Synchronization and random long time dynamics for noisy rotators in interaction

Christophe Poquet

Université Paris Dauphine and CEREMADE

April 8th 2014

In collaboration with Lorenzo Bertini (Sapienza) and Giambattista Giacomin (LPMA)
Rotators in interaction

\[ \phi_j(t) \]

\[ \phi_i(t) \]
The model

We consider the system of $N$ stochastic differential equations

$$d\varphi_j(t) = -\frac{K}{N} \sum_{i=1}^{N} \sin(\varphi_j(t) - \varphi_i(t)) \, dt + \sigma \, d\mathbf{w}_j(t)$$

for $j = 1, 2, \ldots, N$, where

- $\{\mathbf{w}_j(\cdot)\}_{j=1,2,\ldots}$ are IID standard Brownian motions
- $K \geq 0$ and $\sigma > 0$

The $\varphi_j(t)$ are angles ($\rightarrow \varphi_j(t) \mod(2\pi) \in \mathbb{S}$).
The model

We consider the system of \( N \) stochastic differential equations

\[
\mathrm{d}\varphi_j(t) = -\frac{K}{N} \sum_{i=1}^{N} \sin (\varphi_j(t) - \varphi_i(t)) \, \mathrm{d}t + \sigma \, \mathrm{d}w_j(t)
\]

for \( j = 1, 2, \ldots, N \), where

- \( \{w_j(\cdot)\}_{j=1,2,...} \) are IID standard Brownian motions
- \( K \geq 0 \) and \( \sigma > 0 \)

The \( \varphi_j(t) \) are angles (\( \rightarrow \varphi_j(t) \mod(2\pi) \in \mathbb{S} \)).

The real parameter of the model is \( K/\sigma^2 \). In the following we take \( \sigma = 1 \).
Reversibility and Symmetry

System reversible with respect to the Gibbs measure:

\[ \pi_{N,K}(d\varphi) \propto \exp \left( \frac{K}{N} \sum_{i,j=1}^{N} \cos(\varphi_i - \varphi_j) \right) \lambda_N(d\varphi), \]

where \( \lambda_N \) is the uniform measure on \( S^N \).
Reversibility and Symmetry

System reversible with respect to the Gibbs measure:

\[ \pi_{N,K}(d\varphi) \propto \exp \left( \frac{K}{N} \sum_{i,j=1}^{N} \cos(\varphi_i - \varphi_j) \right) \lambda_N(d\varphi), \]

where \( \lambda_N \) is the uniform measure on \( \mathbb{S}^N \).

Rotation symmetry:

- Dynamical: if \( \{\varphi_j(t)\}_{j=1}^{N} \) is solution, \( \{\varphi_j(t) + \psi\}_{j=1}^{N} \) is solution for all \( \psi \in \mathbb{S} \).
- Statical:

\[ \pi_{N,K} \Theta_\psi = \pi_{N,K}, \]

where for all \( \psi \in \mathbb{S} \)

\[ \Theta_\psi(\varphi)_j = \varphi_j + \psi \quad \text{for all} \quad j = 1 \ldots N. \]
The empirical measure

Define the empirical measure

\[ \mu_{N,t}(d\theta) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\varphi_j(t)}(d\theta). \]
The empirical measure

Define the empirical measure

\[ \mu_{N,t}(d\theta) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\varphi_j(t)}(d\theta). \]
The empirical measure

Define the empirical measure

\[ \mu_{N,t}(d\theta) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\varphi_j(t)}(d\theta). \]

The model can be expressed as

\[ d\varphi_j(t) = \left( J \ast \mu_{N,t}\right)(\varphi_j(t)) \, dt + d\omega_j(t), \]

with \( J(\theta) = -K \sin \theta. \)
The empirical measure and the \( N \to \infty \) limit

Recall

\[
d\varphi_j(t) = \frac{1}{N} \sum_{i=1}^{N} J(\varphi_j(t) - \varphi_i(t)) \, dt + dw_j(t),
\]

with \( J(\theta) = -K \sin \theta \).
The empirical measure and the $N \to \infty$ limit

Recall

$$d\varphi_j(t) = \frac{1}{N} \sum_{i=1}^{N} J(\varphi_j(t) - \varphi_i(t)) \, dt + dw_j(t),$$

with $J(\theta) = -K \sin \theta$.

