Dynamics of wave scintillation in random media

Guillaume Bal [*] Olivier Pinaud [†]

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Abstract

This paper concerns the asymptotic structure of the scintillation function in the simplified setting of wave propagation modeled by an Itô-Schrödinger equation. We show that the size of the scintillation function crucially depends on the smoothness of the initial conditions for the wave equation and on the size of the "array of detectors" where the wave fields are measured. In many practical settings, we show that the estimates are optimal and devise an equation for the appropriately rescaled scintillation function. The estimates are based on a careful analysis of Wigner transforms and of linear kinetic equations involving oscillatory integrals.

Keywords: Waves in random media, kinetic model, statistical stability, scintillation function, Itô (Stratonovich) Schrödinger regime, Wigner transform

1 Introduction.

Wave propagation in heterogeneous media and over large distances compared to the wavelength arise e.g. in geophysics with the propagation of seismic waves [25], telecommunications, underwater acoustics, and propagation of light through turbulent atmosphere, see e.g. [26, 29]. Whereas the microscopic dynamics of the wave is fairly complex, macroscopic models may sometimes be derived to simplify the description. These models depend on the relation between the correlation length of the random medium and the wavelength, and also on the strength of the fluctuations. An important feature of many of these models is their *statistical stability*, in the sense that they depend only on some general (macroscopic) characteristics of the medium and not on its local fluctuations. This invocation of ergodicity is valid when the strength of the fluctuations is *weak*, so that the localization phenomenon is avoided, see [15, 26], for then wave may be trapped at some random location depending on the realization of the random media and this will prevent any statistical stability. The so-called weak coupling regime is the regime of interest in this paper.

When the wavelength and the correlation length are of same order and are small compared to the typical distance of propagation, the macroscopic behavior of the wave can be described by *radiative transfer equations* [11, 17]. The rigorous derivation of such a model from high-frequency random wave equations is a challenging mathematical problem which has found solutions only in some simplified settings. A formal derivation can be found for instance in

[*]Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu;

[†]Université de Lyon, Université Lyon 1, CNRS, UMR 5208 Institut Camille Jordan/ISTIL, Bâtiment du Doyen Jean Braconnier, 43, blvd du 11 novembre 1918, F - 69622 Villeurbanne Cedex, France; pinaud@math.univ-lyon1.fr

[24] for acoustic, electromagnetic and elastic waves, while the kinetic limit for discrete wave equations has been demonstrated in the recent paper [20]. In most cases, the rigorous analysis is done within the *paraxial approximation*, see e.g. [28, 10], which occurs when the wave has a privileged direction of propagation and backscattering effects can be neglected.

Let us assume that the beam mainly propagates along the $z \in \mathbb{R}$ axis. Then, starting from the standard scalar wave equation for the pressure potential $p(t, \mathbf{x}, z)$, where t is time, $\mathbf{x} \in \mathbb{R}^d$ (so that the overall spatial dimension is $d+1$), and $c(\mathbf{x}, z)$ is the (random) sound speed,

$$\frac{\partial^2 p}{\partial t^2}(t, \mathbf{x}, z) = c^2(\mathbf{x}, z)\,(\Delta_{\mathbf{x}} + \Delta_z)\,p(t, \mathbf{x}, z),$$

with appropriate initial conditions, we formally obtain [3, 2] for the amplitude $\psi(z, \mathbf{x}, \kappa)$ defined by

$$p(t, \mathbf{x}, z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\kappa(z - c_0 t)}\psi(z, \mathbf{x}, \kappa) c_0 d\kappa,$$

the following high-frequency random Schrödinger equation:

$$i\eta\frac{\partial \psi_\eta}{\partial z}(z, \mathbf{x}, \kappa) = -\frac{\eta^2}{2}\Delta_{\mathbf{x}}\psi_\eta(z, \mathbf{x}, \kappa) - \sqrt{\eta}V\left(\frac{z}{\eta}, \frac{\mathbf{x}}{\eta}\right)\psi_\eta(z, \mathbf{x}, \kappa), \tag{1}$$

augmented with an initial condition $\psi_\eta(z = 0, \mathbf{x}) = \psi_\eta^0(\mathbf{x})$. Above, c_0 is the background sound speed assumed to be constant for simplicity, $\eta \ll 1$ is the rescaled *transverse* wavelength and V is the random potential related to the sound speed c. The variable κ plays no role in the analysis and will therefore be set to $\kappa = 1$. When the sound speed has faster fluctuations in the z direction than in the transverse direction $\mathbf{x}$, the potential V can formally be replaced by a white noise in z, giving rise - after the appropriate Stratonovich correction -, to the *Itô-Schrödinger* equation:

$$d\psi_\eta(z, \mathbf{x}) = \frac{1}{2}\big(i\eta\Delta_{\mathbf{x}} - R(\mathbf{0})\big)\psi_\eta(z, \mathbf{x})dz + i\psi_\eta(z, \mathbf{x})B\Big(\frac{\mathbf{x}}{\eta}, dz\Big). \tag{2}$$

Here, $B(\mathbf{x}, dz)$ is a standard (infinite dimensional) Wiener measure, whose statistics are described by

$$\mathbb{E}\{B(\mathbf{x}, z)B(\mathbf{y}, z')\} = R(\mathbf{x} - \mathbf{y})z \wedge z', \tag{3}$$

where $\mathbb{E}$ is mathematical expectation with respect to the measure of an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $B(\mathbf{x}, dz)$ is defined, $z \wedge z' = \min(z, z')$ and R is the correlation function of the random medium. A rigorous passage from the wave equation to (2) can be found in [1] when $d = 2$ and in stratified media. The radiative transfer equations are then obtained from high-frequency asymptotics of (1) or (2) and the appropriate tool in the analysis of such equations is the *Wigner transform* [30] of the wave function defined as

$$W_\eta[\psi_\eta](z, \mathbf{x}, \mathbf{k}) = W_\eta(z, \mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}}\psi_\eta\Big(z, \mathbf{x} - \frac{\eta\mathbf{y}}{2}\Big)\overline{\psi_\eta}\Big(z, \mathbf{x} + \frac{\eta\mathbf{y}}{2}\Big)d\mathbf{y}, \tag{4}$$

where $\overline{\psi_\eta}$ denotes complex conjugation of ψ_η. The Wigner transform W_η is real-valued and $\int_{\mathbb{R}^d} W_\eta(t, \mathbf{x}, \mathbf{k})d\mathbf{k} = |\psi_\eta(\mathbf{x}, t)|^2$ by inverse Fourier transform so that W_η may be seen as a phase space (microlocal) decomposition of the energy density, even though it is not always positive. We refer the reader to [19, 16] for an extensive study of Wigner transforms with applications to high-frequency limit of hyperbolic or Schrödinger equations. The rigorous limit of the Schrödinger equation (1) to the radiative transfer equations has been investigated, with various hypotheses on the random potential V (e.g. Markovian with respect to time or with finite-range time correlations), for instance in [3, 4, 13, 23, 27]. The main result is the following: under appropriate conditions on the initial condition ψ_η^0, the *ensemble average* of the Wigner transform $a_\eta := \mathbb{E}\{W_\eta\}$ converges weakly in an adapted functional setting to the solution a of the following radiative transfer equation (or linear Boltzmann equation):

$$\Big(\frac{\partial}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}} + R_0 - \mathcal{Q}\Big)a(z,\mathbf{x},\mathbf{k}) = 0, \qquad a(0,\mathbf{x},\mathbf{k}) = a_0(\mathbf{x},\mathbf{k}), \tag{5}$$

where a_0 is the limit of the ensemble average of the Wigner transform of the initial condition ψ_η^0, $R_0 := (2\pi)^d R(\mathbf{0})$ and the scattering operator $\mathcal{Q}$ reads

$$(\mathcal{Q}a)(z,\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} \hat{R}(\mathbf{k}-\mathbf{k}')a(z,\mathbf{x},\mathbf{k}')d\mathbf{k}'.$$

Here, $\hat{R}$ denotes the Fourier transform of R with the convention

$$\hat{R}(\mathbf{k}) = \mathcal{F}R(\mathbf{k}) = \int_{\mathbb{R}^d} e^{-i\mathbf{k}\cdot\mathbf{x}}R(\mathbf{x})d\mathbf{x}.$$

Since $R(\mathbf{x})$ is a correlation function, $\hat{R}(\mathbf{k})$ is non-negative by Bochner's theorem. The derivation of (5) from the Itô-Schrödinger equation (2) is immediate since moments of the wavefunction satisfy closed-form equations. Starting from (2) and writing the stochastic equation for the Wigner transform, a direct application of the Itô calculus yields that a_η solves (5) with an initial condition $a_{\eta 0} := \mathbb{E}\{W_\eta[\psi_\eta^0]\}$, see for instance [22]. It then suffices to pass to the limit in the initial condition to obtain the convergence of a_η to a.

Whereas the limit of $\mathbb{E}\{W_\eta\}$ can be characterized in various settings, much less is known about the limit of the whole process W_η. It is proved in [4], under additional hypotheses on the Wigner transform (basically it is given by a mixed state so as to obtain L^2 estimates), that $W_\eta[\psi_\eta]$, with ψ_η the solution to (1), converges weakly and in probability to its average $\mathbb{E}\{W_\eta[\psi_\eta]\}$, that is

$$\mathbb{P}\Big(|\langle W_\eta(z),\varphi\rangle - \langle a_\eta(z),\varphi\rangle| \geq \delta\Big) \to 0, \qquad \text{uniformly on compact intervals.}$$

Above, φ is a test function in the Schwarz space $\mathcal{S}(\mathbb{R}^{2d})$ and $\langle\cdot,\cdot\rangle$ denotes the $\mathcal{S}'-\mathcal{S}$ duality product, where $\mathcal{S}'$ is the space of tempered distributions. The latter result means that the Wigner transform is *self-averaging*. This is an important property for instance in the analysis of the refocusing properties of time-reversed waves [4, 9, 21, 14] for which it is shown that the quality of refocusing is independent of the local fluctuations of the random medium and hence only depends on macroscopic characteristics. The statistical stability of waves is also a fundamental requirement for applications to imaging or detection in complex media: a heterogeneous medium with unknown local variations is often modeled as a particular realization of a random medium with given macroscopic quantities (which are known or to be estimated). The inverse problem of the reconstruction of an inclusion embedded in the medium is then done using a radiative transfer equation derived from ensemble averages of observables and not from a single realization; see [5, 6, 8]. It is thus important that these observables do not differ significantly for two different realizations of the random medium.

In the Itô-Schrödinger regime, the convergence of W_η to its average can be made precise so as to obtain information on the rate of convergence or on the size of the averaging domain that is needed to obtain statistical stability (typically the size of the support of the test function φ), see e.g. [2, 7, 22]. This is rendered possible by the fact that the *scintillation function* J_η (or covariance function), defined as

$$J_\eta(z,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) = \mathbb{E}\{W_\eta(z,\mathbf{x},\mathbf{k})W_\eta(z,\mathbf{y},\mathbf{p})\} - \mathbb{E}\{W_\eta(z,\mathbf{x},\mathbf{k})\}\mathbb{E}\{W_\eta(z,\mathbf{y},\mathbf{p})\}, \tag{6}$$

solves the closed-form equation

$$\Big(\frac{\partial}{\partial z} + \mathcal{T}_2 + 2R_0 - \mathcal{Q}_2 - \mathcal{K}_\eta\Big) J_\eta = \mathcal{K}_\eta a_\eta \otimes a_\eta, \tag{7}$$

equipped with vanishing initial conditions $J_\eta(0, \mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{p}) = 0$ when the initial condition of the Schrödinger equation is deterministic. Here, $R_0 := (2\pi)^d R(\mathbf{0})$ and we have defined

$$\begin{aligned}
\mathcal{T}_2 &= \mathbf{k} \cdot \nabla_{\mathbf{x}} + \mathbf{p} \cdot \nabla_{\mathbf{y}}, \\
\mathcal{Q}_2 h &= \int_{\mathbb{R}^{2d}} \Big(\hat{R}(\mathbf{k} - \mathbf{k}')\delta(\mathbf{p} - \mathbf{p}') + \hat{R}(\mathbf{p} - \mathbf{p}')\delta(\mathbf{k} - \mathbf{k}')\Big) h(\mathbf{x}, \mathbf{k}', \mathbf{y}, \mathbf{p}') d\mathbf{k}' d\mathbf{p}', \\
\mathcal{K}_\eta h &= \sum_{\epsilon_i, \epsilon_j = \pm 1} \epsilon_i \epsilon_j \int_{\mathbb{R}^{2d}} \hat{R}(\mathbf{u}) e^{i\frac{(\mathbf{x}-\mathbf{y})\cdot \mathbf{u}}{\eta}} h\Big(\mathbf{x}, \mathbf{k} + \epsilon_i \frac{\mathbf{u}}{2}, \mathbf{y}, \mathbf{p} + \epsilon_j \frac{\mathbf{u}}{2}\Big) d\mathbf{u}.
\end{aligned} \tag{8}$$

Above, δ is the Dirac distribution. Equation (7) is obtained by computing the fourth moment of the wave function, see [2]. The analysis of (7) and of the highly oscillating operator $\mathcal{K}_\eta$ shows that J_η converges weakly to zero, which implies convergence of W_η in probability thanks to the Chebyshev inequality

$$\mathbb{P}\Big(|\langle W_\eta(z), \varphi\rangle - \langle a_\eta(z), \varphi\rangle| \geq \varepsilon\Big) \leq \frac{1}{\varepsilon^2} \langle J_\eta(z), \varphi \otimes \varphi\rangle,$$

with $(\varphi \otimes \varphi)(\mathbf{x}, \mathbf{k}, \mathbf{y}, \mathbf{p}) := \varphi(\mathbf{x}, \mathbf{k})\varphi(\mathbf{y}, \mathbf{p})$.

The objectives of the present paper are twofold: (i) refine and complement the convergence estimates for J_η obtained in [7]; and (ii) characterize the dynamics of the statistical instabilities by computing the limit of the first-order corrector of J_η for practically useful (pure state) initial conditions. This requires us to define a functional setting adapted to Wigner transforms and to a precise analysis of (7) and of the oscillating operator $\mathcal{K}_\eta$. The outcome is a complete characterization of the propagation of the statistical instabilities. We show that their dynamics are driven by a transport equation with a non-vanishing initial condition or source term depending on the singularities of the initial condition of the Schrödinger equation.

Note that for the particular form of the initial conditions $W_\eta(z = 0, \mathbf{x}, \mathbf{k}) = \delta(\mathbf{x}) f(\mathbf{k})$, where f is a regular function, the limit in distribution of the random corrector in W_η was recently investigated in [18]. Such an initial condition can be seen as the analog of the special case $\alpha = 1$ in our configuration; see e.g., equation (10) below.

The paper is structured as follows. In section 2, we present our assumptions and describe the main results. Theorem 1 gives a convergence rate of the scintillation function, while theorem 2 shows that the obtained rate is optimal for particular initial conditions and provides us with an asymptotic model for the propagation of the statistical instabilities. In section 3, we introduce the functional setting adapted to the problem and prove preliminary results on the operator $\mathcal{K}_\eta$ and on the well-posedness of both the 2-transport and 4-transport equations (5) and (7), respectively. In section 4, we prove theorems 1 and 2.

2 Main results.

We present in this section the main results of the paper. We give existence and uniqueness results for the Itô-Schrödinger equation (2), present our main assumptions, and state our main results in theorems 2.1 and 2.2.

To be consistent with the usual notation for the time-dependent Schrödinger equation, we relabel the variable z as t. We assume that the initial condition ψ_η^0 is deterministic (i.e., independent of the random medium) and uniformly bounded with respect to η in $L^2(\mathbb{R}^d)$. We assume that our random medium has sufficiently short range correlations so that $\hat{R} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In such a setting, it is proved in [12] that (2) admits a unique solution $\psi_\eta(t,\mathbf{x},\omega) \in \mathcal{C}^0([0,\infty), L^2(\mathbb{R}^d))$, $\mathbb{P}$ a.e., such that, $\forall t>0$,

$$\|\psi_\eta(t,\cdot)\|_{L^2(\mathbb{R}^d)} \le \|\psi_\eta^0\|_{L^2(\mathbb{R}^d)} \le C,$$

with probability one for some constant C independent of η. Moreover, ψ_η admits moments of arbitrary order so that its Wigner transform and related scintillation function are well-defined. Let $a_{\eta 0} := \mathbb{E}\{W_\eta[\psi_\eta^0]\} = W_\eta[\psi_\eta^0]$, where W_η is defined in (4).

Let $\mathcal{F}a_{\eta 0}$ be the Fourier transform of $a_{\eta 0}$ and $\mathcal{F}_{\mathbf{x}} a_{\eta 0}$ (resp. $\mathcal{F}_{\mathbf{k}} a_{\eta 0}$) be its partial Fourier transform with respect to $\mathbf{x}$ (resp. $\mathbf{k}$). Two important quantities are the L^1 norms of $\mathcal{F}_{\mathbf{x}} a_{\eta 0}$ and $\mathcal{F}_{\mathbf{k}} a_{\eta 0}$. Denoting by $a \lesssim b$ the inequality $a \le Cb$, where $C>0$ is some universal constant, this leads us to make the following hypotheses on $a_{\eta 0}$:

Hypotheses H: $\mathcal{F}\nabla_{\mathbf{x}}^p a_{\eta 0} \in L^\infty(\mathbb{R}^{2d})$, $\mathcal{F}_{\mathbf{x}}\nabla_{\mathbf{x}}^p a_{\eta 0} \in L^1(\mathbb{R}^{2d})$, $\mathcal{F}_{\mathbf{k}}\nabla_{\mathbf{x}}^p a_{\eta 0} \in L^1(\mathbb{R}^{2d})$, for $p=0$ or 1 (with the convention that $\nabla_{\mathbf{x}}^0 a_{\eta 0} := a_{\eta 0}$) with the following estimates, for $(\alpha,\beta) \in \mathbb{R}^2$ verifying $0 \le \alpha \le 1$ and $0 \le \beta \le 1$:

$$\|\mathcal{F}\nabla_{\mathbf{x}} a_{\eta 0}\|_{L^\infty(\mathbb{R}^{2d})} \lesssim \eta^{-\alpha},$$

$$\|\mathcal{F}_{\mathbf{x}}\nabla_{\mathbf{x}}^p a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim \eta^{-(d+p)\alpha} \quad \text{and} \quad \|\mathcal{F}_{\mathbf{k}}\nabla_{\mathbf{x}}^p a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim \eta^{-d\beta - p\alpha}.$$

For instance, when $\psi_\eta^0 \in \mathcal{S}(\mathbb{R}^d)$, it follows from

$$\begin{aligned}
\mathcal{F}_{\mathbf{x}} a_{\eta 0}(\mathbf{u},\mathbf{p}) &= \frac{1}{\eta^d}\mathcal{F}\psi_{\eta 0}\left(\frac{\mathbf{p}}{\eta}+\frac{\mathbf{u}}{2}\right)\overline{\mathcal{F}\psi_{\eta 0}}\left(\frac{\mathbf{p}}{\eta}-\frac{\mathbf{u}}{2}\right),\\
\mathcal{F}_{\mathbf{k}} a_{\eta 0}(\mathbf{x},\boldsymbol{\xi}) &= \psi_{\eta 0}\left(\mathbf{x}+\frac{\eta}{2}\boldsymbol{\xi}\right)\overline{\psi_{\eta 0}}\left(\mathbf{x}-\frac{\eta}{2}\boldsymbol{\xi}\right),
\end{aligned}$$

that $\mathcal{F}\nabla_{\mathbf{x}}^p a_{\eta 0} \in L^\infty(\mathbb{R}^{2d})$, $\mathcal{F}_{\mathbf{x}}\nabla_{\mathbf{x}}^p a_{\eta 0} \in L^1(\mathbb{R}^{2d})$, and $\mathcal{F}_{\mathbf{k}}\nabla_{\mathbf{x}}^p a_{\eta 0} \in L^1(\mathbb{R}^{2d})$ for $p=0$ or 1, though the norms are not bounded uniformly in η. The relevance of the above hypothesis is better explained by looking at the following examples.

Typical initial conditions. Let us consider initial conditions $\psi_\eta(\mathbf{x},0)$ oscillating at frequencies of order η^{-1} and with a spatial support of size η^α for $0 \le \alpha \le 1$. The parameter α quantifies the macroscopic concentration of the initial condition. The simplest example is a modulated plane wave of the form:

$$\psi_\eta^{(1)}(\mathbf{x}) = \frac{1}{\eta^{\frac{d\alpha}{2}}}\chi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha}\Big)e^{i\frac{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{k}_0}{\eta}}, \tag{9}$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$. The direction of propagation is given by $\mathbf{k}_0$. Note that the above sequence of initial conditions is indeed uniformly bounded in $L^2(\mathbb{R}^d)$, and that the related Wigner transform reads

$$a_{\eta 0}(\mathbf{x},\mathbf{k}) = \frac{1}{\eta^d}a_0\left(\frac{\mathbf{x}-\mathbf{x}_0}{\eta^\alpha},\frac{\mathbf{k}-\mathbf{k}_0}{\eta^{1-\alpha}}\right), \tag{10}$$

where $a_0(\mathbf{x},\mathbf{k})$ is the Wigner transform of the rescaled initial condition $\psi_1^{(1)}$. Such an initial condition then verifies hypotheses **H** with $\beta = 1-\alpha$. The parameter α measures the concentration of the initial conditions in the spatial variables while β measures that in the momentum variables. We restrict α and β to be less than one to ensure that η^{-1} is the highest frequency in the problem. Allowing for higher frequencies while still considering a Wigner transform at the frequency η^{-1} will lead to vanishing limiting Wigner transforms and would be of little interest for then energy is lost when passing to the limit, see e.g. [16, 19].

As another example of initial conditions, we consider

$$\psi_\eta^{(2)}(\mathbf{x}) = \frac{1}{\eta^{\frac{(d-1)\alpha+1}{2}}}\chi\Big(\frac{\mathbf{x}}{\eta^\alpha}\Big)J_0\Big(\frac{|\mathbf{k}_0||\mathbf{x}|}{\eta}\Big), \tag{11}$$

where J_0 is the zero-th order Bessel function of the first kind. Such an initial condition is supported in the Fourier domain in the vicinity of wavenumbers $\mathbf{k}$ such that $|\mathbf{k}| = |\mathbf{k}_0|$ so that $\psi_\eta^{(2)}$ emits radiation isotropically at wavenumber $|\mathbf{k}_0|$; see [5, 6] for more details. We again verify that the above sequence of initial conditions is indeed uniformly bounded in $L^2(\mathbb{R}^d)$ and satisfies $\mathbf{H}$ with $\alpha = 1-\beta$. For this, we use that $J_0(z) = \sqrt{\frac{2}{\pi z}}\cos(z-\frac{\pi}{4}) + \mathcal{O}(z^{-3/2})$ and the fact that $\nabla_{\mathbf{x}} a_{\eta 0}$ is the Wigner transform of

$$\frac{1}{\eta^{\frac{(d-1)\alpha+3}{2}}}(\nabla\chi)\Big(\frac{\mathbf{x}}{\eta^\alpha}\Big)J_0\Big(\frac{|\mathbf{k}_0||\mathbf{x}|}{\eta}\Big),$$

since $\overline{J_0}(|\mathbf{x}|) = J_0(-|\mathbf{x}|)$ so that the gradients of $\overline{J_0}(|\mathbf{x}|)$ and $J_0(|\mathbf{x}|)$ cancel in the computation.

Since the scintillation function J_η is itself oscillatory, the limit depends at which scale it is measured. We thus define localized test functions of the form:

$$\varphi_{\eta,s_1,s_2}(\mathbf{x},\mathbf{k}) = \frac{1}{\eta^{d(s_1+s_2)}}\varphi\Big(\frac{\mathbf{x}}{\eta^{s_1}},\frac{\mathbf{k}-\mathbf{k}_1}{\eta^{s_2}}\Big), \tag{12}$$

where $(s_1,s_2)\in\mathbb{R}^2$ and $\mathbf{k}_1\in\mathbb{R}^d$ and $\varphi\in\mathcal{S}(\mathbb{R}^{2d})$. In this paper, we do not optimize the convergence rates as a function of s_1 and s_2 so as to obtain statistical stability for averaging domains as small as possible. We refer to [7] for such results, where it is shown for instance that for initial conditions with large support, that is for $\alpha=0$, then we only need $s_1<1$ to obtain statistical stability, which amounts to averaging the energy density over a domain of typical size $\eta^{1-\delta}$, with $\delta>0$.

Our first main result is the following:

Theorem 2.1 *Let $d\geq 2$ and assume that hypotheses **H** are satisfied. Then, the scintillation function J_η verifies the following estimate, uniformly on compact intervals:*

$$\langle J_\eta(t),\varphi_{\eta,s_1,s_2}\otimes\varphi_{\eta,s_1,s_2}\rangle \lesssim g_d(\eta),$$

$$g_d(\eta) = \eta^{d(1-\alpha)-2d(s_1+s_2)}\left[\eta^{2(1-\alpha)-s_1-s_1\vee s_2+(\alpha-\beta)\vee 0}\right]\vee\eta^{1-\beta+((\alpha-\beta)\vee 0)\wedge((d-1)(1-\alpha-\beta)+\alpha)},\ d\geq 3,$$
$$g_2(\eta) = \eta^{2(1-\alpha)-4(s_1+s_2)}\left[\eta^{2(1-\alpha)-s_1-s_1\vee s_2+(\alpha-\beta)\vee 0}\right]\vee\left[\eta^{1-\beta}\left(\eta^{\alpha-\beta}(1+|\log\eta^{\alpha-\beta}|)\right)\wedge 1\right].$$

Here, $\langle\cdot,\cdot\rangle$ denotes the $S'-S$ duality product, $a\wedge b=\min(a,b)$, $a\vee b=\max(a,b)$ and $(\varphi\otimes\varphi)(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) := \varphi(\mathbf{x},\mathbf{k})\varphi(\mathbf{y},\mathbf{p})$.

Theorem 2.1 is a refined version of the result of [7]. It is shown in theorem 2.2 below that the rate of convergence of J_η is optimal when the test function φ is smooth ($s_1=s_2=0$) and for initial conditions of the form (9). Since the proof of theorem 2.1 does not depend on the particular form of the initial conditions, we expect the rate to be optimal for any initial conditions satisfying hypotheses $\mathbf{H}$, although we do not have a complete proof for such a statement.

Our second result on the convergence of scintillation requires that we first define:

$$\begin{aligned}
j_\alpha^1(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= \delta(\mathbf{x}-\mathbf{x}_0-t\mathbf{k})\,\delta(\mathbf{y}-\mathbf{x}_0-t\mathbf{p})(\nabla\delta)^T(\mathbf{k}-\mathbf{k}_0)\,M^\alpha(t)\,(\nabla\delta)(\mathbf{p}-\mathbf{k}_0),\\
(M^{\frac12}(t))_{ij} &= \hat{R}(\mathbf{0})\int_{\mathbb{R}^d}\mathcal{F}\partial_{\mathbf{x}_i}a_0\otimes\partial_{\mathbf{y}_j}a_0(\mathbf{w},t\mathbf{w},-\mathbf{w},-t\mathbf{w})\,d\mathbf{w},\\
(M^\alpha(t))_{ij} &= M_{ij}=(M^{\frac12}(0))_{ij}, \qquad 0\le\alpha<\frac12,\\
(M^\alpha(0))_{ij} &= \int_0^\infty (M^{\frac12}(t))_{ij}\,dt, \qquad \frac12<\alpha<1.
\end{aligned}$$

The above matrices are well-defined and for $0\le\alpha<1$, we have

$$|(M^\alpha)_{ij}| \;\le\; \hat{R}(\mathbf{0})\|\mathcal{F}\partial_{\mathbf{y}_j}a_0\|_{L^\infty(\mathbb{R}^{2d})}\Big(\|\mathcal{F}_\mathbf{x}\partial_{\mathbf{x}_i}a_0\|_{L^1(\mathbb{R}^{2d})}+\|\mathcal{F}_\mathbf{k}\partial_{\mathbf{x}_i}a_0\|_{L^1(\mathbb{R}^{2d})}\Big).$$

We also need to define

$$\begin{aligned}
j_\alpha^2(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= 2\,\delta(\mathbf{x}-\mathbf{y})\Big(\sigma_\alpha(t,\mathbf{x},\mathbf{k}-\mathbf{k}_0)\delta(\mathbf{p}-\mathbf{k})-\sigma_\alpha(t,\mathbf{x},\mathbf{p}-\mathbf{p}_0)\delta(\mathbf{k}-\mathbf{k}_0)\\
&\qquad -\sigma_\alpha(t,\mathbf{x},\mathbf{k}-\mathbf{k}_0)\delta(\mathbf{p}-\mathbf{p}_0)+\delta(\mathbf{k}-\mathbf{k}_0)\delta(\mathbf{p}-\mathbf{p}_0)\int_{\mathbb{R}^d}\sigma_\alpha(t,\mathbf{x},\mathbf{k})d\mathbf{k}\Big),
\end{aligned}$$

where the cross section σ_α depends on the value of α and on the spatial dimension:

$$\begin{aligned}
\sigma_0(t,\mathbf{x},\mathbf{p}) &= (2\pi)^d\,\hat{R}^2(\mathbf{p})\int_0^t d\tau e^{-2R_0(t-\tau)}|\mathcal{F}_\mathbf{k}a_0(\mathbf{x}-\mathbf{x}_0-\mathbf{k}_0t-(t-\tau)\frac12\mathbf{p},-\tau\mathbf{p})|^2,\\
\sigma_\alpha(t,\mathbf{x},\mathbf{k}) &= \delta(\mathbf{x}-\mathbf{x}_0-t\mathbf{k}_0)\,\sigma_\alpha(t,\mathbf{k}), \qquad \alpha>0,\\
\sigma_{\frac12}(t,\mathbf{k}) &= \hat{R}^2(\mathbf{k})\int_0^\infty\int_{\mathbb{R}^d}|\mathcal{F}a_0(\mathbf{w},t\mathbf{w}-\tau\mathbf{k})|^2d\mathbf{w}d\tau,\\
\sigma_\alpha(t,\mathbf{k}) &= \sigma(\mathbf{k})=\sigma_{\frac12}(0,\mathbf{k}), \qquad 0<\alpha<\frac12,\\
\sigma_\alpha(0,\mathbf{k}) &= \int_0^\infty \sigma_{\frac12}(t,\mathbf{k})dt, \qquad \frac12<\alpha<1, \quad d\ge3,\\
\sigma_\alpha(0,\mathbf{k}) &= \hat{R}^2(\mathbf{k})\int_0^\infty\int_{\mathbb{R}^d}|\mathcal{F}a_0(\tau\mathbf{k},\mathbf{w})|^2d\mathbf{w}d\tau, \qquad \frac12<\alpha<1, \quad d=2.
\end{aligned}$$

Besides, $\sigma_0\in\mathcal{C}^0([0,T],L^1(\mathbb{R}^d\times\mathbb{R}^d))$, $\sigma_\alpha(t,\mathbf{k})\in L^1(\mathbb{R}_+\times\mathbb{R}^d)$ for $0<\alpha\le\frac12$ and $\sigma_\alpha(0,\mathbf{k})\in L^1(\mathbb{R}^d)$ for $\frac12<\alpha<1$. We need finally:

$$\begin{aligned}
J_1^{1,0}(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= \Big(\pi\int_{\mathbb{R}^d}d\mathbf{w}\hat{R}(\mathbf{w})\delta(\mathbf{w}\cdot(\mathbf{k}-\mathbf{p}))G(\mathbf{w},\mathbf{k}-\mathbf{k}_0,\mathbf{p}-\mathbf{k}_0)\\
&\quad + i\,\text{p.v.}\int_{\mathbb{R}^d}d\mathbf{w}\hat{R}(\mathbf{w})\frac{1}{\mathbf{w}\cdot(\mathbf{k}-\mathbf{p})}G(\mathbf{w},\mathbf{k}-\mathbf{k}_0,\mathbf{p}-\mathbf{k}_0)\Big)\,\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}-\mathbf{x}_0)
\end{aligned}$$

$$G(\mathbf{w},\mathbf{k},\mathbf{p})=\Big[\mathcal{F}_\mathbf{x}a_0(-\mathbf{w},\mathbf{k}+\frac{\mathbf{w}}{2})-\mathcal{F}_\mathbf{x}a_0(-\mathbf{w},\mathbf{k}-\frac{\mathbf{w}}{2})\Big]\Big[\mathcal{F}_\mathbf{x}a_0(\mathbf{w},\mathbf{p}+\frac{\mathbf{w}}{2})-\mathcal{F}_\mathbf{x}a_0(\mathbf{w},\mathbf{p}-\frac{\mathbf{w}}{2})\Big].$$

$J_1^{1,0}$ is real-valued, and the principal value contribution vanishes when a_0 is even with respect to the variable $\mathbf{x}$.

Then we have the following result for the convergence of the scintillation:

Theorem 2.2 *Assume the initial condition ψ_η^0 has the form (9). Then under the assumptions and notations of theorem 2.1, we have, for $0<\alpha<1$,*

$$J_\eta=\eta^{(d+2)(1-\alpha)+(2\alpha-1)\vee 0}\,J_\alpha^1+\eta^{d(1-\alpha)+\alpha}\big(\big[\eta^{2\alpha-1}f_d(\eta)\big]\wedge 1\big)\,J_\alpha^2+r_\eta,$$

where $f_d = 1$ *when* $d \geq 3$ *and* $f_2 = 1 + |\log \eta^{\alpha-\beta}|$, *where* r_η *is negligible compared to the first two terms in the* $L^\infty((0,T), \mathcal{S}'(\mathbb{R}^{4d})) - *$ *topology, and where we have defined*

$$J_\eta = \eta^d J_0^2 + r_\eta \;\; \textit{when} \;\; \alpha = 0, \qquad \textit{and} \qquad J_\eta = \eta J_1^1 + r_\eta \;\; \textit{when} \;\; \alpha = 1.$$

Here, $J_\alpha^1 \in \mathcal{C}^0([0,T], Z')$ *when* $\alpha < 1$ *and* $J_1^1 \in \mathcal{C}^0([0,T], X_\infty)$ *and* $J_\alpha^2 \in \mathcal{C}^0([0,T], X_\infty)$ *are distributional solutions to the following 4-transport equations,*

$$\Big(\frac{\partial}{\partial t} + \mathcal{T}_2 + 2R_0 - \mathcal{Q}_2\Big) J_\alpha^i = S_\alpha^i, \qquad J_\alpha^i(t=0,\cdot) = J_\alpha^{i,0}. \tag{13}$$

The spaces Z' *and* X_∞ *are defined in section 3. For* $i = 1, 2$, *we have* $S_\alpha^i = 0$ *when* $\alpha > \frac{1}{2}$ *and* $J_\alpha^{i,0} = 0$ *when* $\alpha \leq \frac{1}{2}$, *and*

$$S_\alpha^i = j_\alpha^i \quad \textit{when} \quad 0 \leq \alpha \leq \frac{1}{2} \qquad \textit{and} \qquad J_\alpha^{i,0} = j_\alpha^i(0,\cdot) \quad \textit{when} \quad \frac{1}{2} < \alpha < 1.$$

Theorem 2.2 indicates how the statistical instabilities propagate. Depending on the value of α, either the first term or the second term dominates in the decomposition of J_η. When $d \geq 3$, the critical value of α is $\alpha^\star = \frac{2}{3}$: when $\alpha < \alpha^\star$, then the term involving J_α^2 is the leading one, while the term involving J_α^1 dominates when $\alpha > \alpha^\star$; when $\alpha = \alpha^\star$, both terms are of the same order. Both J_α^1 and J_α^2 satisfy a 4-transport equation. Depending on whether $\alpha \leq \frac{1}{2}$ or $\alpha > \frac{1}{2}$, the instabilities are created either by a source term or by an initial condition.

