# Linear bounds between Cliquewidth and Component twin-width and approximations 

Ambroise Baril, Miguel Couceiro, Victor Lagerkvist
Université de Lorraine, CNRS, LORIA \& Linköpings Universitet
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## k-COLORING



Figure: Instance of 3-COLORING


Figure: Solution of the instance

$$
c: V_{G} \mapsto[k] \text { such that } \forall(u, v) \in E_{G}, c(u) \neq c(v)
$$

## H-COLORING




Example of a $C_{5}$-COLORING
$f: V_{G} \rightarrow V_{H}$
$\forall(u, v) \in E_{G},(f(u), f(v)) \in E_{H}$
$f$ is an Homomorphism
$k$-COLORING $=k_{k}$-COLORING

## NP-complete problems

NP-complete problem

No poly algo unless $\mathrm{P}=\mathrm{NP}$

How to solve in practice ?

## FPT algorithm

## Parameter: $\lambda:\{$ Instances $\} \mapsto \mathbb{N}$

Algo FPT parameterized by $\lambda$ on instance $x$
Complexity: $F(\lambda(x)) \times\|x\|^{O(1)}$
$F$ huge function $\left(F: \lambda \mapsto \lambda^{\lambda^{\lambda \cdots}}\right)$

$$
\|x\|^{\lambda(x)} \text { not allowed }
$$

## Clique-width



Figure: 3-expression of a graph
${ }^{\bullet} i$ : vertex labelled by $i$
$G_{1} \oplus G_{2}$ : disjointed union
$\rho_{j \rightarrow i}(G)$ : relabel the $j$ with $i$
$\eta_{i, j}(G)$ : construct an edge between every $i$ and $j$
$\mathrm{cw}(G)$ number of labels
$k$-COLORING in time
$\left(2^{k}-2\right)^{\mathrm{cw}(G)} \times\left|V_{G}\right|^{2}$

## Exemple of a contraction sequence



Figure: A contraction sequence of a graph

Fundamental Property:
$\left(S_{1}, S_{2}\right)$ is a black edge $\Longrightarrow \forall(u, v) \in S_{1} \times S_{2},(u, v) \in E_{G}$ $\left(S_{1}, S_{2}\right)$ is not an edge $\Longrightarrow \forall(u, v) \in S_{1} \times S_{2},(u, v) \notin E_{G}$

## (Component) twin-width



Figure: Contraction sequence of a graph
$k$-COLORING in time $\left(2^{k}-1\right)^{\text {ctww }(G)} \times\left|V_{G}\right|^{2}$

No FPT algo for 3-COLOR param by tww $(G)$ :

3-COLOR is NP-hard on planar graphs
$\operatorname{tww}(G)$ : Maximal red-degree [BKTW20] ${ }^{a}$ $\operatorname{ctww}(G):$ Max red-component size [BKRT22] ${ }^{b}$
tww is bounded on planar graphs

## Functional Equivalence

Bounded Cliquewidth $\Longleftrightarrow$ Bounded Component twin-width FPT for Cliquewidth $\Longleftrightarrow$ FPT for Component twin-width

## Proof: [BKRT22] ${ }^{1}$

Using boolean-width (func equiv to cliquewidth)

$$
\operatorname{ctww}(G) \leq 2^{\operatorname{boolw}(G)+1} \leq 2^{\mathrm{cw}(G)+1}
$$

AND

$$
\begin{gathered}
\operatorname{cw}(G) \leq 2^{\operatorname{boolw}(G)} \text { and boolw }(G) \leq 2^{\operatorname{ctww}(G)} \\
\text { so } \\
\operatorname{cw}(G) \leq 2^{2^{\operatorname{ctww}(G)}}
\end{gathered}
$$

${ }^{1}$ Bonnet, Kim, Reinald, Thomassé

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## First contribution: Improved bound

$$
\begin{gathered}
\text { I will prove } \\
\mathrm{cw}(G) \leq \operatorname{ctww}(G)+1
\end{gathered}
$$