Suppose $\lim_{N \to \infty} \mu_{N,0} = p_0(\theta) \, d\theta$, and fix a time $T > 0$ independent from $N$. Then $\lim_{N \to \infty} \mu_{N,t}(d\theta) = p_t(\theta) \, d\theta$ in $C^0([0, T]; \mathcal{M}_1)$, with

$$\partial_t p_t(\theta) = \frac{1}{2} \partial_{\theta}^2 p_t(\theta) - \partial_{\theta} [p_t(\theta)(J * p_t)(\theta)] .$$
The empirical measure and the $N \to \infty$ limit

Recall

$$d\varphi_j(t) = \frac{1}{N} \sum_{i=1}^{N} J(\varphi_j(t) - \varphi_i(t)) \, dt + dw_j(t),$$

with $J(\theta) = -K \sin \theta$.

Suppose $\lim_{N \to \infty} \mu_{N,0} = p_0(\theta) \, d\theta$, and fix a time $T > 0$ independent from $N$. Then $\lim_{N \to \infty} \mu_{N,t}(\, d\theta) = p_t(\theta) \, d\theta$ in $C^0([0, T]; M_1)$, with

$$\partial_t p_t(\theta) = \frac{1}{2} \partial_{\theta}^2 p_t(\theta) - \partial_{\theta} [p_t(\theta)(J * p_t)(\theta)].$$

Important observations:

- No space and no time rescaling
The empirical measure and the $N \to \infty$ limit

Recall
\[
d\varphi_j(t) = \frac{1}{N} \sum_{i=1}^{N} J(\varphi_j(t) - \varphi_i(t)) \, dt + d\omega_j(t),
\]
with $J(\theta) = -K \sin \theta$.

Suppose $\lim_{N \to \infty} \mu_{N,0} = p_0(\theta) \, d\theta$, and fix a time $T > 0$ independent from $N$. Then $\lim_{N \to \infty} \mu_{N,t} \, d\theta = p_t(\theta) \, d\theta$ in $C^0([0, T]; \mathcal{M}_1)$, with
\[
\partial_t p_t(\theta) = \frac{1}{2} \partial^2_{\theta} p_t(\theta) - \partial_{\theta} [p_t(\theta)(J * p_t)(\theta)].
\]

Important observations:
- No space and no time rescaling
- Rotation symmetry conserved.
Stationary solutions of the limit PDE

- $q(\theta) = \frac{1}{2\pi}$ is always a stationary solution.
Stationary solutions of the limit PDE

- $q(\theta) = \frac{1}{2\pi}$ is always a stationary solution.

- If $K > 1$, the limit model admits moreover a manifold of synchronized stationary solutions

$$M_0 = \{ q_\psi(\cdot) : \psi \in S^1 \},$$

where $q_\psi(\cdot) = q_0(\cdot - \psi)$. 
Stationary solutions of the limit PDE

- $q(\theta) = \frac{1}{2\pi}$ is always a stationary solution.

- If $K > 1$, the limit model admits moreover a manifold of synchronized stationary solutions
  
  $$M_0 = \{ q_\psi(\cdot) : \psi \in S^1 \},$$

  where $q_\psi(\cdot) = q_0(\cdot - \psi)$.
Local stability : $K > 1$

Define the operator of the linearized evolution at the neighborhood of a stationary profile $q$ :

$$-L_q u(\theta) := \frac{1}{2} u'' - [u J * q + q J * u]' .$$
Local stability: $K > 1$

Define the operator of the linearized evolution at the neighborhood of a stationary profile $q$:

$$-L_q u(\theta) := \frac{1}{2} u'' - [uJ * q + qJ * u]' .$$

$q(\theta) = \frac{1}{2\pi}$ is unstable.
Local stability : $K > 1$

Define the operator of the linearized evolution at the neighborhood of a stationary profile $q$ :

$$- L_q u(\theta) := \frac{1}{2} u'' - [u J * q + q J * u]' .$$

- $q(\theta) = \frac{1}{2\pi}$ is unstable.

