepsilon(t) + \sqrt{\varepsilon}R_\varepsilon(t),
\end{aligned}$$

where $\mathcal{L}_\varepsilon$ is defined in (31), R_ε is an error term, $\mathcal{A}^\varepsilon_L$ and $\mathcal{A}^\varepsilon_{NL}$ are linear and non-linear parts given by

$$\begin{aligned}
\mathcal{A}^\varepsilon_L(t) &= f'_\varepsilon(t)\,\langle W^\varepsilon(t), (k\cdot\nabla_x + \mathcal{L}_\varepsilon)\,\lambda_0\rangle\\
\mathcal{A}^\varepsilon_{NL}(t) &= f'_\varepsilon(t)\,\langle W^\varepsilon(t), \mathcal{L}^\varepsilon_2\lambda_0\rangle\\
R_\varepsilon(t) &= f'_\varepsilon(t)\,\langle W^\varepsilon, k\cdot\nabla_x + \mathcal{L}^\varepsilon_2\lambda_1\rangle\\
&\quad + \langle W^\varepsilon,\lambda_1\rangle f''_\varepsilon(t)\,\langle W^\varepsilon, (k\cdot\nabla_x + \mathcal{L}^\varepsilon_2)\,\lambda_0\rangle\\
&\quad + f'_\varepsilon(t)\,\left\langle W^\varepsilon, \left(\sqrt{\varepsilon}k\cdot\nabla_x + \mathcal{K}[\hat{v}^\varepsilon(t),\eta] + \sqrt{\varepsilon}\mathcal{L}^\varepsilon_2\right)\lambda_2\right\rangle\\
&\quad + \langle W^\varepsilon,\lambda_2\rangle f''_\varepsilon(t)\,\left\langle W^\varepsilon, \left(\sqrt{\varepsilon}k\cdot\nabla_x + \mathcal{K}[\hat{v}^\varepsilon(t),\eta] + \sqrt{\varepsilon}\mathcal{L}^\varepsilon_2\right)\lambda_0\right\rangle\\
&\quad + f''_\varepsilon(t)\,\left\langle W^\varepsilon, \left(\sqrt{\varepsilon}k\cdot\nabla_x + \mathcal{K}[\hat{v}^\varepsilon(t),\eta] + \sqrt{\varepsilon}\mathcal{L}^\varepsilon_2\right)\lambda_1\right\rangle\\
&\quad + (\langle W^\varepsilon,\lambda_1\rangle)^2 f'''_\varepsilon(t)\,\left\langle W^\varepsilon, \left(\sqrt{\varepsilon}k\cdot\nabla_x + \mathcal{K}[\hat{v}^\varepsilon(t),\eta] + \sqrt{\varepsilon}\mathcal{L}^\varepsilon_2\right)\lambda_0\right\rangle\\
&:= \sum_{i=1}^{6} R^i_\varepsilon(t),
\end{aligned}$$

with obvious notations. We need the following Lemma, whose proof is postponed to the Appendix:

Lemma 4.1 *There exists a constant C independent of ε such that*

$$\|\lambda_2\|_{L^\infty(\mathcal{V},L^2)} + \|k\cdot\nabla_x\lambda_2\|_{L^\infty(\mathcal{V},L^2)} + \int_{\mathbb{R}^3} dy|y|^2 \sup_{x\in\mathbb{R}^3} |\mathcal{F}_{k\to y}\lambda_2(x,\frac{x}{\varepsilon},y,\tilde{V}^\varepsilon(t),\lambda_0)|^2 \leq C. \quad (58)$$

We then have the following result:

Lemma 4.2 *There exists a constant C independent of ε such that*

$$\sup_{t\in[0,T]} \mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\{|R_\varepsilon(t)|^2\} \leq C.$$

Proof. Estimating the terms R^ε_1, R^ε_2, R^ε_5 and R^ε_6 (that do not depend on λ_2) is done as in the proof of tightness by using Lemmas 3.1-3.2-3.3, we leave the details to the reader. The linear terms in R^ε_3 and R^ε_4 are treated with the first two estimates on λ_2 of Lemma 4.1, along with (3.2) and Proposition 2. For the non-linear term $\langle W^\varepsilon, \mathcal{L}^\varepsilon_2\lambda_2\rangle$, we use Lemma 3.2 together with the third estimate of (58). $\square$

Let now

$$F(W^\varepsilon, \tilde{V}^\varepsilon(t)) = f^0_\varepsilon(t) + f^1_\varepsilon(t) + f^2_\varepsilon(t) + f^3_\varepsilon(t) := f^0_\varepsilon(t) + r_\varepsilon(t).$$

We know from Lemma (3.1), (58) and the fact that W^ε is bounded in L^2 that

$$\sup_{t\in[0,T]} |r_\varepsilon(t)| \leq \sqrt{\varepsilon}C. \quad (59)$$

We then write

$$M^\varepsilon_{t+s} - M^\varepsilon_t = m^\varepsilon(t,s) + \int_t^{t+s} \mathcal{A}^\varepsilon_{NL}(\tau)d\tau + \mathcal{R}_\varepsilon(t),$$

where

$$
\begin{aligned}
m^\varepsilon(t,s) &= f_\varepsilon^0(t+s) - f_\varepsilon^0(t) - \int_t^{t+s} \mathcal{A}_L^\varepsilon(\tau)d\tau \\
\mathcal{R}_\varepsilon(t) &= r_\varepsilon(t+s) - r_\varepsilon(t) - \sqrt{\varepsilon}\int_t^{t+s} R_\varepsilon(\tau)d\tau.
\end{aligned}
$$

We deduce from Lemma (4.2) and (59) that

$$
\sup_{t\in[0,T]} \mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\{|\mathcal{R}_\varepsilon(t)|^2\} \leq C\varepsilon. \tag{60}
$$

