April 8, 2024

Numerical methods to solve

ODEs (ORDINARY DIFFERENTIAL EQUATIONS)

<u>Reminder</u>: we are going to present some methods to solve ODEs of the forme

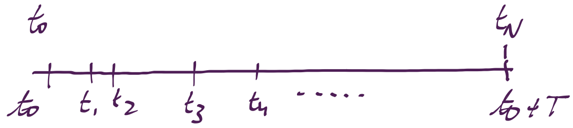
$$X' = f(t, X) \qquad X : I \subset \mathbb{R}^N \to \mathbb{R}^N \quad (N=1 \text{ in general})$$

$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ ($N=1$ in general)

f satisfies the Cauchy-Lipschitz theorem (to get existence & uniqueness of the solutions)

Usually: $t \in \mathbb{R}$, but because of the numerical simulation we are going to take $t \in [t_0, t_0+T]$ where T is finite and $t_0 \in \mathbb{R}$

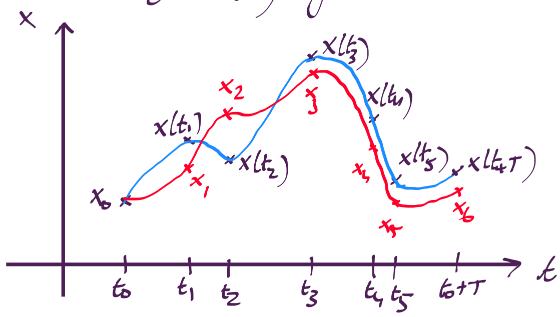
I Numerical simulations of ODEs : explicit methods



subdivision of $[t_0, t_0+T]$

<u>objective</u> : estimate the $x_1, x_2, \dots, x_N$ approaching $x(t_1), x(t_2), \dots, x(t_N)$

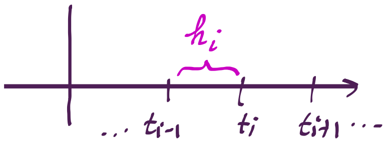
with $x_0 = x(t_0)$ given in $\mathbb{R}$



$x' = f(t, x)$

The t_i $i=0,\dots,N$ points are supposed to approach the $x(t_i)$ (the exact solutions of the ODE $\begin{cases} x'(t) = f(t, x(t)) & t \in [t_0, t_0+T] \\ x(t_0) = x_0 \text{ given} \end{cases}$

we note $h_i = t_i - t_{i-1} > 0$



we note also $h = \max_{i=1,\dots,N} h_i > 0$

Remarks: ① here, we are going to assume that

$h = h_1 = h_2 = \dots = h_N > 0$ for simplicity

② link between T, h and N

$$\boxed{T = N \cdot h} \quad h>0,\ N \in \mathbb{N}^*,\ T>0$$

or $$h = \frac{T}{N}$$

1. Explicit Euler method

Method 1: we consider the ODE $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$, $t \in [t_0, t_0+T]$

we consider $n \in \mathbb{N}$ such that $t_n \in [t_0, t_0+T]$ $\quad n = 0, \dots, N-1$

so $\quad x'(t_n) = f(t_n, x(t_n))$

we want to approach x_{n+1} knowing the approximate value x_n

for this the most natural way is to consider:

$$x'(t_n) \simeq \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq \frac{x_{n+1} - x_n}{t_{n+1} - t_n} = \frac{x_{n+1} - x_n}{h}$$

then the approximate value x_{n+1} of $x(t_{n+1})$

is given by: $\quad \dfrac{x_{n+1} - x_n}{h} = f(t_n, x_n)$

which gives:

$$\boxed{\begin{cases} x_{n+1} = x_n + h\, f(t_n, x_n) \quad , n = 0, \dots, N-1 \\ x_0 \text{ given} \end{cases}}$$

Euler explicit method with 1 step

<u>Method 2</u>

$$\begin{cases} x'(t) = f(t, x(t)) & t \in [t_0, t_0+T] \\ x(t_0) = x_0 & \text{given} \end{cases}$$

