ABSTRACTS

Speaker: Maurice Pouzet
Title: *A topological interpretation of de Jongh -Parikh theorem.*
Abstract: An ordered set P is well quasi-ordered (w.q.o) if it contains no infinite descending chain and if all its antichains are finite. In this case, all the linear extensions of the order are well orders (Wolk 1967), hence their order types are ordinals. Among these ordinals there is a largest one, $\ell(P)$, called the *ordinal length* of P. This fact is a famous result due to de Jongh and Parikh (1977). We interpret this ordinal in terms of the Cantor-Bendixson decomposition of some collections of ideals. The set $\mathbf{Id}(P)$ of ideals (non-empty up-directed initial segments) of a non-empty w.q.o P, once equipped with the topology induced by the product topology on the power set $\wp(P)$ is a compact scattered space. Let $\operatorname{rank}^{-}(\mathbf{Id}(P))$ be the largest ordinal β for which the Cantor-Bendixson derivative $\mathbf{Id}(P)^{(\beta)}$ is nonempty, let $\mathbf{Id}(P)^{(\infty-)}$ be this derivative and $d^0(\mathbf{Id}(P))$ be its cardinality. The *canonical decomposition* of P is the sequence $(P_i)_{i<k}$ of subsets defined by setting $P_0 := \bigcup \mathbf{Id}(P)^{(\infty-)}$, $P_i := \bigcup \mathbf{Id}(P \setminus \bigcup_{j<i} P_j)^{(\infty-)}$ while $P \setminus \bigcup_{j<i} P_j \neq \varnothing$; the integer k is the least integer such that $P = \bigcup_{i<k} P_i$. Let $\beta_i := \operatorname{rank}^{-}(\mathbf{Id}(P_i))$ and $m_i := d^0(\mathbf{Id}(P_i))$ for $i < k$. Let $\mathrm{s}(\mathbf{Id}(P)) := \omega^{\beta_0} \cdot m_0 + \cdots + \omega^{\beta_{k-1}} \cdot m_{k-1}$. We prove that $\mathrm{s}(\mathbf{Id}(P)) = \ell(P)$. We illustrate this result with the poset P made of words over a finite alphabet A.

Speaker: Wieslaw Kubis
Title: *A scattered rank on mathematical structures*
Abstract: Given a class $\mathcal{K}$ of mathematical objects that could be built from a trivial one by simple or prime extensions (which we call transitions), it is natural to define an ordinal rank measuring how complicated a given object is. Under suitable assumptions (including the amalgamation property), we show that the rank of an object X is either a countable ordinal or infinity. The latter case occurs if and only if the Fraïssé limit embeds into X.

We shall discuss some relevant examples and possible applications.

Joint work with S. Shelah.

Speaker: Mirna Džamonja
Title: *On the ABK Conjecture and α-well Quasi Orders*
Abstract: The following is a 2008 conjecture of Abraham, Bonnet and Kubiś.

[ABK Conjecture] Every well quasi order (wqo) is a countable union of better quasi orders (bqo).

The talk is on a joint work with Uri Abraham, Robert Bonnet and Maurice Pouzet. We obtain a partial progress on the conjecture, by showing that the class of orders that are a countable union of better quasi orders (σ-bqo) is closed under various operations. These include diverse products, such as the Dress-Shieffels product. In relation with the main question, we explore the class of α-wqo for countable ordinals α and obtain several closure properties and a Hausdorff-style classification theorem.

Our main contribution is the discovery of various properties of σ-bqos and ruling out potential counterexamples to the ABK Conjecture.

Speaker: Aliaume Lopez
Title: *The Silence of the Powersets*
Abstract: To classify well-quasi-orders, several ordinal invariants have been designed. The *maximal-order-type*, was first introduced by de Jongh and Parikh in 1977, and quickly after the *ordinal height* (Schmidt, 1981) and *ordinal width* (Kříž and Thomas, 1990) were introduced as complementary measures. One of the practical uses of these ordinal invariants in computer science is their application to *length functions theorems*, that provide complexity upper bounds to algorithms whose termination is based on well-quasi-orderings.