We then pass to the limit in the operator $\mathcal{L}_\varepsilon$ defined in (31). As stated in section (2.1), it is given by the Riemann sum approximation of the continuous operator $\mathcal{L}$ introduced in (8). Since $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^6)$, $R_0(s) = R_0 e^{-|s|}$ and $\hat{R}_1 \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, it is then not difficult to see that, almost surely:

$$
\sup_{t\in[0,T]} |\langle W^\varepsilon(t), (\mathcal{L}_\varepsilon - \mathcal{L})\lambda_0\rangle| \leq \sup_{t\in[0,T]} \|W^\varepsilon(t)\|_{L^2}\|(\mathcal{L}_\varepsilon - \mathcal{L})\lambda_0\|_{L^2} \leq Ch_\varepsilon = o(\varepsilon),
$$

so that we can replace $\mathcal{L}_\varepsilon$ by $\mathcal{L}$ in $\mathcal{A}_L^\varepsilon$. Finally, using the fact that $\mathbb{P}^\varepsilon$ converges weakly to a measure $\mathbb{P}$ in $\mathcal{C}^0([0,T],X)$, together with (60) and the fact that h is continuous and bounded, we can pass to the limit in the linear terms so that, $\forall s > 0$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \mathbb{E}^{\tilde{\mathbb{P}}^\varepsilon}\left\{h\left(\langle W^\varepsilon(t_1),\lambda_0\rangle,\cdots,\langle W^\varepsilon(t_n),\lambda_0\rangle\right)[m^\varepsilon(t,s)+\mathcal{R}_\varepsilon(t)]\right\} \\
&= \mathbb{E}^{\mathbb{P}}\left\{h\left(\langle W(t_1),\lambda_0\rangle,\cdots,\langle W(t_n),\lambda_0\rangle\right)m(t,s)\right\},
\end{aligned} \tag{61}
$$

where

$$
\begin{aligned}
m(t,s) &= \phi(\langle W(t+s),\lambda_0\rangle) - \phi(\langle W(s),\lambda_0\rangle) - \int_t^{t+s} \mathcal{A}_L(\tau)d\tau \\
\mathcal{A}_L(t) &= \phi'(\langle W(t),\lambda_0\rangle)\,\langle W(t), (k\cdot\nabla_x + \mathcal{L})\,\lambda_0\rangle\,.
\end{aligned}
$$

It remains therefore to pass to the limit in the non-linear term $\mathcal{A}_{NL}^\varepsilon$.

Non-linear term. We need first a refined version of Lemma 3.2:

Lemma 4.3 *For $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^6)$, we have the relation*

$$
\langle W^\varepsilon, \mathcal{L}_2^\varepsilon\lambda_0\rangle = -\langle W^\varepsilon, \nabla_x U^\varepsilon \cdot \nabla_k\lambda_0\rangle + S_\varepsilon
$$

with

$$
\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}^\varepsilon}|S_\varepsilon(t)| \leq C\varepsilon^{4/15}.
$$

Proof. Using the notation (49), we have

$$
\langle W^\varepsilon, \mathcal{L}_2^\varepsilon\lambda_0\rangle = \frac{i}{(2\pi)^3\varepsilon}\int_{\mathbb{R}^6_{xy}} dxdy(\mathcal{F}_{k\to y}W^\varepsilon(x,y))^*\delta U^\varepsilon(x,\varepsilon y)\mathcal{F}_{k\to y}\lambda_0(x,y).
$$

Using

$$U^\varepsilon(x+\varepsilon y) - U^\varepsilon(x) = \varepsilon \int_0^1 y \cdot \nabla U^\varepsilon(x+\varepsilon t y) dt \tag{62}$$

we find

$$\begin{aligned}
\langle W^\varepsilon, \mathcal{L}_2^\varepsilon \lambda_0\rangle &= -\langle W^\varepsilon, \nabla_x U^\varepsilon \cdot \nabla_k \lambda_0\rangle + S_\varepsilon \\
S_\varepsilon &= \frac{i}{(2\pi)^3}\int_{\mathbb{R}^6_{xy}} dxdy (\mathcal{F}W^\varepsilon(x+\frac{\varepsilon}{2}y,y))^* r^\varepsilon(x,y)\mathcal{F}_{k\to y}\lambda_0(x+\frac{\varepsilon}{2}y,y) \\
r_\varepsilon &= \int_0^1 y\cdot\nabla\left[U^\varepsilon(x+\varepsilon t y) - \nabla U^\varepsilon(x)\right] dt.
\end{aligned}$$

Assume the support in x of λ_0 is included in a bounded domain D. Using standard interpolation inequalities, we have

$$\begin{aligned}
\|r_\varepsilon\|_{L^2_x(D)} &\leq |y|\int_0^1 dt \|\nabla U^\varepsilon(\cdot+\varepsilon t y) - \nabla U^\varepsilon\|_{L^2_x(D)} \\
&\leq C|y|\int_0^1 dt \|\nabla U^\varepsilon(\cdot+\varepsilon t y) - \nabla U^\varepsilon\|^{4/15}_{L^{7/5}_x}\|\nabla U^\varepsilon(\cdot+\varepsilon t y) - \nabla U^\varepsilon\|^{11/15}_{L^{21/8}}.
\end{aligned}\tag{63}$$

From (62), we deduce that

$$\|\partial_{x_i}U^\varepsilon(\cdot+\varepsilon t y) - \partial_{x_i}U^\varepsilon\|_{L^{7/5}_x} \leq \varepsilon|y|t\|\partial_{x_i}\nabla U^\varepsilon\|_{L^{7/5}}$$

and consequently

$$\|\nabla U^\varepsilon(\cdot+\varepsilon t y) - \nabla U^\varepsilon\|_{L^{7/5}_x} \leq C\varepsilon|y|t\sum_{i,j=1}^3\|\partial_{x_i}\partial_{x_j}U^\varepsilon\|_{L^{7/5}}. \tag{64}$$

