Minimal prime ages, words and permutation graphs

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WQO-BQO: What is up?
February 21-23, 2023

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- We classify these classes according to the number of *prime* structures they contain.
- We consider such classes that are *minimal prime*: classes that contain infinitely many primes but every proper hereditary subclass contains only finitely many primes.
- We give a complete description of such classes. In fact, each one of these classes is a *well-quasi-ordered age* and there are uncountably many of them.

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- The completeness of our description is based on classification results of Chudnovsky, Kim, Oum and Seymour 2016 and Malliaris and Terry 2018.

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- At times we may also consider *binary relational structures*, that is ordered pairs $R := (V, (\rho_i)_{i \in I})$ where each ρ_i is a binary relation or a unary relation on V. The sequence $s := (n_i)_{i \in I}$ of arity n_i of ρ_i is the *signature* of R. We denote by Ω_s the collection of finite structures of signature s. In the sequel we will suppose the signature finite, i.e. I finite.

Module

- Let $R := (V, (\rho_i)_{i \in I})$ be a binary relational structure. A *module* of R is any subset A of V such that $(x\rho_i a \Leftrightarrow x\rho_i a')$ and $(a\rho_i x \Leftrightarrow a'\rho_i x)$ for all $a, a' \in A$ and $x \notin A$ and $i \in I$.

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- Let $G = (V, E)$ be a graph. A subset A of V is called *module* in G if for every $v \notin A$, either v is adjacent to all vertices of A or v is not adjacent to any vertex of A.
- Graphs on a set of size at most two are prime. Also, there are no prime graphs on a three-element set.

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- Every prime graph has a path on 4 vertices as an induced subgraph (Sumner 1973 for finite graphs and Kelly 1985 for infinite graphs.
- What are the unavoidable prime induced subgraphs in large (infinite) prime graphs?

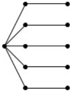
Unavoidable finite prime graphs in large finite graphs

The following is due to Chudnovsky, Kim, Oum and Seymour 2016:

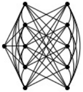
Theorem 1. *For every integer* $n \geq 3$ *there is* N *such that every prime graph with at least* N *vertices contains one of the following graphs or their complements as an induced subgraph.*

1. *The 1-subdivision of* $K_{1,n}$.
2. *The line graph of* $K_{2,n}$.
3. *The thin spider with* n *legs.*
4. *The bipartite half-graph of height* n.
5. *The graph* $H'_{n,I}$.
6. *The graph* H^*_n.
7. *A prime graph induced by a chain of length* n.

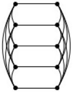
Unavoidable finite prime graphs in large finite graphs



(a) $K^{(1)}_{1,5}$: the 1-subdivision of $K_{1,5}$

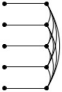


(b) $\overline{K^{(1)}_{1,5}}$

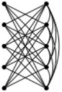


(c) $L(K_{2,5})$

(d) $\overline{L(K_{2,5})}$



(e) A thin spider with 5 legs



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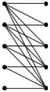
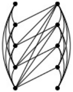
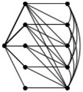


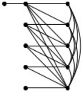
(g) H_5: a half-graph of height 5



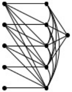
(h) $\overline{H_5}$



(i) $H'_{5,I}$ (isomorphic to $\overline{H'_{5,I}}$)



(j) H^*_5



(k) $\overline{H^*_5}$

An extension of Chudnowski et all result

Malliaris and Terry 2018 prove an infinitary version of the previous theorem for infinite prime graphs.

Theorem 2. *An infinite prime graph G contains one of the following.*

1. *Copies of* H_n, $\overline{H_n}$, H_n^*, $\overline{H_n^*}$, $H'_{n,l}$, $\overline{H'_{n,l}}$ *for arbitrarily large finite n,*
2. *Prime graphs induced by arbitrarily long finite chains,*
3. *The 1-subdivision of* $K_{1,\omega}$ *or its complement,*
4. *The line graph of* $K_{2,\omega}$ *or its complement,*
5. *A spider with* ω *many legs.*

Unavoidable prime graphs in infinite graphs with no infinite cliques

The graphs mentioned in the last three items and some infinite versions of the graphs in Item 1 were already considered by Pouzet and Zaguia in 2009. In addition, the following characterization of unavoidable infinite prime graphs without infinite clique (or infinite independent set) was given.