Take a contraction sequence of $G$ of ctww $k$

Build a $(k+1)$-expression of $G$

## Exemple of a contraction sequence



For $C=\left\{S_{1}, \ldots, S_{p}\right\}$ red-component Build $\varphi_{C}$ a $(k+1)$-expression of $G\left[S_{1} \uplus \cdots \uplus S_{p}\right]$ with $\forall i$, label $\left(S_{i}\right)=i$

Same red-component $=$ Same formula
Same set $=$ Same label
Figure: A contraction sequence of a graph

## Exemple of invariant



## Goal


$\varphi:=$
$G\left[S_{0} \uplus S_{1} \uplus S_{2} \uplus S_{3} \uplus S_{4} \uplus S_{5} \uplus S_{6}\right]$

## Using $\oplus$



$$
\begin{aligned}
& \varphi_{1}:=G\left[S_{1} \uplus S_{2}\right] \\
& \varphi_{2}:=G\left[S_{3} \uplus S_{4} \uplus S_{5} \uplus S_{7}\right] \\
& \varphi_{3}:=G\left[S_{6}\right] \\
& \varphi_{4}:=G\left[S_{8}\right]
\end{aligned}
$$

$\varphi_{1} \oplus \varphi_{2} \oplus \varphi_{3} \oplus \varphi_{4}$

## Adding edges


$\varphi_{1}:=G\left[S_{1} \uplus S_{2}\right]$
$\varphi_{2}:=G\left[S_{3} \uplus S_{4} \uplus S_{5} \uplus S_{7}\right]$
$\varphi_{3}:=G\left[S_{6}\right]$
$\varphi_{4}:=G\left[S_{8}\right]$
$G\left[S_{1} \uplus S_{2} \uplus S_{3} \uplus S_{4} \uplus S_{5} \uplus S_{6} \uplus S_{7} \uplus S_{8}\right]:=$
$\eta_{\mathrm{o},,} \eta_{\mathrm{e},} \eta_{\mathrm{o},} \eta_{\mathrm{o}, \mathrm{e}} \eta_{\mathrm{e},} \eta_{\mathrm{o},}$
$\left(\varphi_{1} \oplus \varphi_{2} \oplus \varphi_{3} \oplus \varphi_{4}\right)$

## Relabelling the bags that will be contracted



$$
\begin{aligned}
& \varphi_{1}:=G\left[S_{1} \uplus S_{2}\right] \\
& \varphi_{2}:=G\left[S_{3} \uplus S_{4} \uplus S_{5} \uplus S_{7}\right] \\
& \varphi_{3}:=G\left[S_{6}\right] \\
& \varphi_{4}:=G\left[S_{8}\right] \\
& \\
& \\
& G\left[S_{1} \uplus S_{2} \uplus S_{3} \uplus S_{4} \uplus S_{5} \uplus S_{6} \uplus S_{7} \uplus S_{8}\right]:= \\
& \rho_{\bullet \rightarrow} \rightarrow \\
& \eta_{\bullet,,} \eta_{\bullet,} \eta_{\bullet,,} \eta_{\bullet, \stackrel{ }{ } \eta_{\bullet,} \eta_{\bullet, \stackrel{ }{ }}}^{\left(\varphi_{1} \oplus \varphi_{2} \oplus \varphi_{3} \oplus \varphi_{4}\right)}
\end{aligned}
$$

## Goal reached!



## Base case

Contraction sequence of $c t w w=3$

We will use 4 labels: •, •• $\cdot, ~ \bullet:$ proves $c w \leq 4$


Red-component are singletons $\{a\},\{b\}, \ldots$

$$
\begin{aligned}
& \varphi_{a}= \\
& \varphi_{b}= \\
& \varphi_{c}= \\
& \varphi_{d}= \\
& \varphi_{e}= \\
& \varphi_{f}= \\
& \varphi_{g}=
\end{aligned}
$$