In order to bound the r.h.s, recall that $U^\varepsilon = (-\Delta)^{-1}n^\varepsilon$, so that, since the operator $\partial_{x_i}\partial_{x_j}(-\Delta)^{-1}$ is bounded in L^p, $1<p<\infty$,

$$\||\partial_{x_i}\partial_{x_j}(-\Delta)^{-1}n^\varepsilon\|_{L^{7/5}} \leq C\|n^\varepsilon\|_{L^{7/5}}. \tag{65}$$

On the other hand, the inequality (35) with $n=1$ yields

$$\|\nabla U^\varepsilon\|_{L^{21/8}_x} \leq C\|n^\varepsilon\|_{L^{7/5}}$$

so that together with (63)-(64)-(65), it comes

$$\|r_\varepsilon\|_{L^2_x} \leq C\varepsilon^{4/15}|y|\|n^\varepsilon\|_{L^{7/5}}.$$

Going back to S_ε, we find

$$|S_\varepsilon| \leq C\varepsilon^{4/15}\|W_0^\varepsilon\|_{L^2}\left(\int_{\mathbb{R}^3} dy|y|^2\sup_{x\in\mathbb{R}^3}|\mathcal{F}_{k\to y}\lambda_0|^2\right)^{1/2}\|n^\varepsilon\|_{L^{7/5}} \leq C\varepsilon^{4/15}\|n^\varepsilon\|_{L^{7/5}},$$

which ends the proof using (52). □

According the previous Lemma, we thus only need to look at the limit of

$$\mathcal{B}_{t,s}^{\varepsilon} := \mathbb{E}^{\mathbb{P}^{\varepsilon}} \left\{ h\left(\langle W^{\varepsilon}(t_1), \lambda_0 \rangle, \cdots, \langle W^{\varepsilon}(t_n), \lambda_0 \rangle\right) \int_t^{t+s} d\tau f_{\varepsilon}'(\tau) \left\langle W^{\varepsilon}, \nabla_x U^{\varepsilon} \cdot \nabla_k \lambda_0 \right\rangle (\tau) \right\}.$$

To this goal, we use the Skohorod representation theorem [18]: since the subsequence $\mathbb{P}^{\varepsilon}$ converges weakly to $\mathbb{P}$ on $\mathcal{C}^0([0,T], X)$ (which is separable), there exist random processes $\widetilde{W^{\varepsilon}}$ and $\widetilde{W}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ (which can be chosen as $([0,1), \mathcal{B}([0,1)), d\omega)$, for $d\omega$ the Lebesgue measure and $\mathcal{B}([0,1))$ the Borel σ-algebra of $[0,1)$, see [18]), such that $\mathbb{P}^{\varepsilon}$ is the law of $\widetilde{W^{\varepsilon}}$, $\mathbb{P}$ is the law of $\widetilde{W}$, and $\widetilde{W^{\varepsilon}}$ converges $\mathcal{P}$ almost surely to $\widetilde{W}$ for the $\mathcal{C}^0(0,T, X - weak)$ topology. In other terms,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} d(\widetilde{W^{\varepsilon}}(t), \widetilde{W}(t)) = 0, \qquad \mathcal{P} \text{ almost surely,} \tag{66}$$

where

$$d(f,g) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} \frac{|(e_i, \widetilde{W^{\varepsilon}} - \widetilde{W})|}{1 + |(e_i, \widetilde{W^{\varepsilon}} - \widetilde{W})|}$$

and $(\cdot,\cdot)$ is the usual $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ inner product and $(e_i)_{i\in\mathbb{N}}$ a dense set in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. We introduce the notations

$$\widetilde{n^{\varepsilon}}(t,x) = \int_{\mathbb{R}^3} \widetilde{W^{\varepsilon}}(t,x,k) dk \qquad ; \qquad \widetilde{U^{\varepsilon}} = \frac{1}{4\pi|x|} * \widetilde{n^{\varepsilon}}.$$

Since the processes $\widetilde{W^{\varepsilon}}$ and W^{ε} have the same laws, tilded quantities satisfy the same estimates as the non-tilded ones, namely, $\mathcal{P}$ almost everywhere:

$$\sup_{t \in [0,T]} \left[\|\widetilde{W^{\varepsilon}}(t)\|_{L^2} + \|\widetilde{n^{\varepsilon}}(t)\|_{L^1} + \widetilde{\mathbb{E}}\{\widetilde{\mathcal{E}^{\varepsilon}}(t)\} + \widetilde{\mathbb{E}}\|\widetilde{n^{\varepsilon}}(t)\|_{L^{\frac{7}{5}}}^{\frac{7}{3}} \right] \leq C, \tag{67}$$

where $\widetilde{\mathbb{E}}$ denotes expectation in $(\Omega, \mathcal{F}, \mathcal{P})$. Moreover, we have the equality

$$\mathcal{B}_{t,s}^{\varepsilon} = \widetilde{\mathbb{E}} \left\{ h\left(\langle \widetilde{W^{\varepsilon}}(t_1), \lambda_0 \rangle, \cdots, \langle \widetilde{W^{\varepsilon}}(t_n), \lambda_0 \rangle\right) \int_t^{t+s} d\tau \tilde{f}_{\varepsilon}'(\tau) \left\langle \widetilde{W^{\varepsilon}}, \nabla_x \widetilde{U^{\varepsilon}} \cdot \nabla_k \lambda_0 \right\rangle (\tau) \right\},$$

where $\tilde{f}_{\varepsilon}'(t) = \phi'(\langle \widetilde{W^{\varepsilon}}(t), \lambda_0 \rangle)$. We pass to the limit in the last expression using (66) and compactness arguments deduced from the various estimates on $\widetilde{n^{\varepsilon}}$ and $\widetilde{U^{\varepsilon}}$. A first result is given by the following Lemma: we use some $W^{1,p}$ type estimates on $\widetilde{U^{\varepsilon}}$ to transform the weak convergence (66) into strong convergence in the appropriate setting. More precisely:

