Mean-field equations reveal synchronization in a 2-populations neural network model

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Abstract.
We study a 2-populations model of analogic recurrent neural network. This model takes into account the influence of inhibitory and excitatory neurons. It is dedicated to study collective dynamical properties of large size fully connected recurrent networks. The evolution of neuron activation states is given in the thermodynamic limit by a set of mean-field equations and the network satisfies a "propagation of chaos" property. All these results are supported by rigorous proofs using large deviations techniques. Moreover, we observe that the bifurcation diagram of these mean-field equations, as well as finite size simulations, reveal a parametric domain where the expectation and variance of the limit law of the activation potentials describe periodic oscillations. Fluctuations of individual neurons around this average may occur, showing the existence of a stochastic non stationary regime for long time. This can be directly related to recent biological discoveries about the role of inhibition in the synchronization of excitatory neurons.

1. Introduction

Synchronization seems to be a very general principle of processing in the brain, especially in the first steps of sensory integration. It has indeed been observed in the processing of visual scenes [8], odor-recognition tasks [11], sensori-motor tasks [1], and even memory recall [3]. The modelling of synchronized dynamics is thus of crucial interest for the comprehension of data processing in the brain. Some theoretical models have recently been proposed [10, 6] and are subjects of growing interest in the field of neural networks.

The complexity of large size recurrent neural networks leads scientists to look for simplifications. In many models, it seems that activation potentials follow gaussian distributions and that individual intercorrelations are vanishing when the size of the network grows to infinity. The characteristics of such gaussian asymptotic distributions are given by mean-field equations. Thus, Sompolinsky [13] used statistical physics methods to obtain mean-field equations and to study the dynamical properties of continuous-time networks in the case of assymetric interactions. Cessac *et al.* [5] used the same approach for discrete-time models and obtained the vanishing correlations of activation states in the thermodynamic limit. For large size networks, they numerically showed the generical occurence of chaos by a quasi-periodicity route. It has also been established in [4] that this chaotic regime is described in the thermodynamic limit by a gaussien process.

Otherwise, mathematicians, as Geman [7], have given important contributions to that domain. Geman proved rigorously the mean-field equations for some linear models. Then, Ben Arous and Guionnet [2] showed many results in a continous time spin glasses context. They proved that the annealed law of the spins empirical measure satisfies a large deviation principle in the high temperature regime. The study of the rate function, which admits a unique minimum and the tightness obtained by Guionnet [9] allowed them to compute the weak convergence of the law of every spin to a measure given by an implicit equation. Thanks to this measures tightness, they had no temperature condition for their "propagation of chaos result", which is closely related to vanishing correlations of activation states in densely connected recurrent networks.

In the first part, we use their methods to study mathematically, in the most general case, discrete time 2-populations neural networks. We prove an exponential tightness property, and therefore establish a large deviation principle without any temperature condition. Moreover, we obtain two limit laws for the neurons of each population. These laws are given explicitly and described by the mean-field equations. In the second part, we precise a particular architecture with opposite inhibitors-excitators influences. We show some parametric domains for which the mean-field stationnary process describes periodic synchronized dynamics. This limit behavior is then compared to finite-size networks dynamics.

2. Model

We consider the following discrete time neural networks, with dynamics:

$$\left\{ \begin{array}{l} x_i^p(t) = f(u_i^p(t)) \\ u_i^p(t) = \sum_{j=1}^{n_1} J_{ij}^{p1} x_j^1(t-1) + \sum_{j=1}^{n_2} J_{ij}^{p2} x_j^2(t-1) + \sigma W_i^p(t) - \theta_i^p \end{array} \right. \tag{1}$$

For $p \in \{1,2\}$, there are n_p neurons of population p. For $p,q \in \{1,2\}^2$, the $(J_{ij}^{pq})'s$ represent the connection weights relative to the influence of population q on population p. Notice that they are not centered, contrary to Ben Arous and Guionnet's spin glasses model. The θ_i^p are the thresholds, and $W_i^p(t)$ is a synaptic noise. f is an arbitrary sigmoidal function.

Our nets are fully connected. We study the statistical behavior of this system when the sizes of the populations grow to infinity without any change in their proportion. We suppose that the distributions of the connection weights, the thresholds and the synaptic noise are gaussian laws $\mathcal{N}(\frac{\bar{J}^{pq}}{n_q}, \frac{(J^{pq})^2}{n_q})$, $\mathcal{N}(\bar{\theta}^p, (\theta^p)^2)$ and $\mathcal{N}(0,1)$. All these random variables are supposed to be independent.