While computing directly the ordinal invariants of a given wqo is error-prone and non-trivial, one usually constructs the wqos used in computer science using an algebra of constructors (sums, products, finite words, finite trees...). Usually, the three ordinal invariants of a given expression are functional in the ordinal invariants of its sub-expressions. This provides an algorithm to compute such ordinal invariants compositionally. In this setting, two constructions stand out as ill-behaved: the cartesian product and the finitary powerset. The former's *ordinal width* is non-functional, and the latter is mostly unexplored territory.

We propose in a joint work with Sergio Abriola, Sylvain Schmitz, Simon Halfon, Philippe Schnoebelen and Isa Vialard a subset of the usual constructors that include the finitary powerset, for which the ordinal invariants are computable. This is a prequel to the talk "Elementary, Dr Powerset !", that focuses on the ordinal invariants of cartesian product.

Speaker: Robert Woodrow
Title: *Counting siblings.*
Abstract: A sibling of a relational structure R is a relational structure of the same signature S such that R embeds in S and S embeds in R. R and S are then said to be equimorphic. A conjecture of Thomassé from 2000 is that if the signature consists of a finite relational symbol, then the number of siblings of a countable relational is 1, $\aleph_0$, or the size of the continuum. Independently Bonato and Tardif (2006) conjectured that a (graph theoretical) Tree T has either one or infinitely many siblings that are trees. Tyomkyn (2009) conjectured that if a tree T has a non-surjective embedding, then T has infinitely many siblings unless T is a ray. Support for the conjectures came from several sources, with arguments that employed the tools of Well-Quasi Order and Better-Quasi Order. For example, Hahn, Pouzet and Woodrow proved that a countable co-graph has either one or infinitely many siblings, and in his PhD thesis, Davoud Abdi proved the same result for countable N free partial orders. While attempting handle locally finite trees, Davoud Abdi learned about a counterexample claimed in the PhD thesis of Atsushi Tateno at Oxford. Abdi, Laflamme, Tateno, and Woodrow have posted a detailed proof of the conjecture and extended the result to partial orders. The examples show that all three of the conjectures fail. However, this leaves open questions about the boundaries on when the conjectures are true. Are there other classes where the tools of WQO-BQO permit positive results?

Speaker: Philippe SCHNOEBELEN
Title: *WQOs and BQOs in automated program verification*
Abstract: Since the 1980s, some algorithms for program verification have relied on fundamental properties of WQOs. This led to the concept of Well-Structured Systems (WSTS) invented by Alain Finkel and later developed by him, Abdulla, Jonsson, Schnoebelen and many others. In this talk we will survey this research domain, with a special focus on the areas I am most familiar with: the early days, the verification of asynchronous protocols, the complexity of verification.

Speaker: Uri Abraham
Title: *Covering of Posets with Chains*
Abstract: In the lecture the following recent theorem of Abraham and Pouzet will be described. The covering number $Cov(P)$ of a poset P is the smallest cardinality of a set of chains of P whose union is all of P. A poset is FAC (Finite Antichain Condition) if it has only finite antichains (an antichain is a set of pairwise incomparable members of P). The dual P^* of a poset has the same universe as P but with a reversed ordering. For any cardinal κ, $[\kappa]^2$ is the Perles poset of all pairs (α, β) where $\alpha < \beta < \kappa$ ordered component-wise.
Theorem: *Let ν be an uncountable cardinal and P a FAC poset.*
(1) If ν is a successor cardinal, then $Cov(P) \geq \nu$ iff P or its dual contains a copy of $[\nu]^2$.
(2) $\Sigma_{\alpha \in C} Q_\alpha$ where C is a linear ordering of cardinality $cf(\nu)$, and for some set of distinct infinite cardinals $\{\kappa_\alpha \mid \alpha \in C\}$ whose supremum is ν, $Q_\alpha = [\kappa_\alpha^+]^2$.