## Contracting $e$ and $f$




$\varphi_{a d e f}=$
$\rho_{\bullet \mapsto}$
$\eta_{\bullet, \stackrel{ }{ } \eta_{\bullet, \stackrel{ }{\prime}} \eta_{\bullet,}}^{\left(\varphi_{a} \oplus \varphi_{d} \oplus\right.}$
$\left.\varphi_{e} \oplus \varphi_{f}\right)$

## Contracting $a$ and $d$


$\varphi_{\text {adef }}$
$\varphi=。$

$\varphi_{\text {adefg }}=$
$\rho_{\text {o• }}$
$\eta_{\circ, \bullet} \eta_{\circ,-}$
$\left(\varphi_{\text {adef }} \oplus \varphi_{g}\right)$

## Contracting $b$ and ef


$\varphi_{\text {ad ef } g}$
$\varphi_{b}$

$\varphi_{\text {adbef } g}=$
$\rho_{\bullet \mapsto}$
$\eta_{\bullet, \bullet} \eta_{\bullet, \circ}$
$\left(\varphi_{\text {adefg }} \oplus \varphi_{b}\right)$

## Contracting ad and $g$


$\varphi$ ad bef g

$\varphi_{\text {adg bef }}=$ $\rho \mapsto$
$\varphi_{\text {ad bef } g}$

## Contracting $c$ and bef


$\varphi_{\text {adg bef }}$
$\varphi_{C}$

$$
\text { bcef }-\mathrm{-} \mathrm{-} \mathrm{-} \text { adg }
$$


$\varphi_{\text {adgbcef }}=$ $\rho \stackrel{ }{ } \rightarrow$
$\eta_{\mathrm{o}, \text { 。 }}$
$\left(\varphi_{\text {adg bef }} \oplus\right.$
$\varphi_{C}$ )

## Consequence

Red-component of size $p$, we need $p+1$ colors ( $p$ colors as a result, 1 temporary color)

Contraction of comp.width $k \Longrightarrow(k+1)$-expression

$$
\mathrm{cw}(\mathrm{G}) \leq \operatorname{ctww}(\mathrm{G})+1
$$

Tight for cographs $(c w=2, c t w w=1)$
No bound possible with linearcliquewidth (cliquewidth and linearcliquewidth are not functionnaly equivalent)

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## Functional equivalence

We already know:

$$
\begin{gathered}
\operatorname{ctww}(G) \leq 2^{\operatorname{boolw}(G)+1} \text { and boolw }(G) \leq \operatorname{cw}(G) \\
\text { so } \\
\operatorname{ctww}(G) \leq 2^{\operatorname{cw}(G)+1}
\end{gathered}
$$

$$
\begin{gathered}
\text { I will prove } \\
\operatorname{ctww}(G) \leq 2 \operatorname{cw}(G)-1 \text { and } \operatorname{ctww}(G) \leq \operatorname{linearcw}(G)
\end{gathered}
$$

Take a (linear) $k$-expression

Build a contraction sequence of $G$, where every red-component has size $\leq 2 k-1$ (resp. $\leq k$ ).

## k-expression



Figure: $k$-expression tree structure

Severe abuse of notation: $\oplus$ must be binary

## Intuition: contract same colors in $\oplus$



Build larger and larger "parks" following the $k$-expressions.

Contract similar colors:

- Parks size $\leq 2 k$
- No red-edges crossing parks


Initial parks are single vertices

## Free contraction of twins

Here, $d, e$ and $f$ (as well as $h$ and $i$ ) are introduced together with the same labels: they are twins

becomes


## Contracting similar colors in a park



- Merge the parks of $a$ and $b$, of $c$ and def and of $g$ and hi.
- Collapse the $k$-expression
- No 2 different colors in the same park: no contraction.



## Joining different colors in a park



- Merge the parks of $\{a, b\}$ and $\{c, d e f\}$ and of $\{g, h i\}$ and $\{j\}$.
- $b$ and $c$ are both blue in the same park: contract them.



