Exercice 6.

$$D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

$$\chi_D(X) = \det \begin{pmatrix} X-1 & -2 & -3 \\ 0 & X-4 & -5 \\ 0 & 0 & X-6 \end{pmatrix} = (X-1)(X-4)(X-6).$$

$\chi_D(X)$ étant scindé à racines simples, $\underline{D \text{ est diagonalisable}}$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E_{D,1} \iff \begin{cases} x_1 = x_1 + 2x_2 + 3x_3 \\ x_2 = 4x_2 + 5x_3 \\ x_3 = 6x_3 \end{cases}$$

$$\iff \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$$

Donc $\underline{E_{D,1} = \mathrm{Vect}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}}$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E_{D,4} \iff \begin{cases} 4x_1 = x_1 + 2x_2 + 3x_3 \\ 4x_2 = 4x_2 + 5x_3 \\ 4x_3 = 6x_3 \end{cases}$$

$$\iff \begin{cases} x_1 = \frac{2}{3} x_2 \\ x_3 = 0 \end{cases}$$

Donc $\underline{E_{D,4} = \mathrm{Vect}_{\mathbb{R}} \left\{ \begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix} \right\}}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E_{D,6} \iff \begin{cases} 6x_1 = x_1 + 2x_2 + 3x_3 \\ 6x_2 = 4x_2 + 5x_3 \\ 6x_3 = 6x_3 \end{cases}$$

$$\iff \begin{cases} 5x_1 = 8x_3 \\ x_2 = \frac{5}{2} x_3 \end{cases}$$

Donc $\underline{E_{D,6} = \mathrm{Vect}_{\mathbb{R}} \left\{ \begin{pmatrix} 8/5 \\ 5/2 \\ 1 \end{pmatrix} \right\}}$

Finalement $\underline{PD_1P^{-1} = D}$ où $\underline{D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}}$ et $\underline{P = \begin{pmatrix} 1 & 2/3 & 8/5 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix}}$.

Calcul de P^{-1} : $\left(\begin{array}{ccc|ccc} 1 & 2/3 & 8/5 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$L_1 \leftarrow L_1 - 2/3\, L_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1/15 & 1 & -2/3 & 0 \\ 0 & 1 & 5/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$L_1 \leftarrow L_1 + 1/15\, L_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1/15 \\ 0 & 1 & 5/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$L_2 \leftarrow L_2 - 5/2\, L_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/3 & 1/15 \\ 0 & 1 & 0 & 0 & 1 & -5/2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Donc $P^{-1} = \begin{pmatrix} 1 & -2/3 & 1/15 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1 \end{pmatrix}$

Exercice 9.

$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$. $P_A(X) = (X-2)(X-6) - 12 = X^2 - 8X + 0 = X(X-8)$

$E_{A,0} = Vect_{\mathbb{R}}\left\{\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right\}$. $E_{A,8} = Vect_{\mathbb{R}}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ (Fait en TD).

$A = PDP^{-1}$ avec $P = \begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}$ et $D = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$

Donc $P^{-1}AP = D = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$.

Les solutions à l'équation $M^n = A$ sont donc les matrices M telles que

$$P^{-1}M^nP = D \iff (P^{-1}MP)^n = D.$$

$$\iff \exists\, \omega \in \mathbb{C},\ \omega^n = 1,\quad M = P\begin{pmatrix} 0 & 0 \\ 0 & \sqrt[n]{8}\,\omega \end{pmatrix}P^{-1}$$

Calcul de P^{-1}:

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \qquad L_1 \leftarrow L_1 + 2L_2 \quad \left(\begin{array}{cc|cc} 1 & 5 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{array}\right)$$

$$L_2 \leftarrow L_2 - 2L_1 \quad \left(\begin{array}{cc|cc} 1 & 5 & 1 & 2 \\ 0 & -8 & -2 & -3 \end{array}\right)$$

$$L_2 \leftarrow \frac{L_2}{-8} \quad \left(\begin{array}{cc|cc} 1 & 5 & 1 & 2 \\ 0 & 1 & \frac{1}{4} & \frac{3}{8} \end{array}\right)$$

$$L_1 \leftarrow L_1 - 5L_2 \quad \left(\begin{array}{cc|cc} 1 & 0 & -1/4 & 1/8 \\ 0 & 1 & 1/4 & 3/8 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -1/4 & 1/8 \\ 1/4 & 3/8 \end{pmatrix}$$

Donc M est de la forme $\begin{pmatrix} -3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{\sqrt[n]{8}\,\omega}{4} & \sqrt[n]{8}\,\omega \times \frac{3}{8} \end{pmatrix} = \underline{\begin{pmatrix} \frac{\sqrt[n]{8}\,\omega}{4} & \sqrt[n]{8}\,\omega \times \frac{3}{8} \\ \frac{\sqrt[n]{8}\,\omega}{2} & \sqrt[n]{8}\,\omega \times \frac{3}{4} \end{pmatrix}}$

Exercice 10

Cas $c \neq 0$, $a+b+\sqrt{2}c = a-b$.

Alors $-2b = \sqrt{2}c$ donc $c = -\sqrt{2}\, b$. et $b \neq 0$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_{a-b,M} \iff \begin{cases} (a-b)x = ax - \sqrt{2}\, by + bz, \\ (a-b)y = -\sqrt{2}\, bx + (a+b)y - \sqrt{2}\, bz \\ (a-b)z = bx - \sqrt{2}\, by + az \end{cases}$$

$$\iff \begin{cases} x = -z + \sqrt{2}\, y \\ 2y = \sqrt{2}\, x + \sqrt{2}\, z \\ z = -x + \sqrt{2}\, y. \end{cases}$$

$$\iff \left\{ x + z = \sqrt{2}\, y. \right.$$

$E_{a-b,M} = \mathrm{Vect}\left\{ \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix} \right\}$ $\dim E_{a-b,M} = 2$ et donc <u>M diagonalisable</u>

Cas $c \neq 0$, $a+b-\sqrt{2}c = a-b$. Alors $c = \sqrt{2}\, b$ et $b \neq 0$.

Les calculs précédents donnent cette fois $E_{a-b,M} = \mathrm{Vect}\left\{ \begin{pmatrix} -\sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\sqrt{2} \end{pmatrix} \right\}$ et <u>M est encore diagonalisable</u>