Lemma 4.4 *We have*

$$\left\langle \widetilde{W^{\varepsilon}}, \nabla_x \widetilde{U^{\varepsilon}} \cdot \nabla_k \lambda_0 \right\rangle (t) = \left\langle \widetilde{W}, \nabla_x \widetilde{U^{\varepsilon}} \cdot \nabla_k \lambda_0 \right\rangle (t) + R^{\varepsilon}(t), \tag{68}$$

where, for all $t \in [0,T]$,

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}} \left\{ \int_0^t |R^{\varepsilon}(\tau)| d\tau \right\} = 0.$$

Proof. Let

$$g_i^\varepsilon(t,x) = \int_{\mathbb{R}^3} dk \partial_{k_i} \lambda_0(x,k)(\widetilde{W^\varepsilon} - \widetilde{W})(t,x,k)$$

as well as

$$\begin{aligned}
\langle \widetilde{W^\varepsilon}, \nabla_x \widetilde{U^\varepsilon} \cdot \nabla_k \lambda_0 \rangle &= \langle \widetilde{W}, \nabla_x \widetilde{U^\varepsilon} \cdot \nabla_k \lambda_0 \rangle + R^\varepsilon \\
R^\varepsilon &= \langle \widetilde{W^\varepsilon} - \widetilde{W}, \nabla_x \widetilde{U^\varepsilon} \cdot \nabla_k \lambda_0 \rangle = \sum_{i=1}^3 (\partial_{x_i} \widetilde{U^\varepsilon}, g_i^\varepsilon)_2
\end{aligned}$$

where $(\cdot,\cdot)_2$ denotes the $L^2(\mathbb{R}^3)$ inner product. From (66), we deduce that, $\forall \psi \in L^2(\mathbb{R}^3)$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} |(\psi, g_i^\varepsilon(t))_2| = 0, \qquad \mathcal{P} \text{ almost surely.}$$

Assume moreover that the support of λ_0 in x, denoted by S_x is stricly included in a ball D. Let χ be a smooth function with support S such that $S_x \subset\subset S \subset\subset D$ and $\chi = 1$ on S_x. Denoting by Δ_D the Dirichlet Laplacian on D, we then have

$$(\partial_{x_i} \widetilde{U^\varepsilon}, g_i^\varepsilon)_2 = ((I - \Delta_D)^{\frac{1}{2}} \chi \partial_{x_i} \widetilde{U^\varepsilon}, (I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon)_2.$$

The classical following estimate holds

$$\|(I - \Delta_D)^{\frac{1}{2}} \chi \partial_{x_i} \widetilde{U^\varepsilon}\|_{L^{\frac{7}{5}}(D)} \leq C \|\chi \partial_{x_i} \widetilde{U^\varepsilon}\|_{W^{1,\frac{7}{5}}(D)},$$

and according to (65),

$$\|\partial_{x_i} \partial_{x_j} \widetilde{U^\varepsilon}\|_{L^{\frac{7}{5}}(D)} \leq C \|\widetilde{n}^\varepsilon\|_{L^{\frac{7}{5}}(D)}$$

so that

$$\|(I - \Delta_D)^{\frac{1}{2}} \chi \partial_{x_i} \widetilde{U^\varepsilon}\|_{L^{\frac{7}{5}}(D)} \leq C \|\nabla \widetilde{U^\varepsilon}\|_{L^2(D)} + C \|\widetilde{n}^\varepsilon\|_{L^{\frac{7}{5}}(D)}. \tag{69}$$

Besides, since $(I - \Delta_D)^{-\frac{1}{2}}$ is a compact operator from $L^2(D)$ to $L^{\frac{7}{2}}(D)$, it is clear that

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^{\frac{7}{2}}(D)} = 0, \qquad \mathcal{P} \text{ almost surely.} \tag{70}$$

This can be seen for instance by interpolating first the $L^{\frac{7}{2}}$ norm as

$$\|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^{\frac{7}{2}}(D)} \leq \|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^2(D)}^\theta \|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^6(D)}^{1-\theta},$$

for appropriate θ. Furthermore, Sobolev embeddings implying that

$$\|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^6(D)} \leq C \|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{H^1(D)} \leq C \|g_i^\varepsilon(t)\|_{L^2(D)} \leq C, \tag{71}$$

for a constant C independent of ε, it is not difficult to show (using for instance a spectral decomposition of $(I - \Delta_D)^{-\frac{1}{2}}$) that

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|(I - \Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^2(D)} = 0, \qquad \mathcal{P} \text{ almost surely.}$$

Finally, using the Hölder inequality and (69)

$$\int_0^t |R^\varepsilon(\tau)|d\tau \leq C \sup_{t\in[0,T]} \|(I-\Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^{\frac{7}{2}}(D)} \int_0^t \left(\|\nabla \widetilde{U^\varepsilon}\|_{L^2(D)} + \|\widetilde{n}^\varepsilon\|_{L^{\frac{7}{5}}(D)} \right) d\tau$$

so that

$$\widetilde{\mathbb{E}}\left\{\int_0^t |R^\varepsilon(\tau)|d\tau\right\} \leq C \left(\widetilde{\mathbb{E}}\left\{ \left(\sup_{t\in[0,T]} \|(I-\Delta_D)^{-\frac{1}{2}} g_i^\varepsilon(t)\|_{L^{\frac{7}{2}}(D)} \right)^2 \right\} \right)^{1/2}$$
$$\times \left(\sup_{t\in[0,T]} \left(\widetilde{\mathbb{E}} \|\widetilde{n}^\varepsilon(t)\|^2_{L^{\frac{7}{5}}(D)} + \widetilde{\mathbb{E}} \|\nabla \widetilde{U^\varepsilon}\|^2_{L^2(D)} \right) \right)^{1/2}.$$