J_α^1 is the most singular term as the corresponding data in the transport equation are proportional to delta distributions both in space and momentum (when $\alpha < 1$) whereas the data corresponding to J_α^2 are more regular in the momentum variables. This should be related to the fact that J_α^1 is linear with respect to the power spectrum $\hat{R}$ while J_α^2 is proportional to $\hat{R}^2$ so that J_α^1 corresponds to the simple scattering contribution to the scintillation while J_α^2 corresponds to the double scattering and is therefore more regular. Moreover, when $\alpha < \alpha^\star$, the double scattering contribution gives the leading order, while it is given by the simple scattering when $\alpha > \alpha^\star$. It can also be noticed that higher order scattering terms are negligible in the limit. Let us now examine the different scenarios depending on the value of α.

Case $0 < \alpha \leq \frac{1}{2}$. The initial condition $a_{\eta 0}$ is more singular in the momentum variables than in the spatial variables, with comparable singularities when $\alpha = \frac{1}{2}$. The instabilities are created by the ballistic part of the wave through the source term j_α^2 supported at the spatial points $\mathbf{x} = \mathbf{y} = \mathbf{x}_0 - t\mathbf{k}_0$ with four configurations for the momentum $\mathbf{k}$ and $\mathbf{p}$: (i) $\mathbf{k} = \mathbf{p}$, the amplitude of $\mathbf{k}$ is given by $\sigma_{\frac{1}{2}}(0, \mathbf{k} - \mathbf{k}_0)$ when $\alpha < \frac{1}{2}$ and by $\sigma_{\frac{1}{2}}(t, \mathbf{p} - \mathbf{p}_0)$ when $\alpha = \frac{1}{2}$; (ii) $\mathbf{k} = \mathbf{k}_0$, the amplitude of $\mathbf{p}$ is given by $\sigma_{\frac{1}{2}}(0, \mathbf{p} - \mathbf{p}_0)$; (iii) $\mathbf{p} = \mathbf{p}_0$, the amplitude of $\mathbf{k}$ is given by $\sigma_{\frac{1}{2}}(0, \mathbf{k} - \mathbf{k}_0)$; (iv) $\mathbf{k} = \mathbf{p} = \mathbf{k}_0$. Instabilities are thus created along the wave propagation in the direction of the initial condition $\mathbf{k}_0$ but also in other directions.

Case $\frac{1}{2} < \alpha < 1$. The initial condition $a_{\eta 0}$ is more singular in the spatial variables than in the momentum variables. This results in a stronger localization of the instabilities. They are generated by an initial condition given by $j_\alpha^1(0,\cdot)$ when $\alpha > \alpha^\star$ and $j_\alpha^2(0,\cdot)$ when $\alpha < \alpha^\star$. When $\alpha < \alpha^\star$, instabilities are created at $\mathbf{x} = \mathbf{y} = \mathbf{x}_0$ with the same momentum configuration as the case $0 < \alpha \leq \frac{1}{2}$. When $\alpha > \alpha^\star$, instabilities are still created at $\mathbf{x} = \mathbf{y} = \mathbf{x}_0$ but with momentum $\mathbf{k} = \mathbf{p} = \mathbf{k}_0$. Note that these instabilities are fairly singular since they are defined in this case by gradients of delta distributions.

Case $\alpha = 1$. This the *most unstable* case since instabilities are of order η. Since in this configuration the initial condition $a_{\eta 0}$ is regular with respect to $\mathbf{k}$, instabilities are created at

$\mathbf{x} = \mathbf{y} = \mathbf{x}_0$ in all directions, which can be seen from the expression of $J_1^{1,0}$, which is more regular in the momentum variables than $J_\alpha^{1,0}$ for $\alpha < 1$.

Case $\alpha = 0$. This is the most stable case since instabilities are of order η^d. The initial condition is regular with respect to the spatial variables so that the source term j_0^2 is also regular. The situation is essentially the same as the case $0 < \alpha \le \frac{1}{2}$. The main difference is that the instabilities are created not only at the ballistic position at time t (that is at $\mathbf{x} = \mathbf{x}_0 - \mathbf{k}t$), but on a larger domain related to the spatial support of a_0.

Finally, when $d = 2$, the situation is similar: only the values of $\alpha^\star$ and σ_α change. Both theorems are proved in section 4. Section 3 concerns important preliminary results needed for the proof.

3 Functional spaces and preliminary results.

In this section, we introduce several functional spaces for the analysis of the operator $\mathcal{K}_\eta$ and of the 2-transport and 4-transport equations. We give some important estimates for $\mathcal{K}_\eta$ and present well-posedness results for the transport equations. The functional spaces are constructed to fulfill several requirements: first, the operator norm of $\mathcal{K}_\eta$ must be small with respect to $\eta \ll 1$ in a space for which the 4-transport equation is stable, so that from a bound on $\mathcal{K}_\eta$ we can deduce a bound on the scintillation function J_η; second, the spaces should be large enough so that $\mathcal{K}_\eta a_\eta \otimes a_\eta$ can be controlled by some norms of a_η well-adapted to Wigner transforms. For the first requirement, a prototype space is X_p introduced below, while for the second, the Y_p spaces are adapted. In particular, the Wigner transform of a η-uniformly L^2-bounded function is bounded in Y_∞ independently of η.

3.1 Functional spaces.

To analyze the 4-transport equation, we define X_p (for $1 \le p \le \infty$), and Z the spaces of tempered distributions h in $\mathcal{S}'(\mathbb{R}^{4d})$ such that

$$\begin{aligned}
\|h\|_{X_p}^p &= \sup_{\mathbf{v},\boldsymbol{\zeta}\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|^p d\mathbf{u} < \infty, \qquad 1 \le p < \infty \\
\|h\|_{X_\infty} &= \sup_{\mathbf{u},\boldsymbol{\zeta},\mathbf{v},\boldsymbol{\xi}\in\mathbb{R}^d} |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})| < \infty, \\
\|h\|_Z &= (2\pi)^{-4d} \int_{\mathbb{R}^{4d}} \omega(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})| d\boldsymbol{\xi} d\mathbf{u} d\mathbf{v} d\boldsymbol{\zeta} < \infty, \\
\omega(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= (1 + |\boldsymbol{\xi}| + |\boldsymbol{\xi}||\mathbf{u}| + |\mathbf{u}|^2)(1 + |\boldsymbol{\zeta}| + |\boldsymbol{\zeta}||\mathbf{v}| + |\mathbf{v}|^2).
\end{aligned}$$

Here $|\mathbf{u}|$ is the Euclidean norm of the vector $\mathbf{u}$. We denote by Z' the dual of Z. Above, we identified the Fourier transform of the distribution h with the function $\mathcal{F}h$. For the analysis of the 2-transport equation, we introduce spaces of tempered distributions defined by

$$\begin{aligned}
\|h\|_{Y_p}^p &= \int_{\mathbb{R}^d} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi})|^p d\mathbf{u} < \infty, \qquad 1 \le p < \infty \\
\|h\|_{Y_\infty} &= \sup_{\mathbf{u},\boldsymbol{\xi}\in\mathbb{R}^d} |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi})| < \infty, \\
\|h\|_Y &= \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi})| d\mathbf{u} < \infty, \qquad \|h\|_{\widetilde{Y}} = \sup_{\mathbf{u}\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}h(\mathbf{u},\boldsymbol{\xi})| d\boldsymbol{\xi} < \infty.
\end{aligned}$$

Note the inclusion $Y_1 \subset Y$. Using the fact the Lebesgue L^p spaces are Banach and that the Fourier transform is an isomorphism from $\mathcal{S}'$ to $\mathcal{S}'$, it can be easily seen that the above spaces are Banach.

3.2 Estimates for $\mathcal{K}_\eta$.

The latter spaces are well-adapted to the estimation of the scintillation operator $\mathcal{K}_\eta$. More precisely, we have the following result:

Lemma 3.1 *Assume that $\hat{R} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then for $1 \le p \le \infty$,*

(i) $\mathcal{K}_\eta$ is bounded in X_p and

$$\|\mathcal{K}_\eta\|_{\mathcal{L}(X_p)} \le 4\|\hat{R}\|_{L^1(\mathbb{R}^d)}. \tag{14}$$

(ii) Let $\mu \in Y_p$, $\nu \in Y$. Then

$$\|\mathcal{K}_\eta\, \mu \otimes \nu\|_{X_p} \quad \le \quad 4\,\eta^d\, \|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \|\mu\|_{Y_p} \|\nu\|_Y, \tag{15}$$

(iii) Let $\mu \in Y_\infty$, $\nabla_{\mathbf{x}}\mu \in Y_\infty$, $\nu \in Y$, $\nabla_{\mathbf{y}}\nu \in Y$. Then

$$\begin{aligned} \|\mathcal{K}_\eta\, \mu \otimes \nu\|_{Z'} \quad \le \quad & \eta^{d+2}\, \|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \left(\|\nabla_{\mathbf{x}}\mu\|_{Y_\infty} \|\nabla_{\mathbf{y}}\nu\|_Y + \|\nabla_{\mathbf{x}}\mu\|_{Y_\infty} \|\nu\|_Y \right. \\ & \left. + \|\mu\|_{Y_\infty} \|\nabla_{\mathbf{y}}\nu\|_Y + \|\mu\|_{Y_\infty} \|\nu\|_Y \right). \end{aligned} \tag{16}$$

Proof. With obvious notation, we recast $K_\eta = \sum_{\epsilon_i, \epsilon_j} \epsilon_i \epsilon_j K_\eta^{ij}$. Let $h \in X_p$. Then we have

$$\mathcal{F}\mathcal{K}_\eta^{ij}\, h \quad = \quad \int_{\mathbb{R}^d} e^{i\mathbf{w}\cdot(\frac{1}{2}\epsilon_i \boldsymbol{\xi} + \frac{1}{2}\epsilon_j \boldsymbol{\zeta})} \hat{R}(\mathbf{w}) \mathcal{F}h\left(\mathbf{u} - \frac{\mathbf{w}}{\eta}, \boldsymbol{\xi}, \mathbf{v} + \frac{\mathbf{w}}{\eta}, \boldsymbol{\zeta}\right) d\mathbf{w},$$

so that using the Hölder inequality with $1 = \frac{1}{p} + \frac{1}{p'}$ and $1 \le p < \infty$,

$$\begin{aligned} \|\mathcal{K}_\eta^{ij}\, h\|_{X_p}^p \le & \sup_{\mathbf{v}, \boldsymbol{\zeta} \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} |\hat{R}(\mathbf{w}) \mathcal{F}h\Big(\mathbf{u} - \frac{\mathbf{w}}{\eta}, \boldsymbol{\xi}, \mathbf{v} + \frac{\mathbf{w}}{\eta}, \boldsymbol{\zeta}\Big) | d\mathbf{w} \right|^p d\mathbf{u}, \\ \le \|\hat{R}\|_{L^1(\mathbb{R}^d)}^{\frac{p}{p'}} & \sup_{\mathbf{v}, \boldsymbol{\zeta} \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{R}(\mathbf{w})| \Big| \mathcal{F}h\Big(\mathbf{u} - \frac{\mathbf{w}}{\eta}, \boldsymbol{\xi}, \mathbf{v} + \frac{\mathbf{w}}{\eta}, \boldsymbol{\zeta}\Big) \Big|^p d\mathbf{w} d\mathbf{u} \le \|\hat{R}\|_{L^1(\mathbb{R}^d)}^p \|h\|_{X_p}^p. \end{aligned}$$

The case $p = \infty$ is addressed similarly. This proves (i). Let now $h := \mu \otimes \nu$. Upon performing the change of variables $\mathbf{w} \to \eta \mathbf{w}$, we have

$$\mathcal{F}\mathcal{K}_\eta^{ij}\, \mu \otimes \nu \quad = \quad \eta^d \int_{\mathbb{R}^d} e^{i\eta\mathbf{w}\cdot(\frac{1}{2}\epsilon_i \boldsymbol{\xi} + \frac{1}{2}\epsilon_j \boldsymbol{\zeta})} \hat{R}(\eta \mathbf{w}) \mathcal{F}\mu \otimes \nu\, (\mathbf{u} - \mathbf{w}, \boldsymbol{\xi}, \mathbf{v} + \mathbf{w}, \boldsymbol{\zeta})\, d\mathbf{w},$$

so that

$$\begin{aligned} \|\mathcal{K}_\eta^{ij}\, h\|_{X_p}^p & \le \eta^d \sup_{\mathbf{v}, \boldsymbol{\zeta} \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} |\hat{R}(\eta(\mathbf{w} - \mathbf{v})) \mathcal{F}\mu \otimes \nu\, (\mathbf{v} + \mathbf{u} - \mathbf{w}, \boldsymbol{\xi}, \mathbf{w}, \boldsymbol{\zeta})\, | d\mathbf{w} \right|^p d\mathbf{u}, \\ & \le \eta^d\, \|\hat{R}\|_{L^\infty(\mathbb{R}^d)}^p\, \|\nu\|_Y^{\frac{p}{p'}} \sup_{\mathbf{v}, \boldsymbol{\zeta} \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}\mu(\mathbf{v} + \mathbf{u} - \mathbf{w}, \boldsymbol{\xi})|^p |\mathcal{F}\nu\, (\mathbf{w}, \boldsymbol{\zeta})\, | d\mathbf{w} d\mathbf{u}, \\ & \le \eta^d\, \|\hat{R}\|_{L^\infty(\mathbb{R}^d)}^p\, \|\mu\|_{Y_p}^p\, \|\nu\|_Y^p, \end{aligned}$$

which proves (ii). To prove (iii), we sum $\epsilon_i \epsilon_j K_\eta^{ij}$ over i and j and combine the exponentials to find:

$$\mathcal{F}\mathcal{K}_\eta\, \mu \otimes \nu \quad = \quad -\eta^{d+2} \int_{\mathbb{R}^d} f^\eta(\mathbf{w}, \boldsymbol{\xi}, \boldsymbol{\zeta})\, (\mathbf{w} \cdot \boldsymbol{\xi})\, (\mathbf{w} \cdot \boldsymbol{\zeta})\, \mathcal{F}\mu \otimes \nu\, (\mathbf{u} - \mathbf{w}, \boldsymbol{\xi}, \mathbf{v} + \mathbf{w}, \boldsymbol{\zeta})\, d\mathbf{w},$$

where

$$f^\eta(\mathbf{w},\boldsymbol{\xi},\boldsymbol{\zeta}) = \frac{\sin\left(\frac{1}{2}\eta\,\mathbf{w}\cdot\boldsymbol{\xi}\right)\sin\left(\frac{1}{2}\eta\,\mathbf{w}\cdot\boldsymbol{\zeta}\right)}{\left(\frac{1}{2}\eta\,\mathbf{w}\cdot\boldsymbol{\xi}\right)\left(\frac{1}{2}\eta\,\mathbf{w}\cdot\boldsymbol{\zeta}\right)}\hat{R}(\eta\mathbf{w}).$$

We then decompose the product $(\mathbf{w}\cdot\boldsymbol{\xi})\,(\mathbf{w}\cdot\boldsymbol{\zeta})$ into fours terms:

$$(\mathbf{w}-\mathbf{u})\cdot\boldsymbol{\xi}\;(\mathbf{w}+\mathbf{v})\cdot\boldsymbol{\zeta}-(\mathbf{w}-\mathbf{u})\cdot\boldsymbol{\xi}\;\mathbf{v}\cdot\boldsymbol{\zeta}+\mathbf{u}\cdot\boldsymbol{\xi}\;(\mathbf{w}+\mathbf{v})\cdot\boldsymbol{\zeta}-\mathbf{u}\cdot\boldsymbol{\xi}\;\mathbf{v}\cdot\boldsymbol{\zeta}.$$

Using this and the fact that $(\mathcal{F}\nabla_{\mathbf{x}}\mu)(\mathbf{u},\boldsymbol{\xi}) = i\mathbf{u}(\mathcal{F}\mu)(\mathbf{u},\boldsymbol{\xi})$, we also decompose $\mathcal{F}\mathcal{K}_\eta\mu\otimes\nu$ accordingly into fours terms $\mathcal{F}\mathcal{K}_\eta^j\mu\otimes\nu$, $j=1,\cdots,4$ that read:

$$\begin{aligned}
\mathcal{F}\mathcal{K}_\eta^1\,\mu\otimes\nu &= -\eta^{d+2}\int_{\mathbb{R}^d} f^\eta(\mathbf{w},\boldsymbol{\xi},\boldsymbol{\zeta})\,\boldsymbol{\xi}\;(\mathcal{F}\nabla_{\mathbf{x}}\mu\otimes\nabla_{\mathbf{y}}\nu\,(\mathbf{u}-\mathbf{w},\boldsymbol{\xi},\mathbf{v}+\mathbf{w},\boldsymbol{\zeta}))\;\boldsymbol{\zeta}\,d\mathbf{w},\\
\mathcal{F}\mathcal{K}_\eta^2\,\mu\otimes\nu &= -i\eta^{d+2}\int_{\mathbb{R}^d} f^\eta(\mathbf{w},\boldsymbol{\xi},\boldsymbol{\zeta})\,\boldsymbol{\xi}\cdot\mathcal{F}\nabla_{\mathbf{x}}\mu\otimes\nu\,(\mathbf{u}-\mathbf{w},\boldsymbol{\xi},\mathbf{v}+\mathbf{w},\boldsymbol{\zeta})\;\mathbf{v}\cdot\boldsymbol{\zeta}\,d\mathbf{w},\\
\mathcal{F}\mathcal{K}_\eta^3\,\mu\otimes\nu &= i\eta^{d+2}\int_{\mathbb{R}^d} f^\eta(\mathbf{w},\boldsymbol{\xi},\boldsymbol{\zeta})\,\mathbf{u}\cdot\boldsymbol{\xi}\,\mathcal{F}\mu\otimes\nabla_{\mathbf{y}}\nu\,(\mathbf{u}-\mathbf{w},\boldsymbol{\xi},\mathbf{v}+\mathbf{w},\boldsymbol{\zeta})\cdot\boldsymbol{\zeta}\,d\mathbf{w},\\
\mathcal{F}\mathcal{K}_\eta^4\,\mu\otimes\nu &= \eta^{d+2}\int_{\mathbb{R}^d} f^\eta(\mathbf{w},\boldsymbol{\xi},\boldsymbol{\zeta})\,\mathbf{u}\cdot\boldsymbol{\xi}\,\mathcal{F}\mu\otimes\nu\,(\mathbf{u}-\mathbf{w},\boldsymbol{\xi},\mathbf{v}+\mathbf{w},\boldsymbol{\zeta})\,\mathbf{v}\cdot\boldsymbol{\zeta}\,d\mathbf{w}.
\end{aligned}$$

The term $\mathcal{F}\nabla_{\mathbf{x}}\mu\otimes\nabla_{\mathbf{y}}\nu$ has to be understood as the matrix $(\mathcal{F}\partial_{x_i}\mu\,\mathcal{F}\partial_{y_j}\nu)_{i,j=1,\cdots,d}$ and $\mathcal{F}\nabla_{\mathbf{x}}\mu\otimes\nu$ as the vector $(\mathcal{F}\partial_{x_i}\mu\,\mathcal{F}\nu)_{i=1,\cdots,d}$. We start to estimate the first term $\mathcal{F}\mathcal{K}_\eta^1\,\mu\otimes\nu$. To simplify the notation, we introduce the matrix-valued function $A(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) := \int_{\mathbb{R}^d} f^\eta(\mathbf{w},\boldsymbol{\xi},\boldsymbol{\zeta})\mathcal{F}\nabla_{\mathbf{x}}\mu\otimes\nabla_{\mathbf{y}}\nu\,(\mathbf{u}-\mathbf{w},\boldsymbol{\xi},\mathbf{v}+\mathbf{w},\boldsymbol{\zeta})\;d\mathbf{w}$. Let then φ be a test function in Z so that,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{4d}}\mathcal{K}_\eta^1\,\mu\otimes\nu\;\overline{\varphi}\;d\mathbf{x}d\mathbf{k}d\mathbf{y}d\mathbf{p}\right| = (2\pi)^{-4d}\left|\int_{\mathbb{R}^{4d}}\mathcal{F}\mathcal{K}_\eta^1\,\mu\otimes\nu\;\overline{\mathcal{F}\varphi},d\mathbf{u}d\boldsymbol{\xi}d\mathbf{v}d\boldsymbol{\zeta}\right|\\
= \;& (2\pi)^{-4d}\eta^{d+2}\left|\int_{\mathbb{R}^{4d}}d\mathbf{u}\,d\mathbf{v}\,d\boldsymbol{\xi}\,d\boldsymbol{\zeta}\;\;\boldsymbol{\xi}\,A(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})\,\boldsymbol{\zeta}\,\overline{\mathcal{F}\varphi}(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})\right|\\
\le \;& (2\pi)^{-4d}\eta^{d+2}\|A\|_{L^\infty(\mathbb{R}^{4d})}\int_{\mathbb{R}^{4d}}|\boldsymbol{\xi}||\boldsymbol{\zeta}||\mathcal{F}\varphi(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|d\mathbf{u}d\boldsymbol{\xi}d\mathbf{v}d\boldsymbol{\zeta}\;\;\le\;\;\eta^{d+2}\|A\|_{L^\infty(\mathbb{R}^{4d})}\|\varphi\|_Z.
\end{aligned}$$

Since $\|f\|_{L^\infty(\mathbb{R}^{3d})}\le\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}$, following exactly the same lines as (ii), we find

$$\|A\|_{L^\infty(\mathbb{R}^{4d})}\le\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\,\|\nabla_{\mathbf{y}}\mu\|_{Y_\infty}\,\|\nabla_{\mathbf{x}}\nu\|_Y.$$

Proceeding in the same way for $\mathcal{K}_\eta^i$, $i=2,3,4$ and using the definition of the space Z to control the different weights involving $\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}$, (16) then follows by duality. □

Remark 3.2 *In items (ii) and (iii) of lemma 3.1, the roles of μ and ν are symmetrical so that they can be interchanged in the above estimates. For instance, for $\mu\in Y$ and $\nu\in Y_p$, we have*

$$\|\mathcal{K}_\eta\,\mu\otimes\nu\|_{X_p}\quad\le\quad 4\,\eta^d\,\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|\nu\|_{Y_p}\|\mu\|_Y.$$

Item (ii) of lemma 3.1 states that when a_η is regular enough, say $a_\eta\in Y_p\cap Y$ with norms not too singular in η, then $\mathcal{K}_\eta a_\eta\otimes a_\eta$ tends to zero in X_p for instance. When the transport equation (7) is well-posed in X_p, this implies that J_η goes to zero as well and therefore we obtain statistical stability. Item (iii) provides us with an optimal rate of convergence needed to capture the behavior of the first-order corrector.

3.3 Well-posedness of the 4-transport equation.

In this section, we show that the 4-transport equation

$$\Big(\frac{\partial}{\partial t} + \mathcal{T}_2 + 2R_0 - \mathcal{Q}_2 - \mathcal{K}_\eta\Big) a = S, \qquad a(t=0,\cdot) = a_0, \tag{17}$$

is well-posed in the X_p spaces and prove related stability estimates. Here, $R_0 := (2\pi)^d R(\mathbf{0})$, $\hat{R} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, where R is the correlation function defined in (3), and $\mathcal{T}_2$, $\mathcal{Q}_2$ and $\mathcal{K}_\eta$ are defined in (8). We show that when the operator $\mathcal{K}_\eta$ vanishes, the equation is also stable (in the sense that the homogeneity in η is the same as for the data) in Z' while this is not the case when $\mathcal{K}_\eta$ is not zero. We first recast (17) as the integral equation

$$a(t) = e^{-2R_0 t}\mathcal{G}_t^2 a_0 + \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2[\mathcal{Q}_2 + \mathcal{K}_\eta]a(s)ds + S_1(t), \tag{18}$$

where $\mathcal{G}_t^2$ is the transport group defined as

$$\mathcal{G}_t^2 a(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) := a(\mathbf{x} - t\mathbf{p}, \mathbf{p}, \mathbf{y} - t\mathbf{q}, \mathbf{q}), \qquad t \in \mathbb{R},$$

and

$$S_1(t) = \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2 S(s)ds.$$

The existence and uniqueness of solutions to (18) is a consequence of the following lemma:

Lemma 3.3 *For $t \in \mathbb{R}_+$, $\mathcal{G}_t^2$ and $\mathcal{Q}_2$ are continuous in X_p and Z' with*

$$\|\mathcal{G}_t^2\|_{\mathcal{L}(X_p)} \leq 1, \qquad \|\mathcal{G}_t^2\|_{\mathcal{L}(Z')} \leq 4(1+t)^2, \qquad \|\mathcal{Q}_2\|_{\mathcal{L}(X_p)} \leq 2R_0, \qquad \|\mathcal{Q}_2\|_{\mathcal{L}(Z')} \leq 2R_0.$$

Proof. We have by Fourier transform:

$$\begin{aligned} \mathcal{F}\mathcal{G}_t^2\varphi &= \mathcal{F}\varphi(\mathbf{u}, \boldsymbol{\xi} + t\mathbf{u}, \mathbf{v}, \boldsymbol{\zeta} + t\mathbf{v}), \\ \mathcal{F}\mathcal{Q}_2\varphi &= (2\pi)^d(R(\boldsymbol{\xi}) + R(\boldsymbol{\zeta}))\mathcal{F}\varphi(\mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}), \end{aligned}$$

so that the continuity of $\mathcal{G}_t^2$ in X_p follows by simple inspection. The same holds for $\mathcal{Q}_2$ in X_p and Z' since $R_0 = (2\pi)^d R(\mathbf{0}) = (2\pi)^d \|R\|_{L^\infty(\mathbb{R}^d)}$, recalling that R is a correlation function. Regarding the continuity in Z' for $\mathcal{G}_t^2$, we have for any $\varphi \in Z$, $t \geq 0$,

$$\begin{aligned} \|\mathcal{G}_{-t}^2\varphi\|_Z &= (2\pi)^{-4d}\int_{\mathbb{R}^{4d}} (1 + |\mathbf{u}|^2 + |\mathbf{u}||\boldsymbol{\xi}| + |\boldsymbol{\xi}|)(1 + |\mathbf{v}|^2 + |\mathbf{v}||\boldsymbol{\zeta}| + |\boldsymbol{\zeta}|) \\ &\qquad |\mathcal{F}\varphi(\mathbf{u}, \boldsymbol{\xi} - t\mathbf{u}, \mathbf{v}, \boldsymbol{\zeta} - t\mathbf{v})| d\boldsymbol{\xi} d\mathbf{u} d\mathbf{v} d\boldsymbol{\zeta}, \\ &\leq \int_{\mathbb{R}^{4d}} (1 + t + (1+2t)|\mathbf{u}|^2 + |\mathbf{u}||\boldsymbol{\xi}| + |\boldsymbol{\xi}|)(1 + t + (1+2t)|\mathbf{v}|^2 + |\mathbf{v}||\boldsymbol{\zeta}| + |\boldsymbol{\zeta}|) \\ &\qquad |\mathcal{F}\varphi(\mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta})| d\boldsymbol{\xi} d\mathbf{u} d\mathbf{v} d\boldsymbol{\zeta}\,(2\pi)^{-4d} \quad \leq \quad 4(1+t)^2\|\varphi\|_Z, \end{aligned}$$

which yields by duality that $\|\mathcal{G}_t^2\|_{\mathcal{L}(Z')} \leq 4(1+t)^2$. ☐

We can now state the following corollary:

Corollary 3.4 *Assume that $a_0 \in X_p$ and $S_1 \in \mathcal{C}^0([0,T], X_p)$, for any $T > 0$ and $1 \leq p \leq \infty$. Then, (18) admits a unique solution in $\mathcal{C}^0([0,T], X_p)$ such that:*

$$\|a\|_{\mathcal{C}^0([0,T],X_p)} \leq \|a_0\|_{X_p} + e^{6R_0 T}\|S_1\|_{\mathcal{C}^0([0,T],X_p)}. \tag{19}$$

When $\mathcal{K}_\eta := 0$, then (17) has a unique solution in $\mathcal{C}^0([0,T], Z')$ such that

$$\|a\|_{\mathcal{C}^0([0,T],Z')} \leq 4(1+T)^2\|a_0\|_{Z'} + e^{8R_0(1+T)^2}\|S_1\|_{\mathcal{C}^0([0,T],Z')}. \tag{20}$$

Proof. According to item (i) of lemma 3.1, $\mathcal{K}_\eta$ is continuous in X_p so that using lemma 3.3, the operator

$$a \mapsto \int_0^t e^{-2R_0(t-s)}\mathcal{G}^2_{t-s}[\mathcal{Q}_2+\mathcal{K}_\eta]a(s)ds$$

is also continuous in $\mathcal{C}^0([0,T],X_p)$. Existence and uniqueness then follow from standard fixed point theorems while estimate (19) follows from the continuity of $\mathcal{G}^2_t$ when $S_1 := 0$. When $a_0 := 0$, (19) is an application of the Gronwall lemma, using the fact that $\|\mathcal{K}_\eta\|_{\mathcal{L}(X_p)} \leq 4\|\hat{R}\|_{L^1(\mathbb{R}^d)} = 4R_0$ since R is a correlation function.

The well-posedness of the 4-transport equation in Z' when $\mathcal{K}_\eta := 0$ and estimate (20) are also easy applications of lemma 3.3, fixed point theorems and the Gronwall lemma. ◻

3.4 Well-posedness of the 2-transport equation.

That section deals with the classical kinetic equation:

$$\begin{aligned} \frac{\partial a}{\partial t} + \mathbf{p}\cdot\nabla_{\mathbf{x}} a + R_0\, a &= \mathcal{Q}a + S, \qquad a(0,\mathbf{x},\mathbf{p}) = a_0(\mathbf{x},\mathbf{p}), \\ (\mathcal{Q}a)(t,\mathbf{x},\mathbf{p}) &= \int_{\mathbb{R}^d} \hat{R}(\mathbf{p}-\mathbf{p}')a(t,\mathbf{x},\mathbf{p}')d\mathbf{p}'. \end{aligned} \tag{21}$$

We show that (21) is well-posed in the spaces Y_p and $\widetilde{Y}$ and that the non-ballistic part of the solution is more regular than its ballistic counterpart. We obtain additional estimates that will be used to prove that the scintillation is dominated by the ballistic component of the wave. We have the following lemma:

Lemma 3.5 *Let $E = Y_p$ or $\widetilde{Y}$ and assume that $a_0 \in E$ and $S \in L^1((0,T),E)$ for any $T>0$ and $1 \leq p \leq \infty$. Then (21) admits a unique solution in $\mathcal{C}^0([0,T],E)$ such that*

$$\|a\|_{\mathcal{C}^0([0,T],E)} \leq \|a_0\|_E + \|S\|_{L^1((0,T),E)}. \tag{22}$$

Let $S := 0$ and let $a^0(t,\mathbf{x},\mathbf{p}) := a_0(\mathbf{x}-t\mathbf{p},\mathbf{p})e^{-R_0 t}$ be the ballistic part of a. Then, assuming that $\mathcal{F}_{\mathbf{k}}a_0 \in L^1(\mathbb{R}^{2d})$, $a_0 \in Y_1 \cap Y_\infty$, we have the following estimates for all $t>0$:

$$\|(a-a^0)(t,\cdot)\|_Y \lesssim t^{1-d}\int_{\mathbb{R}^d} \sup_{\mathbf{v}\in\mathbb{R}^d} |\mathcal{F}a_0(\mathbf{v},\boldsymbol{\xi})|d\boldsymbol{\xi} \lesssim t^{1-d}\|\mathcal{F}_{\mathbf{k}}a_0\|_{L^1(\mathbb{R}^{2d})}, \tag{23}$$

$$\|(a-a^0)(t,\cdot)\|_Y \lesssim \big(\|a_0\|^{1/d}_{Y_\infty}\|a_0\|^{1-1/d}_{Y_1}\big) \wedge \big(t\,\|a_0\|_{Y_1}\big), \tag{24}$$

$$\|(a-a^0)(t,\cdot)\|_{\widetilde{Y}} \lesssim \|a_0\|_{Y_\infty}. \tag{25}$$

Proof. The proof is a direct application of the integral formulation of (21),

$$a(t) = e^{-R_0 t}\mathcal{G}_t a_0 + \int_0^t e^{-R_0(t-s)}\mathcal{G}_{t-s}\mathcal{Q}(a(s)+S(s))ds,$$

where $\mathcal{G}_t$ is the free transport semigroup given by

$$\mathcal{G}_t a(\mathbf{x},\mathbf{p}) := a(\mathbf{x}-t\mathbf{p},\mathbf{p}).$$

The operators $\mathcal{Q}$ and $\mathcal{G}_t$ are both continuous in E. Indeed, for $\varphi \in E$, we have:

$$\mathcal{F}\mathcal{G}_t\varphi = \mathcal{F}\varphi(\mathbf{u},\boldsymbol{\xi}+t\mathbf{u}) \quad \text{and} \quad \mathcal{F}\mathcal{Q}\varphi = R(\boldsymbol{\xi})\mathcal{F}\varphi(\mathbf{u},\boldsymbol{\xi}),$$

so that

$$\|\mathcal{G}_t\varphi\|_E \leq \|\varphi\|_E \qquad \text{and} \qquad \|\mathcal{Q}\varphi\|_E \leq \|R\|_{L^\infty(\mathbb{R}^d)}\|\varphi\|_E.$$

Existence and uniqueness as well as (22) are deduced as in lemma 3.4 from standard fixed point theorems and from separate applications of the maximum principle and the Gronwall lemma.