I will present the significance of the theorem, I will state the open problem concerning limit non-weakly compact cardinals, and say something about the structure of the proof. The proof of Item 1 was motivated in part by an unpublished work of Dorais on successor cardinals.

Speaker: Sylvain Schmitz
Title: *A width function theorem*
Abstract: In computer science, well-quasi-orders are routinely used to prove the termination of algorithms, by mapping any execution to a bad sequence. In turn, such a proof of termination can be instrumented to yield complexity upper bounds through so-called length function theorems. Those provide upper bounds on the length of 'controlled' bad sequences, and are proven by similarly instrumenting the computation of maximal order types.

After recalling the basics of this approach, I will introduce a more recent twist: a factorisation of 'strongly controlled' bad sequences into a forest of antichains, which allows to apply width function theorems instead (on the length of controlled antichains), and results in tighter complexity upper bounds.

This talk will be based on a 2019 ICALP paper available online from https://hal.science/hal-02020728.

Speaker: Gregory McKay
Title: *Better-quasi-ordering classes of partial orders*
Abstract: The notion of σ-scattered can be generalised beyond linear orders and trees to a wider class partial orders. Some large classes of σ-scattered partial orders are indeed better-quasi-ordered under embeddability.

In general one can define σ-scattered partial orders, parameterised by a class of partial orders $\mathcal{P}$ and a class of linear orders $\mathcal{L}$ to be countable unions of scattered orders. A partial order is scattered if its indecomposable subsets in $\mathcal{P}$; linear subsets in $\mathcal{L}$; and it does not embed an infinite binary tree, a reversed infinite binary tree or a "dual" of a binary tree. If $\mathcal{L}$ and $\mathcal{P}$ satisfy a strengthening of bqo, then so does the class of σ-scattered partial orders. Furthermore if $\mathcal{L}$ contains every countable linear order, then the countable partial orders that decompose into $\mathcal{P}$ are all σ-scattered. So this class of countable orders is bqo under embeddability too.

Making the general theorem concrete, for every finite number n, there is an increasingly large bqo class of σ-scattered partial orders. At $n = 2$, this generalises theorems of Laver, Corominas and Thomassé regarding σ-scattered linear orders, σ-scattered trees, countable pseudo-trees and countable N-free partial orders. In fact it is possible to extend this to infinite versions beyond finite n.

Speaker: Imed Zaguia
Title: *Minimal prime ages, words and permutation graphs*
Abstract: We classify hereditary classes of finite graphs according to the number of prime structures they contain. We consider such classes that are *minimal prime*: classes that contain infinitely many primes but every proper hereditary subclass contains only finitely many primes. We give a complete description of such classes. In fact, each one of these classes is a well-quasi-ordered age and there are uncountably many of them. Eleven of these ages are almost multichainable; they remain well-quasi-ordered when labels in a well-quasi-ordering are added and five of them are exhaustible. Among the remaining ones, only countably many remain well-quasi-ordered when one label is added.

Joint work with Djamila Oudrar and Maurice Pouzet.

Speaker: Isa Vialard
Title: *Elementary, Dr Powerset !*
Abstract: To classify well-quasi-orders, several ordinal invariants have been designed. The *maximal-order-type*, was first introduced by de Jongh and Parikh in 1977, and quickly after the *ordinal height* (Schmidt, 1981) and *ordinal width* (Kříž and Thomas, 1990) were introduced as complementary measures. One of the practical uses of these ordinal invariants in computer science is their application to *length functions theorems*, that provide complexity upper bounds to algorithms whose termination is based on well-quasi-orderings.