## Main argument: no red-edge crossing parks


$b$ and $c$ will have eternally the same label
$b$ and $c$ have exactly the same neighbors in $\{g, h, i, j\}$ : no red-edge crossing parks
$b$ and $c$ have been contracted.

$a$ will become blue: contract $a$ and $b c$
$j$ will become green: contract $j$ and $g$

## Renaming in a park: no red-edge crossing parks


$g$ and $j$ will have eternally the same label
$g$ and $j$ have exactly the same neighbors in $\{a, b, c, d, e, f\}$
$a$ and $b c$ have been contracted.


Next step: merge parks.
One park left: Ends.
Finish the contraction sequence randomly

## Largest possible red-component


$k$ labels on both side.
Red-comp of size $k$ on both side.

Peak: Red-comp of size $2 k-1$ Then, contract by color until $k$ vertices left in the park Then, procede to the next $\oplus$


## Case of a linear $k$-expression

Linear $k$-expression: $G_{1} \oplus G_{2}$ is used $\Longrightarrow G_{2}$ has one vertex

$k$ labels on one side.
1 vertex (so 1 label) on the otherside

Peak: Red-comp of size $k$


## Consequence

We have a contraction sequence were every red-comp has size $\leq 2 k-1$ (resp. $k$ ) until we are left with a single park.

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Any park has size $\leq 2 k$ (resp. $\leq k+1$ ). Next contraction: size $2 k-1$ (resp. $k$ ): no red-comp of size $>2 k-1$ (resp. $>k$ ) can emerge.

## Consequence

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(Linear) $k$-expression $\Longrightarrow$ contraction sequence with every red-comp having size $\leq 2 k-1$ (resp. $k$ )

$$
\operatorname{ctww}(G) \leq 2 \operatorname{cw}(G)-1 \text { and } \operatorname{ctww}(G) \leq \operatorname{linearcw}(G)
$$

## Consequence

We have a contraction sequence were every red-comp has size $\leq 2 k-1$ (resp. $k$ ) until we are left with a single park.

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(Linear) $k$-expression $\Longrightarrow$ contraction sequence with every red-comp having size $\leq 2 k-1$ (resp. $k$ )

$$
\operatorname{ctww}(G) \leq 2 \operatorname{cw}(G)-1 \text { and } \operatorname{ctww}(G) \leq \operatorname{linearcw}(G)
$$

$$
\operatorname{tww}(G) \leq 2 \operatorname{cw}(G)-2
$$

## Consequence

We have a contraction sequence were every red-comp has size $\leq 2 k-1$ (resp. $k$ ) until we are left with a single park.

Any park has size $\leq 2 k$ (resp. $\leq k+1$ ). Next contraction: size $2 k-1$ (resp. $k$ ): no red-comp of size $>2 k-1$ (resp. $>k$ ) can emerge.
(Linear) $k$-expression $\Longrightarrow$ contraction sequence with every red-comp having size $\leq 2 k-1$ (resp. $k$ )

```
ctww(G) \leq2cw(G)-1 and ctww(G) \leqlinearcw(G)
```

$$
\operatorname{tww}(G) \leq 2 \operatorname{cw}(G)-2
$$

Tight?

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I will present in general term the associated algorithm

${ }^{2}$ Bonnet, Kim, Reinald, Thomassé

I will present in general term the associated algorithm
It will solve \#k-COLORING, but still works for \#H-COLORING and even \#BINARY-CSP (with edge-labels)

[^0]
## \#BINARY-CSP FPT by component twin-width

I will present in general term the associated algorithm
It will solve \#k-COLORING, but still works for \#H-COLORING and even \#BINARY-CSP (with edge-labels)

Complexity: $\left(2^{k}-1\right)^{c t w w(G)+1} \times\left|V_{G}\right|^{2}$
Very similar to [BKRT22] ${ }^{2}$
${ }^{2}$ Bonnet, Kim, Reinald, Thomassé