The last term in the r.h.s is bounded because of (67). The convergence (70), the bound (71) together with the Lebesgue dominated convergence theorem then end the proof of the Lemma. □

Using the last Lemma, we thus have

$$\mathcal{B}^\varepsilon_{t,s} = \widetilde{\mathbb{E}}\left\{ h\left(\langle \widetilde{W^\varepsilon}(t_1), \lambda_0\rangle, \cdots, \langle \widetilde{W^\varepsilon}(t_n), \lambda_0\rangle \right) \int_t^{t+s} d\tau \tilde{f}'_\varepsilon(\tau) \left\langle \widetilde{W}, \nabla_x \widetilde{U^\varepsilon} \cdot \nabla_k \lambda_0 \right\rangle (\tau) \right\}$$
$$+ g_\varepsilon(t,s), \qquad \text{with} \sup_{t,s\in[0,T]} |g_\varepsilon(t,s)| \to 0,$$

so that we are left to pass to the limit in the term $\langle \widetilde{W}, \nabla_x \widetilde{U^\varepsilon} \cdot \nabla_k \lambda_0\rangle$. This is carried out using weak compactness arguments. We deduce first from (67) that we can extract subsequences (still denoted by $\widetilde{n}^\varepsilon$ and $\nabla \widetilde{U}^\varepsilon$ for simplicity) such that

$$\widetilde{n}^\varepsilon \rightharpoonup \widetilde{n}^0 \qquad L^\infty((0,T), L^{7/3}(\Omega, \mathcal{P}; L^{7/5}(\mathbb{R}^3))) - w* \tag{72}$$
$$\nabla \widetilde{U}^\varepsilon \rightharpoonup F^0 \qquad (L^\infty((0,T), L^2(\Omega, \mathcal{P}; L^2(\mathbb{R}^3))))^3 - w*. \tag{73}$$

We identify now the limits $\widetilde{n}^0$ and F^0. For this, we need to introduce the Husimi transform of $\widetilde{W^\varepsilon}$ [33] defined as

$$\mathcal{W}^\varepsilon = \widetilde{W^\varepsilon} * G_\varepsilon, \qquad \text{where} \qquad G_\varepsilon(x,k) = (\pi\varepsilon)^{-3} e^{-(|x|^2+|k|^2)/\varepsilon}.$$

The main property of the Husimi transform is to be non-negative and it is not difficult to see that $\widetilde{W^\varepsilon}$ and $\mathcal{W}^\varepsilon$ have the same limit $\widetilde{W}$, see [33]. Besides, since $\widetilde{W}$ and W have the same law and W is non-negative, $\widetilde{W}$ is itself non-negative $\mathcal{P}$ almost surely. Moreover

$$\|\widetilde{n}^\varepsilon(t)\|_{L^1(\mathbb{R}^3)} = \|\mathcal{W}^\varepsilon(t)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \leq C,$$

so that we can extract subsequences such that $\mathcal{W}^\varepsilon$ converges in the space

$$L^\infty((0,T), L^\infty(\Omega, \mathcal{P}; \mathcal{M}(\mathbb{R}^3\times\mathbb{R}^3))) - w*$$

to a measure μ, where $\mathcal{M}(\mathbb{R}^3\times\mathbb{R}^3)$ is the cone of bounded positive measures on $\mathbb{R}^3\times\mathbb{R}^3$. The fact that $\widetilde{W^\varepsilon}$ and $\mathcal{W}^\varepsilon$ have the same limit allows to identify $d\mu$ with $\widetilde{W}dxdk$ and therefore $\widetilde{W} \in L^\infty((0,T), L^\infty(\Omega, \mathcal{P}; L^1(\mathbb{R}^3\times\mathbb{R}^3)))$ since it is non-negative.

We have the following lemma, which is a consequence of the energy estimate (36).

Lemma 4.5 *We have the identification*

$$\widetilde{n}^0(t,x) = \int_{\mathbb{R}^\varepsilon} \widetilde{W}(t,x,k)dk = \widetilde{n}(t,x).$$

Proof. For any function $\psi = \psi_1 \otimes \psi_2$, $\psi_1 \equiv \psi_1(t,\omega) \in \mathcal{C}_c^\infty((0,T)\times\Omega)$, $\psi_2 \equiv \psi_2(x) \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, we have from (72)

$$\widetilde{\mathbb{E}} \int_0^T (\psi, \widetilde{n}^\varepsilon)_2 \, ds \to \widetilde{\mathbb{E}} \int_0^T (\psi, \widetilde{n}^0)_2 \, ds. \tag{74}$$

Let

$$\mathcal{N}^\varepsilon(t,x) = \int_{\mathbb{R}^3} \mathcal{W}^\varepsilon(t,x,k)dk = (\pi\varepsilon)^{-3/2} e^{-|x|^2/\varepsilon} * \widetilde{n}^\varepsilon. \tag{75}$$

We have

$$\begin{aligned} |(\psi_2, \widetilde{n}^\varepsilon)_2 - (\psi_2, \mathcal{N}^\varepsilon)_2| &= \left| (\psi_2 - (\pi\varepsilon)^{-3/2} e^{-|x|^2/\varepsilon} * \psi_2, \widetilde{n}^\varepsilon) \right| \\ &\leq \|\psi_2 - (\pi\varepsilon)^{-3/2} e^{-|x|^2/\varepsilon} * \psi_2\|_{L^{7/2}} \|\widetilde{n}^\varepsilon\|_{L^{7/5}}. \end{aligned}$$

Since

$$\lim_{\varepsilon\to 0} \|\psi_2 - (\pi\varepsilon)^{-3/2} e^{-|x|^2/\varepsilon} * \psi_2\|_{L^{7/2}} = 0,$$

we obtain, thanks to the bound (67)