While computing directly the ordinal invariants of a given wqo is error-prone and nontrivial, one usually constructs the wqos used in computer science using an algebra of constructors (sums, products, finite words, finite trees...). Usually, the three ordinal invariants of a given expression are functional in the ordinal invariants of its sub-expressions. This provides an algorithm to compute such ordinal invariants compositionally. In this setting, two constructions stand out as ill-behaved: the cartesian product and the finitary powerset. The former's *ordinal width* is non-functional, and the latter is mostly unexplored territory.

In this sequel of the talk "The Silence of the Powersets", we present a sufficient condition for when the width of the cartesian product reaches its maximal order type. We then leverage this condition, along with tight bounds on the ordinal invariants of the powerset, to compute the ordinal invariants of a family of elementary wqos.

Speaker: Ambroise Baril
Title: *Linear equivalence between component twin-width and clique-width with algorithmic applications.*
Abstract: It is a common strategy to solve efficiently NP-complete problems on graphs by exploiting its structural parameters. This approach motivated the theory of parameterized complexity, in which parameters such as treewidth or cliquewidth emerged as especially efficient to design fast parameterized algorithms. Following the same principle, Bonnet et al. recently introduced the notion of contraction sequence of a graph: the high-level idea is to gain time by treating in the same way vertices with a similar neighborhood. The quality of a contraction sequence can be measured by two (among other) non-functionally equivalent parameters called twin-width and component twin-width, the former being the main focus by Bonnet et al., whereas the latter remained rather unexplored.

It is known that cliquewidth and component twinwidth are functionally equivalent: Bonnet et al. proved this result through functional equivalence with booleanwidth (which is known to be functionally equivalent with cliquewidth). In particular, this entails an exponential bound on component twin-width by cliquewidth,and a double-exponential bound on cliquewidth by component twin-width. In this presentation, we show that the latter bounds can be drastically improved to simple linear bounds. As a concrete application, we will provide an algorithmic approach to counting versions of several graph coloring problems relying on component twin-width (using dynamic programming), and that always beats the best known complexity upper bounds. We will also discuss how these linear bounds can be used to extend known approximations of cliquewidth to approximations of component twinwidth.

Collaborative work with Miguel Couceiro and Victor Lagerkvist.

Speaker: Stephan Thomassé
Title: *Gaps in tournament profiles*
Abstract: The study of the profile of a class C of relational structures (i.e. the growth of the number of non-isomorphic relations of size n in C) has been initiated by Fraïssé and Pouzet. As for wqo, the goal is again to measure how complex is C, with the paradigm that classes with low profile should be simple.

One of the first observation is that profiles usually have gaps. For instance, Boudabbous and Pouzet showed that tournament profiles are either bounded by a polynomial or at least exponential. At the other end of profile, Alekseev, Bollobás and Thomason showed that a class C of graphs either has growth at least $\exp(n^2/4)$ or at most $\exp(n^{(2-c)})$. This gap accounts for the fact that either all (semi) induced bipartite graphs appear in C, or, if not, the VC-dimension is bounded implying a collapse in the profile.

Thus the first and the last gaps seem well-understood, and a natural question is: what are the other ones?

A first answer, coming from the world of permutations, is the Marcus-Tardos theorem: any class of permutations avoiding a fixed pattern has exponential growth (whereas the full class has of course factorial growth). This was extended by Bollobás, Balogh and Morris who proved that ordered sparse graphs have growth either (sub)exponential or (super)factorial. With Bonnet, Giocanti, Ossona de Mendez, Simon and Toruńczyk, we showed this to be true without the sparsity requirement. Thus the two first gaps for ordered graphs are: polynomial/exponential/factorial.

In this talk, I will present a proof that this is also the case for tournaments, in other words the exponential gap proved by Pouzet and Boudabbous is followed by a factorial gap. Interestingly, a class C of tournament has (sub)exponential growth if and only if one cannot interpret the class of all graphs via a first order formula. In model theory terms, C has subexponential growth iff it is NIP (the VC-dimension of any FO-interpretation is bounded). The key-tool for this result is the notion of twin-width, introduced with Bonnet, Kim and Watrigant.