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Theorem 3. *An infinite prime graph which does not contain an infinite clique embeds one of the following:*

1. *The bipartite half-graph of height ω.*
2. *The infinite one way path.*
3. *The 1-subdivision of $K_{1,\omega}$.*
4. *The complement of the line graph of $K_{2,\omega}$.*

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- At the core is the notion of *embeddability*, a quasi-order between relational structures.
- Several important notions in the study of these structures, like *hereditary classes*, *ages*, *bounds*, derive from this quasi-order.

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- The *age* of a relational structure R is the class $\mathrm{Age}(R)$ of all finite relational structures, considered up to isomorphy, which embed into R. This is an *ideal* of Ω_s that is, a nonempty, hereditary and *up-directed* class $\mathcal{C}$ (any pair of members of $\mathcal{C}$ are embeddable in some element of $\mathcal{C}$).

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- A characterization of ages is due to Fraïssé. Namely, a class $\mathcal{C}$ of finite relational structures is the age of some relational structure if and only if $\mathcal{C}$ is an ideal of Ω_s.

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- With this idea, our simplest classes are those who contain finitely many proper subclasses, hence these classes are finite.
- At the next level, there is the classes who contain infinitely many proper subclasses, but every proper subclass contains only finitely many. It is a simple exercise based on Ramsey's theorem that there are only two such classes: the class of finite cliques and the class of their complements.
- Pursuing this idea further, we would like to attach a rank to each class, preferably an ordinal. If we do this, it turns out that a class has a rank if and only if the set of its proper subclasses ordered by set inclusion is well founded. This latter condition amounts to the class being well-quasi-ordered (this follows from Higman's characterization of well-quasi-orders).

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- A hereditary class of finite relational structure is *hereditary w.q.o.* if it remains w.q.o when its elements are labeled with elements from any w.q.o.

Hereditary classes containing finitely many prime structures

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  Theorem 4. *Let $\mathcal{C}$ be a hereditary class of finite binary structures containing only finitely many prime structures. Then $\mathcal{C}$ is hereditarily well-quasi-ordered. In particular, $\mathcal{C}$ has finitely many bounds.*

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- Due to their definition, minimal prime classes ordered by set inclusion form an antichain.

Hereditary classes containing infinitely many prime structures

Theorem 6. *Every hereditary class of finite binary structures (with a given finite signature), which contains infinitely many prime structures contains a minimal prime hereditary subclass.*

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Theorem 7. *Every minimal prime hereditary class is the age of some prime structure; furthermore this age is well-quasi-ordered.*

Minimal prime ages of graphs

Our description of minimal prime ages of graphs is based on several results.

- A previous description of unavoidable prime graphs in large finite prime graphs of Chudnovsky, Kim, Oum and Seymour 2016, see also Malliaris and Terry 2018.

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- Properties of uniformly recurrent words and the associated graphs.
- A description of minimal prime graphs by Pouzet and Zaguia 2009.

Chains

Definition 8. To a word μ on the alphabet $\{0,1\}$ we associate the graph G_μ whose vertex set $V(G_\mu)$ is $\{-1,0,\dots,n-1\}$ if the domain of μ is $\{0,\dots,n-1\}$, $\{-1\}\cup\mathbb{N}$ if the domain of μ is $\mathbb{N}$, and $\mathbb{N}^*$ or $\mathbb{Z}$ if the domain of μ is $\mathbb{N}^*$ or $\mathbb{Z}$. For two vertices i,j with $i<j$ we let $\{i,j\}$ be an edge of G_μ if and only if

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$$\mu_j = 1 \text{ and } j = i + 1, \text{or}$$
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- For instance, if μ is the word defined on $\mathbb{N}$ by setting $\mu_i = 1$ for all $i \in \mathbb{N}$, then G_μ is the infinite one way path on $\{-1\} \cup \mathbb{N}$.

Examples of chains

Figure: 0-1 words of length two and their corresponding graphs.

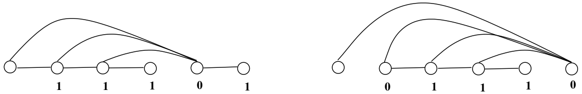
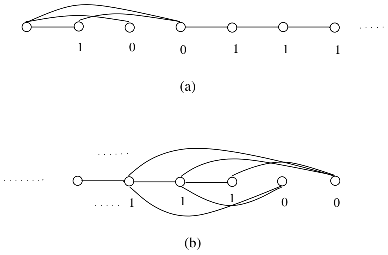


Figure: Two distinct 0-1 sequences with isomorphic corresponding graphs.