For $S = 0$, we have the following Neumann series expansion in terms of multiple scattering:

$$a^n(t) = \int_0^t e^{-R_0(t-s)}\mathcal{G}_{t-s}\mathcal{Q}a^{n-1}(s)ds,$$

with the ballistic part $a^0(t, \mathbf{x}, \mathbf{p}) := e^{-R_0 t}a_0(\mathbf{x} - t\mathbf{p}, \mathbf{p})$. By induction, we find the following expression for the Fourier transform of a^n:

$$\begin{aligned}\mathcal{F}a^n(t, \mathbf{u}, \boldsymbol{\xi}) \quad &= \quad e^{-R_0 t}\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} R(\boldsymbol{\xi} + (t - s_1)\mathbf{u}) \cdots \\ &\qquad R(\boldsymbol{\xi} + (s_{n-1} - s_n)\mathbf{u})\mathcal{F}a_0(\mathbf{u}, \boldsymbol{\xi} + t\mathbf{u})ds_1 \cdots ds_n. \end{aligned} \tag{26}$$

The change of variable $\boldsymbol{\xi} + t\mathbf{u} \to \mathbf{k}$ yields

$$\begin{aligned}\|a^n(t, \cdot)\|_Y \quad &\leq \quad \frac{e^{-R_0 t}}{n!\, t^{d-n}}\|R\|^n_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \sup_{\mathbf{v}\in\mathbb{R}^d} |\mathcal{F}a_0(\mathbf{v}, \mathbf{k})|d\mathbf{k}, \\ &\leq \quad \frac{e^{-R_0 t}}{n!\, t^{d-n}}\|R\|^n_{L^\infty(\mathbb{R}^d)}\|\mathcal{F}_{\mathbf{k}}a_0\|_{L^1(\mathbb{R}^{2d})}.\end{aligned}$$

Summing over $n \geq 1$ gives (23). Regarding (24), we have from (26) and after the change of variable $t - s_1 \to s_1$:

$$|\mathcal{F}a^n(t, \mathbf{u}, \boldsymbol{\xi})| \leq \frac{e^{-R_0 t}}{(n-1)!}\|R\|^{n-1}_{L^\infty(\mathbb{R}^d)}|\mathcal{F}a_0(\mathbf{u}, \boldsymbol{\xi} + t\mathbf{u})| \int_0^t (t - s_1)^{n-1}R(\boldsymbol{\xi} + s_1\mathbf{u})ds_1. \tag{27}$$

In order to control the Y norm, we first need to integrate with respect to $\mathbf{u}$, either $\mathcal{F}a_0$ or R and to obtain a regularization effect, the natural choice is R. Therefore, for $0 < t_0 \leq s_1$, for a $t_0 \in \mathbb{R}^+$ be set latter, we use $R(\boldsymbol{\xi} + s_1\mathbf{u})$ after the change of variable $s_1\mathbf{u} \to \mathbf{u}$ and can thus control a_0 in the Y_∞ norm for which we expect uniform bounds when a_0 is a Wigner transform. When $0 \leq s \leq t_0$, we cannot use R since the time singularity is not integrable and have to control a_0 in Y_1 norm instead which is more singular. Splitting the integral for $s_1 \in [0, t_0]$ and $s_1 \in [t_0, t]$ then leads to

$$\begin{aligned}\|a^n(t, \cdot)\|_Y \quad &\leq \quad \frac{e^{-R_0 t}}{(n-1)!}\|R\|^{n-1}_{L^\infty(\mathbb{R}^d)} \Big((d-1)^{-1}t^{n-1}|t_0^{1-d} - t^{1-d}|\|R\|_{L^1(\mathbb{R}^d)}\, \|a_0\|_{Y_\infty} \\ &\qquad + t^{n-1}t_0\|R\|_{L^\infty(\mathbb{R}^d)}\|a_0\|_{Y_1}\Big).\end{aligned}$$

Setting $t_0 = (\|a_0\|^{1/d}_{Y_\infty}\|a_0\|^{-1/d}_{Y_1}) \wedge t$ and summing over $n \geq 1$ then gives

$$\|a(t, \cdot)\|_Y \lesssim \|a_0\|^{1/d}_{Y_\infty}\|a_0\|^{1-1/d}_{Y_1} + \left(\|a_0\|^{1/d}_{Y_\infty}\|a_0\|^{1-1/d}_{Y_1}\right) \wedge \left(t\, \|a_0\|_{Y_1}\right).$$

From (27), we also have $\|a(t, \cdot)\|_Y \lesssim t\, \|a_0\|_{Y_1}$ so that taking the best estimate between the last two ones gives (24). (25) is obtained by directly integrating $R(\boldsymbol{\xi} + s_1\mathbf{u})$ w.r.t. $\boldsymbol{\xi}$ in (27). □

4 Proof of the theorems.

In this section, we prove theorems 1 and 2. The rather long proof is split into several parts; we first the outline the main ideas of the proof.

4.1 Outline of the proof.

We start with the integral formulation of the 4-transport equation (7) in terms of the transport semi-group $(\mathcal{G}_t^2\varphi)(\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) = \varphi(\mathbf{x}-t\mathbf{k},\mathbf{k},\mathbf{y}-t\mathbf{q},\mathbf{q})$ and the scattering operator $\mathcal{Q}_2$. It reads

$$J_\eta(t) = \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2(\mathcal{Q}_2+\mathcal{K}_\eta)J_\eta(s)ds + \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2\mathcal{K}_\eta a_\eta\otimes a_\eta(s)ds. \tag{28}$$

Defining

$$T^{\mathcal{Q}}\varphi(t) := \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2\mathcal{Q}_2\varphi(s)ds \quad ; \quad T_\eta^{\mathcal{K}}\varphi(t) := \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2\mathcal{K}_\eta\varphi(s)ds,$$

$$T_{2\eta} := T^{\mathcal{Q}} + T_\eta^{\mathcal{K}} \qquad ; \qquad J_\eta^0(t) := \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2\mathcal{K}_\eta a_\eta\otimes a_\eta(s)ds,$$

we recast (28) as

$$J_\eta = T_{2\eta}J_\eta + J_\eta^0.$$

According to corollary 3.4, (28) admits a unique solution in X_p for $1\le p\le\infty$. As a consequence, the dynamics of J_η is basically driven by that of J_η^0. Depending on how singular the initial condition $a_{\eta 0}$ is in the variable η, the behavior of J_η as $\eta\to 0$ can be quite different. A first distinction is whether $\beta>0$ or $\beta=0$. By analogy with (9), the first case corresponds to initial conditions *localized* in the momentum variables while the second corresponds to *smooth* initial conditions in the momentum variables, regardless of the regularity with respect to the spatial variables. The second case is the easier to treat. The oscillatory term $T_\eta^{\mathcal{K}}J_\eta$ is negligible compared to the other terms in this configuration due to regularization effects and so J_η approximately solves

$$J_\eta \approx T^{\mathcal{Q}}J_\eta + J_\eta^0.$$

Since the dominant part of the source term J_η^0 converges in the X_∞ norm and the above equation is stable for the same norm, we can pass to the limit $\eta\to 0$ in the above equation after appropriately rescaling J_η.

When the initial condition is singular in momentum however, *i.e.*, when $\beta>0$, then J_η^0 does not converge in X_∞ but rather in the smaller Fourier weighted space Z'. We cannot pass to the limit directly in the equation since it is not stable in Z', the highly oscillating operator $\mathcal{K}_\eta$ having a norm of order η^{-1} in $\mathcal{L}(Z')$. The term $T_\eta^{\mathcal{K}}J_\eta$ is no longer negligible in some configurations. We are thus lead to studying the convergence of J_η by setting $J_\eta = J_\eta^0 + J_\eta^1$ with J_η^1 the solution to

$$J_\eta^1 = T_{2\eta}J_\eta^1 + T_{2\eta}J_\eta^0.$$

The convergence of J_η^0 can be characterized in Z' and is partly analyzed in section 4.2. The salient feature of the derivation is that, in most configurations, J_η^0 is dominated by its ballistic part, denoted by J_η^{00}, *i.e.*

$$J_\eta^0 \approx J_\eta^{00}.$$

To analyze J_η^1, we distinguish in the source term $T_{2\eta} J_\eta^0$ the smooth part $\mathcal{Q}_2$ from the oscillating part $\mathcal{K}_\eta$ by splitting J_η^1 as $J_\eta^1 = J_\eta^{1,\mathcal{Q}} + J_\eta^{1,\mathcal{K}}$ with

$$
\begin{aligned}
J_\eta^{1,\mathcal{Q}} &= T_{2\eta} J_\eta^{1,\mathcal{Q}} + T^{\mathcal{Q}} J_\eta^0, &(29)\\
J_\eta^{1,\mathcal{K}} &= T_{2\eta} J_\eta^{1,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^0. &(30)
\end{aligned}
$$

The limit of J_η^1 also depends on the singularities of the initial condition, which determine whether $J_\eta^{1,\mathcal{Q}}$ or $J_\eta^{1,\mathcal{K}}$ is the leading term. As long as the initial condition remains sufficiently singular in the momentum variables compared to the spatial variables, which is mathematically expressed by the relation $\beta > 2\alpha - 1$ when $d \geq 3$ (so that $\alpha < \alpha^\star = \frac{2}{3}$ when $\beta = 1 - \alpha$), the dominant term in J_η is given by $J_\eta^{1,\mathcal{K}}$, that is

$$J_\eta \approx J_\eta^{1,\mathcal{K}}, \text{ when } \beta > 2\alpha - 1,$$

the terms J_η^0 and $J_\eta^{1,\mathcal{Q}}$ being negligible. This is due to the fact that the initial condition is too singular in the momentum variables for a regularization effect of the singular source term $T_\eta^{\mathcal{K}} J_\eta^0$ of (30) to take place. This configuration gives rise to a limiting behavior dominated by the double scattering contribution. The main ideas remain the same in dimension $d = 2$. In the case $\beta < 2\alpha - 1$, the dominant term is $J_\eta^0 + J_\eta^{1,\mathcal{Q}}$, that is

$$J_\eta \approx J_\eta^0 + J_\eta^{1,\mathcal{Q}}, \text{ when } \beta < 2\alpha - 1.$$

Here, the initial condition is sufficiently smooth in the momentum variables to render $T_\eta^{\mathcal{K}} J_\eta^0$ negligible compared to J_η^0 and $J_\eta^{1,\mathcal{Q}}$. Such a configuration gives rise to a limiting behavior dominated by the single scattering contribution. When $\beta = 2\alpha - 1$, both dynamics are of the same order and coexist.

Knowing now which term between $J_\eta^{1,\mathcal{K}}$ and $J_\eta^0 + J_\eta^{1,\mathcal{Q}}$ is dominant, it remains to analyze their limit. A distinction is whether $\alpha > \beta$ or not, that is whether the initial condition is more singular in the spatial variables than in the momentum variables. When $\alpha \leq \beta$, the source of scintillation is given by a source term in the limiting equation for the rescaled J_η. When $\alpha > \beta$, it is given by an initial condition. All cases can be treated within similar frameworks.

Regarding the limit of $J_\eta^{1,\mathcal{K}}$, consider first the source term $T_\eta^{\mathcal{K}} J_\eta^0$ in equation (30): when the initial condition is singular in the spatial variables, *i.e.* $\alpha > 0$, we show that the dominant term in $T_\eta^{\mathcal{K}} J_\eta^0$ (which will be denoted by $T_\eta^{\mathcal{K}} J_\eta^{00}$) is induced by the ballistic part of a_η, so that $T_\eta^{\mathcal{K}} J_\eta^0$ can be replaced by $T_\eta^{\mathcal{K}} J_\eta^{00}$ for the X_∞ strong topology in the equation (30) solved by $J_\eta^{1,\mathcal{K}}$. This requires the analysis of a *double* application of the operator $\mathcal{K}_\eta$. When the initial condition is regular in the spatial variables, that is $\alpha = 0$, the ballistic and scattered parts in J_η^0 are of the same order so the full $T_\eta^{\mathcal{K}} J_\eta^0$ has to be considered. The analysis of the term $T_\eta^{\mathcal{K}} J_\eta^0$ is done in section 4.3. Regarding the operator term $T_{2\eta} J_\eta^{1,\mathcal{K}} := T^{\mathcal{Q}} J_\eta^{1,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{K}}$ in (30), we show that $T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{K}}$ is higher order in X_∞ so that the dominant term in $J_\eta^{1,\mathcal{K}}$ basically solves a 4-transport equation with $\mathcal{K}_\eta := 0$ and a source term $T_\eta^{\mathcal{K}} J_\eta^{00}$ (or $T_\eta^{\mathcal{K}} J_\eta^0$ for the particular case $\alpha = 0$), that is

$$J_\eta^{1,\mathcal{K}} \approx T^{\mathcal{Q}} J_\eta^{1,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^{00}.$$

It then suffices to compute the limit of the source term in X_∞ and pass to the limit in the equation. This is partly done in section 4.4.

Regarding the limit of $J_\eta^{1,\mathcal{Q}}$, consider first the source term $T^{\mathcal{Q}} J_\eta^0$ in (29): as J_η^0 will be seen to converge in Z', $T^{\mathcal{Q}} J_\eta^0$ converges in Z' and not in X_∞. It is then not directly possible to pass to the limit in (29) in Z' due to the presence of the operator $\mathcal{K}_\eta$ which is not bounded in $\mathcal{L}(Z')$. Nevertheless, writing $T_{2\eta} J_\eta^{1,\mathcal{Q}} := T^{\mathcal{Q}} J_\eta^{1,\mathcal{Q}} + T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}$, we take advantage of the regularizing properties of the $\mathcal{G}_t^2 \mathcal{Q}_2$ operator in the source term $T^{\mathcal{Q}} J_\eta^0$ to prove that $J_\eta^{1,\mathcal{Q}}$ has enough regularity so that $T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}$ is found to be of higher order and can thus be neglected. This step is not possible when considering the term J_η^0 without the regularization of $T^{\mathcal{Q}}$ as $T^{\mathcal{Q}} J_\eta^0$ and then $J_\eta^{1,\mathcal{Q}}$ would not be sufficiently regular. Hence, the operator $T_{2\eta}$ can be replaced by $T^{\mathcal{Q}}$ and $J_\eta^{1,\mathcal{Q}}$ is a morally a solution to

$$J_\eta^{1,\mathcal{Q}} \approx T^{\mathcal{Q}} J_\eta^{1,\mathcal{Q}} + T^{\mathcal{Q}} J_\eta^0,$$

which is stable in Z' as proved in lemma 3.4 so that we can pass to the limit in the equation. The term $J_\eta^{1,\mathcal{Q}}$ is studied in section 4.5.

In sections 4.6 and 4.7, we give the proofs of theorems 1 and 2. One of the main mathematical tools used in the analysis is the dispersive properties of the transport semi-group $\mathcal{G}_t \varphi(\mathbf{x}, \mathbf{p}) := \varphi(\mathbf{x} - t\mathbf{p}, \mathbf{p})$. For instance, consider an initial condition of the form (9), applying $\mathcal{G}_t$ and Fourier transforming it gives $e^{-i(\mathbf{x}_0 \cdot \mathbf{u} + \mathbf{k}_0 \cdot (\boldsymbol{\xi} + t\mathbf{u}))} \mathcal{F} a_0(\eta^\alpha \mathbf{u}, \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u}))$. To control the Y_p or Y norms, the latter expression needs to be integrated in $\mathbf{u}$. When $t = 0$, this gives a homogeneity of order $\eta^{-\alpha d}$ without any possible refinement. When $t > 0$, that order is optimal as long as $\alpha \leq \frac{1}{2}$. When $\alpha > \frac{1}{2}$, the change of variable $\mathbf{u} = t^{-1}(\mathbf{z} - \boldsymbol{\xi})$ offers a control proportional to $t^{-d} \eta^{(\alpha-1)d}$, which becomes optimal as soon as $t > \eta^{2\alpha - 1}$.

First estimates for a_η. We give here some preliminary estimates for the solution a_η of the transport equation (5) with initial condition $a_{\eta 0}$.

Lemma 4.1 *Let a_η be the solution to (5) with initial condition $a_{\eta 0}$. Assume hypotheses **H** are satisfied and let*

$$F_\eta(t) := \eta^{p\alpha} \|\nabla_{\mathbf{x}}^p a_\eta(t)\|_{Y_\infty} + \eta^{\alpha d + p\alpha} \|\nabla_{\mathbf{x}}^p a_\eta(t)\|_{Y_1} + \eta^{\beta d + p\alpha} \|\nabla_{\mathbf{x}}^p a_\eta(t)\|_{\widetilde{Y}},$$

for $p = 0, 1$, with the convention that $\nabla_{\mathbf{x}}^0 a := a$. Then, for any $T > 0$,

$$\sup_{t \in [0,T]} F_\eta(t) \lesssim F_\eta(0) \lesssim 1. \tag{31}$$

Proof. The case $p = 0$ is a consequence of the definition of the different spaces, the stability of the transport equation proved in lemma 3.5 and of the fact that a_η is the Wigner transform of a regular L^2-bounded function ψ_η. According to (22), we have, for $E = Y_1$, Y_∞, $\widetilde{Y}$:

$$\sup_{t \in [0,T]} \|a_\eta(t)\|_E \leq \|a_\eta(0)\|_E,$$

so that it remains to control $\|a_\eta(0)\|_E$. We have

$$a_\eta(0, \mathbf{x}, \mathbf{k}) = \mathbb{E}\left[\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{y}} \psi_\eta(0, \mathbf{x} - \frac{\eta}{2}\mathbf{y}, \omega) \overline{\psi_\eta}(0, \mathbf{x} + \frac{\eta}{2}\mathbf{y}, \omega) d\mathbf{y}\right],$$

with $\|\psi_\eta(0)\|^2_{L^2(\mathbb{R}^d \times \Omega, d\mathbf{x} \times \mathbb{P})} = \mathbb{E}\left[\int_{\mathbb{R}^d} |\psi_\eta(0, \mathbf{x}, \omega)|^2 d\mathbf{x}\right] \leq C$, with C bounded independently of η. Applying Fubini, we find, for the Fourier transform of a_η,

$$\mathcal{F} a_\eta(0, \mathbf{u}, \boldsymbol{\xi}) = e^{-i\eta \frac{1}{2} \mathbf{u}\cdot\boldsymbol{\xi}}\, \mathbb{E}\left[\int_{\mathbb{R}^d} \mathcal{F}\psi_\eta(t, \mathbf{u} - \mathbf{v}, \omega) \overline{\mathcal{F}\psi_\eta}(t, \mathbf{v}, \omega) d\mathbf{v}\right],$$

so that the Fourier-Plancherel equality yields

$$\|a_\eta(0)\|_{Y_\infty} \lesssim \|\psi_\eta(0)\|^2_{L^2(\mathbb{R}^d\times\Omega, d\mathbf{x}\times\mathbb{P})} \le C,$$

which gives the bound in Y_∞. For the other estimates in Y_1 and $\widetilde{Y}$, we have directly, according to hypotheses **H**:

$$\begin{aligned}\|a_\eta(0)\|_{Y_1} &\le \|\mathcal{F}_{\mathbf{x}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim \eta^{-\alpha d},\\ \|a_\eta(0)\|_{\widetilde{Y}} &\le \|\mathcal{F}_{\mathbf{k}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim \eta^{-\beta d}.\end{aligned}$$

Regarding the case $p=1$, it suffices to notice that $\nabla_{\mathbf{x}} a_\eta$ satisfies the same transport equation as a_η but with an initial condition $\nabla_{\mathbf{x}} a_{\eta 0}$, so that following the same lines as above yields the result. □

4.2 The J^0_η term.

We recall that J^0_η and its ballistic part J^{00}_η read

$$J^0_\eta(t) = \int_0^t e^{-2R_0(t-s)} \mathcal{G}^2_{t-s} \mathcal{K}_\eta a_\eta \otimes a_\eta(s) ds, \qquad J^{00}_\eta(t) = \int_0^t e^{-2R_0(t-s)} \mathcal{G}^2_{t-s} \mathcal{K}_\eta a^0_\eta \otimes a^0_\eta(s) ds,$$

where $a^0_\eta(t,\mathbf{x},\mathbf{k}) = e^{-R_0 t} a_{\eta 0}(\mathbf{x}-t\mathbf{k},\mathbf{k})$ is the ballistic part of a_η and $a_{\eta 0}$ the initial condition. The main result of the section is the following:

Lemma 4.2 *Assume hypotheses **H** are verified. Then, J^0_η and J^{00}_η satisfy the estimates, for any $T>0$:*

$$\sup_{t\in[0,T]} \|J^0_\eta(t) - J^{00}_\eta(t)\|_{X_\infty} \lesssim \begin{cases} \eta^{d(1-\alpha)+\alpha+\frac{1}{d-1}(d(\alpha-\beta)-\alpha)\vee 0} & \text{when } d\ge 3,\\ \eta^{2(1-\alpha)+\alpha}(\eta^{\alpha-2\beta}|\log\eta|)\wedge 1 & \text{when } d=2,\end{cases} \tag{32}$$

$$\sup_{t\in[0,T]} \left(\|J^0_\eta(t)\|_{X_\infty} + \|J^{00}_\eta(t)\|_{X_\infty}\right) \lesssim \eta^{d(1-\alpha)+(\alpha-\beta)\vee 0}, \tag{33}$$

$$\sup_{t\in[0,T]} \|J^0_\eta(t) - J^{00}_\eta(t)\|_{Z'} \lesssim \begin{cases} \eta^{d(1-\alpha)+2-\alpha+\frac{1}{d-1}((d(\alpha-\beta)-\alpha)\vee 0} & \text{when } d\ge 3,\\ \eta^{2(1-\alpha)+2-\alpha}(\eta^{\alpha-2\beta}|\log\eta|)\wedge 1 & \text{when } d=2,\end{cases} \tag{34}$$

$$\sup_{t\in[0,T]} \left(\|J^0_\eta(t)\|_{Z'} + \|J^{00}_\eta(t)\|_{Z'}\right) \lesssim \eta^{(d+2)(1-\alpha)+(\alpha-\beta)\vee 0}, \tag{35}$$

Here, $a\wedge b = \min(a,b)$ and $a\vee b = \max(a,b)$.

Proof. We start with (32) and write

$$J^0_\eta(t) - J^{00}_\eta(t) = \int_0^t e^{-2R_0(t-s)} \mathcal{G}^2_{t-s} \mathcal{K}_\eta \left((a_\eta - a^0_\eta)\otimes a_\eta(s) + a^0_\eta \otimes (a_\eta - a^0_\eta)\right) ds. \tag{36}$$

We then apply item (ii) of lemma 3.1 with first $\mu = a^0_\eta$ and $\nu = a_\eta - a^0_\eta$, and secondly with $\nu = a_\eta$ and $\mu = a_\eta - a^0_\eta$ according to remark 3.2, to find, for any $s\le T$,

$$\|\mathcal{K}_\eta\left((a_\eta - a^0_\eta)\otimes a_\eta + a^0_\eta\otimes(a_\eta - a^0_\eta)\right)(s)\|_{X_\infty} \lesssim \eta^d \|(a_\eta - a^0_\eta)(s)\|_Y \left(\|a_\eta(s)\|_{Y_\infty} + \|a^0_\eta\|_{Y_\infty}\right).$$

From lemma 4.1, we know that $a_\eta(s)$ is uniformly bounded in Y_∞ with respect to η for all $s\le T$, and so does a^0_η. This leaves us with the norm of $a_\eta - a^0_\eta$ in Y. According to (24) of lemma 3.5, we have

$$\|(a_\eta - a^0_\eta)(s)\|_Y \lesssim (\|a_{\eta 0}\|^{1/d}_{Y_\infty} \|a_{\eta 0}\|^{1-1/d}_{Y_1}) \wedge (s\|a_{\eta 0}\|_{Y_1}) \lesssim \eta^{-d\alpha+\alpha} \wedge (s\eta^{-\alpha d}) \lesssim \eta^{-\alpha d}(\eta^\alpha \wedge s),$$

since $\|a_{\eta 0}\|_{Y_\infty} + \eta^{\alpha d}\|a_{\eta 0}\|_{Y_1}$ is uniformly bounded according to estimate (31). Consequently, for all $t \leq T$, owing the fact that the semi-group $\mathcal{G}^2_{t-s}$ is continuous in X_∞, see lemma 3.3,

$$\begin{aligned} \|J^0_\eta(t) - J^{00}_\eta(t)\|_{X_\infty} &\lesssim \eta^d \int_0^t e^{-2R_0(t-s)} \|(a_\eta - a^0_\eta)(s)\|_Y \left(\|a_\eta(s)\|_{Y\infty} + \|a^0_\eta\|_{Y\infty}\right) ds, \\ &\lesssim t\,\eta^{d(1-\alpha)+\alpha} + \eta^{d(1-\alpha)+2\alpha}. \end{aligned} \tag{37}$$

This proves (32) when $d(\alpha-\beta)-\alpha \leq 0$. The estimate is not optimal for the remaining cases. Therefore, instead of using (24) and to take advantage of the fact that the scattered part $a_\eta - a^0_\eta$ is smoother than the ballistic part a^0_η, we use (23) to estimate $a_\eta - a^0_\eta$ and split the time integral in the definition of $J^0_\eta(t) - J^{00}_\eta$ into contributions in $[0, t_0]$ and $[t_0, t]$. For short times, we proceed in the same manner as (37) and obtain the bound $t_0\,\eta^{d(1-\alpha)+\alpha} + \eta^{d(1-\alpha)+2\alpha}$. For times larger than t_0, we apply (23) and finally obtain, for all $t \leq T$ and $t \geq t_0$,

$$\begin{aligned} \|J^0_\eta(t) - J^{00}_\eta(t)\|_{X_\infty} &\lesssim \eta^{d(1-\alpha)+2\alpha} + t_0\eta^{d(1-\alpha)+\alpha} + \eta^d \|\mathcal{F}_{\mathbf{k}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \int_{t_0}^t s^{1-d} ds, \\ &\lesssim \eta^{d(1-\alpha)+2\alpha} + t_0\eta^{d(1-\alpha)+\alpha} + h_d(t_0)\eta^{d(1-\beta)} + \eta^{d(1-\beta)}, \end{aligned}$$

where $h_d(t_0) = t_0^{2-d}$ when $d \geq 3$ and $h_d(t_0) = |\log t_0|$ if $d = 2$. Recall that $\|\mathcal{F}_{\mathbf{k}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim \eta^{-\beta d}$ according to hypotheses **H**. Setting then $t_0 = \eta^{\frac{1}{d-1}(d(\alpha-\beta)-\alpha)}$ when $d \geq 3$ and $t_0 = \eta^{2(\alpha-\beta)-\alpha}|\log \eta|$ when $d = 2$ gives (32) when $t \geq t_0$. When $t < t_0$, we simply use (37). We proceed analogously to estimate J^{00}_η. We first have:

$$\|J^{00}_\eta(t)\|_{X_\infty} \leq \eta^d \int_0^t e^{-2R_0(t-s)} \|a^0_\eta(s)\|_Y \|a^0_\eta(s)\|_{Y_\infty} ds,$$

so that, since a^0_η is uniformly bounded in Y_∞ and $\forall s \leq T$,

$$\|a^0_\eta(s)\|_Y \leq \|a^0_\eta(s)\|_{Y_1} \lesssim \eta^{-\alpha d},$$

we find, for $t \leq T$,

$$\|J^{00}_\eta(t)\|_{X_\infty} \leq t\,\eta^{d(1-\alpha)}.$$

The latter estimate is used for the case $\alpha \leq \beta$; when $\beta < \alpha$ we need to split the time integral over s in $[0, t_0]$ and $[t_0, t]$. Using the fact that

$$\begin{aligned} \|a^0_\eta(s)\|_Y &\leq \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F} a_{\eta 0}(\mathbf{u}, \boldsymbol{\xi} + s\mathbf{u})| d\mathbf{u}, \\ &\leq s^{-d} \int_{\mathbb{R}^d} \sup_{\mathbf{z} \in \mathbb{R}^d} |\mathcal{F} a_{\eta 0}(\mathbf{z}, \mathbf{u})| d\mathbf{u} \leq s^{-d} \|\mathcal{F}_{\mathbf{k}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim s^{-d}\eta^{-\beta d}, \end{aligned}$$

we have for all $t \leq T$,

$$\|J^{00}_\eta(t)\|_{X_\infty} \lesssim t_0 \eta^{d(1-\alpha)} + t_0^{1-d}\eta^{d(1-\beta)} \lesssim \eta^{d(1-\alpha)+\alpha-\beta},$$

by setting $t_0 = \eta^{\alpha-\beta}$. This gives (33) for the J^{00}_η part. Regarding the J^0_η part, we simply remark that, for any $\alpha, \beta \geq 0$, $J^0_\eta - J^{00}_\eta$ is of order higher or equal than J^{00}_η. Thus (33) is proved.

The proof of (34) goes along the same lines as above, so that we just underline the differences. We start from (36) and use the stability of $\mathcal{G}^2_t$ in Z' proved in lemma 3.3. Moreover, item (iii) of lemma (3.1) gives

$$\begin{aligned} \|\mathcal{K}_\eta\,(a_\eta - a^0_\eta) \otimes a_\eta\|_{Z'} \lesssim \eta^{d+2} \big(&\|\nabla_{\mathbf{x}}(a_\eta - a^0_\eta)\|_Y \|\nabla_{\mathbf{x}} a_\eta\|_{Y_\infty} + \|\nabla_{\mathbf{x}}(a_\eta - a^0_\eta)\|_Y \|a_\eta\|_{Y_\infty} \\ &\|a_\eta - a^0_\eta\|_Y \|\nabla_{\mathbf{x}} a_\eta\|_{Y_\infty} + \|a_\eta - a^0_\eta\|_Y \|a_\eta\|_{Y_\infty} \big). \end{aligned}$$

The leading term in the latter expression is the first one since it involves two derivatives $\nabla_{\mathbf{x}}(a_\eta - a_\eta^0)$ and $\nabla_{\mathbf{x}} a_\eta$ and is thus expected to be at least a factor $\eta^{-\alpha}$ greater than the other terms (see lemma 4.1) which we subsequently neglect. We then proceed exactly as for (32) by splitting the time integration for short and long times. The function $\nabla_{\mathbf{x}} a_\eta$ is solution to the same transport equation as a_η with the initial condition replaced by $\nabla_{\mathbf{x}} a_{\eta 0}$, and so we can use (23) and (24) to find, for $s > 0$,

$$\begin{aligned}\|\nabla_{\mathbf{x}}(a_\eta - a_\eta^0)(s)\|_Y &\lesssim s^{1-d}\|\mathcal{F}_{\mathbf{k}}\nabla_{\mathbf{x}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim s^{1-d}\eta^{-d\beta-\alpha},\\ \|\nabla_{\mathbf{x}}(a_\eta - a_\eta^0)(s)\|_Y &\lesssim (\|\nabla_{\mathbf{x}} a_{\eta 0}\|_{Y_\infty}^{1/d}\|\nabla_{\mathbf{x}} a_{\eta 0}\|_{Y_1}^{1-1/d}) \wedge (s\,\|\nabla_{\mathbf{x}} a_{\eta 0}\|_{Y_1}) \lesssim \eta^{-d\alpha}(1\wedge[\eta^{-\alpha}s]),\end{aligned}$$

thanks to lemma 4.1 and hypotheses **H**. This finally yields, together with $\eta^\alpha\|\nabla_{\mathbf{x}} a_\eta(t)\|_{Y_\infty} \lesssim 1$ according to (31), for any $t \le T$,

$$\begin{aligned}\|J_\eta^0(t) - J_\eta^{00}(t)\|_{Z'} &\lesssim \eta^{d(1-\alpha)+2} + t_0\eta^{d(1-\alpha)+2-\alpha} + \eta^{d(1-\beta)+2(1-\alpha)}\int_{t_0}^t s^{1-d}ds,\\ &\lesssim \eta^{d(1-\alpha)+2} + t_0\eta^{d(1-\alpha)+2-\alpha} + (h_d(t_0)+1)\eta^{d(1-\beta)+2(1-\alpha)},\end{aligned}$$

where h_d is the same as before. Setting $t_0 = \eta^{\frac{1}{d-1}(d(\alpha-\beta)-\alpha)}$ when $d \ge 3$ and $t_0 = \eta^{2(\alpha-\beta)-\alpha}|\log\eta|$ when $d = 2$ gives (34). Regarding (35), using the estimates of lemma 4.1, we find

$$\begin{aligned}\|J_\eta^{00}(t)\|_{Z'} &\lesssim \eta^{d+2}\int_0^t \|\nabla_{\mathbf{x}} a_\eta^0(s)\|_Y\|\nabla_{\mathbf{x}} a_\eta^0(s)\|_{Y\infty}ds\\ &\lesssim t_0\eta^{(d+2)(1-\alpha)} + t_0^{1-d}\eta^{d(1-\beta)+2(1-\alpha)},\end{aligned}$$

and setting $t_0 = \eta^{\alpha-\beta}$ yields the estimate on J_η^{00} in (35). Regarding J_η^0, it suffices to notice that $J_\eta^0 = J_\eta^0 - J_\eta^{00} + J_\eta^{00}$ and that for any $\alpha, \beta \ge 0$, $J_\eta^0 - J_\eta^{00}$ is at best the same order as J_η^{00}. This concludes the proof of the lemma. □

4.3 The terms $T_\eta^{\mathcal{K}} J_\eta^0$.

We recall that $T_\eta^{\mathcal{K}} J_\eta^0$ reads

$$\begin{aligned}T_\eta^{\mathcal{K}} J_\eta^0(t) &:= \int_0^t e^{-2R_0(t-s)}\mathcal{G}_{t-s}^2\mathcal{K}_\eta J_\eta^0(s)ds,\\ &= \int_0^t\int_0^s e^{-2R_0(t-\tau)}\mathcal{G}_{t-s}^2\mathcal{K}_\eta\mathcal{G}_{s-\tau}^2\,\mathcal{K}_\eta a_\eta\otimes a_\eta(\tau)dsd\tau, \qquad (38)\end{aligned}$$

and involves a double application of the operator $\mathcal{K}_\eta$ that needs to be treated carefully in order to find optimal estimates. The Fourier transform of $T_\eta^{\mathcal{K}} J_\eta^0$ is given by

$$\begin{aligned}&(\mathcal{F}T_\eta^{\mathcal{K}} J_\eta^0)(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = \eta^d\int_0^t\int_0^s\int_{\mathbb{R}^{2d}} dsd\tau d\mathbf{w}d\mathbf{w}'\,e^{-2R_0(t-\tau)}\hat{R}(\mathbf{w}')\hat{R}(\eta\mathbf{w}) \qquad (39)\\ &\times g(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\mathcal{F}a_\eta\otimes a_\eta\big(\tau,\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}',\\ &\qquad\boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}',\mathbf{v}+\mathbf{w}+\eta^{-1}\mathbf{w}',\boldsymbol{\zeta}+(t-\tau)\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}'\big)\end{aligned}$$

with

$$\begin{aligned}&g(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') = 16\,\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2)\\ &\times\sin(\eta\mathbf{w}\cdot(\boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}')/2)\times\sin(\eta\mathbf{w}\cdot(\boldsymbol{\zeta}+(t-\tau)\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}')/2).\end{aligned}$$

The term $\mathcal{F}a_\eta \otimes a_\eta(\tau, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta})$ stands for $\mathcal{F}a_\eta(\tau, \mathbf{u}, \boldsymbol{\xi})\mathcal{F}a_\eta(\tau, \mathbf{v}, \boldsymbol{\zeta})$. The Fourier transform is obtained by using that

$$\begin{aligned}
(\mathcal{F}\mathcal{K}_\eta h)(\mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}) &= -4\eta^d \int_{\mathbb{R}^d} \sin\left(\frac{\eta \mathbf{w}\cdot\boldsymbol{\xi}}{2}\right) \sin\left(\frac{\eta \mathbf{w}\cdot\boldsymbol{\zeta}}{2}\right) \hat{R}(\eta\mathbf{w})\mathcal{F}h\left(\mathbf{u}-\mathbf{w}, \boldsymbol{\xi}, \mathbf{v}+\mathbf{w}, \boldsymbol{\zeta}\right) d\mathbf{w},\\
(\mathcal{F}\mathcal{G}_t^2 h)(\mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}) &= \mathcal{F}h(\mathbf{u}, \boldsymbol{\xi}+t\mathbf{u}, \mathbf{v}, \boldsymbol{\zeta}+t\mathbf{v}),
\end{aligned}$$

see the proof of lemma 3.1 for the first relation and that of lemma 3.3 for the second. Let us consider functions of the form

$$G_\eta(t) = \int_0^t \int_0^s e^{-2R_0(t-\tau)} \mathcal{G}_{t-s}^2 \mathcal{K}_\eta \mathcal{G}_{s-\tau}^2 \mathcal{K}_\eta b_\eta \otimes c_\eta(\tau) ds d\tau, \qquad t \le T,$$

for two functions b_η and c_η with c_η satisfying the estimate (31) of lemma 4.1. The following result will be used several times in the forthcoming sections:

Lemma 4.3 *Let b_η and c_η be two functions in $\mathcal{C}^0([0,T], Y_1 \cap Y_\infty \cap \widetilde{Y})$ with c_η satisfying* (31) *for $p = 0$. Then, for any $0 \le t \le \eta^\alpha$,*

$$\|G_\eta(t)\|_{X_\infty} \lesssim \left(\eta^{d+1} \sup_{s\in[0,T]} \|b_\eta(s)\|_{\widetilde{Y}}^{\frac{1}{d}} \sup_{s\in[0,T]} \|b_\eta(s)\|_Y^{1-\frac{1}{d}}\right) \vee \left(\eta^{2\alpha+2d(1-\alpha)} \sup_{z\in[0,T]} \|b_\eta(z)\|_{\widetilde{Y}}\right),$$

and for any $\eta^\alpha \le t_0 \le t \le T$, $\|G_\eta(t)\|_{X_\infty} \lesssim A(t) \wedge B(t,t_0) \wedge C(t)$, where

$$\begin{aligned}
A(t) &= \left(t\,\eta^{d+1-\alpha} \sup_{s\in[0,T]} \|b_\eta(s)\|_{\widetilde{Y}}^{\frac{1}{d}} \sup_{s\in[0,T]} \|b_\eta(s)\|_Y^{1-\frac{1}{d}}\right) \vee \left(t\,\eta^{\alpha+2d(1-\alpha)} \sup_{z\in[0,T]} \|b_\eta(z)\|_{\widetilde{Y}}\right),\\
B(t,t_0) &= A(t_0) + \eta^{d+1-\alpha} \int_{t_0}^t \int_0^1 \|b_\eta(s(1-\eta^{1-\alpha}\tau_1))\|_Y \, s ds d\tau_1\\
&\qquad + \eta^{d+1-\alpha}(h_d(t_0) \vee h_d(t)) \sup_{z\in[0,T]} \|b_\eta(z)\|_{\widetilde{Y}},\\
C(t) &= t^2\eta^d \sup_{z\in[0,T]} \|b_\eta(z)\|_Y,
\end{aligned}$$

where for $x > 0$, $h_d(x) = x^{2-d}$ when $d \ge 3$, $h_d(x) = |\log x|$ when $d = 2$, $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

Proof. For a given time $t_0 \in [\eta^\alpha, t]$, we split the integral over $[0,t]$ into the two parts $[0,t_0]$ and $[t_0,t]$ and denote by G_η^1 and G_η^2 the corresponding terms. When $t_0 = t$, we only need to treat G_η^1 since G_η^2 vanishes. This will give the $A(t)$ part of the lemma. When $t_0 \ne t$, we need to estimate G_η^2 as well and obtain the $B(t,t_0)$ part of the estimate. The $C(t)$ part is direct consequence of the continuity of $\mathcal{G}_t^2$ in X_∞.