## Solving \#k-COLORING FPT by component twin-width



For all $C_{1}, C_{2}$ subsets of colors we know
$\left|\operatorname{COL}_{S_{1}, S_{2}}^{c_{1}, C_{2}}\right|=$
$\mid\left\{f: G\left[S_{1} \uplus S_{2}\right] \rightarrow{ }_{c o l} C_{1} \uplus C_{2}\right.$
$\left.f\left(S_{1}\right)=C_{1}, f\left(S_{2}\right)=C_{2}\right\} \mid$
$\mid \operatorname{COL}_{S_{3}, S_{4}, S_{5}, S_{7}}^{c_{3}, C_{4}, C_{5}, C_{7}}$
$\left|\mathrm{COL}_{S_{8}}^{\mathrm{C}_{8}}\right|$
$\left|\operatorname{COL}_{S_{6}}^{C_{6}}\right|$

$$
G_{k+1}
$$

## Base Case

No red-edges: red-components are singletons $\{u\}$ for $u \in V_{G}$ :

$$
\left|C O L_{\{u\}}^{C}\right|=\left\{\begin{array}{lc}
1 & \text { if } C \text { is a singleton } \\
0 & \text { otherwise }
\end{array}\right.
$$

## Dealing with a contraction



Let $\left(C_{0}, \ldots, C_{6}\right)$ subsets of colors We have $\left|\operatorname{COL}_{S_{1}, S_{2}}^{C_{1}, C_{2}}\right|,\left|C O L_{S_{8}}^{C_{8}}\right|$ $\left|C O L L_{S_{6}}^{C_{6}}\right|$ and $\mid \mathrm{COL}_{S_{3}, S_{4}, S_{5}, S_{7}}^{C_{3}, C_{4}, C_{5},}$ We need to compute $\left|C O L_{S_{0}, \ldots, S_{6}}^{C_{0}, \ldots, C_{6}}\right|$

Problem: No $S_{0}$ in the term above.

## Dealing with a contraction



Let $\left(C_{0}, \ldots, C_{6}\right)$ subsets of colors
We have $\left|\operatorname{COL}_{S_{1}, S_{2}}^{C_{1}, C_{2}}\right|,\left|C O L_{S_{8}}^{C_{8}}\right|$
$\left|C O L L_{S_{6}}^{C_{6}}\right|$ and $\mid \mathrm{COL}_{S_{3}, S_{4}, S_{5}, S_{7}}^{C_{3}, C_{4}, C_{5},}$
We need to compute $\left|C O L_{S_{0}, \ldots, S_{6}}^{C_{0}, \ldots, C_{6}}\right|$
Problem: No $S_{0}$ in the term above.
Solution: Partition by image of $S_{7}$ and $S_{8}$
$\operatorname{COL}_{S_{0}, \ldots, S_{6}}^{C_{0}, \ldots, C_{6}}={ }_{C_{7} \cup C_{8}=C_{0}}^{\uplus} \operatorname{COL}_{S_{1}, \ldots, S_{7}, S_{8}}^{C_{1}, \ldots C_{7}, C_{8}}$
$G_{k}$

## Non feasibility: Empty cases



3-COLORING:
$C_{1}=\{0,0\}, C_{3}=\{0,0, \cdot\}$
Then: $\operatorname{COL}_{S_{1}, \ldots, S_{8}}^{\mathcal{C}_{1}, \ldots C_{8}}=\varnothing$

## Proof:

By contradiction $f \in \operatorname{COL}_{S_{1}, \ldots, S_{8}}^{C_{1}, \ldots, C_{8}}$
$\exists\left(s_{1}, s_{3}\right) \in S_{1} \times S_{3}$
$f\left(s_{1}\right)=\cdot, f\left(s_{3}\right)=0$
Then $\left(s_{1}, s_{3}\right) \in E_{G}$ (because
( $S_{1}, S_{3}$ ) is a black edge)
But $f\left(s_{1}\right)=f\left(s_{3}\right)$ !
$f$ is not a valid 3-coloring