3. Mathematical advances

We consider the evolution of the system between 0 and a fixed time T. We suppose $\sigma > 0$. We note P^N the law of $(x_1^1, ..x_{n_1}^1, x_1^2, .., x_{n_2}^2)$. Let $(\hat{\mu}_N^1, \hat{\mu}_N^2)$, defined for $p \in \{1,2\}$ by : $\hat{\mu}_N^p = \frac{1}{n_p}\sum_{i=1}^{n_p}\delta_{x_i^p}$. In [12], we prove the exponential tightness of the $(\hat{\mu}_N^1, \hat{\mu}_N^2)$ without any temperature condition. Moreover, we show that when N grows to infinity, these laws satisfy a large deviations principle, whose *good rate function* admits a unique minimum.

We deduce that the distribution of every $(u_i^p(t))_{1\le t\le T}$ converges towards a gaussian law Q^p. The characteristics of these two limit laws are given by the following mean-field equations : we consider $(\mu^p(t), \Delta^p(t,t'))_{1\le t,t'\le T}$ the expectation and covariance matrix of Q^p (Δ^p represents the time covariance of each population's generic neuron). In particular, $\nu^p(t) = \Delta^p(t,t)$. We note $Dh = 1/\sqrt{2\pi}exp(-h^2/2)$. Then we have :

Theorem 1 : ***Mean-field equations.***
For $p \in \{1,2\}$, and $1 \le t, t' \le T-1 (t \neq t')$:

$$\mu^p(t+1) = -\bar{\theta}^p + \bar{J}^{p1}\int f(\sqrt{\nu^1(t)}+\mu^1(t))Dh + \bar{J}^{p2}\int f(\sqrt{\nu^2(t)}+\mu^2(t))Dh \quad (2)$$

$$\nu^p(t+1) = (\sigma)^2 + (\theta^p)^2 + (J^{p1})^2\int f^2(\sqrt{\nu^1(t)}+\mu^1(t))Dh + (J^{p2})^2\int f^2(\sqrt{\nu^2(t)}+\mu^2(t))Dh \quad (3)$$

$$\Delta^p(t+1, t'+1) = (J^{p1})^2 C^1(t,t') + (J^{p2})^2 C^2(t,t') + (\theta^p)^2 \quad (4)$$

with

$$C^p(t,t') = \int\int DhDh' f\left(\frac{\sqrt{\nu^p(t)\nu^p(t') - (\Delta^p(t,t'))^2}}{\sqrt{\nu^p(t')}}h + \frac{\Delta^p(t,t')}{\sqrt{\nu^p(t')}} + \mu^p(t)\right) f(h'\sqrt{\nu^p(t')}+\mu^p(t'))$$

As they depend on a small set of parameters (in particular they don't depend on the size N), these equations are of great help for anticipating the dynamics of large neuronal assemblies (see next section).

The other mathematical main result we proved in [12] is the **propagation of chaos** : at time 0, the activation states of the neurons are chosen independent from each other. But from time 1 to T, as the net is fully connected, many relations take place between the neurons. We call propagation of chaos the property of the activation states (x_i^p) to become **independent** random vectors (of size $T+1$) when the size N of the network grows to infinity.

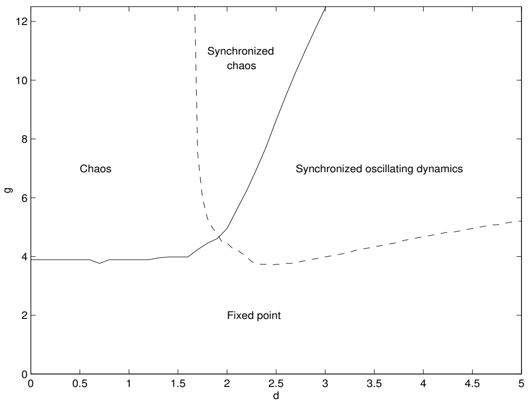
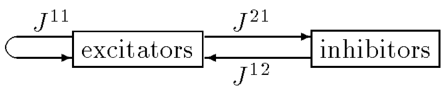


Figure 1: **Bifurcation map derived from mean-field equations**. The map describes 4 different dynamical regions when varying d (mean shift) and g (gain). Other parameters are $J = 1$, $\bar{\theta}^1 = 0$, $\theta^1 = 0$, $\bar{\theta}^2 = 0.3$, $\theta^2 = 0$