Chains

This correspondence between 0-1 words and graphs was considered by Sobrani in his Thèse de 3ème cycle 1992 and his Thèse d'État 2002. See also Zverovich 2003.

0-1 graphs nonrealizable by a sequence on $\mathbb{Z}$.



Alert Message

Given a 0-1 graph defined on $\mathbb{N} \cup \{-1\}$ or on $\mathbb{N}^*$ there does not exist necessarily a 0-1 graph on $\mathbb{Z}$ with the same age.

Modules in chains

G_μ not being prime forces μ to have a large factor of 0's or of 1's.

Theorem 9. *Let $\mu = \mu_0 \dots \mu_{n-1}$ be a finite 0-1 word, $n \geq 4$. Suppose G_μ is not prime and let M be a nontrivial module of G_μ. Then M has cardinality 2 or n.*

- *M has cardinality 2. Then <u>either</u> $M = \{-1, n-1\}$ and ($\mu = 1\underbrace{00\dots0}_{n-3}10$ or $\mu = 0\underbrace{11\dots1}_{n-3}01$), <u>or</u> $M = \{0, n-1\}$, and ($\mu = 11\underbrace{00\dots0}_{n-4}10$ or $\mu = 00\underbrace{11\dots1}_{n-4}01$).*
- *M has cardinality n. Then <u>either</u> $M = \{0, \dots, n-1\}$ and ($\mu = 1\underbrace{00\dots0}_{n-1}$ or $\mu = 0\underbrace{11\dots1}_{n-1}$), <u>or</u> $M = \{-1, 1, \dots, n-1\}$ and ($\mu = 00\underbrace{11\dots1}_{n-2}$ or $\mu = 11\underbrace{00\dots0}_{n-2}$).*

Prime chains

We note that all 0-1-graphs G_μ on $\mathbb{N}$, except four, are prime.

Theorem 10. *If μ is a word with domain $\mathbb{N}$ then G_μ is prime if and only if $\mu \notin \{011111\ldots, 100000\ldots, 0011111\ldots, 1100000\ldots\}$.*

Permutation graph

Definition 11. A graph $G := (V, E)$ is a *permutation graph* if there is a linear order $\leq$ on V and a permutation σ of V such that the edges of G are the pairs $\{x, y\} \in [V]^2$ which are reversed by σ.

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Theorem 12. *For every* 0-1 *word* μ *the age* $\mathrm{Age}(G_\mu)$ *consists of permutation graphs.*

Recurrent words

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- Let μ be an infinite sequence of 0's and 1's. The sequence μ is *Sturmian* if for some $x \in [0,1)$ and some irrational $\theta \in (0,\infty)$, w is realized as the cutting sequence of the line $f(t) = \theta t + x$.

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- There are $2^{\aleph_0}$ uniformly recurrent words with distinct sets of factors (e.g. Sturmian words or billiard sequence with different slopes).

- In general, it is not true that two words with different sets of finite factors give different ages. For instance, if μ is the word defined on $\mathbb{N}$ by setting $\mu_i = 1$ for all $i \in \mathbb{N}$ and if μ' is the word defined on $\mathbb{N}$ by setting $\mu'_i = 1$ for all $i \in \mathbb{N} \setminus \{1\}$ and $\mu'_1 = 0$, then G_μ and $G_{\mu'}$ are the infinite one way path. In particular, the graphs G_μ and $G_{\mu'}$ have the same age but μ and μ' do not have the same sets of finite factors. But, we prove:

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  Theorem 13. *Let μ and μ' be two words. If μ is recurrent and $\mathrm{Age}(G_\mu) \subseteq \mathrm{Age}(G_{\mu'})$, then $Fac(\mu) \subseteq Fac(\mu')$.*

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  Theorem 13. *Let μ and μ' be two words. If μ is recurrent and $\mathrm{Age}(G_\mu) \subseteq \mathrm{Age}(G_{\mu'})$, then $Fac(\mu) \subseteq Fac(\mu')$.*

- Using this result and the fact that there are $2^{\aleph_0}$ 0-1 recurrent words with distinct sets of factors, we obtain the following.