Joint work with Colin Geniet.

Speaker: Robert Bonnet
Title: *Skula spaces over a well-quasi-ordering*
Abstract: A **Skula space** X is a compact space such that there is a family $\mathcal{U} := \{U_x : x \in X\}$ of closed and open (clopen) sets such that $x \in U_x$ and for any distinct $x, y \in X$ either $x \notin U_y$ or $y \notin U_x$ ($\mathcal{U}$ separates the points of X) and:
(S): If $y \in X$ then $U_y \subseteq U_x$.
Such a family $\mathcal{U} := \{U_x : x \in X\}$ is called a **clopen selector** for X. Any clopen selector $\mathcal{U}$ is well-founded and the space X is scattered. We study the relationships between the well-founded rank (rk_{WF}) and the Cantor-Bendixson height (ht_{CB}) of a point x of X: for instance

$$\mathrm{ht}_{CB_X}(x) \leq \mathrm{rk}_{WF_X}(x) < \omega^{\mathrm{ht}CB(X)+1}.$$

Case of posets Let P be a poset. We denote by $\mathrm{FS}(P)$ its set of final subsets (so $\varnothing, P \in \mathrm{FS}(P)$) endowed with the pointwise topology. A base of the pointwise topology is the set of the following clopen sets:

$$U_{\sigma,\tau} := \{x \in \mathrm{FS}(P) : \sigma \subseteq x \text{ and } \tau \cap x = \varnothing\}$$

where σ and τ are finite subsets of P. For simplicity we set $U_x := U_{x,\varnothing}$.

Fact 1. *Let P be a well-quasi-ordering (wqo). For each $x \in \mathrm{FS}(P)$ set $U_x := \{y \in \mathrm{FS}(P) : y \supseteq x\}$. Then $\mathcal{U} := \{U_x : x \in \mathrm{FS}(P)\}$ is a clopen selector for* $\mathrm{FS}(P)$. □

We consider also the **Vietoris hyperspace** $H(X)$ of a Skula (in fact Priesly) space X:
- the universe of $H(X)$ is the set of non-empty final subsets of X and
- the topology on $H(X)$ is generated by the sets

$$U^+ := \{K \in H(X) : K \supseteq U\} \text{ and } V^- := \{K \in H(X) : K \cap V \neq \varnothing\}$$

declared to be clopen sets in $H(X)$ where U is any clopen final subset in X and V is any clopen initial subset of X. If $V = K \smallsetminus U$ then $H(X) \smallsetminus U^+ = V^-$.

If X is Skula, then so is for $H(X)$, and for instance

$$\mathrm{rk}_{WF}(H(X)) \leq \omega^{\mathrm{rk}WF(X)}.$$

A space Z is **unitary** if the last nonempty Cantor-Bendixson derivative $D^\zeta(Z)$ is a singleton. We say that $\mathcal{U}$ is a **canonical clopen selector** for X if $\mathcal{U}$ is a clopen selector such that

(CS): Each U_x satisfies one of the equivalent properties

(i) U_x satisfies $D^{\mathrm{ht}CB}(U_x) = \{x\}$ (and thus U_x is unitary).

(ii) U_x is unitary and $\mathrm{ht}_{CB}(U_x) = \mathrm{rk}_{WF}(U_x)$.

(iii) x is the maximum of U_x and $\mathrm{ht}_{CB}(U_x) = \mathrm{rk}_{WF}(U_x)$.

We can prove that if X is Skula then $(*)$: $\mathrm{ht}_{CB}(X) \leq \mathrm{rk}_{WF}(X) < \omega^{\mathrm{ht}CB(X)+1}$.

Main Question (Pouzet) *Let P be a wqo. Is there a canonical clopen selector for* $\mathrm{FS}(P)$*?*