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \widetilde{\mathbb{E}} \, |(\psi_2, \widetilde{n}^\varepsilon)_2 - (\psi_2, \mathcal{N}^\varepsilon)_2| = 0, \tag{76}$$

so that we only need to consider $(\psi, \mathcal{N}^\varepsilon)$ and can replace $(\psi, \widetilde{n}^\varepsilon)$ by $(\psi, \mathcal{N}^\varepsilon)$ in (74). Since $\widetilde{W}^\varepsilon$ and $\mathcal{W}^\varepsilon$ have the same limit $\widetilde{W}$, we clearly have for some $R > 0$, according to (66), $\mathcal{P}$ almost surely:

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^3} \int_{|k|\leq R} \psi_2(x) \mathcal{W}^\varepsilon(t,x,k) dx dk - \int_{\mathbb{R}^3} \int_{|k|\leq R} \psi_2(x) \widetilde{W}(t,x,k) dx dk \right| = 0. \tag{77}$$

To obtain a control for large R, we observe that we have the relation:

$$\int_{\mathbb{R}^6_{xk}} |k|^2 \mathcal{W}^\varepsilon dx dk = \int_{\mathbb{R}^6_{xk}} |k|^2 \widetilde{W}^\varepsilon dx dk + \frac{3\varepsilon}{2} \|\widetilde{n}^\varepsilon\|_{L^1},$$

which implies, thanks to (67)

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^6_{xk}} |k|^2 \widetilde{\mathbb{E}}\{\mathcal{W}^\varepsilon\} dx dk \leq C. \tag{78}$$

Therefore, since $\mathcal{W}^\varepsilon$ is non-negative,

$$\begin{aligned} &\lim_{R\to\infty} \lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^3} \int_{|k|>R} \psi_2(x) \mathbb{E}\{\mathcal{W}^\varepsilon(t,x,k)\} dx dk \right| \\ &\leq \|\psi_2\|_{L^\infty} \lim_{R\to\infty} \lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \frac{1}{R^2} \int_{\mathbb{R}^6_{x,k}} |k|^2 \widetilde{\mathbb{E}}\{\mathcal{W}^\varepsilon(t,x,k)\} dx dk = 0. \end{aligned} \tag{79}$$

Let now

$$n_R(t,x)=\int_{|k|\le R}\widetilde{W}(t,x,k)dk.$$

We have

$$\begin{aligned}\widetilde{\mathbb{E}}\int_0^T(\psi,n_R)_2\,ds &= \widetilde{\mathbb{E}}\int_0^T\int_{\mathbb{R}^3}\int_{|k|\le R}\psi(\widetilde{W}-\mathcal{W}^\varepsilon)\,dxdkds+\widetilde{\mathbb{E}}\int_0^T(\psi,\mathcal{N}^\varepsilon-\widetilde{n}^\varepsilon)_2\,ds\\ &\quad -\widetilde{\mathbb{E}}\int_0^T\int_{\mathbb{R}^3}\int_{|k|>R}\psi\mathcal{W}^\varepsilon\,dxdkds+\widetilde{\mathbb{E}}\int_0^T(\psi,\widetilde{n}^\varepsilon)_2\,ds.\end{aligned}$$

Gathering (74), (76), (77) and (79) then show that

$$\lim_{R\to\infty}\widetilde{\mathbb{E}}\int_0^T(\psi,n_R)_2\,ds=\widetilde{\mathbb{E}}\int_0^T(\psi,\widetilde{n}^0)_2\,ds.$$

Since we proved earlier that $\widetilde{W}\in L^\infty((0,T),L^\infty(\Omega,\mathcal{P};L^1(\mathbb{R}^3\times\mathbb{R}^3)))$, the Lebesgue dominated convergence theorem shows that the first term on the left above is equal to $\widetilde{\mathbb{E}}\int_0^T(\psi,\widetilde{n})_2\,ds$. By density of functions of the form $\psi=\psi_1\otimes\psi_2$ in $\mathcal{C}_c^\infty((0,T)\times\Omega\times\mathbb{R}^3)$, see [41] Chapter 39, we deduce that $\widetilde{n}^0$ is equal to $\widetilde{n}$. This ends the proof of the Lemma. □

Owing Lemma (4.5), the identification of F^0 is now straightforward: for all $\psi\in\mathcal{C}_c^\infty((0,T)\times\Omega\times\mathbb{R}^3)$, we have from (73)

$$\widetilde{\mathbb{E}}\int_0^T(\psi,\nabla\widetilde{U}^\varepsilon)_2\,ds\to\widetilde{\mathbb{E}}\int_0^T(\psi,F^0)_2\,ds.$$

But

$$(\psi,\nabla\widetilde{U}^\varepsilon)_2=(\psi,\nabla\frac{1}{4\pi|x|}*\widetilde{n}^\varepsilon)_2=-(\frac{1}{4\pi|x|}*(\nabla\psi),\widetilde{n}^\varepsilon)_2,$$

and from inequality (35), $\frac{1}{4\pi|x|}*(\nabla\psi)\in(L^1((0,T),L^{7/4}(\Omega,\mathcal{P};L^{7/2}(\mathbb{R}^3))))^3$. This then allows to pass to the limit using (72) and Lemma 4.5 and to conclude that $F^0=\nabla\frac{1}{4\pi|x|}*\widetilde{n}$.