First part: G_η^1. Starting from (39), we make the change of variables $\mathbf{w} = \mathbf{u} - \mathbf{w}_1 - \eta^{-1}\mathbf{w}'$ and $\tau = s - \tau_1\eta^\alpha$ to get:

$$\begin{aligned}
(\mathcal{F}G_\eta^1)(t, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}) &= \eta^{d+\alpha} \int_0^{t_0} \int_0^{\frac{s}{\eta^\alpha}} \int_{\mathbb{R}^{2d}} ds d\tau_1 d\mathbf{w}_1 d\mathbf{w}' \, e^{-2R_0(t-s+\eta^\alpha\tau_1)} \hat{R}(\mathbf{w}')\hat{R}(\eta(\mathbf{u}-\mathbf{w}_1)-\mathbf{w}')\\
&\times g(t, s, s-\eta^\alpha\tau_1, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}, \mathbf{u}-\mathbf{w}_1-\eta^{-1}\mathbf{w}', \mathbf{w}') \, \mathcal{F}b_\eta \otimes c_\eta\big(s-\eta^\alpha\tau_1, \mathbf{w}_1,\\
&\qquad \boldsymbol{\xi} + (t-s+\eta^\alpha\tau_1)\mathbf{u} - \eta^{\alpha-1}\tau\mathbf{w}', \mathbf{v}+\mathbf{u}-\mathbf{w}_1, \boldsymbol{\zeta} + (t-s+\eta^\alpha\tau_1)\mathbf{v} + \eta^{\alpha-1}\tau\mathbf{w}'\big)\\
&= \int_{\eta^\alpha}^{t_0} \int_0^{t_1} (\cdot)\, ds d\tau_1 + \int_{\eta^\alpha}^{t_0} \int_{t_1}^{s\eta^{-\alpha}} (\cdot)\, ds d\tau_1 + \int_0^{\eta^\alpha} \int_0^{s\eta^{-\alpha}} (\cdot)\, ds d\tau_1 \; := \; I + II + III.
\end{aligned}$$

for a constant $0 < t_1 \le 1$ to be fixed later. Since $|g| \le 16$ uniformly in all variables, we find

$$\sup_{[0,T]\times\mathbb{R}^{4d}} |I| \le 16\,\eta^{d+\alpha}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \int_{\eta^\alpha}^{t_0}\int_0^{t_1} \sup_{z\le T}\int_{\mathbb{R}^{2d}} ds_1 d\tau d\mathbf{w}_1 d\mathbf{w}'\,\hat{R}(\mathbf{w}')$$
$$\big|\mathcal{F}b_\eta\otimes c_\eta\big(z,\mathbf{w}_1,\boldsymbol{\xi}+(t-s+\eta^\alpha\tau_1)\mathbf{u}-\eta^{\alpha-1}\tau\mathbf{w}',$$
$$\mathbf{v}+\mathbf{u}-\mathbf{w}_1,\boldsymbol{\zeta}+(t-s+\eta^\alpha\tau_1)\mathbf{v}+\eta^{\alpha-1}\tau_1\mathbf{w}'\big)\big|$$

$$\le 16\,t_0\,t_1\eta^{d+\alpha}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|\hat{R}\|_{L^1(\mathbb{R}^d)} \sup_{z\in[0,T]}\|b_\eta(z)\|_Y \sup_{z\in[0,T]}\|c_\eta(z)\|_{Y_\infty} \lesssim t_0\,t_1\eta^{d+\alpha}\sup_{z\in[0,T]}\|b_\eta(z)\|_Y,$$

since c_η is uniformly bounded in Y_∞ as it satisfies lemma 4.1. Concerning II, we perform the change of variable $\mathbf{w}' = f(\mathbf{w}_1') := \eta^{1-\alpha}\tau_1^{-1}(-\mathbf{w}_1'+\boldsymbol{\xi}+(t-s+\eta^\alpha\tau_1)\mathbf{u})$ and $\mathbf{w}_1 = \mathbf{v}+\mathbf{u}-\mathbf{w}$ so that:

$$II(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = \eta^{d+\alpha+d(1-\alpha)}\int_{\eta^\alpha}^{t_0}\int_{t_1}^{s\eta^{-\alpha}}\int_{\mathbb{R}^{2d}} dsd\tau_1 d\mathbf{w}d\mathbf{w}_1'\,e^{-2R_0(t-s+\eta^\alpha\tau_1)}\tau_1^{-d}$$
$$\times\hat{R}(f(\mathbf{w}_1'))\hat{R}(\eta(\mathbf{w}-\mathbf{v}-f(\mathbf{w}_1'))g(t,s,s-\eta^\alpha\tau_1,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w}-\mathbf{v}-\eta^{-1}f(\mathbf{w}_1'),f(\mathbf{w}_1'))$$
$$\times\mathcal{F}b_\eta\otimes c_\eta\big(s-\eta^\alpha\tau_1,\mathbf{v}+\mathbf{u}-\mathbf{w},\mathbf{w}_1',\mathbf{w},\boldsymbol{\zeta}+\boldsymbol{\xi}+(t-s+\eta^\alpha\tau_1)(\mathbf{v}+\mathbf{u})-\mathbf{w}_1'\big).$$

We then find, since $s\eta^{-\alpha}\ge 1$ and $t_1\le 1$, $\int_{\eta^\alpha}^{t_0}\int_{t_1}^{s\eta^{-\alpha}} dsd\tau_1\tau_1^{-d} \le t_1^{1-d}t_0$, so that,

$$\sup_{[0,T]\times\mathbb{R}^{4d}} |II| \le 16\,\eta^{d+\alpha+d(1-\alpha)}\|\hat{R}\|^2_{L^\infty(\mathbb{R}^d)}\int_{\eta^\alpha}^{t_0}\int_{t_1}^{s\eta^{-\alpha}}\sup_{z\le T}\int_{\mathbb{R}^{2d}} dsd\tau_1 d\mathbf{w}d\mathbf{w}_1'\,\tau_1^{-d}$$
$$\big|\mathcal{F}b_\eta\otimes c_\eta\big(z,\mathbf{v}+\mathbf{u}-\mathbf{w},\mathbf{w}_1',\mathbf{w},\boldsymbol{\zeta}+\boldsymbol{\xi}+(t-s+\eta^\alpha\tau_1)(\mathbf{v}+\mathbf{u})-\mathbf{w}_1'\big)\big|$$

$$\lesssim t_1^{1-d}t_0\,\eta^{d+\alpha+d(1-\alpha)}\sup_{z\in[0,T]}\|b_\eta(z)\|_{\widetilde{Y}}\sup_{z\in[0,T]}\|c_\eta(z)\|_{Y_1} \lesssim t_1^{1-d}t_0\eta^{\alpha+2d(1-\alpha)}\sup_{z\in[0,T]}\|b_\eta(z)\|_{\widetilde{Y}},$$

since $\|c_\eta(z)\|_{Y_1}\lesssim \eta^{-\alpha d}$ for $z\le T$ by lemma 4.1. Setting

$$t_1 = \Big[\eta^{1-2\alpha}\Big(\sup_{z\in[0,T]}\|b_\eta(z)\|_{\widetilde{Y}}\Big)^{\frac{1}{d}}\Big(\sup_{z\in[0,T]}\|b_\eta(z)\|_Y\Big)^{-\frac{1}{d}}\Big]\wedge 1,$$

gives:

$$\sup_{[0,T]\times\mathbb{R}^{4d}}(|I|+|II|) \lesssim \Big(t_0\eta^{d+1-\alpha}\sup_{s\in[0,T]}\|b_\eta(s)\|_{\widetilde{Y}}^{\frac{1}{d}}\sup_{s\in[0,T]}\|b_\eta(s)\|_Y^{1-\frac{1}{d}}\Big)$$
$$\vee\Big(t_0\,\eta^{\alpha+2d(1-\alpha)}\sup_{z\in[0,T]}\|b_\eta(z)\|_{\widetilde{Y}}\Big). \tag{40}$$

It remains to treat III. After the change of variable $s=\eta^\alpha s_1$, we find

$$\eta^{-\alpha}III = \int_0^{t_1}\int_0^{s_1}(\cdot)\,ds_1d\tau_1 + \int_{t_1}^1\int_0^{t_1}(\cdot)\,ds_1d\tau_1 + \int_{t_1}^1\int_{t_1}^{s_1}(\cdot)\,ds_1d\tau_1, := III_1+III_2+III_3,$$

for the t_1 defined earlier. III_1 and III_2 are treated as I and III_3 as II. This yields

$$\sup_{[0,T]\times\mathbb{R}^{4d}}|III_1| \lesssim t_1^2\,\eta^{d+\alpha}\sup_{z\in[0,T]}\|b_\eta(z)\|_Y,$$
$$\sup_{[0,T]\times\mathbb{R}^{4d}}|III_2| \lesssim t_1\,\eta^{d+\alpha}\sup_{z\in[0,T]}\|b_\eta(z)\|_Y,$$
$$\sup_{[0,T]\times\mathbb{R}^{4d}}|III_3| \lesssim t_1^{1-d}\eta^{\alpha+2d(1-\alpha)}\sup_{z\in[0,T]}\|b_\eta(z)\|_{\widetilde{Y}}.$$

We then find the estimate:

$$\sup_{[0,T]\times\mathbb{R}^{4d}} |III| \lesssim \Big(\eta^{d+1} \sup_{s\in[0,T]} \|b_\eta(s)\|_{\widetilde{Y}}^{\frac{1}{d}} \sup_{s\in[0,T]} \|b_\eta(s)\|_Y^{1-\frac{1}{d}}\Big) \vee \Big(\eta^{2\alpha+2d(1-\alpha)} \sup_{z\in[0,T]} \|b_\eta(z)\|_{\widetilde{Y}}\Big). \quad (41)$$

Therefore, III is either negligible compared to $I+II$ or is of the same order when $\alpha=0$. We turn now to the second part.

Second part: G_η^2. Starting from a similar expression as (39), we make the change of variable $\tau = s(1-\eta^{1-\alpha}\tau_1)$ and find

$$\begin{aligned}(\mathcal{F}G_\eta^2)(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= \eta^{d+(1-\alpha)} \int_{t_0}^t \int_0^{\eta^{\alpha-1}} \int_{\mathbb{R}^{2d}} ds d\tau_1 d\mathbf{w} d\mathbf{w}'\, e^{-2R_0(t-s(1-\eta^{1-\alpha}\tau_1))} \\ &\times \hat{R}(\mathbf{w}')\hat{R}(\eta\mathbf{w}) s g(t,s,s(1-\eta^{1-\alpha}\tau_1),\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') \quad (42)\\ &\times \mathcal{F}b_\eta\otimes c_\eta\big(s(1-\eta^{1-\alpha}\tau_1),\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}',\boldsymbol{\xi}+(t-s(1-\eta^{1-\alpha}\tau_1))\mathbf{u}-\eta^{-\alpha}s\tau_1\mathbf{w}', \\ &\qquad \mathbf{v}+\mathbf{w}+\eta^{-1}\mathbf{w}',\boldsymbol{\zeta}+(t-s(1-\eta^{1-\alpha}\tau_1))\mathbf{v}+\eta^{-\alpha}s\tau_1\mathbf{w}'\big).\end{aligned}$$

We split the integral over τ_1 in $[0,1]$ and $[1,\eta^{\alpha-1}]$ and denote by I_1 and I_2 the associated terms. Regarding I_1, we make the change of variable $\mathbf{w} = -\mathbf{w}_1 + \mathbf{u} - \eta^{-1}\mathbf{w}'$. We then control c_η by its Y_∞ norm, b_η by its Y norm and integrate $\hat{R}$ with respect to $\mathbf{w}'$. This yields the following bound for I_1, with g uniformly bounded in all variables:

$$\begin{aligned}\sup_{[0,T]\times\mathbb{R}^{4d}} |I_1| &\lesssim \eta^{d+(1-\alpha)} \|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \|\hat{R}\|_{L^1(\mathbb{R}^d)} \sup_{z\in[0,T]} \|c_\eta(z)\|_{Y_\infty} \\ &\quad \times \int_{t_0}^t \int_0^1 \|b_\eta(s(1-\eta^{1-\alpha}\tau_1))\|_Y s ds d\tau_1, \\ &\lesssim \eta^{d+(1-\alpha)} \int_{t_0}^t \int_0^1 \|b_\eta(s(1-\eta^{1-\alpha}\tau_1))\|_Y s ds d\tau_1, \quad (43)\end{aligned}$$

since c_η is uniformly bounded in Y_∞. Regarding now I_2, we perform in (42) (with the second time integral replaced by $[1,\eta^{\alpha-1}]$) the change of variables $\mathbf{w} = \mathbf{w}_1 - \mathbf{v} - \eta^{-1}\mathbf{w}'$ and $\mathbf{w}' = \eta^\alpha (s\tau_1)^{-1}(-\mathbf{w}_1' + \boldsymbol{\xi} + (t-s(1-\eta^{1-\alpha}\tau_1))\mathbf{u})$. Controlling c_η by its Y_1 norm and b_η by its $\widetilde{Y}$ norm we find the estimate

$$\begin{aligned}\sup_{[0,T]\times\mathbb{R}^{4d}} |I_2| &\lesssim \eta^{d(1+\alpha)+(1-\alpha)} \|\hat{R}\|^2_{L^\infty(\mathbb{R}^d)} \sup_{z\in[0,T]} \|c_\eta(z)\|_{Y_1} \sup_{z\in[0,T]} \|b_\eta(z)\|_{\widetilde{Y}} \int_{t_0}^t s^{1-d} ds \int_1^\infty \frac{d\tau_1}{\tau_1^d} \\ &\lesssim (h_d(t_0)\vee h_d(t))\, \eta^{d+(1-\alpha)} \sup_{z\in[0,T]} \|b_\eta(z)\|_{\widetilde{Y}}, \quad (44)\end{aligned}$$

since $\sup_{z\in[0,T]} \|c_\eta(z)\|_{Y_1} \lesssim \eta^{-\alpha d}$. Above, $h_d(x)$ is the same as before.

Last part: the C term. Starting from the definition of G_η, using the continuity of $\mathcal{G}_t^2$, together with item (ii) of lemma 3.1, we obtain, uniformly in t:

$$\|G_\eta(t)\|_{X_\infty} \lesssim t^2\eta^d \sup_{z\in[0,T]} \|b_\eta(z)\|_Y \sup_{z\in[0,T]} \|c_\eta(z)\|_{Y_\infty} \le t^2\eta^d \sup_{z\in[0,T]} \|b_\eta(z)\|_Y.$$

Conclusion. Setting first $t_0 = t \geq \eta^\alpha$ so that G^2_η vanishes yields the $A(t)$ and $C(t)$ part of the result thanks to (40), (41) and the estimate above. The second part when $\eta^\alpha \leq t_0 < t \leq T$ is obtained by gathering (40), (41), (43) and (44). When $t \leq \eta^\alpha$, the estimate is obtained along the same lines as for (41). This ends the proof. □

We now state the main result of this section, which provides us with an estimate, which will be shown to be optimal for certain initial conditions, for $T^{\mathcal{K}}_\eta J^{00}_\eta$ and shows that the non-ballistic part is higher order.

Proposition 4.4 *We have the following estimates:*

$$\sup_{t\in[0,T]} \|T^{\mathcal{K}}_\eta J^{00}_\eta(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+1-\beta}\left(\left[\eta^{\alpha-\beta} f_d(\eta)\right]\wedge 1\right)\vee\left(\eta^{(d-1)(1-\alpha-\beta)+\alpha}\right), \tag{45}$$

$$\sup_{t\in[0,T]} \|T^{\mathcal{K}}_\eta (J^0_\eta - J^{00}_\eta)(t)\|_{X_\infty} \lesssim \begin{cases} \eta^{d(1-\alpha)+1+(\alpha-\frac{d}{d-1}\beta)\vee 0} & \textit{when } d\geq 4,\\ \eta^{3(1-\alpha)+1}(\eta^\alpha \vee (|\log\eta|\eta^{2\alpha-3\beta}))\wedge 1, & \textit{when } d=3,\\ \eta^{2(1-\alpha)+1}(\eta^\alpha(|\log\eta|\vee\eta^{-2\beta}))\wedge 1, & \textit{when } d=2,\end{cases} \tag{46}$$

Above, $f_d(\eta)=1$ when $d\geq 3$ and $f_2(\eta) = 1+|\log\eta^{\beta-\alpha}|$, $a\wedge b=\min(a,b)$ and $a\vee b = \max(a,b)$.

Proof. We first separate the ballistic part from the scattered part by writing

$$T^{\mathcal{K}}_\eta J^0_\eta = T^{\mathcal{K}}_\eta J^{00}_\eta + T^{\mathcal{K}}_\eta (J^0_\eta - J^{00}_\eta),$$

and estimate the ballistic part $T^{\mathcal{K}}_\eta J^{00}_\eta$.

The ballistic part. The expression of $T^{\mathcal{K}}_\eta J^{00}_\eta$ is given by (38) with a_η replaced by $a^0_\eta(t,\mathbf{x},\mathbf{k}) = e^{-R_0 t} a_{\eta 0}(\mathbf{x}-t\mathbf{k},\mathbf{k})$, where $a_{\eta 0}$ is the initial condition. The ballistic part a^0_η trivially satisfies estimate (31). In particular, we have

$$\sup_{t\in[0,T]}\left(\eta^{\alpha d}\|a^0_\eta(t)\|_{Y_1} + \eta^{\beta d}\|a^0_\eta(t)\|_{\widetilde{Y}}\right)\lesssim 1.$$

We now apply lemma 4.3 to the case $a^0_\eta = b_\eta = c_\eta$. Controlling the Y norm of a^0_η by its Y_1 norm, it comes from the first estimate of the lemma for $t\leq\eta^\alpha$:

$$\sup_{t\in[0,\eta^\alpha]} \|T^{\mathcal{K}}_\eta J^{00}_\eta(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+1-\beta+\alpha}\vee\eta^{2\alpha+2d(1-\alpha)-d\beta}. \tag{47}$$

For longer times $t\geq\eta^\alpha$, we use first the $A(t)$ part of the lemma. It comes, $\forall t\in[\eta^\alpha, T]$:

$$\|T^{\mathcal{K}}_\eta J^{00}_\eta(t)\|_{X_\infty} \quad\lesssim\quad t(\eta^{d(1-\alpha)+1-\beta})\vee(\eta^{2\alpha+2d(1-\alpha)-d\beta}). \tag{48}$$

That estimate is optimal $\beta\geq\alpha$, that is when the initial condition is more singular in the momentum variables than in the spatial variables. It is not optimal in the reverse setting when $\alpha>\beta$, for which we need to use the "B" term in lemma 4.3. For this, setting $\eta^\alpha\leq t_0\leq t$ and assuming $\alpha>\beta$, we control the different terms in B according to η to obtain the leading contribution. Since $A(t_0)$ has already been estimated before, the only remaining term is

$$\begin{aligned}
&\int_{t_0}^t\int_0^1 \|a^0_\eta(s(1-\eta^{1-\alpha}\tau_1))\|_Y\, sdsd\tau_1\\
=\;& \int_{t_0}^t\int_0^1 e^{-R_0 s(1-\eta^{1-\alpha}\tau_1)} sdsd\tau_1 \sup_{\boldsymbol{\xi}\in\mathbb{R}^d}\int_{\mathbb{R}^d}|a_{\eta 0}(\mathbf{u},\boldsymbol{\xi}+s(1-\eta^{1-\alpha}\tau_1)\mathbf{u})|d\mathbf{u},\\
\leq\;& \int_{t_0}^t s^{1-d}ds\int_0^1(1-\eta^{1-\alpha}\tau_1)^{-d}d\tau_1\int_{\mathbb{R}^d}\sup_{\mathbf{z}\in\mathbb{R}^d}|\mathcal{F}a_{\eta 0}(\mathbf{z},\mathbf{u})|d\mathbf{u},\\
\lesssim\;& h_d(t_0)\vee h_d(t)\int_{\mathbb{R}^{2d}}|\mathcal{F}_{\mathbf{k}}a_{\eta 0}(\mathbf{x},\mathbf{u})|d\mathbf{x}d\mathbf{u} \quad\lesssim\; h_d(t_0)\vee h_d(t)\,\eta^{-d\beta},
\end{aligned}$$

since $\|\mathcal{F}_{\mathbf{k}} a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})} \lesssim \eta^{-\beta d}$ according to the hypotheses **H**. Above, for $x>0$, $h_d(x) = x^{2-d}$ when $d \geq 3$ and $h_2(x) = |\log x|$. Let $t_0 = \eta^{\alpha-\beta}$. Then using (48) and the B part of lemma 4.3, we find

$$\begin{aligned}\sup_{t\in[t_0,T]} \|T_\eta^{\mathcal{K}} J_\eta^{00}(t)\|_{X_\infty} &\lesssim \left(t_0 \eta^{d(1-\alpha)+1-\beta}\right) \vee (t_0 \eta^{2\alpha+2d(1-\alpha)-d\beta}) + h_d(t_0)\eta^{d(1-\beta)+1-\alpha}, \\ &\lesssim (\eta^{d(1-\alpha)+1-\beta+\alpha-\beta} f_d(\eta)) \vee (\eta^{2\alpha+2d(1-\alpha)-d\beta+\alpha-\beta}). \qquad (49)\end{aligned}$$

Above, $f_d(\eta) = 1$ when $d \geq 3$ and $f_2(\eta) = 1 + |\log \eta^{\alpha-\beta}|$. Using (48), we finally verify that the contribution of times $\eta^\alpha \leq t \leq t_0$ is included in the previous cases. Selecting the best estimate between the latter, (47), (48) and (49) then ends the proof of estimate (45) for the ballistic part.

The non-ballistic part. The proof follows along the same lines as that for the ballistic part so that we simply underline the key differences in the analysis. We have, $\forall t \in [0,T]$:

$$\begin{aligned}T_\eta^{\mathcal{K}}(J_\eta^0 - J_\eta^{00})(t) &= \int_0^t \int_0^s e^{-2R_0(t-\tau)} \mathcal{G}_{t-s}^2 \mathcal{K}_\eta \mathcal{G}_{s-\tau}^2 \\ &\qquad \big((a_\eta - a_\eta^0)\otimes a_\eta + a_\eta^0 \otimes (a_\eta - a_\eta^0)\big)\, ds d\tau \;:=\; T_1 + T_2.\end{aligned}$$

Since a_η and a_η^0 have the same homogeneity in η in the various spaces needed to estimate T_1 and T_2 (i.e., Y_∞, Y and $\widetilde{Y}$), the two terms T_1 and T_2 are treated in the same manner and we consider only T_1. We use lemma 4.3 with $b_\eta = a_\eta - a_\eta^0$ and $c_\eta = a_\eta$. This requires us to estimate $a_\eta - a_\eta^0$ in $\widetilde{Y}$ and Y. From estimate (25), we have for all $t \leq T$,

$$\|(a_\eta - a_\eta^0)(t)\|_{\widetilde{Y}} \lesssim \|a_{\eta 0}\|_{Y_\infty} \lesssim 1,$$

and moreover $\|(a_\eta - a_\eta^0)(t)\|_Y \leq \|(a_\eta - a_\eta^0)(t)\|_{Y_1} \lesssim \eta^{-\alpha d}$ according to (31). Assume first that $t \geq \eta^\alpha$. Then, the A part of lemma 4.3 gives for such t's:

$$\sup_{\mathbb{R}^{4d}} |T_1(t)| \lesssim (\eta^{d(1-\alpha)+1} t) \vee (t\eta^{2\alpha+2d(1-\alpha)}) = \eta^{d(1-\alpha)+1} t. \qquad (50)$$

The above result is not optimal for all possible values of α and β. To refine it, we use the B part of lemma 4.3. Assume first that $d=2$ or $d=3$. Then, for $\eta^\alpha \leq t_0 \leq t$, (23) gives

$$\begin{aligned}&\int_{t_0}^t \int_0^1 \|(a_\eta - a_\eta^0)(s(1-\eta^{1-\alpha}\tau_1))\|_Y\, s ds d\tau_1 \\ \lesssim\; &\int_{t_0}^t s^{2-d} \int_0^1 (1-\eta^{1-\alpha}\tau_1)^{1-d} d\tau_1 \int_{\mathbb{R}^{2d}} |\mathcal{F}_{\mathbf{k}} a_{\eta 0}(\mathbf{x},\mathbf{u})| d\mathbf{x} d\mathbf{u} \quad\lesssim\quad [i_d(t_0) \vee i_d(t)]\eta^{-d\beta},\end{aligned}$$

where for $x>0$, $i_2(x) = x$ and $i_3(x) = |\log x|$. When $d=2$, we can use lemma 4.3 with $t_0 = \eta^{\alpha-2\beta}$ together with (50). This yields, $\forall t \in [\eta^\alpha, T]$,

$$\begin{aligned}\sup_{\mathbb{R}^{4d}} |T_1(t)| &\lesssim t_0 \eta^{2(1-\alpha)+1} + (i_2(t_0) \vee i_2(t))\eta^{2(1-\beta)+1-\alpha} + h_2(t_0)\eta^{3-\alpha} \\ &\lesssim \eta^{2(1-\alpha)+1}(\eta^{\alpha-2\beta} + |\log \eta|\eta^\alpha).\end{aligned}$$

The latter result gives, together with (50), estimate (46) for times $t \geq \eta^\alpha$ when $d=2$. When $d=3$, we choose $t_0 = \eta^\alpha$ and obtain with the B part of lemma 4.3, together with (50):

$$\sup_{\mathbb{R}^{4d}} |T_1(t)| \lesssim \eta^{3(1-\alpha)+1}(\eta^\alpha + |\log \eta|\eta^{2\alpha-3\beta}).$$

Along with (50), this proves (46) when $d = 3$ and $t \geq \eta^\alpha$. Consider now the case $d \geq 4$. Still using (23), we rather estimate $a_\eta - a_\eta^0$ in Y as, for $\eta^\alpha \leq t_0 \leq t$,

$$\begin{aligned}
&\int_{t_0}^t \int_0^1 \|(a_\eta - a_\eta^0)(s(1-\eta^{1-\alpha}\tau_1))\|_Y \, sdsd\tau_1 \\
\lesssim &\int_{t_0}^t s^{2-d} \int_0^1 (1-\eta^{1-\alpha}\tau_1)^{1-d} d\tau_1, \int_{\mathbb{R}^{2d}} |\mathcal{F}_{\mathbf{k}} a_{\eta 0}(\mathbf{x},\mathbf{u})| d\mathbf{x} d\mathbf{u} \quad \lesssim \quad (t-t_0)\, h_d(t_0)\, \eta^{-d\beta},
\end{aligned}$$

so that lemma 4.3, yields with (50) and $t_0 = \eta^{\alpha - \frac{d}{d-1}\beta}$

$$\sup_{\mathbb{R}^{4d}} |T_1(t)| \quad \lesssim \quad t_0 \eta^{d(1-\alpha)+1} + h_d(t_0)\eta^{d(1-\beta)+1-\alpha} + h_d(t_0)\eta^{d+1-\alpha} \lesssim \eta^{d(1-\alpha)+1+\alpha-\frac{d}{d-1}\beta}.$$

This proves (46) when $d \geq 4$ and $t \geq \eta^\alpha$. It remains to treat the times $t < \eta^\alpha$ and $\eta^\alpha \leq t \leq t_0$ for any dimension $d \geq 2$. In the latter case, we use (50) with $t \leq t_0$ for the different values of t_0 defined earlier when $d = 2, 3$ and $d \geq 4$. The obtained results are included in the previous case $t_0 \leq t$. When $t < \eta^\alpha$, we use lemma 4.3 and find a bound of order $\eta^{d(1-\alpha)+1+\alpha}$, which is higher order than the other terms. This concludes the proof. □

4.4 The term $J_\eta^{1,\mathcal{K}}$

We recall that $J_\eta^{1,\mathcal{K}} = T_{2\eta} J_\eta^{1,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^0$, so that its homogeneity in η is basically given by that of the source term $T_\eta^{\mathcal{K}} J_\eta^0$.

The ballistic part gives the leading order. Using the results of the preceding section on $T_\eta^{\mathcal{K}} J_\eta^0$, we first show that the leading order in $J_\eta^{1,\mathcal{K}}$ is given by that of $T_\eta^{\mathcal{K}} J_\eta^{00}$. Let $J_\eta^{1,\mathcal{K}} := J_\eta^{2,\mathcal{K}} + J_\eta^{3,\mathcal{K}}$, where $J_\eta^{2,\mathcal{K}}$ and $J_\eta^{3,\mathcal{K}}$ solve

$$J_\eta^{2,\mathcal{K}} = T_{2\eta} J_\eta^{2,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^{00} \qquad ; \qquad J_\eta^{3,\mathcal{K}} = T_{2\eta} J_\eta^{3,\mathcal{K}} + T_\eta^{\mathcal{K}} (J_\eta^0 - J_\eta^{00}).$$

In the sequel, A negligible compared to B in X means the norm of A in X verifies an estimate with higher degree in η than does B. We say they are of the same order when the degree of η is the same for both estimates. We have the following proposition:

Proposition 4.5 *$J_\eta^{3,\mathcal{K}}$ is negligible compared to $J_\eta^{2,\mathcal{K}}$ in $\mathcal{C}^0([0,T], X_\infty)$ when $\beta > 0$. When $\beta = 0$, $J_\eta^{3,\mathcal{K}}$ is of the same order for any $\alpha \in [0,1]$.*

Proof. When $\beta > 0$ and for any $d \geq 2$, the term $J_\eta^{3,\mathcal{K}}$ is negligible in X_∞ compared to $J_\eta^{2,\mathcal{K}}$ since the corresponding source term in the integral equation is higher order. Indeed, on the one hand the stability of the 4-transport equation in X_∞ expressed through (19) and estimate (46) yield

$$\sup_{t\in[0,T]} \|J_\eta^{3,\mathcal{K}}(t)\|_{X_\infty} \lesssim \begin{cases} \eta^{d(1-\alpha)+1+(\alpha-\frac{d}{d-1}\beta)\vee 0}, & \text{when } d \geq 4, \\ \eta^{3(1-\alpha)+1}(\eta^\alpha \vee (|\log\eta|\eta^{2\alpha-3\beta})) \wedge 1, & \text{when } d = 3, \\ \eta^{2(1-\alpha)+1}(\eta^\alpha(|\log\eta| \vee \eta^{-2\beta})) \wedge 1, & \text{when } d = 2. \end{cases} \tag{51}$$

On the other hand, (45) gives

$$\sup_{t\in[0,T]} \|J_\eta^{2,\mathcal{K}}(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+1-\beta} \left(\left[\eta^{\alpha-\beta} f_d(\eta)\right] \wedge 1 \right) \vee \left(\eta^{(d-1)(1-\alpha-\beta)+\alpha} \right), \tag{52}$$

with $f_d(x) = 1$ when $d \geq 3$ and $f_2(x) = 1 + |\log x^{\alpha-\beta}|$. It is enough to show that the order of $J_\eta^{3,\mathcal{K}}$ is higher than $\eta^{d(1-\alpha)+1-\beta}\left[\eta^{\alpha-\beta} f_d(\eta)\right] \wedge 1$. Assume therefore that the order of $J_\eta^{2,\mathcal{K}}$ is $\eta^{d(1-\alpha)+1-\beta}\left[\eta^{\alpha-\beta} f_d(\eta)\right] \wedge 1$. Under the hypotheses $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, let us compare the orders in η of $J_\eta^{2,\mathcal{K}}$ and $J_\eta^{3,\mathcal{K}}$. Assume first that $d \geq 4$. When $\alpha \leq \beta$, the order of $J_\eta^{2,\mathcal{K}}$ is $d(1-\alpha)+1-\beta$ and that of $J_\eta^{3,\mathcal{K}}$ is $d(1-\alpha)+1$ so that the order of $J_\eta^{3,\mathcal{K}}$ is always greater since $\beta > 0$. When $\beta \leq \alpha \leq \frac{d}{d-1}\beta$, the orders are $d(1-\alpha)+1+\alpha-2\beta$ and still $d(1-\alpha)+1$. $J_\eta^{3,\mathcal{K}}$ is thus negligible when $\alpha < 2\beta$ which is the case in this configuration since $\alpha \leq \frac{d}{d-1}\beta$, with $d \geq 4$. When $\frac{d}{d-1}\beta \leq \alpha$, the orders are $d(1-\alpha)+1+\alpha-2\beta$ and $d(1-\alpha)+1+\alpha-\frac{d}{d-1}\beta$ so that the order of $J_\eta^{3,\mathcal{K}}$ is always greater when $\beta > 0$ since $d \geq 3$. Assume now that $d = 3$. The case $\alpha \leq \beta$ is the same as for $d = 4$. Suppose that $\beta < \alpha \leq \frac{3}{2}\beta$. The orders are $3(1-\alpha)+1+\alpha-2\beta$ and still $3(1-\alpha)+1$. Since $\alpha \leq \frac{3}{2}\beta < 2\beta$, the ballistic part dominates. It remains the case $\alpha > \frac{3}{2}\beta$. As long as $\alpha < 2\beta$, we are in the same configuration as before, when $\alpha \geq 2\beta > 0$, the scattered part is of order $\eta^{3(1-\alpha)+1}(\eta^\alpha \vee (|\log \eta|\eta^{2\alpha-3\beta})))$ which is greater than $\eta^{3(1-\alpha)+1+\alpha-2\beta}$ when $\alpha \geq 2\beta > 0$. Assume now that $d = 2$. The case $\alpha \leq \beta$ is similar to the treatment above. When $\beta < \alpha \leq 2\beta$, $J_\eta^{2,\mathcal{K}}$ is of order $\eta^{2(1-\alpha)+1+\alpha-2\beta}|\log \eta|$ and $J_\eta^{3,\mathcal{K}}$ of order $\eta^{2(1-\alpha)+1}$. Since $\eta \ll \eta^{1+\alpha-2\beta}|\log \eta|$, $J_\eta^{2,\mathcal{K}}$ dominates. When $\alpha > 2\beta$, the ballistic part is of order $\eta^{2(1-\alpha)+1+\alpha-2\beta}|\log \eta| \gg \eta^{2(1-\alpha)+1+\alpha-2\beta}$ which is the order the scattered part.