4. Dynamical properties of the mean-field process

The mean-field equations (2) (3) (4) describe a dynamical process which is easy to display with the use of computer simulations, and helps to study these equations when time t grows to infinity. They have been found to fit very well the behavior of large-size neural assemblies [12]. In this general framework, we restrict the range of the parameters in order to describe statistically inhibitory and excitatory influences and to rise up specific dynamical behaviors. We take as transfer function $f_g(x) = \frac{1+\tanh(gx)}{2}$, of gain g, for the states $x_i^p(t)$ to be comprised between 0 and 1. The synaptic weights are described with only two parameters J (reference standard deviation) and d (mean shift), according to the following statements: $\bar{J}^{11} = Jd$, $J^{11} = J$, $\bar{J}^{12} = -2Jd$, $J^{12} = \sqrt{2}J$, $\bar{J}^{21} = Jd$, $J^{21} = J$, $\bar{J}^{22} = 0$, $J^{22} = 0$. At finite size, inhibition and excitation characterize the mean influence of a population towards their receptors. As the individual laws of the synaptic weights are $\mathcal{N}(\frac{\bar{J}^{pq}}{n_q}, \frac{(J^{pq})^2}{n_q})$, they tend in any case towards a centered Dirac when $n_1, n_2 \to \infty$. Moreover, inhibitory neurons have no connections towards inhibitory population. This last point is motivated by biological considerations (see [11]).



Given some parameters (namely $J = 1$, $\bar{\theta}^1 = 0$, $\theta^1 = 0$, $\bar{\theta}^2 = 0.3$, $\theta^2 = 0$), we iterate the mean-field dynamics for different values of g and d. Two kinds of bifurcations can take place when g and d vary. These bifurcations are displayed on Figure 1. The continuous line gives the destabilization of equation (4) as a transition between a fixed point dynamics and a gaussian process (seen as a

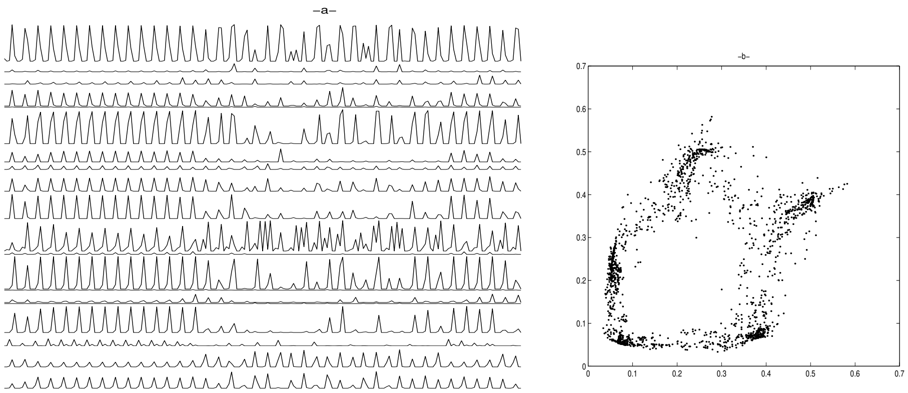


Figure 2: **Synchronized chaotic dynamics on a finite-sized network**. -a- Stationnary dynamics $x_i^1(t)$ of 20 excitatory neurons, on 200 time steps. Inner periodicity is 5. -b- $m^1(t) = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i^1(t)$ displayed in the $(t,t+1)$ space on 4000 time steps. Parameters are $n_1 = 100$, $n_2 = 100$, $J = 1$, $d = 2$, $g = 6$, $\bar{\theta}^1 = 0$, $\theta^1 = 0$, $\bar{\theta}^2 = 0.3$, $\theta^2 = 0.1$.

description of chaos at the thermodynamic limit). The dashed line, based on the evolution equation (2), gives the transition between asynchronous dynamics ($\mu^p(t)$ converges towards a fixed point μ^p) and synchronous dynamics ($\mu^p(t)$'s asymptotics oscillate). In the second case, the period of $\mu^p(t)$ is found to be comprised between 4 and 6. These two lines define four dynamical regions, namely fixed point dynamics, plain chaotic dynamics, synchronized chaotic dynamics and plain synchronized dynamics. All these dynamical regions can be found on finite sized networks in the same range of parameters. An example of synchronized chaotic dynamics is displayed Figure 2. In this particular example, intermittent synchrony is followed by more chaotic dynamics. This kind of dynamics evokes rythmic behaviours observed in the first steps of information processing in brain.

5. Conclusion

The rigorous proof obtained for the mean-field equations and the propagation of chaos confirms the simulations and gives a solid basis to our work. These equations describe with good accuracy the behaviour of large size random networks, whose dynamics can be given according to a small set of parameters. Biological considerations helped us to define the law of the synaptic weights in order to describe inhibitory and excitatory influences. In such a model, mean-field equations reveal strong phenomena of synchrony, and give the range of parameters where they can be found. Such dynamics, observed on finite-size networks, could now be compared to real brain neuronal dynamics, in order to clarify the role of inhibitors in synchronization processes.

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