End of the proof. We have all needed now to end the proof and to pass to the limit in $\mathcal{B}^\varepsilon_{t,s}$. Since h and $\tilde{f}'_\varepsilon$ are continuous with respect to $\widetilde{W}^\varepsilon$ and uniformly bounded, since

$$g(t,x,\omega)=\int_{\mathbb{R}^3}\widetilde{W}(t,x,k)\nabla_k\lambda_0(x,k)\in L^\infty(\Omega,\mathcal{P};\mathcal{C}^0([0,T],L^2(\mathbb{R}^3)))^3$$

with a uniform bound, we can pass to the limit in $\mathcal{B}^\varepsilon(t,s)$ using (73) to obtain that, $\forall t,s\in[0,T]$:

$$\lim_{\varepsilon\to0}\mathcal{B}^\varepsilon_{t,s}=\widetilde{\mathbb{E}}\left\{h\left(\langle\widetilde{W}(t_1),\lambda_0\rangle,\cdots,\langle\widetilde{W}(t_n),\lambda_0\rangle\right)\int_t^{t+s}d\tau\tilde{f}'(\tau)\left\langle\widetilde{W},\nabla_x\widetilde{U}\cdot\nabla_k\lambda_0\right\rangle(\tau)\right\},$$

where

$$\tilde{f}'(\tau)=\phi'(\langle\widetilde{W}(\tau),\lambda_0\rangle)\qquad;\qquad\widetilde{U}=\frac{1}{4\pi|x|}*\widetilde{n}.$$

Gathering this latter result with (61), we find, for all $s > 0$:

$$\widetilde{\mathbb{E}}\left\{h\left(\langle\widetilde{W}(t_1),\lambda_0\rangle,\cdots,\langle\widetilde{W}(t_n),\lambda_0\rangle\right)[\widetilde{M}_{t+s}-\widetilde{M}_t]\right\}=0,$$

with

$$\widetilde{M}_t=\tilde{f}(t)-\int_0^t\widetilde{\mathcal{A}}\tilde{f}(s)ds,\qquad \widetilde{\mathcal{A}}\tilde{f}(t)=\tilde{f}'(t)\left\langle\widetilde{W},\left(k\cdot\nabla_x-\nabla_x\widetilde{U}\cdot\nabla_k+\mathcal{L}\right)\lambda_0\right\rangle.$$

Since $\widetilde{W}$ and W have the same law, we deduce that

$$\mathbb{E}^{\mathbb{P}}\left\{h\left(\langle W(t_1),\lambda_0\rangle,\cdots,\langle W(t_n),\lambda_0\rangle\right)[M_{t+s}-M_t]\right\}=0,$$

where the definitions of the untilded quantities are the same as the tilded quantities with the ˜ removed. We therefore conclude that M_t is a $\mathbb{P}$-martingale. We use this fact for the functions $\phi(u)=u$ and $\phi(u)=u^2$, and denote by M_t^1 and M_t^2 the corresponding martingales. Classical arguments related to martingale theory, see for instance [30] section 5.4, then show that the quadratic variation of M_t^1 is null, so that, for all $t\in[0,T]$

$$M_t^1=\langle W(t),\lambda_0\rangle-\int_0^t\langle W,(k\cdot\nabla_x-\nabla_xU\cdot\nabla_k+\mathcal{L})\lambda_0\rangle(s)ds=\langle W_0,\lambda_0\rangle.$$

It is then direct to see that $W\in\mathcal{C}^0([0,T],L^2)$ is a solution in the distribution sense of the Vlasov-Poisson-Boltzmann system (9). The solution being unique according to Assumption **H'**, W is therefore this unique solution and deterministic. Now that we know that W is deterministic, the convergence in probability is a consequence of the fact that, for any $\psi\in L^2$

$$\mathbb{E}^{\mathbb{P}^\varepsilon}\sup_{t\in[0,T]}|\langle W^\varepsilon(t)-W(t),\psi\rangle|^2=0.$$

Convergence in probability of the density. We prove now the convergence in probability of the density knowing that W is deterministic. For this, we remark as shown in the proof of Lemma 4.5, that it is enough to prove the convergence of $\mathcal{N}^\varepsilon(t,x)$ defined in (75). Let $\psi\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ and write

$$\begin{aligned}(\psi,\mathcal{N}^\varepsilon-n)_2 &= \int_{\mathbb{R}^3}\int_{|k|\leq R}(\mathcal{W}^\varepsilon-W)\psi(x)dxdk+\int_{\mathbb{R}^3}\int_{|k|>R}(\mathcal{W}^\varepsilon-W)\psi(x)dxdk\\ &:= T_1^\varepsilon+T_2^\varepsilon.\end{aligned}$$

We show first that, $\forall R\in(0,\infty)$:

$$\lim_{\varepsilon\to0}\mathbb{E}^{\mathbb{P}^\varepsilon}\sup_{t\in[0,T]}|T_1^\varepsilon|=0.\qquad(80)$$

Notice that the above quantity is perfectly defined since W is deterministic. Let $F(W^\varepsilon)$ be defined by

$$F(W^\varepsilon)=\sup_{t\in[0,T]}|T_1^\varepsilon|,$$

where the Husimi transform is seen as a function of W^ε. It is a continuous function of W^ε for the $\mathcal{C}^0([0,T], X-weak)$ topology, i.e., for any $u^\varepsilon \in \mathcal{C}^0([0,T], X)$ such that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} d(u^\varepsilon(t), W(t)) = 0, \qquad \text{then} \qquad \lim_{\varepsilon\to 0} F(u^\varepsilon) = F(W).$$

Above, the distance $d(\cdot,\cdot)$ is defined in (66). Since moreover F is uniformly integrable as

$$|F(W^\varepsilon)| \le C(\|W_0^\varepsilon\|_{L^2} + \|W_0\|_{L^2}) \le C,$$

we conclude from the convergence in law of W_ε in $\mathcal{C}^0([0,T], X)$ that

$$\lim_{\varepsilon\to 0} \mathbb{E}^{\mathbb{P}^\varepsilon}\{F(W^\varepsilon)\} = \mathbb{E}^{\mathbb{P}}\{F(W)\} = 0.$$