In all cases, the contribution of $J_\eta^{3,\mathcal{K}}$ is negligible as soon as $\beta > 0$. When $\beta = 0$, a simple examination shows that $J_\eta^{3,\mathcal{K}}$ and $J_\eta^{2,\mathcal{K}}$ have the same order for any $d \geq 2$ ∎

The $T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}$ term is higher order. We show now that the term $T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}$ can be neglected when computing the limit. We first decompose $J_\eta^{2,\mathcal{K}}$ into $J_\eta^{4,\mathcal{K}} + J_\eta^{5,\mathcal{K}}$, where

$$J_\eta^{4,\mathcal{K}} = T^{\mathcal{Q}} J_\eta^{4,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^{00}, \tag{53}$$

$$J_\eta^{5,\mathcal{K}} = T^{\mathcal{Q}} J_\eta^{5,\mathcal{K}} + T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}. \tag{54}$$

We have the following proposition:

Proposition 4.6 *When $d \geq 3$, $J_\eta^{5,\mathcal{K}}$ is negligible in $\mathcal{C}^0([0,T], X_\infty)$ compared to $J_\eta^{4,\mathcal{K}}$ as soon as $\alpha + \beta \geq \frac{d-1}{d-2}$, or $\alpha + \beta < \frac{d-1}{d-2}$ with $\alpha < 1$ and $\beta < 1$. When $\alpha + \beta < \frac{d-1}{d-2}$ and $\alpha = 1$ or $\beta = 1$, both terms are of the same order. When $d = 2$, $J_\eta^{2,\mathcal{K}}$ and $J_\eta^{5,\mathcal{K}}$ are of the same order when $\beta = 1$ or when $\alpha = 1$ with $\beta < 1$. In all other cases, $J_\eta^{5,\mathcal{K}}$ can be neglected.*

Proof. The core of the proof is estimating $T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}$. To do so, we start by applying the operator $\mathcal{K}_\eta$ to the integral equation solved by $J_\eta^{2,\mathcal{K}}$. This yields

$$\mathcal{K}_\eta J_\eta^{2,\mathcal{K}} = \mathcal{K}_\eta T_{2\eta} J_\eta^{2,\mathcal{K}} + \mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{00} = \mathcal{K}_\eta T^{\mathcal{Q}} J_\eta^{2,\mathcal{K}} + \mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}} + \mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{00}. \tag{55}$$

First step: the term $\mathcal{K}_\eta T^{\mathcal{Q}} J_\eta^{2,\mathcal{K}}$. We will need the following lemma:

Lemma 4.7 *Let $h \in \mathcal{C}^0([0,T], X_\infty)$. Then, we have the estimate:*

$$\sup_{t\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{Q}} h(t)\|_{X_\infty} \lesssim \eta \sup_{s\in[0,T]} \|h(s)\|_{X_\infty}.$$

Proof. We have:

$$\begin{aligned}
\mathcal{K}_\eta T^{\mathcal{Q}} h &= \int_0^t e^{-2R_0(t-s)} \mathcal{K}_\eta \mathcal{G}_{t-s}^2 \mathcal{Q}_2 h(s) ds,\\
\mathcal{F}\mathcal{K}_\eta T^{\mathcal{Q}} h &= -4\eta^d \int_0^t \int_{\mathbb{R}^d} e^{-2R_0(t-s)} \sin\left(\frac{1}{2}\eta\, \mathbf{w}\cdot\boldsymbol{\xi}\right) \sin\left(\frac{1}{2}\eta\, \mathbf{w}\cdot\boldsymbol{\zeta}\right) \hat{R}(\eta \mathbf{w})\\
&\quad \times\left[R(\boldsymbol{\xi} + (t-s)(\mathbf{u}-\mathbf{w})) + R(\boldsymbol{\zeta} + (t-s)(\mathbf{v}+\mathbf{w}))\right]\\
&\quad \times \mathcal{F}h(s, \mathbf{u}-\mathbf{w}, \boldsymbol{\xi} + (t-s)(\mathbf{u}-\mathbf{w}), \mathbf{v}+\mathbf{w}, \boldsymbol{\zeta} + (t-s)(\mathbf{v}+\mathbf{w})) d\mathbf{w} ds := I + II.
\end{aligned}$$

The terms I and II are treated in the same way so we focus on I. We first split the integral in s on $[0, t-t_0]$ and $[t-t_0, t]$, where $0 \le t_0 \le t$, and denote the corresponding terms by I_1 and I_2. For I_1, we make the change of variable $\mathbf{w} = \mathbf{u} + (t-s)^{-1}(\boldsymbol{\xi} - \mathbf{w}_1)$ and obtain, uniformly for $t \in [t_0, T]$,

$$\begin{aligned}
\sup_{\mathbb{R}^{4d}} |I_1(t)| &\le 4\eta^d \, \|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \sup_{s\in[0,T]} \|h(s)\|_{X_\infty} \int_0^{t-t_0} \int_{\mathbb{R}^d} R(\mathbf{w}_1)(t-s)^{-d} d\mathbf{w}_1 ds, \\
&\lesssim \eta^d \, t_0^{1-d} \sup_{s\in[0,T]} \|h(s)\|_{X_\infty}.
\end{aligned}$$

To handle I_2, we cannot use the regularization of the operator $\mathcal{G}^2_{t-2}\mathcal{Q}_2$ and make the same change of variable since the singularity in time is not integrable in the vicinity of t. Rather, we make the change of variable $\mathbf{w} = \eta^{-1}\mathbf{w}_1$ and integrate $\hat{R}$ with respect to $\mathbf{w}_1$. Thus, $\forall t \in [t_0, T]$:

$$\sup_{\mathbb{R}^{4d}} |I_2(t)| \le 4\, t_0 \, \|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \|\hat{R}\|_{L^1(\mathbb{R}^d)} \sup_{s\in[0,T]} \|h(s)\|_{X_\infty}.$$

To conclude the proof, we set $t_0 = t$ when $t \le \eta$ so that I_1 vanishes and only I_2 remains. When $\eta < t$, we set $t_0 = \eta$ so that I_1 and I_2 have the same order. $\square$

We now apply the preceding lemma to $h = J^{2,\mathcal{K}}_\eta$. We find, using estimate (45),

$$\sup_{t\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{Q}} J^{2,\mathcal{K}}_\eta(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+2-\beta} \left(\left[\eta^{\alpha-\beta} f_d(\eta) \right] \wedge 1 \right) \vee \left(\eta^{(d-1)(1-\alpha-\beta)+\alpha} \right), \tag{56}$$

with $f_d(x) = 1$ when $d \ge 3$ and $f_2(x) = 1 + |\log x^{\alpha-\beta}|$.

Second step: the term $\mathcal{K}_\eta T^{\mathcal{K}}_\eta J^{00}_\eta$. We have the following lemma:

Lemma 4.8 *$\mathcal{K}_\eta T^{\mathcal{K}}_\eta J^{00}_\eta$ satisfies the estimate:*

$$\sup_{t\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{K}}_\eta J^{00}_\eta(t)\|_{X_\infty} \lesssim \begin{cases} \eta^{d(1-\alpha)+2(1-\beta)}, & \text{when } d \ge 3, \\ \eta^{2(1-\alpha)}(\eta^{2(1-\beta)}|\log \eta|) \wedge 1, & \text{when } d = 2. \end{cases}$$

Proof. We have, for any $0 < t_0 \le t$:

$$\mathcal{K}_\eta T^{\mathcal{K}}_\eta J^{00}_\eta(t) = \int_0^t e^{-2R_0(t-s)} \mathcal{K}_\eta \mathcal{G}^2_{t-s} \mathcal{K}_\eta J^{00}_\eta(s) ds = \int_0^{t-t_0} + \int_{t-t_0}^t := I + II.$$

For times s less than $t - t_0$, we are able to use dispersive properties of the operator $\mathcal{G}^2_t$. This cannot be done for times close to t because of a non-integrable singularity in time. To estimate the long time contribution, we rather use the continuity of $\mathcal{K}_\eta$ in X_∞ and the estimate (45) for $T^{\mathcal{K}}_\eta J^{00}_\eta$. Regarding I, we have:

$$\begin{aligned}
(\mathcal{F}I)(t, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}) = \eta^{2d} \int_0^{t-t_0} \int_0^s \int_{\mathbb{R}^{3d}} & ds d\tau d\mathbf{w} d\mathbf{w}' d\mathbf{w}'' \, e^{-2R_0(t-\tau)} \hat{R}(\eta\mathbf{w}'')\hat{R}(\mathbf{w}')\hat{R}(\eta\mathbf{w}) \\
& \times \tilde{g}(t, s, \tau, \mathbf{u} - \mathbf{w}'', \boldsymbol{\xi}, \mathbf{v} + \mathbf{w}'', \boldsymbol{\zeta}, \mathbf{w}, \mathbf{w}', \mathbf{w}'') \\
& \times \mathcal{F}a^0_\eta \otimes a^0_\eta\big(\tau, \mathbf{u} - \mathbf{w} - \eta^{-1}\mathbf{w}' - \mathbf{w}'', \boldsymbol{\xi} + (t-\tau)(\mathbf{u} - \mathbf{w}'') - \eta^{-1}(s-\tau)\mathbf{w}', \\
& \qquad \mathbf{v} + \mathbf{w} + \eta^{-1}\mathbf{w}' + \mathbf{w}'', \boldsymbol{\zeta} + (t-\tau)(\mathbf{v} + \mathbf{w}'') + \eta^{-1}(s-\tau)\mathbf{w}'\big),
\end{aligned}$$

where

$$\begin{aligned}\tilde{g}(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}',\mathbf{w}'') &= 64\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2)\\ &\times\sin(\eta\mathbf{w}\cdot(\boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}')/2)\\ &\times\sin(\eta\mathbf{w}\cdot(\boldsymbol{\zeta}+(t-\tau)\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}')/2)\\ &\times\sin(\mathbf{w}''\cdot\boldsymbol{\xi}/2)\sin(\mathbf{w}''\cdot\boldsymbol{\zeta}/2). \qquad (57)\end{aligned}$$

The latter expression is obtained by applying the operator $\mathcal{K}_\eta$ to (39) with a_η replaced by a^0_η. Using the fact that $(\mathcal{F}a^0_\eta)(\tau,\mathbf{u},\boldsymbol{\xi}) = e^{-R_0\tau}(\mathcal{F}a_{\eta 0})(\mathbf{u},\boldsymbol{\xi}+\tau\mathbf{u})$, we find

$$\begin{aligned}&\mathcal{F}a^0_\eta\otimes a^0_\eta\big(\tau,\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}'-\mathbf{w}'',\boldsymbol{\xi}+(t-\tau)(\mathbf{u}-\mathbf{w}'')-\eta^{-1}(s-\tau)\mathbf{w}',\\ &\qquad\mathbf{v}+\mathbf{w}+\eta^{-1}\mathbf{w}'+\mathbf{w}'',\boldsymbol{\zeta}+(t-\tau)(\mathbf{v}+\mathbf{w}'')+\eta^{-1}(s-\tau)\mathbf{w}'\big)\\ =\ & e^{-2R_0\tau}\mathcal{F}a_{\eta 0}\otimes a_{\eta 0}\big(\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}'-\mathbf{w}'',\boldsymbol{\xi}+t(\mathbf{u}-\mathbf{w}'')-\eta^{-1}s\mathbf{w}'-\tau\mathbf{w},\\ &\qquad\mathbf{v}+\mathbf{w}+\eta^{-1}\mathbf{w}'+\mathbf{w}'',\boldsymbol{\zeta}+t(\mathbf{v}+\mathbf{w}'')+\eta^{-1}s\mathbf{w}'+\tau\mathbf{w}\big).\end{aligned}$$

After the change of variable $\mathbf{w}=\mathbf{u}-\mathbf{w}_1-\eta^{-1}\mathbf{w}'-\mathbf{w}''$ and $\mathbf{w}''=(t-\tau)^{-1}(\mathbf{w}''_1-\boldsymbol{\zeta}-t\mathbf{v}-\eta^{-1}(s-\tau)\mathbf{w}'-\tau\mathbf{u}+\tau\mathbf{w}_1)$, we find since $\tilde{g}$ is uniformly bounded in all variables that $\forall t\in[t_0,T]$,

$$\begin{aligned}\sup_{\mathbb{R}^{4d}}|I(t)| &\le \eta^{2d}\|\hat{R}\|^2_{L^\infty}\int_0^{t-t_0}\int_0^s\int_{\mathbb{R}^{3d}} dsd\tau d\mathbf{w}_1 d\mathbf{w}' d\mathbf{w}''_1\,\hat{R}(\mathbf{w}')(t-\tau)^{-d}\\ &\qquad \sup_{\boldsymbol{\xi}\in\mathbb{R}^d}|\mathcal{F}a_{\eta 0}(\mathbf{w}_1,\boldsymbol{\xi})|\sup_{\mathbf{v}\in\mathbb{R}^d}|\mathcal{F}a_{\eta 0}(\mathbf{v},\mathbf{w}''_1)|.\end{aligned}$$

We have:

$$\int_0^{t-t_0}\int_0^s(t-\tau)^{-d}dsd\tau\lesssim\begin{cases} t_0^{2-d}, & \text{when } d\ge 3,\\ |\log t|+|\log t_0|, & \text{when } d=2.\end{cases}$$

Since $t_0\le t\le T$, the double integral above is controlled by a constant times $1+|\log t_0|$ when $d=2$. This finally yields, when $d\ge 3$, $\forall t\in[t_0,T]$,

$$\begin{aligned}\sup_{\mathbb{R}^{4d}}|I(t)| &\lesssim \eta^{2d}\,t_0^{2-d}\|\hat{R}\|^2_{L^\infty(\mathbb{R}^d)}\|\hat{R}\|_{L^1(\mathbb{R}^d)}\|a_{\eta 0}\|_{Y_1}\int_{\mathbb{R}^d}\sup_{\mathbf{v}\in\mathbb{R}^d}|\mathcal{F}a_{\eta 0}(\mathbf{v},\mathbf{w}'')|d\mathbf{w}'',\\ &\lesssim \eta^{d(2-\alpha-\beta)}\,t_0^{2-d},\end{aligned}$$

since $\int_{\mathbb{R}^d}\sup_{\mathbf{v}\in\mathbb{R}^d}|\mathcal{F}a_{\eta 0}(\mathbf{v},\mathbf{w}'')|d\mathbf{w}''\le\|\mathcal{F}_{\mathbf{k}}a_{\eta 0}\|_{L^1(\mathbb{R}^{2d})}\lesssim\eta^{-\beta d}$ and when $d=2$,

$$\sup_{\mathbb{R}^{4d}}|I(t)|\lesssim\eta^{(2-\alpha-\beta)}\,|\log t_0|.$$

Concerning II, we have

$$II(t)=\mathcal{K}_\eta\int_{t-t_0}^t e^{-2R_0(t-s)}\mathcal{G}^2_{t-s}\mathcal{K}_\eta J^{00}_\eta(s)ds,$$

so that the stability of $\mathcal{K}_\eta$ in X_∞ gives,

$$\sup_{\mathbb{R}^{4d}}|II(t)|\lesssim\left\|\int_{t-t_0}^t e^{-2R_0(t-s)}\mathcal{G}^2_{t-s}\mathcal{K}_\eta J^{00}_\eta(s)ds\right\|_{X_\infty}.$$

We now apply lemma 4.3 with $b_\eta=c_\eta=a^0_\eta$, (the first time integral $[0,t]$ needs to replaced by $[t-t_0,t]$ without any change in the analysis) and find, using the "C" estimate:

$$\sup_{\mathbb{R}^{4d}}|II(t)|\ \lesssim\ t_0^2\,\eta^d\sup_{s\in[0,T]}\|a^0_\eta(s)\|_Y\ \lesssim\ t_0^2\,\eta^{d(1-\alpha)}.$$

When $\beta < 1$, setting $t_0 = \eta^{1-\beta} \ll 1$ when $d \geq 3$ gives the estimate of the lemma. When $d = 2$, we set $t_0 = \eta^{1-\beta}\sqrt{|\log \eta|}$. When $\beta = 1$, we simply choose $t_0 = t$ so that the term I vanishes and the estimate stems from II. When $t \leq t_0$ for the previously defined t_0 for $d = 2$ and $d \geq 3$, we proceed as for II. This ends the proof of the lemma □

End of the proof of proposition (4.6). To estimate $\mathcal{K}_\eta J_\eta^{2,\mathcal{K}}$, we go back to (55), use the fact that the operators $\mathcal{K}_\eta$ and $\mathcal{G}_t^2$ are bounded in X_∞ according to lemma 3.3 to write, $\forall t \in [0, T]$,

$$\|\mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}(t)\|_{X_\infty} = \left\| \int_0^t e^{-2R_0(t-s)} \mathcal{K}_\eta \mathcal{G}_{t-s}^2 \mathcal{K}_\eta J_\eta^{2,\mathcal{K}}(s) ds \right\|_{X_\infty} \lesssim \int_0^t \|\mathcal{K}_\eta J_\eta^{2,\mathcal{K}}(s)\|_{X_\infty} ds, \quad (58)$$

so that, according to (55), $\forall t \in [0, T]$:

$$\|\mathcal{K}_\eta J_\eta^{2,\mathcal{K}}(t)\|_{X_\infty} \lesssim \sup_{s\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{Q}} J_\eta^{2,\mathcal{K}}(s)\|_{X_\infty} + \sup_{s\in[0,T]} \|\mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{00}(s)\|_{X_\infty} + \int_0^t \|\mathcal{K}_\eta J_\eta^{2,\mathcal{K}}(s)\|_{X_\infty} ds.$$

From (56) and lemma 4.8, we compare $\mathcal{K}_\eta T^{\mathcal{Q}} J_\eta^{2,\mathcal{K}}$ and $\mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{00}$ and find that the leading order is $j_d = \eta^{d(1-\alpha)+2-\beta}(\eta^{-\beta}) \vee \left(\eta^{(d-1)(1-\alpha-\beta)+\alpha}\right)$ when $d \geq 3$ and $j_2 = \eta^{2(1-\alpha)}(\eta^{2(1-\beta)}|\log \eta|) \wedge 1$ when $d = 2$, so that the Gronwall lemma finally yields

$$\sup_{t\in[0,T]} \|\mathcal{K}_\eta J_\eta^{2,\mathcal{K}}(t)\|_{X_\infty} \lesssim j_d. \quad (59)$$

The latter results allow us to control $T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}$ from the continuity of $\mathcal{G}_t^2$ in X_∞ since

$$\|T_\eta^{\mathcal{K}} J_\eta^{2,\mathcal{K}}(t)\|_{X_\infty} \lesssim \int_0^t \|\mathcal{K}_\eta J_\eta^{2,\mathcal{K}}(s)\|_{X_\infty} ds.$$

We know from (53) and lemma 3.4 that $J_\eta^{4,\mathcal{K}}$ satisfies the same estimate as $J_\eta^{2,\mathcal{K}}$. On the other hand, we get from (54) and (59) that $J_\eta^{5,\mathcal{K}}$ satisfies the estimate

$$\sup_{t\in[0,T]} \|J_\eta^{5,\mathcal{K}}(t)\|_{X_\infty} \lesssim j_d.$$

When $d \geq 3$, a direct inspection then shows that $J_\eta^{5,\mathcal{K}}$ is negligible compared to $J_\eta^{4,\mathcal{K}}$ as soon as $\alpha + \beta \geq \frac{d-1}{d-2}$, or $\alpha + \beta < \frac{d-1}{d-2}$ with $\alpha < 1$ and $\beta < 1$. When $\alpha + \beta < \frac{d-1}{d-2}$ and $\alpha = 1$ or $\beta = 1$, both terms are of the same order. When $d = 2$, $J_\eta^{2,\mathcal{K}}$ and $J_\eta^{5,\mathcal{K}}$ are of the same order when $\beta = 1$ or when $\alpha = 1$ with $\beta < 1$. In all other cases, $J_\eta^{5,\mathcal{K}}$ can be neglected. This ends the proof of the proposition. □

4.5 The $J_\eta^{1,\mathcal{Q}}$ term.

We recall that $J_\eta^{1,\mathcal{Q}}$ is solution to

$$J_\eta^{1,\mathcal{Q}} = T_{2\eta} J_\eta^{1,\mathcal{Q}} + T^{\mathcal{Q}} J_\eta^0. \quad (60)$$

The $T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}$ term is higher order. As for $J_\eta^{2,\mathcal{K}}$, this fact is of crucial importance when computing the limit of $J_\eta^{1,\mathcal{Q}}$ since this implies that $T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}$ can be neglected. Indeed, up to some renormalization factors, the source term $T^{\mathcal{Q}} J_\eta^0$ converges in the space Z'. This *does not* directly imply convergence of $J_\eta^{1,\mathcal{Q}}$ since the $\mathcal{L}(Z')$ norm of $\mathcal{K}_\eta$ is of order ε^{-1} and the equation becomes unstable in Z'. We thus need to decompose first $J_\eta^{1,\mathcal{Q}}$ in $J_\eta^{2,\mathcal{Q}} + J_\eta^{3,\mathcal{Q}}$, where

$$J_\eta^{2,\mathcal{Q}} = T^{\mathcal{Q}} J_\eta^{2,\mathcal{Q}} + T^{\mathcal{Q}} J_\eta^0, \tag{61}$$
$$J_\eta^{3,\mathcal{Q}} = T^{\mathcal{Q}} J_\eta^{3,\mathcal{Q}} + T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}. \tag{62}$$

The limit of $J_\eta^{2,\mathcal{Q}}$ can then be computed since the operator $\mathcal{K}_\eta$ is not involved in the equation. It thus remains to show that $J_\eta^{3,\mathcal{Q}}$ can be neglected in X_∞. This is the object of the next proposition:

Proposition 4.9 *$J_\eta^{3,\mathcal{Q}}$ is negligible compared to $J_\eta^{4,\mathcal{K}}$ in $\mathcal{C}^0([0,T], X_\infty)$ when $\beta > 0$ and of the same order when $\beta = 0$ when $d \geq 3$ or when $\alpha = \beta = 0$ when $d = 2$.*

Proof. We apply the operator $\mathcal{K}_\eta$ to (60) to find:

$$\mathcal{K}_\eta J_\eta^{1,\mathcal{Q}} = \mathcal{K}_\eta T_{2\eta} J_\eta^{1,\mathcal{Q}} + \mathcal{K}_\eta T^{\mathcal{Q}} J_\eta^0 = \mathcal{K}_\eta T^{\mathcal{Q}}(J_\eta^{1,\mathcal{Q}} + J_\eta^0) + \mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}. \tag{63}$$

We treat the first term $\mathcal{K}_\eta T^{\mathcal{Q}}(J_\eta^{1,\mathcal{Q}} + J_\eta^0)$ by applying lemma 4.7 to $h = J_\eta^{1,\mathcal{Q}} + J_\eta^0$ and thus need to estimate h in X_∞. From lemma 4.2 and estimate (33), we know that:

$$\sup_{t\in[0,T]} \|J_\eta^0(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+(\alpha-\beta)\vee 0}.$$

Since $\mathcal{G}_t^2$ and $\mathcal{Q}_2$ are continuous in X_∞, then so is the operator $T^{\mathcal{Q}}$ and we have

$$\sup_{t\in[0,T]} \|T^{\mathcal{Q}} J_\eta^0(t)\|_{X_\infty} \lesssim \sup_{t\in[0,T]} \|J_\eta^0(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+(\alpha-\beta)\vee 0}.$$

In the same way, since the 4-transport is well-posed in X_∞ (lemma 3.4), we have from (60),

$$\sup_{t\in[0,T]} \|J_\eta^{1,\mathcal{Q}}(t)\|_{X_\infty} \lesssim \sup_{t\in[0,T]} \|T^{\mathcal{Q}} J_\eta^0(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+(\alpha-\beta)\vee 0}.$$

Using the above estimates for $J_\eta^{1,\mathcal{Q}}$ and J_η^0 and lemma 4.7, we find

$$\sup_{t\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{Q}}(J_\eta^{1,\mathcal{Q}} + J_\eta^0)(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+1+(\alpha-\beta)\vee 0}.$$

It remains to treat the second term of (63): $\mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}$. For this, we use the fact that the operators $\mathcal{K}_\eta$ and $\mathcal{Q}_t^2$ are bounded in X_∞ and obtain, as in (58),

$$\|\mathcal{K}_\eta T_\eta^{\mathcal{K}} J_\eta^{1,\mathcal{Q}}(t)\|_{X_\infty} \lesssim \int_0^t \|\mathcal{K}_\eta J_\eta^{1,\mathcal{Q}}(s)\|_{X_\infty} ds.$$

Gathering the previous results and getting back to (63) gives, $\forall t \in [0,T]$,

$$\|\mathcal{K}_\eta J_\eta^{1,\mathcal{Q}}(t)\|_{X_\infty} \lesssim \sup_{s\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{Q}}(J_\eta^{1,\mathcal{Q}} + J_\eta^0)(s)\|_{X_\infty} + \int_0^t \|\mathcal{K}_\eta J_\eta^{1,\mathcal{Q}}(s)\|_{X_\infty} ds,$$

so that the Gronwall lemma yields

$$\sup_{t\in[0,T]} \|\mathcal{K}_\eta J_\eta^{1,\mathcal{Q}}(t)\|_{X_\infty} \lesssim \sup_{t\in[0,T]} \|\mathcal{K}_\eta T^{\mathcal{Q}}(J_\eta^{1,\mathcal{Q}} + J_\eta^0)(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+1+(\alpha-\beta)\vee 0}.$$

We finally deduce from the continuity of $\mathcal{G}^2_t$ in X_∞ that, $\forall t \in [0,T]$,

$$\|T^{\mathcal{K}}_\eta J^{1,\mathcal{Q}}_\eta(t)\|_{X_\infty} \lesssim \int_0^t \|\mathcal{K}_\eta J^{1,\mathcal{Q}}_\eta(s)\|_{X_\infty} ds$$

which implies the following estimate for $J^{3,\mathcal{Q}}_\eta$, together with the stability of (62) in X_∞,

$$\sup_{t\in[0,T]} \|J^{3,\mathcal{Q}}_\eta(t)\|_{X_\infty} \lesssim \eta^{d(1-\alpha)+1+(\alpha-\beta)\vee 0}. \tag{64}$$

We recall that $J^{4,\mathcal{K}}_\eta$ is of order $\eta^{d(1-\alpha)+1-\beta}\left(\left[\eta^{\alpha-\beta} f_d(\eta)\right] \wedge 1\right) \vee \left(\eta^{(d-1)(1-\alpha-\beta)+\alpha}\right)$. When $d \geq 3$, this is lower order than $J^{3,\mathcal{Q}}_\eta$ when $\beta > 0$ and of the same order when $\beta = 0$. When $d = 2$, $J^{3,\mathcal{Q}}_\eta$ is of the same order when $\alpha = \beta = 0$. This concludes the proof. □

The ballistic part gives the leading order. Since (61) does not involve the operator $T^{\mathcal{K}}_\eta$, we can use the stability of that equation in Z' to find the leading term in $J^{2,\mathcal{Q}}_\eta$. Hence, we decompose $J^{2,\mathcal{Q}}_\eta$ as $J^{2,\mathcal{Q}}_\eta = J^{4,\mathcal{Q}}_\eta + J^{5,\mathcal{Q}}_\eta$, where

$$J^{4,\mathcal{Q}}_\eta = T^{\mathcal{Q}} J^{4,\mathcal{Q}}_\eta + T^{\mathcal{Q}} J^{00}_\eta, \tag{65}$$

$$J^{5,\mathcal{Q}}_\eta = T^{\mathcal{Q}} J^{5,\mathcal{Q}}_\eta + T^{\mathcal{Q}} (J^0_\eta - J^{00}_\eta), \tag{66}$$

where J^{00}_η is the ballistic part of J^0_η defined in section 4.2. We have the following proposition:

Proposition 4.10 *When $\alpha > 0$, $J^{5,\mathcal{Q}}_\eta$ can be neglected in $\mathcal{C}^0([0,T], Z')$ compared to $J^{4,\mathcal{Q}}_\eta$; when $\alpha = 0$ both terms are of the same order.*

Indeed, we know from lemma (3.4) that both equations on $J^{4,\mathcal{Q}}_\eta$ and $J^{5,\mathcal{Q}}_\eta$ are well-posed in Z', so that (34) and (35) imply the following estimates:

$$\sup_{t\in[0,T]} \|J^{4,\mathcal{Q}}_\eta(t)\|_{Z'} \lesssim \eta^{(d+2)(1-\alpha)+(\alpha-\beta)\vee 0}, \tag{67}$$

$$\sup_{t\in[0,T]} \|J^{5,\mathcal{Q}}_\eta(t)\|_{Z'} \lesssim \begin{cases} \eta^{d(1-\alpha)+2-\alpha+\frac{1}{d-1}((d(\alpha-\beta)-\alpha)\vee 0} & \text{when } d \geq 3, \\ \eta^{2(1-\alpha)+2-\alpha}(\eta^{\alpha-2\beta}|\log\eta|) \wedge 1 & \text{when } d = 2. \end{cases}$$

We finally verify that as soon as $0 < \alpha$, $J^{5,\mathcal{Q}}_\eta$ is higher order than $J^{4,\mathcal{Q}}_\eta$ and can be neglected. When $\alpha = 0$, both terms are of the same order. In all cases, the leading order is thus given by that of $J^{4,\mathcal{Q}}_\eta$. This ends the analysis of $J^{1,\mathcal{Q}}_\eta$.

4.6 Proof of theorem 2.1.

The proof is now a simple application of the results of sections 4.2, 4.3, 4.4 and 4.5. We recall that the total scintillation J_η is decomposed in

$$J_\eta = J^0_\eta + J^{1,\mathcal{K}}_\eta + J^{1,\mathcal{Q}}_\eta.$$

From section 4.2 and (34) (or by analogy with proposition 4.10), we obtain that when $0 < \alpha \leq 1$, J^0_η is dominated by J^{00}_η and that $J^0_\eta - J^{00}_\eta$ can be neglected in Z'. When $\alpha = 0$, both terms are of the same order. Let $\varphi \in \mathcal{S}(\mathbb{R}^{4d})$ be a test function and φ_{η,s_1,s_2} be the related localized version as in (12). We have the scaling properties:

$$\|\mathcal{F}\varphi_{\eta,s_1,s_2}\|_{L^1(\mathbb{R}^{4d})} = \frac{1}{\eta^{2d(s_1+s_2)}} \|\mathcal{F}\varphi\|_{L^1(\mathbb{R}^{4d})},$$

$$\|\varphi_{\eta,s_1,s_2}\|_Z \lesssim \frac{1}{\eta^{2(d(s_1+s_2)+s_1+s_1\vee s_2)}} \|\varphi\|_Z,$$

where $a \vee b = \max(a,b)$. Hence, it stems from (35), uniformly for $t \in [0,T]$, denoting by $\langle \cdot, \cdot \rangle$ the $\mathcal{S}'(\mathbb{R}^{4d}) - \mathcal{S}(\mathbb{R}^{4d})$ duality product, that

$$\begin{aligned} |\langle J_\eta^0(t), \varphi_{\eta,s_1,s_2}\rangle| &\leq |\langle J_\eta^{00}(t), \varphi_{\eta,s_1,s_2}\rangle| + |\langle (J_\eta^0 - J_\eta^{00})(t), \varphi_{\eta,s_1,s_2}\rangle|, \\ &\leq (\|J_\eta^{00}(t)\|_{Z'} + \|(J_\eta^0 - J_\eta^{00})(t)\|_{Z'})\|\varphi_{\eta,s_1,s_2}\|_Z, \\ &\lesssim \eta^{(d+2)(1-\alpha)+(\alpha-\beta)\vee 0 - 2(d(s_1+s_2)+s_1+s_1\vee s_2)}. \end{aligned} \tag{68}$$

Regarding $J_\eta^{1,\mathcal{K}}$, it is decomposed following section 4.4 as $J_\eta^{2,\mathcal{K}} + J_\eta^{3,\mathcal{K}}$, where $J_\eta^{3,\mathcal{K}}$ is higher order in $\mathcal{C}^0([0,T], X_\infty)$ when $\beta > 0$ and same order when $\beta = 0$, according to proposition 4.5. Then, $J_\eta^{2,\mathcal{K}}$ is split into $J_\eta^{4,\mathcal{K}} + J_\eta^{5,\mathcal{K}}$, see (53) and (54). According to proposition 4.6, $J_\eta^{5,\mathcal{K}}$ can be neglected in $\mathcal{C}^0([0,T], X_\infty)$ when $\alpha + \beta \geq \frac{d-1}{d-2}$, or $\alpha + \beta < \frac{d-1}{d-2}$ with $\alpha < 1$ and $\beta < 1$. When $\alpha + \beta < \frac{d-1}{d-2}$ and $\alpha = 1$ or $\beta = 1$, both terms have the same order. When $d = 2$, they are of the same order when $\beta = 1$ or when $\alpha = 1$ and $\beta < 1$. Otherwise $J_\eta^{5,\mathcal{K}}$ can be neglected. Therefore, the dominant order of $J_\eta^{1,\mathcal{K}}$ is given by that of $J_\eta^{4,\mathcal{K}}$ and we find, according to (52) and the scaling of the test function, uniformly for $t \in [0,T]$,

$$\begin{aligned} |\langle J_\eta^{1,\mathcal{K}}(t), \varphi_{\eta,s_1,s_2}\rangle| = \frac{1}{(2\pi)^{4d}}|\langle \mathcal{F}J_\eta^{1,\mathcal{K}}(t), \mathcal{F}\varphi_{\eta,s_1,s_2}\rangle| &\leq \frac{1}{(2\pi)^{4d}}\|J_\eta^{1,\mathcal{K}}(t)\|_{X_\infty}\|\mathcal{F}\varphi_{\eta,s_1,s_2}\|_{L^1(\mathbb{R}^{4d})}, \\ &\lesssim \eta^{d(1-\alpha)+1-\beta-2d(s_1+s_2)}\left(\left[\eta^{\alpha-\beta}f_d(\eta)\right]\wedge 1\right) \vee \left(\eta^{(d-1)(1-\alpha-\beta)+\alpha}\right), \end{aligned} \tag{69}$$

with $f_d(x) = 1$ when $d \geq 3$ and $f_2(x) = 1 + |\log x^{\alpha-\beta}|$.