## Feasibility: Combining the partial colorings

Otherwise, taking $f_{1} \in \operatorname{COL}_{S_{1}, S_{2}}^{C_{1}, C_{2}}, f_{2} \in C O L_{S_{8}}^{C_{8}}$

$$
f_{3} \in C O L_{S_{6}}^{C_{6}} \text { and } f_{4}=C O L L_{S_{3}, S_{4}, S_{5}, S_{7}}^{C_{3}, C_{4}, C_{5}, C_{7}} .
$$

Joinning $f_{1}, f_{2}, f_{3}, f_{4}$ into $f_{1} \not f_{2} * f_{3} * f_{4} \in C O L_{S_{1}, \ldots, S_{8}}^{C_{1}, \ldots,,_{8}}$.


Figure 1: Partial color- Figure 2: Partial color- Figure 3: Partial color$\operatorname{ing} f_{1} \quad \operatorname{ing} f_{2} \quad$ ing $f_{1} \bowtie f_{2}$

Figure: Joining 2 partial colorings

## Third contribution: Dynamic Programming algorithm

$$
\operatorname{COL}_{\substack{S_{0}, \ldots, S_{6}}}^{C_{0}, \ldots, C_{6}}=\underset{\substack{C_{7} \cup \\ \text { feasible }}}{\stackrel{\leftrightarrow}{C_{8}}=C_{0}} \operatorname{COL}_{S_{1}, S_{2}}^{C_{1}, C_{2}} \times \operatorname{COL}_{S_{8}}^{C_{8}} \times \operatorname{COL}_{S_{6}}^{C_{6}} \times \operatorname{COL}_{S_{3}, S_{4}, S_{5}, S_{7}}^{C_{3}, C_{4}, C_{5}, C_{7}}
$$

$$
\begin{aligned}
& \operatorname{COL}_{S_{0}, \ldots, S_{6}}^{C_{0}, \ldots, C_{6}}=C_{7} \cup \stackrel{\uplus}{C_{8}}=C_{0} . \operatorname{COL}_{S_{1}, S_{2}}^{C_{1}, C_{2}} \times \operatorname{COL}_{S_{8}}^{C_{8}} \times \operatorname{COL}_{S_{6}}^{C_{6}} \times \operatorname{COL}_{S_{3}, S_{4}, S_{5}, S_{7}}^{C_{3}, C_{4}, C_{5}, C_{7}} \\
& \text { feasible } \\
& \text { Taking the cardinal: }
\end{aligned}
$$

## Third contribution: Dynamic Programming algorithm

$$
\begin{aligned}
& \operatorname{COL}_{S_{0}, \ldots, S_{6}}^{C_{0}, \ldots, C_{6}}=C_{7} \cup \stackrel{\uplus}{C_{8}}=C_{0} \operatorname{COL}_{S_{1}, S_{2}}^{C_{1}, C_{2}} \times \operatorname{COL}_{S_{8}}^{C_{8}} \times \operatorname{COL}_{S_{6}}^{C_{6}} \times \operatorname{COL}_{S_{3}, S_{4}, S_{5}, S_{7}}^{C_{3}, C_{4}, C_{5}, C_{7}} \\
& \text { feasible } \\
& \text { Taking the cardinal: }
\end{aligned}
$$

To solve \#BINARY-CSP, return:

$$
\sum_{C \subseteq[k]}\left|C O L_{V_{G}}^{C}\right|
$$

## H-COLORING with (oriented) edge-labelled graphs



Rule: White $\mapsto$ White, Yellow or Purple Blue $\mapsto$ White or Purple Pink $\mapsto$ Purple
$\operatorname{PINK}=\{(A, A),(A, B),(B, A)$, $(C, B),(C, C),(E, D)\}$

BINARY-CSP translation: $\operatorname{PINK}\left(x_{2}, x_{1}\right) \wedge \operatorname{BLUE}\left(x_{1}, x_{2}\right) \wedge \operatorname{WHITE}\left(x_{3}, x_{3}\right) \ldots$

We can express BINARY-CSP(\{WHITE,BLUE,PINK\})