- In general, it is not true that two words with different sets of finite factors give different ages. For instance, if μ is the word defined on $\mathbb{N}$ by setting $\mu_i = 1$ for all $i \in \mathbb{N}$ and if μ' is the word defined on $\mathbb{N}$ by setting $\mu'_i = 1$ for all $i \in \mathbb{N} \setminus \{1\}$ and $\mu'_1 = 0$, then G_μ and $G_{\mu'}$ are the infinite one way path. In particular, the graphs G_μ and $G_{\mu'}$ have the same age but μ and μ' do not have the same sets of finite factors. But, we prove:

  Theorem 13. *Let μ and μ' be two words. If μ is recurrent and $\mathrm{Age}(G_\mu) \subseteq \mathrm{Age}(G_{\mu'})$, then $Fac(\mu) \subseteq Fac(\mu')$.*

- Using this result and the fact that there are $2^{\aleph_0}$ 0-1 recurrent words with distinct sets of factors, we obtain the following.

Corollary 14. *There are $2^{\aleph_0}$ ages of permutation graphs.*

A result which has been obtained by other means.

Uncountably many w.q.o. ages of permutation graphs.

The ages we obtain are not necessarily well-quasi-ordered. To obtain well-quasi-ordered ages, we consider graphs associated to uniformly recurrent sequences.

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Theorem 15. *Let μ be a 0-1 sequence on an infinite interval of $\mathbb{Z}$. The following propositions are equivalent.*

1. (i) *μ is uniformly recurrent.*
2. (ii) *μ is recurrent and $\mathrm{Age}(G_\mu)$ is minimal prime.*

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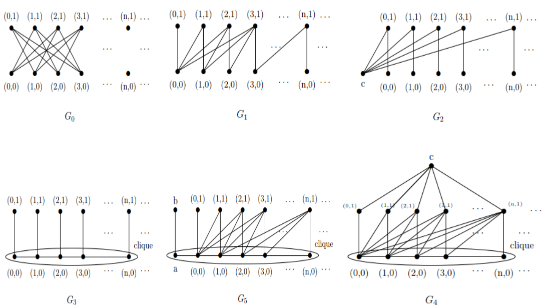
Theorem 15. *Let μ be a 0-1 sequence on an infinite interval of $\mathbb{Z}$. The following propositions are equivalent.*

- (i) *μ is uniformly recurrent.*
- (ii) *μ is recurrent and $\mathrm{Age}(G_\mu)$ is minimal prime.*

Corollary 16. *There are $2^{\aleph_0}$ ages of permutation graphs which are minimal prime, and hence w.q.o.*

Minimal prime ages of graphs

Theorem 17. *A hereditary class $\mathcal{C}$ of finite graphs is minimal prime if and only if $\mathcal{C} = \mathrm{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or $\mathcal{C}$ is the age of one of the following graphs or their complement.*



Bounds of ages of chains

- A *bound* of a hereditary class $\mathcal{C}$ of finite structures (e.g. graphs) is any structure $R \notin \mathcal{C}$ such that every proper induced substructure of R belongs to $\mathcal{C}$.

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Bounds of ages of chains

- A *bound* of a hereditary class $\mathcal{C}$ of finite structures (e.g. graphs) is any structure $R \notin \mathcal{C}$ such that every proper induced substructure of R belongs to $\mathcal{C}$.
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Theorem 18. *Let μ be a* 0-1 *uniformly recurrent word.*

1. *If μ is non periodic, then $\mathrm{Age}(G_\mu)$ has infinitely many bounds;*
2. *If μ is periodic and non constant, then $\mathrm{Age}(G_\mu)$ has finitely many bounds.*

- Brignall, Engen and Vatter 2018 provided an example of a hereditary class of permutation graphs which is w.q.o., have finitely many bounds, but are not labelled w.q.o. solving negatively a conjecture of Korpelainen, Lozin and Razgon 2013.

- Brignall, Engen and Vatter 2018 provided an example of a hereditary class of permutation graphs which is w.q.o., have finitely many bounds, but are not labelled w.q.o. solving negatively a conjecture of Korpelainen, Lozin and Razgon 2013.
- As stated in (2) of the previous theorem, ages of 0-1 graphs corresponding to periodic and non constant words provide infinitely many examples of such classes.

Thank you!

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