It remains to analyze the term $J_\eta^{1,\mathcal{Q}}$. In the same way as for $J_\eta^{1,\mathcal{K}}$, it is decomposed following section 4.5 as $J_\eta^{3,\mathcal{Q}} + J_\eta^{4,\mathcal{Q}} + J_\eta^{5,\mathcal{Q}}$. Proposition 4.9 states that $J_\eta^{3,\mathcal{Q}}$ is negligible compared to $J_\eta^{4,\mathcal{K}}$ in $\mathcal{C}^0([0,T], X_\infty)$ when $d \geq 3$, and $\beta > 0$, and same order when $\beta = 0$. When $d = 2$, $J_\eta^{3,\mathcal{Q}}$ is of the same order when $\alpha = \beta = 0$. Finally, proposition 4.10 states that when $\alpha > 0$, $J_\eta^{5,\mathcal{Q}}$ can be neglected in $\mathcal{C}^0([0,T], Z')$ compared to $J_\eta^{4,\mathcal{Q}}$ and that, when $\alpha = 0$, both terms are of the same order. Therefore, the dominant order in $J_\eta^{1,\mathcal{Q}}$ is that of $J_\eta^{4,\mathcal{Q}}$ and we have, following (67), $\forall t \in [0,T]$,

$$|\langle J_\eta^{1,\mathcal{Q}}(t), \varphi_{\eta,s_1,s_2}\rangle| \leq \eta^{(d+2)(1-\alpha)+(\alpha-\beta)\vee 0 - 2(d(s_1+s_2)+s_1+s_1\vee s_2)}. \tag{70}$$

Gathering (68), (69) and (70), we conclude the proof of theorem 1.

4.7 Proof of theorem 2.2.

We assume here that the initial condition of the Schrödinger equation is a coherent state $\psi_\eta^{(1)}$ of the form (9). This translates into an initial condition for the Wigner transform reading

$$a_{\eta 0}(\mathbf{x}, \mathbf{k}) = \frac{1}{\eta^d} a_0\left(\frac{\mathbf{x} - \mathbf{x}_0}{\eta^\alpha}, \frac{\mathbf{k} - \mathbf{k}_0}{\eta^{1-\alpha}}\right),$$

where $a_0(\mathbf{x}, \mathbf{k})$ is the Wigner transform of the rescaled initial condition $\psi_{\eta=1}^{(1)}$. Its Fourier transform reads

$$\mathcal{F}a_{\eta 0}(\mathbf{u}, \boldsymbol{\xi}) = e^{-i(\mathbf{x}_0\cdot\mathbf{u} + \mathbf{k}_0\cdot\boldsymbol{\xi})}\mathcal{F}a_0(\eta^\alpha \mathbf{u}, \eta^{1-\alpha}\boldsymbol{\xi}). \tag{71}$$

We thus have $\beta = 1 - \alpha$ and theorem 2.1 gives, for $s_1 = s_2 = 0$,

$$\langle J_\eta(t), \varphi_{\eta,s_1,s_2} \otimes \varphi_{\eta,s_1,s_2}\rangle \lesssim \eta^{d(1-\alpha)}\left[\eta^{2(1-\alpha)+(2\alpha-1)\vee 0}\right] \vee \left[\eta^\alpha\left(\eta^{2\alpha-1}f_d(\eta)\right)\wedge 1\right].$$

The proof is split into two cases, $0 \leq \alpha < 1$ and $\alpha = 1$.

4.7.1 The case $0 \le \alpha < 1$.

Following the proof of theorem 2.1, the leading terms in J_η are J_η^{00}, $J_\eta^{4,\mathcal{K}}$ and $J_\eta^{4,\mathcal{Q}}$ (one needs to add $J_\eta^{5,\mathcal{Q}}$ when $\alpha = 0$ by proposition 4.10 since $J_\eta^{4,\mathcal{Q}}$ and $J_\eta^{5,\mathcal{Q}}$ are of the same order). Computing the limit of J_η then boils down to computing that of the source terms J_η^{00} and $T_\eta^{\mathcal{K}} J_\eta^{00}$. We start with J_η^{00}.

First step: the term J_η^{00}. We assume here that $\alpha \neq 0$. When $\alpha = 0$, J_η^{00} is of order η^{d+2} in $\mathcal{C}^0([0,T], Z')$ while $T_\eta^{\mathcal{K}} J_\eta^{00}$ is of order η^d in $\mathcal{C}^0([0,T], X_\infty)$ so that J_η^{00} is negligible compared to $T_\eta^{\mathcal{K}} J_\eta^{00}$. We recall that

$$J_\eta^{00}(t) = \int_0^t e^{-2R_0(t-s)} \mathcal{G}_{t-s}^2 \mathcal{K}_\eta a_\eta^0 \otimes a_\eta^0(s) ds,$$

where $a_\eta^0(s, \mathbf{x}, \mathbf{k}) = e^{-R_0 s} a_{0\eta}(\mathbf{x} - s\mathbf{k}, \mathbf{k})$, so that its Fourier transform reads $\mathcal{F} a_\eta^0(s, \mathbf{u}, \boldsymbol{\xi}) = e^{-R_0 s} \mathcal{F} a_{0\eta}(\mathbf{u}, \boldsymbol{\xi} + s\mathbf{u})$. This gives the following expression for the Fourier transform of J_η^{00}, together with (71):

$$\begin{aligned}
(\mathcal{F} J_\eta^{00})(t, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}) = & -\eta^d \int_0^t \int_{\mathbb{R}^d} ds d\mathbf{w}\, e^{-2R_0(t-s)} f(t-s, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}, \eta \mathbf{w}) \\
& \times (\eta\, \mathbf{w} \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u}))\, (\eta\, \mathbf{w} \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v})) \\
& \times \mathcal{F} a_{\eta 0} \otimes a_{\eta 0} \big(\mathbf{u} - \mathbf{w}, \boldsymbol{\xi} + t\mathbf{u} - s\mathbf{w}, \mathbf{v} + \mathbf{w}, \boldsymbol{\zeta} + t\mathbf{v} + s\mathbf{w}\big), \\
= & -\eta^d \int_0^t \int_{\mathbb{R}^d} ds d\mathbf{w}\, e^{-2R_0(t-s)} f(t-s, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}, \eta \mathbf{w}) \\
& \times (\eta\, \mathbf{w} \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u}))\, (\eta\, \mathbf{w} \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v})) \\
& \times e^{-i(\mathbf{x}_0 \cdot (\mathbf{u} - \mathbf{w}) + \mathbf{k}_0 \cdot (\boldsymbol{\xi} + t\mathbf{u} - s\mathbf{w}))} e^{-i(\mathbf{x}_0 \cdot (\mathbf{v} + \mathbf{w}) + \mathbf{k}_0 \cdot (\boldsymbol{\zeta} + t\mathbf{v} + s\mathbf{w}))} \\
& \times a_0 \otimes a_0 \big(\eta^\alpha (\mathbf{u} - \mathbf{w}), \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u} - s\mathbf{w}), \eta^\alpha(\mathbf{v} + \mathbf{w}), \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + s\mathbf{w})\big),
\end{aligned} \tag{72}$$

with

$$f(t, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}, \mathbf{w}) = \frac{\sin\left(\frac{1}{2}\mathbf{w} \cdot (\boldsymbol{\xi} + t\mathbf{u})\right) \sin\left(\frac{1}{2}\mathbf{w} \cdot (\boldsymbol{\zeta} + t\mathbf{v})\right)}{\left(\frac{1}{2}\mathbf{w} \cdot (\boldsymbol{\xi} + t\mathbf{u})\right)\left(\frac{1}{2}\mathbf{w} \cdot (\boldsymbol{\zeta} + t\mathbf{v})\right)} \hat{R}(\mathbf{w}).$$

As for item (iii) of lemma 3.1, we decompose $(\mathbf{w} \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u}))\, (\mathbf{w} \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v}))$ into fours terms:

$$\begin{aligned}
&(\mathbf{w} - \mathbf{u}) \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u})\; (\mathbf{w} + \mathbf{v}) \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v}) - (\mathbf{w} - \mathbf{u}) \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u})\; \mathbf{v} \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v}) \\
&\quad + \mathbf{u} \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u})\; (\mathbf{w} + \mathbf{v}) \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v}) - \mathbf{u} \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u})\; \mathbf{v} \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v}).
\end{aligned}$$

This leads to four different terms in J_η^{00}. The first one involves

$$\begin{aligned}
& (\mathbf{w} - \mathbf{u}) \cdot (\boldsymbol{\xi} + (t-s)\mathbf{u})\; (\mathbf{w} + \mathbf{v}) \cdot (\boldsymbol{\zeta} + (t-s)\mathbf{v}) \\
& \mathcal{F} a_0 \otimes a_0 \big(\eta^\alpha (\mathbf{u} - \mathbf{w}), \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u} - s\mathbf{w}), \eta^\alpha(\mathbf{v} + \mathbf{w}), \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + s\mathbf{w})\big), \\
= \; & \eta^{-2\alpha} (\boldsymbol{\xi} + (t-s)\mathbf{u})^T\; \big[\mathcal{F} \nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}} a_0 \big(\eta^\alpha (\mathbf{u} - \mathbf{w}), \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u} - s\mathbf{w}), \\
& \quad \eta^\alpha(\mathbf{v} + \mathbf{w}), \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + s\mathbf{w})\big)\big]\; (\boldsymbol{\zeta} + (t-s)\mathbf{v}),
\end{aligned}$$

where $\mathcal{F} \nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}} a_0$ has to be understood as the matrix $(\mathcal{F} \partial_{x_i} a_0\, \mathcal{F} \partial_{y_j} a_0)_{i,j=1,\cdots,d}$. The other three terms involve $\eta^{-\alpha} \mathcal{F} \nabla_{\mathbf{x}} a_0 \otimes a_0$, $\eta^{-\alpha} \mathcal{F} a_0 \otimes \nabla_{\mathbf{y}} a_0$ and $\mathcal{F} a_0 \otimes a_0$, so that they are at least a factor η^α smaller as soon as $\alpha > 0$. Following the same analysis below for the first and dominant term, it is easy to prove that these terms are negligible and do not affect the limit, we thus only focus on the leading one in the sequel.

Performing the change of variable $\mathbf{w} = \eta^{-\alpha}\mathbf{w}_1$ in (72) leads to an integrand proportional to

$$\mathcal{F}\nabla_{\mathbf{x}}a_0 \otimes \nabla_{\mathbf{y}}a_0\big(\eta^\alpha \mathbf{u} - \mathbf{w}_1, \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u}) - \eta^{1-2\alpha}s\mathbf{w}_1, \eta^\alpha \mathbf{v} + \mathbf{w}_1, \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v}) + \eta^{1-2\alpha}s\mathbf{w}_1\big).$$

When $0 < \alpha < \frac{1}{2}$, that term converges, uniformly in all variables to

$$\mathcal{F}\nabla_{\mathbf{x}}a_0 \otimes \nabla_{\mathbf{y}}a_0(-\mathbf{w}_1, 0, \mathbf{w}_1, 0),$$

and to

$$\mathcal{F}\nabla_{\mathbf{x}}a_0 \otimes \nabla_{\mathbf{y}}a_0(-\mathbf{w}_1, -s\mathbf{w}_1, \mathbf{w}_1, s\mathbf{w}_1)$$

when $s = \frac{1}{2}$. The case $\alpha > \frac{1}{2}$ needs more work since $\eta^{2\alpha-1} \to \infty$. We set $s = \eta^{2\alpha-1}s_1$, so that the leading term in $\mathcal{F}J_\eta^{00}$ -denoted by I_η- obtained from (72) reads

$$\begin{aligned} I_\eta(t, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}) = &-\eta^{(d+2)(1-\alpha)+2\alpha-1} \int_0^{t\eta^{1-2\alpha}} \int_{\mathbb{R}^d} ds_1 d\mathbf{w}_1\, e^{-2R_0(t-s_1\eta^{2\alpha-1})} \\ &\times f(t - s_1\eta^{2\alpha-1}, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta}, \eta^{1-\alpha}\mathbf{w}_1) \times e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u})}e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))} \\ &\times(\boldsymbol{\xi} + (t - s_1\eta^{2\alpha-1})\mathbf{u})^T \mathcal{F}\nabla_{\mathbf{x}}a_0 \otimes \nabla_{\mathbf{y}}a_0\big(\eta^\alpha\mathbf{u} - \mathbf{w}_1, \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u}) - s_1\mathbf{w}_1, \\ &\qquad \eta^\alpha\mathbf{v} + \mathbf{w}_1, \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v}) + s_1\mathbf{w}_1\big)(\boldsymbol{\zeta} + (t - s_1\eta^{2\alpha-1})\mathbf{v}). \end{aligned}$$

We consider first the case $\alpha > \frac{1}{2}$ and pass to the limit in the latter term. To do so, let $\varphi \in \mathcal{C}^0([0,T], Z)$ be a test function and consider

$$\eta^{-(d+2)(1-\alpha)-2\alpha+1} \int_0^T \langle I_\eta(t,\cdot), \varphi(t,\cdot)\rangle dt := \int_0^{\eta^{2\alpha-1}} + \int_{\eta^{2\alpha-1}}^{\eta^{2\alpha-1-\gamma}} + \int_{\eta^{2\alpha-1-\gamma}}^T := I_1 + I_2 + I_3,$$

where $0 < \gamma < 2\alpha - 1$ and $\langle\cdot,\cdot\rangle$ denotes here the $Z' - Z$ duality product. For I_1, we make the change of variable $t = \eta^{2\alpha-1}t_1$ and $\mathbf{w}_1 = -\mathbf{w}_2 + \eta^\alpha\mathbf{u}$. Since f is uniformly bounded by $\hat{R}(\mathbf{w})$, this yields

$$\begin{aligned} |I_1| \le &\eta^{2\alpha-1}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|\nabla_{\mathbf{y}}a_0\|_{Y_\infty} \int_{\mathbb{R}^d} \sup_{\mathbf{z}\in\mathbb{R}^d} |\mathcal{F}\nabla_{\mathbf{x}}a_0(\mathbf{w}_2, \mathbf{z})| d\mathbf{w}_2 \int_0^1 \int_0^{t_1} \int_{\mathbb{R}^{4d}} ds_1 dt_1 d\mathbf{u} d\boldsymbol{\xi} d\mathbf{v} d\boldsymbol{\zeta} \\ &\times(|\boldsymbol{\xi}| + (T + s_1\eta^{2\alpha-1})|\mathbf{u}|)(|\boldsymbol{\zeta}| + (T + s_1\eta^{2\alpha-1})|\mathbf{v}|)|\mathcal{F}\varphi(\eta^{2\alpha-1}t_1, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta})|, \\ \lesssim\ &\eta^{2\alpha-1}\|\mathcal{F}_{\mathbf{x}}\nabla_{\mathbf{x}}a_0\|_{L^1(\mathbb{R}^{2d})}\|\varphi\|_{\mathcal{C}^0([0,T],Z)} \quad \lesssim \quad \eta^{2\alpha-1}, \end{aligned}$$

so that I_1 goes to zero. Regarding I_2, we make the change of variable $t = \eta^{2\alpha-1-\gamma}t_1$ and $\mathbf{w}_1 = -\mathbf{w}_2 + \eta^\alpha\mathbf{u}$ which gives

$$\begin{aligned} &|I_2| \le \eta^{2\alpha-1-\gamma}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|\nabla_{\mathbf{y}}a_0\|_{Y_\infty} \int_{\eta^\gamma}^1 \int_0^{t_1\eta^{-\gamma}} dt_1 ds_1 \\ &\times \sup_{(\mathbf{u},\boldsymbol{\xi})\in\mathbb{R}^{2d}} \int_{\mathbb{R}^d} |\mathcal{F}\nabla_{\mathbf{x}}a_0(\mathbf{w}_2, \eta^{1-\alpha}(\boldsymbol{\xi} + \eta^{2\alpha-1-\gamma}t_1\mathbf{u}) - \eta^\alpha\mathbf{u} + s_1\mathbf{w}_2)| d\mathbf{w}_2 \\ &\times\int_{\mathbb{R}^{4d}} d\mathbf{u} d\boldsymbol{\xi} d\mathbf{v} d\boldsymbol{\zeta}\, \big(|\boldsymbol{\xi}| + (T + s_1\eta^{2\alpha-1})|\mathbf{u}|\big)\, \big(|\boldsymbol{\zeta}| + (T + s_1\eta^{2\alpha-1})|\mathbf{v}|\big)\, |\mathcal{F}\varphi(\eta^{2\alpha-1-\gamma}t_1, \mathbf{u}, \boldsymbol{\xi}, \mathbf{v}, \boldsymbol{\zeta})|. \end{aligned}$$

The integral over s_1 runs from 0 to $t_1\eta^{-\gamma}$, and since $\eta^\gamma \le t_1 \le 1$, it is controlled by the integral over $[0, \eta^{-\gamma}]$ and we thus need to integrate for large s_1 to obtain a bound independent of η.

This is done by splitting the integral in $[0,1]$ and $[1,\eta^{-\gamma}]$. We denote by II_2 and III_2 the corresponding terms. Controlling II_2 is straightforward and done in the same manner as I_1; we obtain

$$II_2 \lesssim \eta^{2\alpha-1-\gamma}.$$

Concerning III_2, we make the change of variable $\mathbf{w}_2 = s_1^{-1}(\mathbf{w}_3 - \eta^{1-\alpha}(\boldsymbol{\xi}+\eta^{2\alpha-1-\gamma}t_1\mathbf{u})+\eta^\alpha\mathbf{u})$ and find

$$\begin{aligned}
III_2 &\lesssim \eta^{2\alpha-1-\gamma}\int_{\eta^\gamma}^1\int_1^{t_1\eta^{-\gamma}} s_1^{-d}ds_1dt_1\int_{\mathbb{R}^d}\sup_{\mathbf{z}\in\mathbb{R}^d}|\mathcal{F}\nabla_\mathbf{x}a_0(\mathbf{z},\mathbf{w}_3)|d\mathbf{w}_3\\
&\quad\times\int_{\mathbb{R}^{4d}} d\mathbf{u}d\boldsymbol{\xi}d\mathbf{v}d\boldsymbol{\zeta}\left(|\boldsymbol{\xi}|+(T+s_1\eta^{2\alpha-1})|\mathbf{u}|\right)\left(|\boldsymbol{\zeta}|+(T+s_1\eta^{2\alpha-1})|\mathbf{v}|\right)\\
&\quad\times\quad|\mathcal{F}\varphi(\eta^{2\alpha-1-\gamma}t_1,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|.\\
&\lesssim \eta^{2\alpha-1-\gamma}\|\mathcal{F}_\mathbf{k}\nabla_\mathbf{x}a_0\|_{L^1(\mathbb{R}^{2d})}\int_{\eta^\gamma}^1\int_1^{t_1\eta^{-\gamma}}\left(s_1^{-d}+\eta^{2\alpha-1}s_1^{1-d}+\eta^{4\alpha-2}s_1^{2-d}\right)ds_1dt_1.
\end{aligned}$$

Since $d\geq 2$, the right hand side is an $\mathcal{O}(\eta^{2\alpha-1-\gamma})$ and consequently I_2 converges as well to zero. It remains to analyze I_3 which reads

$$\begin{aligned}
I_3 &= -\int_{\eta^{2\alpha-1-\gamma}}^T\int_0^{t\eta^{1-2\alpha}}\int_{\mathbb{R}^{5d}} dtds_1d\mathbf{w}_1d\mathbf{u}d\boldsymbol{\xi}d\mathbf{v}d\boldsymbol{\zeta}\, e^{-2R_0(t-s\eta^{2\alpha-1})}\mathcal{F}\varphi(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})\\
&\quad\times f(t-s_1\eta^{2\alpha-1},\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\eta^{1-\alpha}\mathbf{w}_1)e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))}e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))}\\
&\quad\times(\boldsymbol{\xi}+(t-s_1\eta^{2\alpha-1})\mathbf{u})^T\left[\mathcal{F}\nabla_\mathbf{x}a_0\otimes\nabla_\mathbf{x}a_0\big(\eta^\alpha\mathbf{u}-\mathbf{w}_1,\eta^{1-\alpha}(\boldsymbol{\xi}+t\mathbf{u})-s_1\mathbf{w}_1\big),\right.\\
&\qquad\left.\eta^\alpha\mathbf{v}+\mathbf{w}_1,\eta^{1-\alpha}(\boldsymbol{\zeta}+t\mathbf{v})+s_1\mathbf{w}_1\big)\right](\boldsymbol{\zeta}+(t-s_1\eta^{2\alpha-1})\mathbf{v}).
\end{aligned}$$

As III_2, we need to control the integral over s_1 for large s_1 since $t\eta^{1-2\alpha}\geq\eta^{-\gamma}$ for $t\geq\eta^{2\alpha-1-\gamma}$. As a consequence, to apply the Lebesgue dominated convergence theorem to pass to the limit in I_3, we split the integral in s_1 into $[0,1]$ and $[1,t\eta^{1-2\alpha}]$. We denote by II_3 and III_3 the corresponding terms. Regarding II_3, we make the change of variable $\mathbf{w}_1=\eta^\alpha\mathbf{u}-\mathbf{w}_2$ and choose as majorizing function the function

$$\sup_{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbb{R}^{3d}}|\mathcal{F}\nabla_\mathbf{x}a_0\otimes\nabla_\mathbf{y}a_0\big(\mathbf{w}_2,\mathbf{x},\mathbf{y},\mathbf{z})|\left(|\boldsymbol{\xi}|+T|\mathbf{u}|\right)(|\boldsymbol{\zeta}|+T|\mathbf{v}|)\,|\mathcal{F}\varphi(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|.$$

Since

$$f(t-s_1\eta^{2\alpha-1},\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\eta\mathbf{u}-\eta^{1-\alpha}\mathbf{w}_2)\to\hat{R}(\mathbf{0}),\qquad\text{a.e. in }(0,T)\times\mathbb{R}^{5d},$$

we then obtain for the limit of II_3:

$$\begin{aligned}
&-\int_0^T dtd\mathbf{u}d\boldsymbol{\xi}d\mathbf{v}d\boldsymbol{\zeta}\,e^{-2R_0t}e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))}e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))}\\
&\qquad(\boldsymbol{\xi}+t\mathbf{u})^TM_1(\boldsymbol{\zeta}+t\mathbf{v})\mathcal{F}\varphi(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}),
\end{aligned}$$

where the matrix M_1 is defined by

$$M_1=\hat{R}(\mathbf{0})\int_0^1\int_{\mathbb{R}^d}\mathcal{F}\nabla_\mathbf{x}a_0\otimes\nabla_\mathbf{y}a_0\big(\mathbf{w}_2,s_1\mathbf{w}_2,-\mathbf{w}_2,-s_1\mathbf{w}_2\big)\,d\mathbf{w}_2\,ds_1.$$

The latter matrix is well-defined since

$$|(M_1)_{ij}|\leq\hat{R}(\mathbf{0})\|\partial_{\mathbf{x}_i}a_0\|_{Y_1}\|\partial_{\mathbf{y}_j}a_0\|_{Y_\infty},\tag{73}$$

and is real-valued since a_0 is real and

$$\mathcal{F}\partial_{\mathbf{x}_i} a_0 \otimes \partial_{\mathbf{x}_j} a_0 \big(\mathbf{w}_2, s_1\mathbf{w}_2, -\mathbf{w}_2, -s_1\mathbf{w}_2\big) = s_1(\mathbf{w}_2)_i(\mathbf{w}_2)_j|\mathcal{F}\partial_{\mathbf{x}_i} a_0(\mathbf{w}_2, s_1\mathbf{w}_2)|^2.$$

For the second part III_3, we make the change of variable $\mathbf{w}_1 = s_1^{-1}(-\mathbf{w}_2 + \eta^{1-\alpha}(\boldsymbol{\xi} + t\mathbf{u}))$ and split the integrand into three terms: one proportional to $(\boldsymbol{\xi}+t\mathbf{u})^T\,[\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{x}} a_0](\boldsymbol{\zeta}+t\mathbf{v})$, the second one proportional to $\eta^{2\alpha-1}s_1([\mathbf{u}^T\,\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{x}} a_0](\boldsymbol{\zeta}+t\mathbf{v}) + (\boldsymbol{\xi}+t\mathbf{u})^T\,[\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{x}} a_0]\mathbf{v})$ and the last one proportional to $\eta^{4\alpha-2}s_1^2\mathbf{u}^T\,[\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{x}} a_0]\mathbf{v}$. Proceeding as for III_2, the last two terms vanish at the limit. To pass to the limit in the first one, we use the majorizing function

$$s_1^{-d} \sup_{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbb{R}^{3d}} |\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}} a_0\big(\mathbf{x},\mathbf{w}_2,\mathbf{y},\mathbf{z}\big)|\,(|\boldsymbol{\xi}| + T|\mathbf{u}|)(|\boldsymbol{\zeta}| + T|\mathbf{v}|)\,|\mathcal{F}\varphi(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})|,$$

and obtain the expression for the limit of III_3:

$$-\int_0^T dt d\mathbf{u} d\boldsymbol{\xi} d\mathbf{v} d\boldsymbol{\zeta}\, e^{-2R_0 t} e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))} e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))}$$
$$(\boldsymbol{\xi}+t\mathbf{u})^T M_2(\boldsymbol{\zeta}+t\mathbf{v})\mathcal{F}\varphi(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}),$$

where the matrix M_2 is given by

$$\begin{aligned} M_2 &= \hat{R}(\mathbf{0})\int_1^\infty\int_{\mathbb{R}^d} s_1^{-d}\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}} a_0(s_1^{-1}\mathbf{w}_2, \mathbf{w}_2, -s_1^{-1}\mathbf{w}_2, -\mathbf{w}_2)\,d\mathbf{w}_2\,ds_1,\\ &= \hat{R}(\mathbf{0})\int_1^\infty\int_{\mathbb{R}^d} \mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}}\ a_0(\mathbf{w}_2, s_1\mathbf{w}_2, -\mathbf{w}_2, -s_1\mathbf{w}_2)\,d\mathbf{w}_2\,ds_1.\end{aligned}$$

The latter is real-valued and well-defined since

$$|(M_2)_{ij}| \le \hat{R}(\mathbf{0})\|\mathcal{F}_{\mathbf{k}}\partial_{\mathbf{x}_i} a_0\|_{L^1(\mathbb{R}^{2d})}\|\partial_{\mathbf{y}_j} a_0\|_{Y_\infty}.$$

Gathering both parts of the integral, we finally conclude that, when $\frac{1}{2} < \alpha < 1$,

$$\eta^{-(d+2)(1-\alpha)-2\alpha+1}\int_0^T \langle I_\eta(t,\cdot),\varphi(t,\cdot)\rangle dt \to \int_0^T \langle \mathcal{F}J^{00}(t,\cdot),\varphi(t,\cdot)\rangle dt$$

where

$$\mathcal{F}J^{00}(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = -e^{-2R_0 t} e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))} e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))}(\boldsymbol{\xi}+t\mathbf{u})^T M(\boldsymbol{\zeta}+t\mathbf{v}),$$

with

$$M = M_1 + M_2. \tag{74}$$

Computing the inverse Fourier transform of $\mathcal{F}J^{00} \in \mathcal{C}^0([0,T],\mathcal{S}'(\mathbb{R}^{4d}))$ gives finally

$$\begin{aligned} J^{00}(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= e^{-2R_0 t}\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{k}t)\,\delta(\mathbf{y}-\mathbf{x}_0-\mathbf{p}t)\\ &\quad (\nabla\delta)^T(\mathbf{k}-\mathbf{k}_0)\,M\,(\nabla\delta)(\mathbf{p}-\mathbf{k}_0),\end{aligned}$$

where δ is the Dirac distribution.

The cases $0 < \alpha < \frac{1}{2}$ and $\alpha = \frac{1}{2}$ are simpler to treat and follow along the same lines as above. This yields the limits for J^{00}_η:

$$\begin{aligned} \mathcal{F}J^{00}(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= -\int_0^t ds\, e^{-2R_0(t-s)} e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))} e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))}\\ &\quad (\boldsymbol{\xi}+(t-s)\mathbf{u})^T M^\alpha(s)(\boldsymbol{\zeta}+(t-s)\mathbf{v}),\\ M^\alpha(s) &= \hat{R}(\mathbf{0})\int_{\mathbb{R}^d}\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}} a_0(\mathbf{w}, s\mathbf{w}, -\mathbf{w}, -s\mathbf{w})\,d\mathbf{w},\ \text{when } \alpha = \frac{1}{2}, \quad (75)\\ M^\alpha(s) &= \hat{R}(\mathbf{0})\int_{\mathbb{R}^d}\mathcal{F}\nabla_{\mathbf{x}} a_0 \otimes \nabla_{\mathbf{y}} a_0(\mathbf{w}, 0, -\mathbf{w}, 0)\,d\mathbf{w}, \quad \text{when } 0 < \alpha < \frac{1}{2}, \quad (76)\end{aligned}$$

which is equivalent in the physical space to

$$J^{00}(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) = \int_0^t ds\, e^{-2R_0(t-s)}\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{k}_0 s-\mathbf{k}(t-s))$$
$$\times\delta(\mathbf{y}-\mathbf{x}_0-\mathbf{k}_0 s-\mathbf{p}(t-s))(\nabla\delta)^T(\mathbf{k}-\mathbf{k}_0)\, M^\alpha(s)\,(\nabla\delta)(\mathbf{p}-\mathbf{k}_0).$$

Moreover, M^α satisfies as well (73) when $0<\alpha\leq\frac{1}{2}$. To summarize this section, we have proved the:

Proposition 4.11 *Let $\varphi\in\mathcal{C}^0([0,T],Z)$ and $0<\alpha<1$. Then, as η goes to zero,*

$$\eta^{-(d+2)(1-\alpha)-(2\alpha-1)\vee 0}\int_0^T \langle J^{00}_\eta(t,\cdot),\varphi(t,\cdot)\rangle_{Z',Z}dt \to \int_0^T \langle J^{00}(t,\cdot),\varphi(t,\cdot)\rangle_{Z',Z}dt,$$

where $J^{00}\in\mathcal{C}^0([0,T],Z')$ and

$$\begin{aligned} J^{00}(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= \int_0^t ds\, e^{-2R_0(t-s)}(\mathcal{G}^2_{t-s}J^1(s,\cdot))(t-s,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) \quad \textit{when } 0<\alpha\leq\frac{1}{2},\\ &= e^{-2R_0 t}(\mathcal{G}^2_t J^1(0,\cdot))(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}), \quad \textit{when } \frac{1}{2}<\alpha<1,\\ J^1(s,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= \delta(\mathbf{x}-\mathbf{x}_0-\mathbf{k}_0 s)\,\delta(\mathbf{y}-\mathbf{x}_0-\mathbf{k}_0 s)(\nabla\delta)^T(\mathbf{k}-\mathbf{k}_0)\,M^\alpha(s)\,(\nabla\delta)(\mathbf{p}-\mathbf{k}_0).\end{aligned}$$

The matrix M^α is real-valued and given by (74) *when $\frac{1}{2}<\alpha<1$ and by* (75)-(76) *when $0<\alpha\leq\frac{1}{2}$. It is well-defined since*

$$|(M^\alpha)_{ij}|\leq \hat{R}(\mathbf{0})\|\partial_{\mathbf{y}_j}a_0\|_{Y_\infty}\left(\|\partial_{\mathbf{x}_i}a_0\|_{Y_1}+\|\mathcal{F}_{\mathbf{k}}\partial_{\mathbf{x}_i}a_0\|_{L^1(\mathbb{R}^{2d})}\right).$$

Second step : the term $T^{\mathcal{K}}_\eta J^{00}_\eta$. From expression (39) and the fact that $\mathcal{F}a^0_\eta(s,\mathbf{u},\boldsymbol{\xi}) = e^{-R_0 s}\mathcal{F}a_{0\eta}(\mathbf{u},\boldsymbol{\xi}+s\mathbf{u})$, we deduce that the Fourier transform of $T^{\mathcal{K}}_\eta J^{00}_\eta$ reads, after the change of variable $\mathbf{w}=-\mathbf{w}_1+\mathbf{u}-\eta^{-1}\mathbf{w}'$:

$$\begin{aligned}(\mathcal{F}T^{\mathcal{K}}_\eta J^{00}_\eta)(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = \eta^d\int_0^t\int_0^s\int_{\mathbb{R}^{2d}} ds d\tau d\mathbf{w}_1 d\mathbf{w}'\, e^{-2R_0(t-\tau)} \\ \times\hat{R}(\mathbf{w}')\hat{R}(\eta(\mathbf{u}-\mathbf{w}_1)-\mathbf{w}')g(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{u}-\mathbf{w}_1-\eta^{-1}\mathbf{w}',\mathbf{w}') \\ \times\mathcal{F}a_{\eta 0}\otimes a_{\eta 0}\big(\mathbf{w}_1,\boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}'+\tau\mathbf{w}_1, \\ \mathbf{v}+\mathbf{u}-\mathbf{w}_1,\boldsymbol{\zeta}+t\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}'+\tau(\mathbf{u}-\mathbf{w}_1)\big),\end{aligned} \tag{77}$$

where g is defined in (40). Using the formula $\sin(a)\sin(b)=\frac{1}{2}(\cos(a-b)-\cos(a+b))$, we decompose g in g_1+g_2 accordingly with

$$\begin{aligned} g_1(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') &= -8\,\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2)\\ &\quad\times\cos\left[\eta\mathbf{w}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}+\mathbf{v}))/2\right],\\ g_2(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') &= 8\,\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2)\\ &\quad\times\cos\left[\eta\mathbf{w}\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v})/2-\eta^{-1}(s-\tau)\mathbf{w}')\right].\end{aligned}$$

The g_1 term is smooth and admits a limit as η goes to zero, and the related part yields the dominant term at the limit. The g_2 term involves a highly oscillating function that renders the term negligible after a integration by part and a careful analysis of the integrand. We first separate $T^{\mathcal{K}}_\eta J^{00}_\eta$ accordingly in $G^1_\eta+G^2_\eta$ and compute the limit of G^1_η.
The term G^1_η. We have the following proposition:

Proposition 4.12 *Let $\varphi \in \mathcal{C}^0([0,T], \mathcal{S}(\mathbb{R}^{4d}))$, $0 \leq \alpha < 1$ and $g_d(\eta) = \eta^{-d(1-\alpha)-\alpha-(2\alpha-1)\vee 0}$ if $d \geq 3$ and $g_2(\eta) = \eta^{-2(1-\alpha)-\alpha}(\eta^{2\alpha-1}(1+|\log \eta^{1-2\alpha}|))^{-1} \vee 1$. Then, as η goes to zero, denoting by $(\cdot,\cdot)$ the $L^2(\mathbb{R}^{4d})$ scalar product,*

$$g_d(\eta)\int_0^T (G^1_\eta(t,\cdot),\varphi(t,\cdot))dt \to \int_0^T (G^1_0(t,\cdot),\varphi(t,\cdot))dt,$$

where $G^1_0 \in \mathcal{C}^0([0,T], X_\infty)$ and

$$\begin{aligned}
G^1_0(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= \int_0^t ds e^{-2R_0(t-s)}(\mathcal{G}^2_{t-s}\mathcal{K}^\alpha_s J^2(s,\cdot))(t-s,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}), \quad \text{when } 0\leq\alpha\leq\frac{1}{2},\\
&= e^{-2R_0 t}(\mathcal{G}^2_t\,\mathcal{K}_d J^2(0,\cdot))(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) \quad \text{when } \frac{1}{2}<\alpha<1,\\
J^2(s,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) &= \mathcal{G}^2_s[\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{y}-\mathbf{x}_0)\delta(\mathbf{k}-\mathbf{k}_0)\delta(\mathbf{p}-\mathbf{k}_0)],
\end{aligned}$$

where $\mathcal{K}^\alpha_s$ and $\mathcal{K}_d$ are operators defined in the Fourier space by the multipliers k^α_s and k_d, that is, for a tempered distribution J, $\mathcal{K}^\alpha_s J = \mathcal{F}^{-1}(k^\alpha_s \mathcal{F}J)$ and $\mathcal{K}_d J = \mathcal{F}^{-1}(k_d\mathcal{F}J)$. $\mathcal{K}^\alpha_s$ and $\mathcal{K}_d$ act on the momentum variables $\mathbf{k}$ and $\mathbf{p}$ when $\alpha>0$ and on all variables when $\alpha=0$, and

$$\begin{aligned}
k_s^{0<\alpha\leq\frac{1}{2}}(\mathbf{v},\boldsymbol{\zeta}) &= -8\int_0^\infty\int_{\mathbb{R}^d} d\tau d\mathbf{w}'\mathcal{M}^\alpha(\mathbf{w}',\tau,s)\cos\Big(\frac{\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta})}{2}\Big)\sin\Big(\frac{\mathbf{w}'\cdot\boldsymbol{\xi}}{2}\Big)\sin\Big(\frac{\mathbf{w}'\cdot\boldsymbol{\zeta}}{2}\Big),\\
k^0_s(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= -8\int_0^s\int_{\mathbb{R}^d} e^{-2R_0(s-\tau)}d\tau d\mathbf{w}'\mathcal{M}^0(\mathbf{w}',\tau,\mathbf{u}+\mathbf{v})\\
&\qquad\times\cos\Big(\frac{1}{2}\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+(s-\tau)(\mathbf{u}+\mathbf{v}))\Big)\sin\Big(\frac{\mathbf{w}'\cdot\boldsymbol{\xi}}{2}\Big)\sin\Big(\frac{\mathbf{w}'\cdot\boldsymbol{\zeta}}{2}\Big),
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{\frac{1}{2}}(\mathbf{w}',\tau,s) &= \hat{R}^2(\mathbf{w}')\int_{\mathbb{R}^d}|\mathcal{F}a_0(\mathbf{w},s\mathbf{w}-\tau\mathbf{w}')|^2 d\mathbf{w},\\
\mathcal{M}^\alpha(\mathbf{w}',\tau,s) &= \mathcal{M}^{\frac{1}{2}}(\mathbf{w}',\tau,0), \quad 0<\alpha<\frac{1}{2},\\
\mathcal{M}^0(\mathbf{w}',\tau,\mathbf{z}) &= \hat{R}^2(\mathbf{w}')\int_{\mathbb{R}^d}\mathcal{F}a_0\otimes a_0(\mathbf{w},-\tau\mathbf{w}',\mathbf{z}-\mathbf{w},\tau\mathbf{w}')d\mathbf{w},\\
k_d(\mathbf{v},\boldsymbol{\zeta}) &= \int_0^\infty k_s^{\frac{1}{2}}(\mathbf{v},\boldsymbol{\zeta})ds, \qquad \text{when } d\geq 3,\\
k_2(\mathbf{v},\boldsymbol{\zeta}) &= -8\int_0^\infty\int_{\mathbb{R}^d} d\tau d\mathbf{w}'\mathcal{M}^{2D}(\mathbf{w}',\tau)\cos\Big(\frac{1}{2}\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta})\Big)\sin\Big(\frac{\mathbf{w}'\cdot\boldsymbol{\xi}}{2}\Big)\sin\Big(\frac{\mathbf{w}'\cdot\boldsymbol{\zeta}}{2}\Big),\\
\mathcal{M}^{2D}(\mathbf{w}',\tau) &= \hat{R}^2(\mathbf{w}')\int_0^\infty\int_{\mathbb{R}^d}|\mathcal{F}a_0(\tau\mathbf{w}',\mathbf{w})|^2 d\mathbf{w},
\end{aligned}$$

The operators $\mathcal{K}^\alpha_s$ and $\mathcal{K}_d$ are well-defined from $\mathcal{S}'$ to $\mathcal{S}'$ since k^α_s belongs to $L^\infty(\mathbb{R}^{2d})$ for $0<\alpha<1$, k^0_s belongs to $L^\infty(\mathbb{R}^{4d})$, $\forall s\in\mathbb{R}$ and $k_d \in L^\infty(\mathbb{R}^{2d})$.