## Component twin-width applied to BINARY-CSP



$$
\begin{gathered}
\operatorname{label}(1,2)=\operatorname{label}(1,3)=: \operatorname{label}(1,23) \text { and } \\
\operatorname{label}(2,1)=\operatorname{label}(3,1)=: \operatorname{label}(23,1)
\end{gathered}
$$

label $(4,2) \neq$ label $(4,3):(4,23)$ is a red-edge (unoriented)
Fundamental Property: label $(1,23)=$ blue $\Longrightarrow$
label $(1,2)=$ label $(1,3)=$ blue
Component-twin width: Size of the largest red-component

Algo for 3 -COLOR in time $f(\operatorname{tww}(G)) \times\left|V_{G}\right|^{O(1)}$ ?

Planar graphs: $\quad$ www $\leq 7$

3-COLOR is NP-complete on planar graphs: $\mathrm{P}=\mathrm{NP}$

## Fine-grained algorithm

If we have an optimal contraction sequence for $H$ (instead of $G$ )

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Algorithm in time $(\operatorname{ctww}(H)+2)^{\left|V_{G}\right|}$

## Fine-grained algorithm

If we have an optimal contraction sequence for $H$ (instead of $G$ )
Algorithm in time $(\operatorname{ctww}(H)+2)^{\left|V_{G}\right|}$
Beats many best known upper bounds:

- \# $C_{k}$-COLORING in time $O^{*}\left(5^{\left|V_{G}\right|}\right)$ instead of $O^{*}\left(6^{\left|V_{G}\right|}\right)$
- \#H-COLORING ( $H$ any cograph) in time $O^{*}\left(3^{\left|V_{G}\right|}\right)$ instead of $O^{*}\left(5^{\left|V_{G}\right|}\right)$
Previous bounds $(\min (\operatorname{linearcw}(H)+2,2 c w(H)+1))^{\left|V_{G}\right|}[W a h 11]^{3}$.

[^1]
## Fine-grained algorithm

If we have an optimal contraction sequence for $H$ (instead of $G$ )
Algorithm in time $(\operatorname{ctww}(H)+2)^{\left|V_{G}\right|}$
Beats many best known upper bounds:

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Previous bounds $(\min (\operatorname{linearcw}(H)+2,2 \mathrm{cw}(\mathrm{H})+1))^{\left|V_{G}\right|}[\mathrm{Wah} 11]^{3}$.
We know $\operatorname{ctww}(H)+2 \leq \min (2 c w(H)+1$, linearcw $(H)+2)$.


## Fine-grained algorithm

If we have an optimal contraction sequence for $H$ (instead of $G$ )
Algorithm in time $(\operatorname{ctww}(H)+2)^{\left|V_{G}\right|}$
Beats many best known upper bounds:

- \# $C_{k}$-COLORING in time $O^{*}\left(5^{\left|V_{G}\right|}\right)$ instead of $O^{*}\left(6^{\left|V_{G}\right|}\right)$
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Previous bounds $(\min (\operatorname{linearcw}(H)+2,2 \mathrm{cw}(\mathrm{H})+1))^{\left|V_{G}\right|}[\mathrm{Wah} 11]^{3}$.
We know $\operatorname{ctww}(H)+2 \leq \min (2 \operatorname{cw}(H)+1$, linearcw $(H)+2)$.
Claim: We can solve weighted, list... variants

[^2]
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## Approximating component twin-width

$$
\mathrm{cw}(\mathrm{G}) \leq \operatorname{ctww}(\mathrm{G})+1 \leq 2 \mathrm{cw}(\mathrm{G})
$$

Approx $\operatorname{cw}(G)$ with ratio $\lambda \Longrightarrow \operatorname{Approx} \operatorname{ctww}(G)$ with ratio $2 \lambda$
Positive results for $\mathrm{cw} \Longrightarrow$ Positive restults for ctww

## Exponential approximation

Algo 1: $[\mathrm{HOO}]^{4}$
Input: G, $k$
Output: A $\left(2^{3 k}-1\right)$-expression of $G$, or a witness that $\mathrm{cw}(G)>k$