It remains to treat T_2^ε. For this, we use the energy estimate for the Husimi transform (78) together with the estimate (21) satisfied the W the solution to the Vlasov-Poisson-Boltzmann equation. We find

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}^\varepsilon}\{|T_2^\varepsilon|\} \le \sup_{\varepsilon\in(0,1)} \sup_{t\in[0,T]} \frac{C}{R^2} \int_{\mathbb{R}^6_{x,k}} |k|^2 (\mathbb{E}^{\mathbb{P}^\varepsilon}\{\mathcal{W}^\epsilon\} + W)(t,x,k) dx dk \le \frac{C}{R^2}.$$

Together with (76)-(80), this shows that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \mathbb{E}^{\mathbb{P}^\varepsilon}\{|(\psi, n^\varepsilon - n)_2|\} = 0,$$

which implies the convergence in probability thanks to the Markov inequality. This ends the proof of the theorem.

Appendix

4.1 Proof of lemma 3.3.

With the notation $\eta = \frac{x}{\varepsilon}$, we deduce from (29) that

$$\mathcal{F}_{k\to y}\lambda_1(x,\eta,y) = \frac{1}{i(2\pi)^3} \sum_{\sigma_1=\pm 1} \sigma_1 \int_0^\infty dr e^{-r} \int_{\mathbb{R}^3} d\hat{V}(p) e^{ir\sigma_1|p|^2/2 + i\eta\cdot p} \mathcal{F}_{k\to y}\lambda_0(x, y-rp) \tag{81}$$

and therefore

$$|\mathcal{F}_{k\to y}\lambda_1(x,\eta,y)| \le C \int_0^\infty dr e^{-r} \int_{\mathbb{R}^3} |d\hat{V}| \|\mathcal{F}_{k\to y}\lambda_0\|_{L^\infty} \le C.$$

Hence, we only need to consider the set $|y| \ge 1$ to prove the proposition. For $r > |y|/(2L)$, the corresponding contribution in right hand side of (81) is controlled by

$$Ce^{-|y|/(2L)} \int_{\mathbb{R}^3} |d\hat{V}| \|\mathcal{F}_{k\to y}\lambda_0\|_{L^\infty} \le Ce^{-|y|/(2L)}. \tag{82}$$

When $r \le |y|/(2L)$, since $\lambda_0 \in \mathcal{S}$, there exists C such that

$$|\mathcal{F}_{k\to y}\lambda_0(x,y)| \le \frac{C}{(1+|x|^{10})(1+|y|^{10})},$$

which implies, since $r \leq |y|/(2L)$ and $|p| \leq L$ (recall that the measure $\hat{V}$ has a bounded support included the ball of radius L):

$$|y - rp| \geq ||y| - r|p|| \geq |y|/2.$$

The second part of the r.h.s of (81) is thus bounded by $C|y|^{-10}$. As a consequence, it comes together with (82)

$$\int_{\mathbb{R}^3} dy|y|^2 \sup_{x\in\mathbb{R}^3} |\mathcal{F}_{k\to y}\lambda_1(x,\eta,y)|^2 \quad \leq \quad C + C\int_{|y|\geq 1} dy|y|^2(e^{-|y|/(L)} + |y|^{-20}) \leq C.$$

This ends the proof.

4.2 Proof of lemma 4.1.

We start from the definition of λ_2 given in (32). Define the shorthand

$$T := (\mathcal{L}_\varepsilon\lambda_0)(x,k) - (\mathcal{K}[\hat{v},\eta + rk]\lambda_1)(x,\eta + rk,k,\hat{v})$$

so that $\lambda_2(x,\eta,k,\hat{v}) = -\int_0^\infty dr e^{rQ_\varepsilon} T$. Using Lemma 3.1, the fact that $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^6)$ and the definition of $\mathcal{L}_\varepsilon$ as a Riemann sum, it is not difficult to obtain that

$$\|T\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leq C\|\lambda_0\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}, \qquad C \text{ independent of } \varepsilon. \tag{83}$$

Some lengthy but simple calculations show that, for $k \in \mathbb{N}^*$:

$$(Q_\varepsilon)^k T = (-1)^k T + (-1)^{k+1} kR, \qquad R := \sum_{m\in N_\varepsilon} \sum_{\substack{n\,\in\, N_\varepsilon \\ n\,\neq\, -m}} \alpha_m \beta_{n,m} \hat{v}_m \hat{v}_n$$

where

$$\begin{aligned}
\alpha_m &= \frac{h_\varepsilon^3}{i(2\pi)^3}\sqrt{\hat{R}_1(p_m^\varepsilon)}e^{ip_m^\varepsilon\cdot\eta}, \\
\beta_{n,m} &= \frac{h_\varepsilon^3}{i}\sqrt{\hat{R}_1(p_n^\varepsilon)} \sum_{\sigma_1,\sigma_2=\pm 1} \sigma_1\sigma_2 e^{ip_n^\varepsilon\cdot(r(k-\frac{\sigma_1}{2}p_m^\varepsilon)+\eta)}\lambda_0(x,k - \frac{\sigma_1}{2}p_m^\varepsilon - \frac{\sigma_2}{2}p_n^\varepsilon).
\end{aligned}$$

In the same spirit as (83), we can show that $\|R\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leq C\|\lambda_0\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}$, and therefore

$$e^{rQ_\varepsilon}T = e^{-r}T + re^{-r}R, \qquad \text{and} \quad \|\lambda_2\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leq C\|\lambda_0\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)},$$

$\pi-$ almost surely. This proves the first estimate of the Lemma, as well as the second one by remembering that the derivative is taken with respect to the slow variable x and not η. The third estimate is obtained by using the above decomposition of $e^{rQ_\varepsilon}T$ and by proceeding as in Lemma 3.3 by splitting the integral in r into two contributions. We leave the details to the reader. This ends the proof.

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