Proof. Using (71) and (77), the Fourier transform of G^1_η reads, after the change of variable $\mathbf{w}_1 = \eta^{-\alpha}\mathbf{w}_2$ and $\tau = s-\eta^\alpha\tau_1$:

$$\begin{aligned}
(\mathcal{F}G^1_\eta)(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = \eta^{d(1-\alpha)+\alpha}&\int_0^t\int_0^{s\eta^{-\alpha}}\int_{\mathbb{R}^{2d}} dsd\tau_1 d\mathbf{w}_2 d\mathbf{w}'\, e^{-2R_0(t-(s-\eta^\alpha\tau_1))}\\
&\times\hat{R}(\mathbf{w}')\hat{R}(\eta(\mathbf{u}-\eta^{-\alpha}\mathbf{w}_2)-\mathbf{w}')g_1(t,s,s-\eta^\alpha\tau_1,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{u}-\eta^{-\alpha}\mathbf{w}_2-\eta^{-1}\mathbf{w}',\mathbf{w}')\\
&\times e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\mathcal{F}a_0\otimes a_0(\mathbf{w}_2,\mathbf{z}_1,\eta^\alpha(\mathbf{v}+\mathbf{u})-\mathbf{w}_2,\mathbf{z}_2) \qquad (78)
\end{aligned}$$

$$
\begin{aligned}
\mathbf{z}_1 &= \eta^{1-\alpha}(\boldsymbol{\xi} + (t-s+\eta^\alpha\tau_1)\mathbf{u}) - \tau_1\mathbf{w}' + \eta^{1-\alpha}(\eta^{-\alpha}s - \tau_1)\mathbf{w}_2,\\
\mathbf{z}_2 &= \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + (s-\eta^\alpha\tau_1)\mathbf{u}) + \tau_1\mathbf{w}' - \eta^{1-\alpha}(\eta^{-\alpha}s - \tau_1)\mathbf{w}_2.
\end{aligned}
$$

Assume first that $0 \le \alpha \le \frac{1}{2}$. Then, $\mathbf{z}_1 \to -\tau_1\mathbf{w}'$ and $\mathbf{z}_2 \to \tau_1\mathbf{w}'$ when $\alpha < \frac{1}{2}$ and $\mathbf{z}_1 \to -\tau_1\mathbf{w}' + s\mathbf{w}_2$ and $\mathbf{z}_2 \to \tau_1\mathbf{w}' - s\mathbf{w}_2$ when $\alpha = \frac{1}{2}$. Proceeding as for the limit of J_η^{00} by splitting the time integrals conveniently and applying the Lebesgue dominated convergence theorem, we verify that, for any test function in $\mathcal{C}^0([0,T],\mathcal{S}(\mathbb{R}^{4d}))$,

$$
\eta^{-d(1-\alpha)-\alpha}\int_0^T (G_\eta^1(t,\cdot),\varphi(t,\cdot))dt \to \int_0^T (G_0^1(t,\cdot),\varphi(t,\cdot))dt,
$$

Here, we have defined when $0 < \alpha \le \frac{1}{2}$:

$$
\begin{aligned}
(\mathcal{F}G_0^1)(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\\
&\int_0^t\int_0^\infty\int_{\mathbb{R}^d} ds d\tau d\mathbf{w}' e^{-2R_0(t-s)}\mathcal{M}^\alpha(\mathbf{w}',\tau,s)\cos\big(\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+(t-s)(\mathbf{u}+\mathbf{v}))/2\big)\\
&\qquad\times\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2),\\
&= e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\int_0^t ds\, e^{-2R_0(t-s)}\, k_s^\alpha(\mathbf{u},\boldsymbol{\xi}+(t-s)\mathbf{u},\mathbf{v},\boldsymbol{\zeta}+(t-s)\mathbf{v}),
\end{aligned}
$$

and, when $\alpha = 0$:

$$
\begin{aligned}
(\mathcal{F}G_0^1)(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\\
&\int_0^t\int_0^s\int_{\mathbb{R}^d} ds d\tau d\mathbf{w}' e^{-2R_0(t-\tau)}\mathcal{M}^0(\mathbf{w}',\tau,\mathbf{u}+\mathbf{v})\cos\big(\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}+\mathbf{v}))/2\big)\\
&\qquad\times\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2),\\
&= e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\int_0^t ds\, e^{-2R_0(t-s)}\, k_s^0(\mathbf{u},\boldsymbol{\xi}+(t-s)\mathbf{u},\mathbf{v},\boldsymbol{\zeta}+(t-s)\mathbf{v}),
\end{aligned}
$$

where $\mathcal{M}^\alpha$, $\mathcal{M}^0$, and k_s^α are defined in the proposition. This proves the proposition by identification when $0 < \alpha \le \frac{1}{2}$. Regarding the fact that the multipliers are bounded, we split the integral on τ for $\tau \in [0,1]$ and $\tau \in [1,\infty)$. This yields

$$
\begin{aligned}
\|k_s^\alpha\|_{L^\infty(\mathbb{R}^{2d})} &\le \|\hat{R}\|_{L^1(\mathbb{R}^d)}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|a_0\|_{Y_\infty}\|a_0\|_{Y_1} + \|\hat{R}\|^2_{L^\infty(\mathbb{R}^d)}\|a_0\|_{Y_1}\|\mathcal{F}_{\mathbf{k}}a_0\|_{L^1(\mathbb{R}^{2d})},\\
\|k_s^0\|_{L^\infty(\mathbb{R}^{4d})} &\le s\|\hat{R}\|_{L^1(\mathbb{R}^d)}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|a_0\|_{Y_\infty}\|a_0\|_{Y_1}.
\end{aligned}
$$

We treat now the case $\frac{1}{2} < \alpha < 1$. For such values of α, $\mathbf{z}_1$ and $\mathbf{z}_2$ diverge when $\eta \to 0$ so that we need to perform in (78) the additional change of variable $s = \eta^{2\alpha-1}s_1$ and split $\mathcal{F}G_\eta^1$ as

$$
\begin{aligned}
\eta^{-d(1-\alpha)-3\alpha+1}\mathcal{F}G_\eta^1 &= \int_0^{\eta^{1-\alpha}}\int_0^{s_1\eta^{\alpha-1}} + \int_{\eta^{1-\alpha}}^1\int_0^1 + \int_{\eta^{1-\alpha}}^1\int_1^{s_1\eta^{\alpha-1}} + \int_1^{t\eta^{1-2\alpha}}\int_0^{s_1\eta^{\alpha-1}},\\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

After the change of variable $s_1 = \eta^{1-\alpha}s_2$, it is straightforward to see that I_1 converges to zero in $L^\infty((0,T)\times\mathbb{R}^{4d})$. Regarding I_2, using the majorizing function

$$
\hat{R}(\mathbf{w}')\sup_{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbb{R}^{3d}}|\mathcal{F}a_0\otimes a_0(\mathbf{w}_2,\mathbf{x},\mathbf{y},\mathbf{z})|,
$$

and the Lebesgue dominated convergence theorem, we verify that, in $L^\infty((0,T)\times\mathbb{R}^{4d})$:

$$
\begin{aligned}
I_2(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) \to & -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})} e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})} \\
& \int_0^1\int_0^1\int_{\mathbb{R}^d} ds_1 d\tau d\mathbf{w}' e^{-2R_0 t}\mathcal{M}^{\frac{1}{2}}(\mathbf{w}',\tau,s_1)\cos\big(\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+t(\mathbf{u}+\mathbf{v}))/2\big) \\
& \qquad \times\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+t\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+t\mathbf{v})/2),
\end{aligned}
$$

where $\mathcal{M}^{\frac{1}{2}}$ is defined in the theorem. Concerning I_3, we have in the second time integral $s\eta^{\alpha-1}\geq 1$ since $s\geq \eta^{1-\alpha}$, so that we need a control for large τ_1. We thus perform the change of variable $\mathbf{w}'=\tau_1^{-1}(\mathbf{w}-\eta^{1-\alpha}(\boldsymbol{\zeta}+t\mathbf{v}+(s_1\eta^{2\alpha-1}-\eta^\alpha\tau_1)\mathbf{u})+(s_1-\eta^{1-\alpha}\tau_1)\mathbf{w}_2)$, and use the majorizing function

$$
\tau_1^{-d}\sup_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}}|\mathcal{F}a_0\otimes a_0(\mathbf{w}_2,\mathbf{x},\mathbf{y},\mathbf{w})|
$$

to pass to the limit. We find, in $L^\infty((0,T)\times\mathbb{R}^{4d})$:

$$
\begin{aligned}
I_3(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) \to & -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})} e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})} \\
& \int_0^1\int_1^\infty\int_{\mathbb{R}^d} ds_1 d\tau d\mathbf{w}\,\tau^{-d} e^{-2R_0 t}\mathcal{M}^{\frac{1}{2}}(\tau^{-1}\mathbf{w},\tau,s_1)\cos\big(\tau^{-1}\mathbf{w}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+t(\mathbf{u}+\mathbf{v}))/2\big) \\
& \qquad \times\sin(\tau^{-1}\mathbf{w}\cdot(\boldsymbol{\xi}+t\mathbf{u})/2)\sin(\tau^{-1}\mathbf{w}\cdot(\boldsymbol{\zeta}+t\mathbf{v})/2).
\end{aligned}
$$

Concerning I_4, we only consider times such that $t\geq\eta^{2\alpha-1}$ since the contribution for times less than $\eta^{2\alpha-1}$ converges to zero in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$. Setting $\tau_1=s_1\tau$, we split the integral on τ as $\int_0^{\eta^{\alpha-1}}(\cdot)=\int_0^1(\cdot)+\int_1^{\eta^{\alpha-1}}(\cdot)$ and denote by II_4 an III_4 the corresponding terms. Assume first that $d\geq 3$. For II_4, performing the change of variable

$$
\mathbf{w}_2=\tilde{h}(\mathbf{w}):=(s_1(1-\eta^{1-\alpha}\tau))^{-1}[\mathbf{w}-\eta^{1-\alpha}(\boldsymbol{\xi}+(t-s_1(\eta^{2\alpha-1}-\eta^\alpha\tau)\mathbf{u})+s_1\tau\mathbf{w}'], \tag{79}
$$

and using the majorizing function

$$
s_1^{1-d}\hat{R}(\mathbf{w}')\sup_{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbb{R}^{3d}}|\mathcal{F}a_0\otimes a_0(\mathbf{x},\mathbf{w},\mathbf{y},\mathbf{z})|,
$$

we find in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$,

$$
\begin{aligned}
II_4(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) \to & -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})} e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})} \\
& \int_1^\infty\int_0^1\int_{\mathbb{R}^d} s_1^{1-d} ds_1 d\tau d\mathbf{w}' e^{-2R_0 t}\widetilde{\mathcal{M}}(\mathbf{w}',\tau,s_1)\cos\big(\mathbf{w}'\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+t(\mathbf{u}+\mathbf{v}))/2\big) \\
& \qquad \times\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+t\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+t\mathbf{v})/2), \\
\widetilde{\mathcal{M}}(\mathbf{w}',\tau,s_1) = & \hat{R}^2(\mathbf{w}')\int_{\mathbb{R}^d}\mathcal{F}a_0\otimes a_0(s_1^{-1}\mathbf{w}+\tau\mathbf{w}',\mathbf{w},-s_1^{-1}\mathbf{w}-\tau\mathbf{w}',-\mathbf{w})d\mathbf{w}.
\end{aligned}
$$

Regarding the term III_4, we need to integrate for large τ and large s_1. Setting, in addition to (79),

$$
\mathbf{w}'=h(\mathbf{w}_2'):=(s_1\tau)^{-1}[\mathbf{w}_2'-\eta^{1-\alpha}(\boldsymbol{\zeta}+t\mathbf{v}+s_1(\eta^{2\alpha-1}-\eta^\alpha\tau)\mathbf{u})+s_1(1-\eta^{1-\alpha}\tau)\mathbf{w}_2], \tag{80}
$$

and using the majorizing function

$$
s_1^{1-d}\tau^{-d}\sup_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2d}}|\mathcal{F}a_0\otimes a_0(\mathbf{w}_2,\mathbf{x},\mathbf{y},\mathbf{w}_2')|,
$$

this yields in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$:

$$III_4(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) \to -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}$$
$$\int_1^\infty\int_1^\infty\int_{\mathbb{R}^d} s^{1-d}\tau^{-d}\, ds d\tau d\mathbf{w} e^{-2R_0 t}\mathcal{M}^{\frac{1}{2}}((s\tau)^{-1}\mathbf{w},\tau s,s)\cos\left(\frac{\mathbf{w}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+t(\mathbf{u}+\mathbf{v}))}{2\tau s}\right)$$
$$\times\sin((\tau s)^{-1}\mathbf{w}\cdot(\boldsymbol{\xi}+t\mathbf{u})/2)\sin((\tau s)^{-1}\mathbf{w}\cdot(\boldsymbol{\zeta}+t\mathbf{v})/2).$$

To recover the expression of G_0^1 given in the proposition, it then suffices to add I_2, I_3 and I_4 and to notice that

$$\begin{aligned}
-\frac{1}{8}k_d(\boldsymbol{\xi},\boldsymbol{\zeta}) &= \int_0^\infty\int_0^\infty\int_{\mathbb{R}^d} ds d\tau d\mathbf{w}'\mathcal{M}^{\frac{1}{2}}(\mathbf{w}',\tau,s)F(0,\mathbf{w}',\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}),\\
&= \int_0^1\int_0^1\int_{\mathbb{R}^d} ds d\tau d\mathbf{w}'\mathcal{M}^{\frac{1}{2}}(\mathbf{w}',\tau,s)F(0,\mathbf{w}',\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta})\\
&\quad+\int_0^1\int_1^\infty\int_{\mathbb{R}^d}\tau^{-d} ds d\tau d\mathbf{w}\mathcal{M}^{\frac{1}{2}}(\tau^{-1}\mathbf{w},\tau,s)F(0,\tau^{-1}\mathbf{w},\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta})\\
&\quad+\int_1^\infty\int_0^1\int_{\mathbb{R}^d} s^{1-d} ds d\tau d\mathbf{w}'\widetilde{\mathcal{M}}^{\frac{1}{2}}(\mathbf{w}',\tau,s)F(0,\mathbf{w}',\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta})\\
&\quad+\int_1^\infty\int_1^\infty\int_{\mathbb{R}^d} s^{1-d}\tau^{-d} ds d\tau d\mathbf{w}\mathcal{M}^{\frac{1}{2}}((s\tau)^{-1}\mathbf{w},s\tau,s)F(0,\frac{\mathbf{w}}{s\tau},\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}),\\
F(t,\mathbf{w},\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}) &= \cos\left(\frac{1}{2}\mathbf{w}\cdot(\boldsymbol{\xi}+\boldsymbol{\zeta}+t(\mathbf{u}+\mathbf{v})\right)\sin\left(\frac{1}{2}\mathbf{w}\cdot(\boldsymbol{\xi}+t\mathbf{u})\right)\sin\left(\frac{1}{2}\mathbf{w}\cdot(\boldsymbol{\zeta}+t\mathbf{v})\right).
\end{aligned}$$

The fact that $f_d\in L^\infty(\mathbb{R}^{2d})$ stems from the latter decomposition.

The case $d=2$ is more difficult since s^{-1} is not integrable and computing the limit of the term I_4 is more involved. We give here the main lines of the analysis and skip some details. We have to compute the limit of terms of the form

$$\mathcal{I}^j=\int_1^{t\eta^{1-2\alpha}} s_1^{-1}f_\eta^i(s_1)ds_1,\qquad t\geq\eta^{2\alpha-1},\qquad j=1,2,$$

with either ($t,\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}$ are fixed here), case 1 (term similar to III_4 when $d\geq 3$):

$$f_\eta^1(s_1)=\int_1^{\eta^{\alpha-1}}\int_{\mathbb{R}^{2d}} d\tau d\mathbf{w} d\mathbf{w}_2'\, e^{-2R_0(t-s_1(\eta^{2\alpha-1}-\eta^\alpha\tau))}\frac{1}{\tau^2}\hat{R}(h(\mathbf{w}_2'))\hat{R}(\eta(\mathbf{u}-\eta^{-\alpha}\mathbf{w}_2)-h(\mathbf{w}_2'))$$
$$\times g_1(t,s_1\eta^{2\alpha-1},s_1(\eta^{2\alpha-1}-\eta^\alpha\tau),\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{u}-\eta^{-\alpha}\mathbf{w}_2-\eta^{-1}h(\mathbf{w}_2'),h(\mathbf{w}_2'))e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}$$
$$\times e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\mathcal{F}a_0\otimes a_0(\mathbf{w}_2,\eta^{1-\alpha}(\boldsymbol{\zeta}+\boldsymbol{\xi}+t(\mathbf{v}+\mathbf{u}))-\mathbf{w}_2',\eta^\alpha(\mathbf{v}+\mathbf{u})-\mathbf{w}_2,\mathbf{w}_2'),$$

or, case 2 (term similar to II_4):

$$f_\eta^2(s_1)=\int_0^1\int_{\mathbb{R}^{2d}} d\tau d\mathbf{w} d\mathbf{w}'\, e^{-2R_0(t-s_1(\eta^{2\alpha-1}-\eta^\alpha\tau))}\frac{\hat{R}(\mathbf{w}')}{(1-\eta^{1-\alpha}\tau)^2}\hat{R}(\eta\mathbf{u}-\eta^{1-\alpha}\tilde{h}(\mathbf{w})-\mathbf{w}')$$
$$\times g_1(t,s_1\eta^{2\alpha-1},s_1(\eta^{2\alpha-1}-\eta^\alpha\tau),\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{u}-\eta^{-\alpha}\tilde{h}(\mathbf{w})-\eta^{-1}\mathbf{w}',\mathbf{w}')e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})}$$
$$\times e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})}\mathcal{F}a_0\otimes a_0(\tilde{h}(\mathbf{w}),\mathbf{w},\eta^\alpha(\mathbf{v}+\mathbf{u})-\tilde{h}(\mathbf{w}),\eta^{1-\alpha}(\boldsymbol{\zeta}+\boldsymbol{\xi}+t(\mathbf{v}+\mathbf{u}))-\mathbf{w}).$$

Above, h and $\tilde{h}$ are defined in (79)-(80). Since the function f_η^1 is uniformly bounded in all variables, the integral $\mathcal{I}^1$ is expected to be of order $\log\eta^{1-2\alpha}$. To see that, we integrate by parts and obtain that

$$\int_1^{t\eta^{1-2\alpha}} s^{-1}f_\eta^1(s)ds=\log(t\eta^{1-2\alpha})f_\eta^1(t\eta^{1-2\alpha})-\int_1^{t\eta^{1-2\alpha}}\log s\,(f_\eta^1)'(s)ds. \tag{81}$$

Since in particular $h(\mathbf{w}_2') \to (s\tau)^{-1}\mathbf{w}_2' + \tau^{-1}\mathbf{w}_2$, we first verify that $f_\eta^1(t\eta^{1-2\alpha}) \to f_0^1(t)$ in $L^\infty((0,T)\times\mathbb{R}^{4d}) - *$, with

$$\begin{aligned} f_0^1(z) &= -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})} e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})} \\ &\quad \int_1^\infty \int_{\mathbb{R}^{2d}} \tau^{-2}\, d\tau d\mathbf{w}_2 d\mathbf{w}_2'\, e^{-2R_0(t-z)}\, \hat{R}^2(\tau^{-1}\mathbf{w}_2) \\ &\quad \times F(t-z,\tau^{-1}\mathbf{w}_2,\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta})\mathcal{F}a_0\otimes a_0(\mathbf{w}_2,-\mathbf{w}_2',-\mathbf{w}_2,\mathbf{w}_2'), \end{aligned}$$

where F is defined as above. Therefore, the first term of r.h.s of (81) is of order $\log\eta^{1-2\alpha}$. It remains the second term involving $(f_\eta^1)'$. We claim it can be written as $(f_\eta^1)'(s) = \eta^{2\alpha-1}2R_0 f_\eta^1(s) + h_\eta^1(s) + r_\eta(s)$, where h_η^1 has the same expression as f_η^1 except g_1 is replaced by $\tilde{g}_1$ with

$$\begin{aligned} &\tilde{g}_1(t, s_1\eta^{2\alpha-1}, s_1(\eta^{2\alpha-1}-\eta^\alpha\tau), \mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') = \\ &\eta^{2\alpha-1}(\partial_s g_1)(t, s_1\eta^{2\alpha-1}, s_1(\eta^{2\alpha-1}-\eta^\alpha\tau), \mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') \\ &+(\eta^{2\alpha-1}-\eta^\alpha\tau)(\partial_\tau g_1)(t, s_1\eta^{2\alpha-1}, s_1(\eta^{2\alpha-1}-\eta^\alpha\tau), \mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}'), \end{aligned}$$

and some lengthy calculations show that

$$|r_\eta(s)| \lesssim \eta^\alpha|\log\eta| + \frac{1}{s^2}(1+|\mathbf{u}|+|\mathbf{v}|+|\boldsymbol{\xi}|+|\boldsymbol{\zeta}|)^2. \tag{82}$$

This requires in particular to regularize $\hat{R}$ since r_η involves $\nabla\hat{R}$. This has no incidence on the leading term since r_η is negligible and the limit does not depend on $\nabla\hat{R}$. We thus have:

$$\begin{aligned} \int_1^{t\eta^{1-2\alpha}} \log s\,(f_\eta^1)'(s)ds &= \eta^{2\alpha-1}2R_0\int_1^{t\eta^{1-2\alpha}} \log s\, f_\eta^1(s) + \int_1^{t\eta^{1-2\alpha}} \log s\, h_\eta^1(s)ds \\ &\quad + \int_1^{t\eta^{1-2\alpha}} \log s\, r_\eta(s)ds. \end{aligned} \tag{83}$$

Estimate (82) implies, for any $\varphi\in\mathcal{C}^0([0,T],\mathcal{S}(\mathbb{R}^{4d}))$,

$$(\log\eta^{1-2\alpha})^{-1}\int_0^T\int_1^{t\eta^{1-2\alpha}} \log s\, r_\eta\, \varphi\, dsdtd\mathbf{u}d\mathbf{v}d\boldsymbol{\xi}d\boldsymbol{\zeta} \to 0.$$

The term related to r_η can thus be neglected. It remains the terms in the r.h.s of (83) for which we perform the change of variable $s\to s\eta^{1-2\alpha}$. This yields in $L^\infty((0,T)\times\mathbb{R}^{4d}) - *$:

$$\begin{aligned} (\log\eta^{1-2\alpha})^{-1}\eta^{2\alpha-1}\int_1^{t\eta^{1-2\alpha}} \log s\, f_\eta^1(s)ds &= (\log\eta^{1-2\alpha})^{-1}\int_{\eta^{2\alpha-1}}^t \log(s\eta^{1-2\alpha})\, f_\eta^1(s\eta^{1-2\alpha})ds \\ &\to \int_0^t f_0^1(s)ds. \end{aligned}$$

Regarding the term involving h_η^1, we verify that

$$(\log\eta^{1-2\alpha})^{-1}\int_1^{t\eta^{1-2\alpha}} \log s\, h_\eta^1(s)ds \quad\to\quad \int_0^t h_0^1(s)ds,$$

where h_0^1 has the same expression as f_0^1 except that $F(t-z,\cdot)$ is replaced by $-\partial_z F(t-z,\cdot)$. Gathering the previous results, we thus find, for any $\varphi\in\mathcal{C}^0([0,T],\mathcal{S}(\mathbb{R}^{4d}))$,

$$\begin{aligned} &(\log\eta^{1-2\alpha})^{-1}\int_0^T\int_{\mathbb{R}^d} \mathcal{I}^1\,\varphi\, dtd\mathbf{u}d\mathbf{v}d\boldsymbol{\xi}d\boldsymbol{\zeta} \qquad (84) \\ \to &\int_0^T\int_{\mathbb{R}^d}\left[f_0^1(t) - \int_0^t(2R_0 f_0^1(s) + h_0^1(s))ds\right]\varphi\, dtd\mathbf{u}d\mathbf{v}d\boldsymbol{\xi}d\boldsymbol{\zeta} = \int_0^T\int_{\mathbb{R}^d} f_0^1(0)\varphi\, dtd\mathbf{u}d\mathbf{v}d\boldsymbol{\xi}d\boldsymbol{\zeta}. \end{aligned}$$

Regarding $\mathcal{I}^2$, we verify that $f_\eta^2(t\eta^{1-2\alpha}) \to f_0^2(t)$ in $L^\infty((0,T)\times\mathbb{R}^{4d}) - *$, with

$$\begin{aligned} f_0^2(z) &= -8\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})} e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})} \int_0^1 \int_{\mathbb{R}^{2d}} d\tau d\mathbf{w}' d\mathbf{w} e^{-2R_0(t-z)} \hat{R}^2(\mathbf{w}') \\ & \quad F(t-z,\mathbf{w}',\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta})\mathcal{F}a_0\otimes a_0(\tau\mathbf{w}',\mathbf{w},-\tau\mathbf{w}',\mathbf{w}). \end{aligned}$$

In the same manner as $\mathcal{I}^1$, we write $(f_\eta^2)'(s) = \eta^{2\alpha-1}2R_0 f_\eta^2(s) + h_\eta^2(s) + r_\eta(s)$, where r_η yields a negligible term. Following along the same lines, we find the same relation as (84) for $\mathcal{I}^2$, with f_0^1 replaced by f_0^2. Summing f_0^1 and f_0^2 then gives in the end, since now I_1, I_2 and I_3 are negligible compared to I_4:

$$(\log\eta^{1-2\alpha})\mathcal{F}G_0^1(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = e^{-2R_0 t}\, e^{-i\mathbf{x}_0\cdot(\mathbf{u}+\mathbf{v})} e^{-i\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})} k_2(\boldsymbol{\xi}+t\mathbf{u},\boldsymbol{\zeta}+t\mathbf{v}).$$

The fact that $k_2 \in L^\infty(\mathbb{R}^{2d})$ follows from separate estimates of f_0^1 and f_0^2. This ends the proof of the proposition. □

The term G_η^2. We have the following proposition:

Proposition 4.13 *Let $\varphi \in \mathcal{C}^0([0,T],\mathcal{S}(\mathbb{R}^{4d}))$, $0 \le \alpha < 1$ and $g_d(\eta) = \eta^{-d(1-\alpha)-\alpha-(2\alpha-1)\vee 0}$ if $d \ge 3$ and $g_2(\eta) = \eta^{-2(1-\alpha)-\alpha}(\eta^{2\alpha-1}(1+|\log\eta^{1-2\alpha}|))^{-1}\vee 1$. Then, as η goes to zero, denoting by $(\cdot,\cdot)$ the $L^2(\mathbb{R}^{4d})$ scalar product,*

$$g_d(\eta)\int_0^T (G_\eta^2(t,\cdot),\varphi(t,\cdot))dt \to 0.$$

Proof. We have:

$$\begin{aligned} & g_2(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}',\mathbf{w}') \\ & \quad = 8\,\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2) \\ & \quad \times\cos\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2+\eta^{-1}(s-\tau)|\mathbf{w}'|^2\right]. \end{aligned}$$

We decompose the cosine as

$$\begin{aligned} & \cos\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2+\eta^{-1}(s-\tau)|\mathbf{w}'|^2\right] = \\ & \cos\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2-\eta^{-1}\tau|\mathbf{w}'|^2\right]\cos\left(\eta^{-1}s|\mathbf{w}'|^2\right) \\ & -\sin\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2-\eta^{-1}\tau|\mathbf{w}'|^2\right]\sin\left(\eta^{-1}s|\mathbf{w}'|^2\right), \end{aligned}$$

and split g_1 accordingly so that $G_\eta^2 = G_\eta^{21} + G_\eta^{22}$. Both terms are treated similarly, so that we only focus on the first one. Introducing the notation

$$\begin{aligned} & 8\,\sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2)\sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2)\cos\left(\eta^{-1}s|\mathbf{w}'|^2\right) \\ & \cos\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2-\eta^{-1}\tau|\mathbf{w}'|^2\right] \\ & := h(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\cos\left(\eta^{-1}s|\mathbf{w}'|^2\right), \end{aligned}$$

and integrating by parts the cosine, this yields from (77):

$$\begin{aligned} (\mathcal{F}G_\eta^{21})(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= \eta^d \int_0^t\int_0^s\int_{\mathbb{R}^{2d}} H(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') \\ & \qquad \times\cos\left(\eta^{-1}s|\mathbf{w}'|^2\right) ds d\tau d\mathbf{w} d\mathbf{w}' \quad := \quad I+II+III, \end{aligned}$$

$$
\begin{aligned}
I &= \eta^{d+1}\left[\int_0^s\int_{\mathbb{R}^{2d}}\frac{1}{|\mathbf{w}'|^2}H(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\,\sin\left(\eta^{-1}s|\mathbf{w}'|^2\right)d\tau d\mathbf{w}d\mathbf{w}'\right]_{s=0}^{s=t},\\
II &= -\eta^{d+1}\int_0^t\int_{\mathbb{R}^{2d}}\frac{1}{|\mathbf{w}'|^2}H(t,s,s,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\sin\left(\eta^{-1}s|\mathbf{w}'|^2\right)dsd\mathbf{w}d\mathbf{w}',\\
III &= -\eta^{d+1}\int_0^t\int_0^s\int_{\mathbb{R}^{2d}}\frac{1}{|\mathbf{w}'|^2}\partial_s H(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\sin\left(\eta^{-1}s|\mathbf{w}'|^2\right)dsd\tau d\mathbf{w}d\mathbf{w}',
\end{aligned}
$$

with

$$
\begin{aligned}
&H(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') = e^{-2R_0(t-\tau)}\hat{R}(\mathbf{w}')\hat{R}(\eta(\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}'))\\
&\quad\times h(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\mathcal{F}a_{\eta0}\otimes a_{\eta0}\big(\mathbf{w},\boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}'+\tau\mathbf{w},\\
&\qquad\mathbf{v}+\mathbf{u}-\mathbf{w},\boldsymbol{\zeta}+t\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}'+\tau(\mathbf{u}-\mathbf{w})\big).
\end{aligned}
$$

Let us consider first the term I that reads

$$
I = \eta^{d+1}\int_0^t\int_{\mathbb{R}^{2d}}\frac{1}{|\mathbf{w}'|^2}H(t,t,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\sin\left(\eta^{-1}t|\mathbf{w}'|^2\right)d\tau d\mathbf{w}d\mathbf{w}',
$$

and assume in the beginning that $d\geq 3$. For the case $0\leq\alpha\leq\frac{1}{2}$, we perform the change of variable $\mathbf{w}=\eta^{-\alpha}\mathbf{w}_1$ and using (71) we obtain, uniformly in $t,\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}$:

$$
|I|\lesssim t\,\eta^{d(1-\alpha)+1}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\|a_0\|_{Y_1}\|a_0\|_{Y_\infty}\int_{\mathbb{R}^d}\frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|^2}d\mathbf{w}'. \tag{85}
$$

When $d\geq 3$, the latter integral is finite (since $\hat{R}\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$) so that I is controlled by $\eta^{d(1-\alpha)+1}$ and consequently $g_d(\eta)I$ by $\eta^{1-\alpha}$ which goes to zero since we are in the case $0\leq\alpha\leq\frac{1}{2}$. When $\alpha>\frac{1}{2}$, we proceed as usual by setting $\tau=\eta^{2\alpha-1}\tau_1$ and splitting the time integral on τ_1 into short times $[0,1]$ and long times $[1,t\eta^{1-2\alpha}]$. We assume here that $t>\eta^{2\alpha-1}$ since when $t\leq\eta^{2\alpha-1}$, we already know from (85) that I is of order $\eta^{d(1-\alpha)+1+2\alpha-1}$ so that $g_d(\eta)I$ tends to zero. Following (85), the short times part $[0,1]$ is controlled by $\eta^{d(1-\alpha)+1+2\alpha-1}$. The long time contribution on $[1,t\eta^{1-2\alpha}]$ is bounded by

$$
\begin{aligned}
&\eta^{d(1-\alpha)+1+(2\alpha-1)}\|\hat{R}\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^{2d}}\int_1^{t\eta^{1-2\alpha}}d\mathbf{w}'d\tau_1 d\mathbf{w}_1\frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|^2}\\
&|\mathcal{F}a_0\big(\mathbf{w}_1,\eta^{1-\alpha}(\boldsymbol{\xi}+(t-\eta^{2\alpha-1}\tau_1)\mathbf{u})-\eta^{-\alpha}(s-\eta^{2\alpha-1}\tau_1)\mathbf{w}'+\tau_1\mathbf{w}_1\big)|.
\end{aligned}
$$

The change of variable $\mathbf{w}_1=\tau_1^{-1}(\mathbf{w}_2-\eta^{1-\alpha}(\boldsymbol{\xi}+(t-\eta^{2\alpha-1}\tau_1)\mathbf{u})+\eta^{-\alpha}(s-\eta^{2\alpha-1}\tau_1)\mathbf{w}')$ allows us to control the time integral and we obtain that the long time integral is bounded by $\eta^{d(1-\alpha)+1+(2\alpha-1)}$. Therefore $g_d(\eta)I$ is of order $\eta^{1-\alpha}$ and goes to zero. So far, we have thus seen that for any $0<\alpha<1$ and $d\geq 3$, $g_d(\eta)I$ can be neglected. We turn now to the case $d=2$ which requires more work since the function $|\mathbf{w}'|^{-2}\hat{R}(\mathbf{w}')$ is no longer integrable. We are thus led to introducing a cut-off and perform the integration by part in G^{21}_η only on the complementary of a ball $B(r)\subset\mathbb{R}^2$ so that in addition to $I+II+III$ adds up a term of the form

$$
IV = \eta^2\int_0^t\int_0^s\int_{\mathbb{R}^2}\int_{B(r)}H(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}')\cos\left(\eta^{-1}s|\mathbf{w}'|^2\right)dsd\tau d\mathbf{w}d\mathbf{w}',
$$

where the integration on $\mathbf{w}'$ in $I+II+III$ is performed in $\mathbb{R}^2\backslash\overline{B(r)}$. Proceeding with the standard splitting of the time integral when considering the cases $\alpha\leq\frac{1}{2}$ and $\alpha>\frac{1}{2}$, we verify that IV can be uniformly controlled in all variables by $\eta^{2(1-\alpha)+(2\alpha-1)\vee 0}r^2$. The term I is just treated as for the case $d \geq 3$, unless $\mathbf{w}'$ is integrated on the complementary of $B(r)$. We have, for any $1 < p < \infty$,

$$\int_{\mathbb{R}^2 \setminus \overline{B(r)}} \frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|^2} d\mathbf{w}' \leq C \left(\int_r^\infty \frac{1}{|\mathbf{w}'|^{2p-1}} d|\mathbf{w}'| \right)^{1/p} \leq C r^{2(\frac{1}{p}-1)} := C r^{-\delta},$$

for any $0 < \delta < 1$. This finally gives the following bound for I when $d = 2$:

$$|I| \lesssim \eta^{2(1-\alpha)+1+(2\alpha-1)\vee 0} r^{-\delta}.$$

The bounds on I and IV are same order when $r = \eta^{\frac{1}{2+\delta}}$ so that we find the estimate, uniformly in $t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}$:

$$|I| + |IV| \lesssim \eta^{2(1-\alpha)+(\alpha-\beta)\vee 0+\frac{2}{2+\delta}}.$$

Since $\alpha < 1$, it is possible to find δ such that $\frac{2}{2+\delta} > \alpha$, which in turns imply $g_2(\eta)(I + IV)$ is of order $\eta^{\frac{2}{2+\delta}-\alpha}$ and therefore tends to zero in $\mathcal{C}^0([0,T], X_\infty)$.