Algo 2:
Input: A $q$-expression of a graph $G$
Output: A $(2 q-1)$-contraction sequence of $G$

Algo $1(k=p+1)$ then algo $2\left(q=2^{3 p+3}-1\right)$
Input: $G, p$
Output: A $\left(2^{3 p+4}-3\right)$-contraction sequence or a witness that $\mathrm{cw}(G)>p+1 \Rightarrow \operatorname{ctww}(G)>p$
Complexity: FPT in $p$

## Improvement: Comparing with rank-width

$$
\begin{aligned}
& \qquad \operatorname{rw}(G) \leq \mathrm{cw}(G) \leq \operatorname{ctww}(G)+1 \\
& \text { (tight for graphs with no edges, is that all ?) }
\end{aligned}
$$

$\operatorname{ctww}(G) \leq 2^{\operatorname{rw}(G)+1}$ (unsure, proof for booleanwidth also works for rankwidth ?)

## Improved exponential approximation ?

Algo 1 [Oum05]
Input: G,r
Output: A $(3 r-1)$-rank decomposition or a witness that $r w(G)>r$

Algo 2
Input: A $q$-rank decomposition of a graph $G$
Output: A $2^{q+1}$-contraction sequence of $G$

Algo $1(r=p+1)$ then Algo $2(q=3 p+2)$
Input: $G, p$
Output: A $2^{3 p+3}$-contraction sequence or a witness that $\operatorname{rw}(G)>p+1 \Longrightarrow \operatorname{ctww}(G)>p$
Complexity: FPT in $p$

$$
\frac{1}{2} \operatorname{ctww}(G)+1 \leq \operatorname{cw}(G) \leq \operatorname{ctww}(G)+1
$$

Approx $\operatorname{ctww}(G)$ with ratio $\lambda \Rightarrow \operatorname{Approx} \mathrm{cw}(G)$ with ratio $2 \lambda$
Negative results for $\mathrm{cw} \Rightarrow$ Negative results for ctww

## Non-approximation of cliquewidth

## $0 \leq \varepsilon<1$ <br> NO ALGO: (unless $\mathrm{P}=\mathrm{NP}$ ) [FRRS09] ${ }^{5}$ <br> Input: $G, k$

Ouput: A witness that $\mathrm{cw}(G) \leq k+\left|V_{G}\right|^{\varepsilon}$ or of that $\mathrm{cw}(G)>k$. Complexity: P

Does not exclude a 2-approx...
Unless $\exists \varepsilon<1, \forall G, \mathrm{cw}(G) \leq\left|V_{G}\right|^{\epsilon}$ ?

[^3]
## References

Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, and Stéphan Thomassé, Twin-width vi: the lens of contraction sequences, SODA-2022, SIAM, 2022, pp. 1036-1056.

Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant, Twin-width I: tractable FO model checking, FOCS-2020, IEEE, 2020.

Michael R Fellows, Frances A Rosamond, Udi Rotics, and Stefan Szeider, Clique-width is np-complete, SIAM Journal on Discrete Mathematics 23 (2009), no. 2, 909-939.

Petr Hliněnỳ and Sang-il Oum, Finding branch-decompositions and rank-decompositions, SIAM Journal on Computing 38 (2008), no. 3, 1012-1032.

Sang-il Oum, Approximating rank-width and clique-width quickly, Graph-Theoretic Concepts in Computer Science, Springer, 2005, pp. 49-58.

Magnus Wahlström, New plain-exponential time classes for graph homomorphism, Theory of Computing Systems 49 (2011), no. 2, 273-282.

Thank you for your attention!
Questions?


[^0]:    ${ }^{2}$ Bonnet, Kim, Reinald, Thomassé

[^1]:    ${ }^{3}$ Wahlström

[^2]:    ${ }^{3}$ Wahlström

[^3]:    ${ }^{5}$ Fellows, Rosamond, Rotics, Szeider