- $M_0$ is locally stable :

**Theorem [Bertini, Giacomin, Pakdaman, 2010]**

$L_q$ is self-adjoint in $H_{-1, 1/q}$ and has a discrete non-negative spectrum, has no effect on the tangent space of $M_0$ at $q$ ($L_q q' = 0$), and admits a spectral gap on the normal space.
Global behaviour

Define

$$U = \left\{ p \in M_1, \int_S \exp(i\theta)p(d\theta) = 0 \right\}.$$  

If $p_0 \in U$, then $p_t \to 1/2\pi$ (heat equation).
Global behaviour

Define

\[ U = \left\{ p \in \mathcal{M}_1, \int_{\mathcal{S}} \exp(i\theta)p(d\theta) = 0 \right\}. \]

If \( p_0 \in U \), then \( p_t \to 1/2\pi \) (heat equation).

**Theorem [Giacomin, Pakdaman, Pellegrin, 2012]**

If \( p_0 \in \mathcal{M}_1 \setminus U \), then there exists \( \psi \in \mathcal{S} \) such that \( \lim_{t \to \infty} p_t =: p_\infty = q_\psi \) in \( C^k(\mathcal{S}, \mathbb{R}) \) (for all \( k \)).
Gradient flow point of view

The Fokker-Planck PDE can be rewritten in a gradient form:

$$\partial_t p_t(\theta) = \nabla \left[ p_t(\theta) \nabla \left( \frac{\delta F(p_t)}{\delta p_t} (\theta) \right) \right],$$

where (with $\tilde{J}(\cdot) = K \cos(\cdot)$)

$$F(p) := \frac{1}{2} \int_S p(\theta) \log p(\theta) \, d\theta + \frac{1}{2} \int_{S^2} \tilde{J}(\theta - \theta') p(\theta) p(\theta') \, d\theta \, d\theta'.$$
Gradient flow point of view

The Fokker-Planck PDE can be rewritten in a gradient form:

\[ \partial_t p_t(\theta) = \nabla \left[ p_t(\theta) \nabla \left( \frac{\delta F(p_t)}{\delta p_t}(\theta) \right) \right], \]

where (with \( \tilde{J}(\cdot) = K \cos(\cdot) \))

\[ F(p) := \frac{1}{2} \int_S p(\theta) \log p(\theta) \, d\theta + \frac{1}{2} \int_{S^2} \tilde{J}(\theta - \theta') p(\theta) p(\theta') \, d\theta \, d\theta'. \]

\( F \) satisfies

\[ \frac{d}{dt} F(p_t) = - \int_S p_t(\theta) \left[ \nabla \left( \frac{\delta F(p_t)}{\delta p_t}(\theta) \right) \right]^2 \, d\theta. \]
$N = 1000$, $K = 2$, $\sigma = 1$

Movie of the evolution of the empirical measure up to time $t = 15$

0.0 0.2 0.4 0.6

Christophe Poquet (CEREMADE)  Colloque Jeunes Probabilistes et Statisticiens
$N = 1000, \ K = 2, \ \sigma = 1, \ but \ much \ faster$

Movie of the evolution of the empirical measure up to time $t = 8000$

000 time units
Long time fluctuations

**Theorem [Bertini, Giacomin, P. (2013)]**

Fix a constant \(\tau_f\) and a phase \(\psi_0 \in S^1\). If for all \(\varepsilon > 0\)

\[
\lim_{N \to \infty} \mathbb{P} \left( \| \mu_{N,0} - q_{\psi_0} \|_{-1} \leq \varepsilon \right) = 1,
\]

then

\[
\lim_{N \to \infty} \mathbb{P} \left( \sup_{\tau \in [0,\tau_f]} \| \mu_{N,\tau} - q_{\psi_N^\tau} \|_{-1} \leq \varepsilon \right) = 1,
\]

where

- \(\psi_N^\tau = \psi_0 + D_K W_N^\tau\),
- \(W^N\) converges to a standard Brownian motion,
- \(D_K = \|q'\|_{-1,1/q}^{-1}\).