The term II is treated exactly in the same manner as I and requires no additional work. The term III is more involved. We first write $\partial_s H = \partial_s H_1 + \partial_s H_2$ with

$$\begin{aligned}
\partial_s H_1(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') &= e^{-2R_0(t-\tau)} \hat{R}(\mathbf{w}') \hat{R}(\eta(\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}')) \\
&\times 4 \big[-(\mathbf{w}'\cdot\mathbf{u}) \cos(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2) \sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2) \\
&\quad -(\mathbf{w}'\cdot\mathbf{v}) \sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2) \cos(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2) \big] \\
&\times \cos\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2 - \eta^{-1}\tau|\mathbf{w}'|^2 \right] \\
&\times \mathcal{F}a_{\eta 0} \otimes a_{\eta 0}\big(\mathbf{w}, \boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}'+\tau\mathbf{w} \\
&\quad , \mathbf{v}+\mathbf{u}-\mathbf{w}, \boldsymbol{\zeta}+t\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}'+\tau(\mathbf{u}-\mathbf{w})\big),
\end{aligned}$$

$$\begin{aligned}
\partial_s H_2(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') &= e^{-2R_0(t-\tau)} \hat{R}(\mathbf{w}') \hat{R}(\eta(\mathbf{u}-\mathbf{w}-\eta^{-1}\mathbf{w}')) \\
&\times \sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2) \sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2) \\
&\times \sin(\mathbf{w}'\cdot(\boldsymbol{\xi}+(t-s)\mathbf{u})/2) \sin(\mathbf{w}'\cdot(\boldsymbol{\zeta}+(t-s)\mathbf{v})/2) \\
&\times \cos\left[\eta(\mathbf{u}-\mathbf{w})\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}+(t-\tau)(\mathbf{u}-\mathbf{v}))/2 - \eta^{-1}\tau|\mathbf{w}'|^2 \right] \\
&\times \eta^{-1}\mathbf{w}'\cdot(-\nabla_2+\nabla_4)\,\mathcal{F}a_{\eta 0} \otimes a_{\eta 0}\big(\mathbf{w}, \boldsymbol{\xi}+(t-\tau)\mathbf{u}-\eta^{-1}(s-\tau)\mathbf{w}'+\tau\mathbf{w} \\
&\quad , \mathbf{v}+\mathbf{u}-\mathbf{w}, \boldsymbol{\zeta}+t\mathbf{v}+\eta^{-1}(s-\tau)\mathbf{w}'+\tau(\mathbf{u}-\mathbf{w})\big),
\end{aligned}$$

and set $III := III_1 + III_2$ accordingly. Above, we separated the derivatives of $\mathcal{F}a_{0\eta} \otimes a_{\eta 0}$ from the rest, $\nabla_2 h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = \nabla_{\boldsymbol{\xi}} h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})$ and $\nabla_4 h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = \nabla_{\boldsymbol{\zeta}} h(\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta})$. The III_1 term is treated almost as I unless the singularity is now $|\mathbf{w}'|^{-1}$ so that $|\mathbf{w}'|^{-1}\hat{R}(\mathbf{w}')$ is integrable for any $d \geq 2$, and a change of topology is needed since the s derivative yields terms proportional to $\mathbf{w}'\cdot\mathbf{u}$ and $\mathbf{w}'\cdot\mathbf{v}$. We thus find, $\forall (t,\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}) \in [0,T]\times\mathbb{R}^{4d}$:

$$|III_1| \lesssim \eta^{d(1-\alpha)+1+(2\alpha-1)\vee 0} \left(|\mathbf{u}|+|\mathbf{v}|\right) \|\hat{R}\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|} d\mathbf{w}'.$$

so that III_1 is of order $\eta^{d(1-\alpha)+1+(2\alpha-1)\vee 0}$ for any $d \geq 2$ in the $\mathcal{C}^0([0,T], Z')$ norm to account for the weight $|\mathbf{u}|+|\mathbf{v}|$. The III_2 term is the most technical to deal with and we consider only the term involving ∇_2 as the contribution ∇_4 can be estimated analogously. We rewrite $\partial_s H_2$ as

$$\partial_s H_2 = \eta^{-1} Q(t,s,\tau,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta},\mathbf{w},\mathbf{w}') \hat{R}(\mathbf{w}') \mathbf{w}'\cdot\nabla_2 \mathcal{F}a_{\eta 0} \otimes a_{\eta 0} + \text{term proportional to } \nabla_4,$$

with obvious identification for Q with the property $|Q| \leq \|\hat{R}\|_{L^\infty(\mathbb{R}^d)}$ uniformly in all variables. Following the expression of III, we are thus led to studying the integral

$$\begin{aligned} V &= \eta^d \int_0^t \int_0^s \int_{\mathbb{R}^{2d}} ds d\tau d\mathbf{w} d\mathbf{w}' \frac{1}{|\mathbf{w}'|^2} \hat{R}(\mathbf{w}') \sin\left(\eta^{-1} s |\mathbf{w}'|^2\right) Q \\ & \quad \mathbf{w}' \cdot \nabla_2 \mathcal{F} a_{\eta 0} \otimes a_{\eta 0}\big(\mathbf{w}, \boldsymbol{\xi} + (t-\tau)\mathbf{u} - \eta^{-1}(s-\tau)\mathbf{w}' + \tau \mathbf{w}, \\ & \qquad \mathbf{v} + \mathbf{u} - \mathbf{w}, \boldsymbol{\zeta} + t\mathbf{v} + \eta^{-1}(s-\tau)\mathbf{w}' + \tau(\mathbf{u} - \mathbf{w})\big). \end{aligned} \tag{86}$$

The approach is very close to that of the proof of lemma 4.3. The main difference lies in the presence of the singular factor $\mathbf{w}'|\mathbf{w}'|^{-2}$ which requires particular care. Using first the expression of the Fourier transform of $a_{\eta 0}$ given in (71), we have

$$\begin{aligned} \nabla_2 \mathcal{F} a_{\eta 0}(\mathbf{u}, \boldsymbol{\xi}) &= \eta^{1-\alpha} e^{-i(\mathbf{u}\cdot\mathbf{x}_0 + \boldsymbol{\xi}\cdot\mathbf{k}_0)} (\nabla_2 \mathcal{F} a_0)(\eta^\alpha \mathbf{u}, \eta^{1-\alpha}\boldsymbol{\xi}), \\ &= -i\eta^{1-\alpha} e^{-i(\mathbf{u}\cdot\mathbf{x}_0 + \boldsymbol{\xi}\cdot\mathbf{k}_0)} (\mathcal{F}\mathbf{k} a_0)(\eta^\alpha \mathbf{u}, \eta^{1-\alpha}\boldsymbol{\xi}). \end{aligned}$$

And after the change of variable $\mathbf{w}_1 = \eta^{-\alpha}\mathbf{w}$, we find the straightforward estimate, uniformly for $(t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in [0, T] \times \mathbb{R}^{4d}$:

$$|V| \lesssim \eta^{d(1-\alpha)+1-\alpha}\, t^2 \|\mathbf{k} a_0\|_{Y_1} \|a_0\|_{Y_\infty} \int_{\mathbb{R}^d} \frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|} d\mathbf{w}'. \tag{87}$$

Above, $\mathbf{k} a_0$ is bounded in Y_1 since

$$\|\mathbf{k} a_0\|_{Y_1} \leq \|\mathcal{F}_\mathbf{x} \mathbf{k} a_0\|_{L^1(\mathbb{R}^{2d})} \leq \|\mathcal{F}\nabla_\mathbf{x} \psi_1^{(1)}\|^2_{L^1(\mathbb{R}^d)} \leq C,$$

where $\psi_1^{(1)}$ is the rescaled initial condition deduced from (9). Hence, when $t \leq \eta^\alpha$, V is of order $\eta^{d(1-\alpha)+1+\alpha}$ so that $\mathbb{1}_{t \leq \eta^\alpha} g_d(\eta) V \to 0$ in $L^\infty((0,T) \times \mathbb{R}^{4d})$ when $\alpha < 1$ for any $d \geq 2$. From now on, assume therefore that $t \geq \eta^\alpha$. In (86), we then separate times $s \leq \eta^\alpha$ and times $\eta^\alpha \leq s \leq t$ and perform the change of variable $\tau = s - \eta^\alpha \tau_1$ in the part $s > \eta^\alpha$. Splitting the integral over τ_1 in $[0,1]$ and $[1, s\eta^{-\alpha}]$, we recast V as

$$\begin{aligned} V &= \eta^{d(1-\alpha)+1-\alpha} \int_0^{\eta^\alpha} \int_0^s (\cdot) + \eta^{d(1-\alpha)+1} \int_{\eta^\alpha}^t \int_0^1 (\cdot) + \eta^{d(1-\alpha)+1} \int_{\eta^\alpha}^t \int_1^{s\eta^{-\alpha}} (\cdot), \\ &:= V_0 + V_1 + V_2. \end{aligned}$$

V_0 is estimated using (87) with $t = \eta^\alpha$. Similarly, we find for V_1:

$$\forall (t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in [\eta^\alpha, T] \times \mathbb{R}^{4d}, \qquad |V_1| \lesssim \eta^{d(1-\alpha)+1}\, t\, \|\mathbf{k} a_0\|_{Y_1} \|a_0\|_{Y_\infty} \int_{\mathbb{R}^d} \frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|} d\mathbf{w}'.$$

For the long times part V_2, we make in addition the change of variable

$$\mathbf{w}' = \tau_1^{-1}(\mathbf{w}_1' - \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + (s - \eta^\alpha \tau_1)\mathbf{u}) + \eta^{1-2\alpha}(s - \eta^\alpha \tau_1)\mathbf{w}_1) := \tau_1^{-1} f(\mathbf{w}_1')$$

and obtain the bound, $\forall (t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in [\eta^\alpha, T] \times \mathbb{R}^{4d}$:

$$|V_2| \lesssim \eta^{d(1-\alpha)+1} \|\mathbf{k} a_0\|_{Y_1} \int_{\eta^\alpha}^t \int_1^{s\eta^{-\alpha}} \int_{\mathbb{R}^d} ds d\tau_1 d\mathbf{w}_1' \tau_1^{1-d} \frac{\hat{R}(\tau_1^{-1} f(\mathbf{w}_1'))}{|f(\mathbf{w}_1')|} \sup_{\mathbf{z} \in \mathbb{R}^d} |\mathcal{F} a_0(\mathbf{z}, \mathbf{w}_1')|.$$

The function $|f(\mathbf{w}_1')|^{-1}$ is integrable in the vicinity of the origin for any $d \geq 2$. So, splitting the integral over $\mathbf{w}_1'$ for $|f(\mathbf{w}_1')| < 1$ and $|f(\mathbf{w}_1')| \geq 1$ finally gives, $\forall (t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in [\eta^\alpha, T] \times \mathbb{R}^{4d}$:

$$|V_2| \lesssim \eta^{d(1-\alpha)+1}\, t, \qquad \text{for } d \geq 3, \qquad |V_2| \lesssim \eta^{2(1-\alpha)+1} |\log \eta|\, t, \qquad \text{for } d = 2.$$

This gives a first estimate for V suitable when $\alpha \le \frac{1}{2}$. Indeed, in this case, we verify that $\mathbb{1}_{t>\eta^\alpha} g_d(\eta) V \to 0$ in $L^\infty((0,T)\times\mathbb{R}^{4d})$. When $\alpha > \frac{1}{2}$, we need a refined estimate. Hence, we perform in addition the change of variable $s = \eta^{2\alpha-1} s_1$ in V_1 and V_2 and write

$$
\begin{aligned}
V_1 + V_2 &= \eta^{d(1-\alpha)+1+2\alpha-1} \int_{\eta^{1-\alpha}}^{t\eta^{1-2\alpha}} \int_0^{s_1\eta^{\alpha-1}} (\cdot), \\
&= \eta^{d(1-\alpha)+1+2\alpha-1} \left(\int_{\eta^{1-\alpha}}^1 \int_0^1 (\cdot) + \int_{\eta^{1-\alpha}}^1 \int_1^{s_1\eta^{\alpha-1}} (\cdot) + \int_1^{t\eta^{1-2\alpha}} \int_0^{s_1\eta^{\alpha-1}} (\cdot) \right), \\
&:= V_3 + V_4 + V_5, \\
V_3 &= -i\, e^{-i((\mathbf{u}+\mathbf{v})\cdot\mathbf{x}_0 + (\boldsymbol{\xi}+t\mathbf{u}+\boldsymbol{\zeta}+t\mathbf{v})\cdot\mathbf{k}_0)}\, \eta^{d(1-\alpha)+1+2\alpha-1} \\
&\quad \times \int_{\eta^{1-\alpha}}^1 \int_0^1 ds_1 d\tau_1 d\mathbf{w}_1 d\mathbf{w}' \frac{1}{|\mathbf{w}'|^2} \hat{R}(\mathbf{w}') \sin\left(\eta^{-1} s |\mathbf{w}'|^2\right) Q \\
&\quad \mathbf{w}' \cdot (\mathcal{F}\mathbf{k}a_0)(\mathbf{w}_1, \eta^{1-\alpha}(\boldsymbol{\xi} + (t - s_1\eta^{2\alpha-1} + \eta^\alpha\tau_1)\mathbf{u} - \tau_1\mathbf{w}' + (s_1 - \eta^{1-\alpha}\tau_1)\mathbf{w}_1) \\
&\quad , \eta^\alpha(\mathbf{u}+\mathbf{v}) - \mathbf{w}_1, \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + (s_1\eta^{2\alpha-1} - \eta^\alpha\tau_1)\mathbf{u} + \tau_1\mathbf{w}' - (s_1 - \eta^{1-\alpha}\tau_1)\mathbf{w}_1),
\end{aligned}
$$

with similar expressions for V_4 and V_5. Estimating V_3 is straightforward and we find, $\forall (t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in [\eta^\alpha, T] \times \mathbb{R}^{4d}$,

$$
|V_3| \lesssim \eta^{d(1-\alpha)+1+2\alpha-1} \|\mathbf{k}a_0\|_{Y_1} \|a_0\|_{Y_\infty} \int_{\mathbb{R}^d} \frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|} d\mathbf{w}'.
$$

Regarding V_4, we set $\mathbf{w}' = h(\mathbf{w}_2') := (\tau_1)^{-1}(\mathbf{w}_2' - \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + (s_1\eta^{2\alpha-1} - \eta^\alpha\tau_1)\mathbf{u}) + (s_1 - \eta^{1-\alpha}\tau_1)\mathbf{w}_1)$. It comes, using the fact that $|h(\mathbf{w}_2')|^{-1}$ is integrable around the origin, $\forall (t, \mathbf{u}, \mathbf{v}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in [\eta^\alpha, T] \times \mathbb{R}^{4d}$:

$$
\begin{aligned}
|V_4| &\lesssim \eta^{d(1-\alpha)+1+2\alpha-1} \int_{\mathbb{R}^d} \frac{\hat{R}(h(\mathbf{w}_2'))}{|h(\mathbf{w}_2')|} \sup_{\mathbf{x}\in\mathbb{R}^d} |\mathcal{F}a_0(\mathbf{x}, \mathbf{w}_2')| d\mathbf{w}_2' \\
&\quad \times \int_{\mathbb{R}^d} \sup_{\mathbf{x}\in\mathbb{R}^d} |\mathcal{F}\mathbf{k}a_0(\mathbf{w}, \mathbf{x})| d\mathbf{w} \int_{\eta^{1-\alpha}}^1 \int_1^{s_1\eta^{\alpha-1}} \tau^{-d} d\tau ds_1, \quad \lesssim \quad \eta^{d(1-\alpha)+1+2\alpha-1}.
\end{aligned}
$$

It remains to analyze V_5. We set $\tau_1 = s_1\tau$ and write

$$
V_5 = \eta^{d(1-\alpha)+1+2\alpha-1} \left(\int_1^{t\eta^{1-2\alpha}} \int_0^1 (\cdot) + \int_1^{t\eta^{1-2\alpha}} \int_1^{\eta^{\alpha-1}} (\cdot) \right) \quad := \quad V_5^1 + V_5^2.
$$

In V_5^1, we perform the change of variable $\mathbf{w} = s_1(1-\eta^{1-\alpha}\tau)^{-1}(-\mathbf{w}_1 + \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + s_1(\eta^{2\alpha-1} - \eta^\alpha\tau)\mathbf{u}) + s_1\tau\mathbf{w}')$. This yields:

$$
\begin{aligned}
|V_5^1| &\lesssim \eta^{d(1-\alpha)+1+2\alpha-1} \|\mathbf{k}a_0\|_{Y_\infty} \int_{\mathbb{R}_d} \sup_{\mathbf{x}\in\mathbb{R}^d} |\mathcal{F}a_0(\mathbf{x}, \mathbf{w})| d\mathbf{w} \\
&\quad \times \int_{\mathbb{R}^d} \frac{\hat{R}(\mathbf{w}')}{|\mathbf{w}'|} d\mathbf{w}' \int_1^{t\eta^{1-2\alpha}} \int_0^1 s_1^{1-d}(1-\eta^{1-\alpha}\tau)^{-d} ds_1 d\tau, \\
&\lesssim \begin{cases} \eta^{d(1-\alpha)+1+2\alpha-1}, & \text{when } d \ge 3, \\ \eta^{2(1-\alpha)+1+2\alpha-1} |\log\eta|, & \text{when } d = 2. \end{cases}
\end{aligned}
$$

Regarding V_5^2, we set $\mathbf{w}' = h(\mathbf{w}_2') := (s_1\tau)^{-1}(\mathbf{w}_2' - \eta^{1-\alpha}(\boldsymbol{\zeta} + t\mathbf{v} + s_1(\eta^{2\alpha-1} - \eta^\alpha\tau)\mathbf{u}) + s_1(-\eta^{1-\alpha}\tau)\mathbf{w}_1)$. It comes, using the fact that $|h(\mathbf{w}_2')|^{-1}$ is integrable around the origin,

$\forall (t,\mathbf{u},\mathbf{v},\boldsymbol{\xi},\boldsymbol{\zeta}) \in [\eta^\alpha, T]\times \mathbb{R}^{4d}$:

$$
\begin{aligned}
|V_5^2| &\lesssim \eta^{d(1-\alpha)+1+2\alpha-1} \int_{\mathbb{R}^d} \frac{\hat{R}(h(\mathbf{w}_2'))}{|h(\mathbf{w}_2')|} |\mathcal{F}a_0(\mathbf{x},\mathbf{w}_2')| d\mathbf{w}_2' \\
&\quad \times \int_{\mathbb{R}^d} \sup_{\mathbf{x}\in\mathbb{R}^d} |\mathcal{F}\mathbf{k}a_0(\mathbf{w}_1,\mathbf{x})| d\mathbf{w}_1 \int_1^{t\eta^{1-2\alpha}} s^{1-d} ds \int_1^\infty \tau^{-d} d\tau, \\
&\lesssim \begin{cases} \eta^{d(1-\alpha)+1+2\alpha-1}, & \text{when } d\geq 3, \\ \eta^{2(1-\alpha)+1+2\alpha-1} |\log \eta|, & \text{when } d=2. \end{cases}
\end{aligned}
$$

Gathering the different estimates on I, II, III, IV and V then ends the proof of the proposition. □

4.7.2 The case $\alpha = 1$.

The Fourier transform of J_η^{00} is given in (72). After the change of variable $\mathbf{w} = \eta^{-1}\mathbf{w}_1$ and $s = \eta s_1$, this yields:

$$
\begin{aligned}
(\mathcal{F}J_\eta^{00})(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) = -4\eta \int_0^{t\eta^{-1}} &\int_{\mathbb{R}^d} ds_1 d\mathbf{w}_1\, e^{-2R_0(t-\eta s_1)} \hat{R}(\mathbf{w}_1) \\
\times e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))} e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))} &\sin\Big(\frac{1}{2}\mathbf{w}_1\cdot(\boldsymbol{\xi}+t\mathbf{u})\Big) \sin\Big(\frac{1}{2}\mathbf{w}_1\cdot(\boldsymbol{\zeta}+t\mathbf{v})\Big) \\
\times \mathcal{F}a_0\otimes a_0\big(\eta\mathbf{u}-\mathbf{w}_1, \boldsymbol{\xi}+t\mathbf{u}-s_1\mathbf{w}_1, &\eta\mathbf{v}+\mathbf{w}_1, \boldsymbol{\zeta}+t\mathbf{v}+s_1\mathbf{w}_1\big).
\end{aligned}
$$

When $t\leq\eta$, it is easy to see that $\eta^{-1}\mathbb{1}_{t\leq\eta}\mathcal{F}J_\eta^{00} \rightharpoonup 0$ in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$. For times $t\geq\eta$, we split the integral over s_1 for $s_1\in[0,1]$ and $s_1\in[1,t\eta^{-1}]$. Passing to the limit in the first integral is straightforward. For the second integral, the change of variable $\mathbf{w}_1 = s_1^{-1}\mathbf{w}$ allows to use the Lebesgue dominated convergence theorem so as to obtain that $\eta^{-1}(\mathcal{F}J_\eta^{00})(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) \to J^{00}$ in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$, where

$$
\begin{aligned}
(\mathcal{F}J^{00})(t,\mathbf{u},\boldsymbol{\xi},\mathbf{v},\boldsymbol{\zeta}) &= e^{-2R_0t} e^{-i(\mathbf{x}_0\cdot\mathbf{u}+\mathbf{k}_0\cdot(\boldsymbol{\xi}+t\mathbf{u}))} e^{-i(\mathbf{x}_0\cdot\mathbf{v}+\mathbf{k}_0\cdot(\boldsymbol{\zeta}+t\mathbf{v}))}\, k(\boldsymbol{\xi}+t\mathbf{u},\boldsymbol{\zeta}+t\mathbf{v}), \\
k(\boldsymbol{\xi},\boldsymbol{\zeta}) &= -4\int_0^\infty \int_{\mathbb{R}^d} ds_1 d\mathbf{w}_1\, \hat{R}(\mathbf{w}_1) \sin\Big(\frac{1}{2}\mathbf{w}_1\cdot\boldsymbol{\xi}\Big) \sin\Big(\frac{1}{2}\mathbf{w}_1\cdot\boldsymbol{\zeta}\Big) \\
&\qquad \times \mathcal{F}a_0\otimes a_0\big(-\mathbf{w}_1, \boldsymbol{\xi}-s_1\mathbf{w}_1, \mathbf{w}_1, \boldsymbol{\zeta}+s_1\mathbf{w}_1\big).
\end{aligned}
$$

We verify that k is indeed well-defined since

$$
\|k\|_{L^\infty(\mathbb{R}^{2d})} \lesssim \|\hat{R}\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}} \|\hat{R}\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{1}{d}} \|a_0\|_{Y_1}^{\frac{1}{d}} \|a_0\|_{Y_\infty}^{1-\frac{1}{d}},
$$

and also that J^{00} can be written as

$$
J^{00} = e^{-2R_0t}\mathcal{G}_t^2 J, \qquad J = \delta(\cdot-\mathbf{x}_0)\delta(\cdot-\mathbf{x}_0)\,\mathcal{K}(\delta(\cdot-\mathbf{k}_0)\delta(\cdot-\mathbf{k}_0)), \tag{88}
$$

where $\mathcal{K}$ is the operator defined for a tempered distribution J by $\mathcal{K}J = \mathcal{F}^{-1}(k\mathcal{F}J)$.

4.7.3 Proof of theorem 2.2: conclusion.

We recall that $J_\eta = J_\eta^0 + J_\eta^{1,\mathcal{Q}} + J_\eta^{1,\mathcal{K}}$ and compute the limit of $J_\eta^0 + J_\eta^{1,\mathcal{Q}}$ and $J_\eta^{1,\mathcal{K}}$ separately. Consider first the term $J_\eta^0 + J_\eta^{1,\mathcal{Q}}$ and assume $0<\alpha<1$. We have already seen in the proof of theorem 2.1 that the leading term in $J_\eta^0 + J_\eta^{1,\mathcal{Q}}$ is $\widetilde{J}_\eta := J_\eta^{00} + J_\eta^{4,\mathcal{Q}}$. According to (65), $\widetilde{J}_\eta$ solves the integral equation

$$
\widetilde{J}_\eta = T^{\mathcal{Q}}\widetilde{J}_\eta + J_\eta^{00},
$$

and following (35)-(67), $\eta^{-(d+2)(1-\alpha)-(2\alpha-1)\vee 0}\widetilde{J}_\eta$ is bounded in the Banach space $\mathcal{C}^0([0,T],Z')$. We can thus extract a subsequence such that

$$\eta^{-(d+2)(1-\alpha)-(2\alpha-1)\vee 0}\widetilde{J}_\eta \rightharpoonup J^1_\alpha, \qquad \text{in } L^\infty((0,T),Z')-*.$$

Let $\varphi \in \mathcal{C}^0([0,T],Z)$ and

$$T^{\mathcal{Q},*}:\mathcal{C}^0([0,T],Z)\to\mathcal{C}^0([0,T],Z), \qquad \left(T^{\mathcal{Q},*}\varphi\right)(s)=\int_s^T e^{-2R_0(t-s)}\mathcal{Q}_2\mathcal{G}^2_{s-t}\varphi(t)dt.$$

Then:

$$\int_0^T \langle \widetilde{J}_\eta,\varphi\rangle_{Z',Z}\,dt=\int_0^T \langle \widetilde{J}_\eta,T^{\mathcal{Q},*}\varphi\rangle_{Z',Z}\,dt+\int_0^T\langle J^{00}_\eta,\varphi\rangle_{Z',Z}\,dt.$$

Rescaling the latter equation by $\eta^{-(d+2)(1-\alpha)-(2\alpha-1)\vee 0}$ and passing to the limit, we find that $J^1_\alpha\in L^\infty((0,T),Z')$ satisfies

$$\int_0^T \langle J^1_\alpha,\varphi\rangle_{Z',Z}\,dt=\int_0^T \langle J^1_\alpha,T^{\mathcal{Q},*}\varphi\rangle_{Z',Z}\,dt+\int_0^T\langle J^{00},\varphi\rangle_{Z',Z}\,dt,$$

where J^{00} is defined in proposition 4.11. J^1_α is thus solution to

$$J^1_\alpha=T^{\mathcal{Q}}J^1_\alpha+J^{00}, \tag{89}$$

which admits a unique solution in $\mathcal{C}^0([0,T],Z')$ according to corollary 3.4 since $J^{00}\in\mathcal{C}^0([0,T],Z')$. This implies that the whole sequence $\eta^{-(d+2)(1-\alpha)-(2\alpha-1)\vee 0}\widetilde{J}_\eta$ converges to J^1_α.

Consider now the term $J^{1,\mathcal{K}}_\eta$ and assume $0\le\alpha<1$. The leading term in $J^{1,\mathcal{K}}_\eta$ is $J^{4,\mathcal{K}}_\eta$, solution to (53). $J^{4,\mathcal{K}}_\eta$ is of order $\eta^{d(1-\alpha)+\alpha}\left[\eta^{2\alpha-1}f_d(\eta)\right]\wedge 1$ in $\mathcal{C}^0([0,T],X_\infty)$, with $f_d(x)=1$ when $d\ge 3$ and $f_2(x)=1+|\log x^{1-2\alpha}|$. We can thus extract a subsequence such that $\eta^{-d(1-\alpha)-\alpha}(\left[\eta^{2\alpha-1}f_d(\eta)\right]^{-1}\vee 1)\,\mathcal{F}J^{4,\mathcal{K}}_\eta\rightharpoonup\mathcal{F}J^2_\alpha$ in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$. Considering a test function $\varphi\in\mathcal{C}^0([0,T],\mathcal{S}(\mathbb{R}^{4d}))$, denoting by $(\cdot,\cdot)$ the $L^2(\mathbb{R}^{4d})$ scalar product and verifying that $\mathcal{F}T^{\mathcal{Q},*}\varphi\in\mathcal{C}^0([0,T],L^1(\mathbb{R}^{4d}))$, we have

$$\int_0^T(\mathcal{F}J^{4,\mathcal{K}}_\eta,\mathcal{F}\varphi)\,dt=\int_0^T(\mathcal{F}J^{4,\mathcal{K}}_\eta,\mathcal{F}T^{\mathcal{Q},*}\varphi)\,dt+\int_0^T(\mathcal{F}T^{\mathcal{K}}_\eta J^{00}_\eta,\mathcal{F}\varphi)\,dt.$$

Recalling that $T^{\mathcal{K}}_\eta J^{00}_\eta=G^1_\eta+G^2_\eta$, rescaling the latter equation by $\eta^{-d(1-\alpha)-\alpha}(\left[\eta^{2\alpha-1}f_d(\eta)\right]^{-1}\vee 1)$ and passing to the limit using propositions 4.12 and 4.13, we find

$$\int_0^T(\mathcal{F}J^2_\alpha,\mathcal{F}\varphi)\,dt=\int_0^T(\mathcal{F}J^2_\alpha,\mathcal{F}T^{\mathcal{Q},*}\varphi)\,dt+\int_0^T(\mathcal{F}G^1_0,\mathcal{F}\varphi)\,dt,$$

where G^1_0 is defined in proposition 4.12. J^2_α is thus solution to

$$J^2_\alpha=T^{\mathcal{Q}}J^2_\alpha+G^1_0, \tag{90}$$

which admits unique solution in $\mathcal{C}^0([0,T],X_\infty)$ according to corollary 3.4 since $G^1_0\in\mathcal{C}^0([0,T],X_\infty)$. Hence the whole sequence converges.

It remains the limit of $\widetilde{J}_\eta$ when $\alpha=1$. Proceeding exactly as above, we find that the whole sequence $\eta^{-1}\mathcal{F}\widetilde{J}_\eta$ converges in $L^\infty((0,T)\times\mathbb{R}^{4d})-*$ to $\mathcal{F}J^1_1$, where J^1_1 is the unique solution to $J^1_1=T^{\mathcal{Q}}J^1_1+J^{00}$ and $J^{00}\in\mathcal{C}^0([0,T],X_\infty)$ is now given by (88).

We have proved that, when $0<\alpha<1$,

$$J^0_\eta=\eta^{(d+2)(1-\alpha)+(2\alpha-1)\vee 0}J^1_\alpha+\eta^{d(1-\alpha)+\alpha}(\left[\eta^{2\alpha-1}f_d(\eta)\right]\wedge 1)J^2_\alpha+r_\eta,$$

where r_η is negligible compared to the two first terms in the $L^\infty((0,T),\mathcal{S}'(\mathbb{R}^{4d}))-*$ topology. To obtain the expressions of the theorem, it suffices to recast (89) and (90) as partial differential equations and to rewrite (after lengthy calculations) the operators $\mathcal{K}_s^\alpha$ and $\mathcal{K}_d$ in terms of the physical variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{k}$ and $\mathbf{p}$. We verify as well that $\sigma_\alpha(t,\mathbf{k})\in L^1(\mathbb{R}_+\times\mathbb{R}^d)$ for the different values of α and that $\sigma_\alpha(0,\mathbf{k})\in L^1(\mathbb{R}^d)$. When $\alpha=0$, the leading term is proportional to J_0^2 so that $J_\eta^0=\eta^d J_0^2+r_\eta$ and the theorem follows by recasting $\mathcal{K}_s^0$ and by noticing that $\sigma_0\in\mathcal{C}^0([0,T],L^1(\mathbb{R}_+\times\mathbb{R}^d))$. When $\alpha=1$, the leading term is proportional to J_0^1 so that $J_\eta^0=\eta J_0^1+r_\eta$. The fact that $J_1^{1,0}$ is real stems from separating the G term of the theorem into real and imaginary parts and by using that $\mathcal{F}_{\mathbf{x}}a_0(-\mathbf{w},\mathbf{k})=\overline{\mathcal{F}_{\mathbf{x}}a_0(\mathbf{w},\mathbf{k})}$. When a_0 is even, $\mathcal{F}_{\mathbf{x}}a_0(\mathbf{w},\mathbf{k})=\mathcal{F}_{\mathbf{x}}a_0(-\mathbf{w},\mathbf{k})$ and the integral in principal value sense vanishes. This concludes the proof.

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