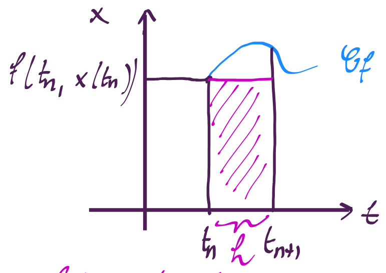
consider $n \in \{0, \dots, N-1\}$

<u>idea</u> 💡 consider $\displaystyle\int_{t_n}^{t_{n+1}} x'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, x(s))\,ds$

which gives: $\displaystyle x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f(s, x(s))\,ds$

The idea is to estimate the integral $\int^{t_n}$

The idea is to estimate the integral $\int_{t_n}^{t_{n+1}} f(s,x(s))ds$ with the rectangle method:



left rectangle

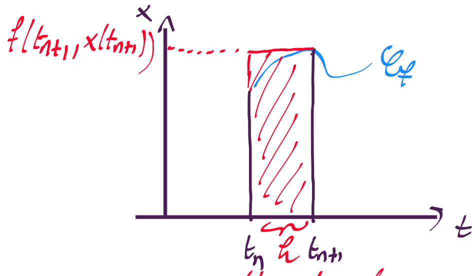
$$\int_{t_n}^{t_{n+1}} f(s,x(s))ds \simeq h \cdot f(t_n, x(t_n))$$

by approximating this gives:

$$x_{n+1} - x_n = h\, f(t_n, x_n)$$

$$\Leftrightarrow \begin{cases} x_{n+1} = x_n + h\, f(t_n, x_n), \; n = 0, \dots, N-1 \\ x_0 \text{ given} \end{cases}$$

$\rightarrow$ explicit Euler method



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$$\int_{t_n}^{t_{n+1}} f(s,x(s))ds \simeq h \cdot f(t_{n+1}, x(t_{n+1}))$$

$$x_{n+1} - x_n = h \cdot f(t_{n+1}, x_{n+1})$$

$$\Leftrightarrow \begin{cases} x_{n+1} = x_n + h f(t_{n+1}, x_{n+1}) \\ x_0 \text{ given} \end{cases}$$

$\rightarrow$ implicit Euler method

2. One step explicit methods: a generalization

Definition: one step explicit method

A one step explicit method approaching the solution of the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)) & , t \in [t_0, t_0+T] \quad (t_N = t_0+T) \\ x(t_0) = x_0 \end{cases}$$

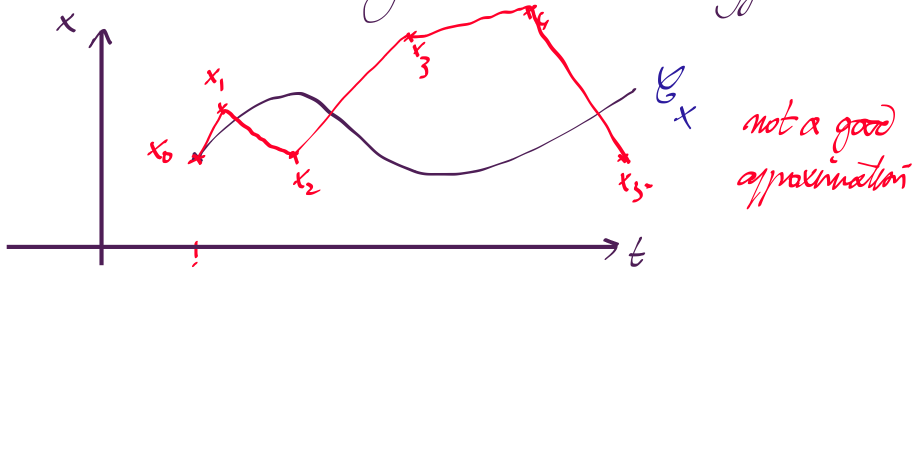
is a numerical scheme written as follows:

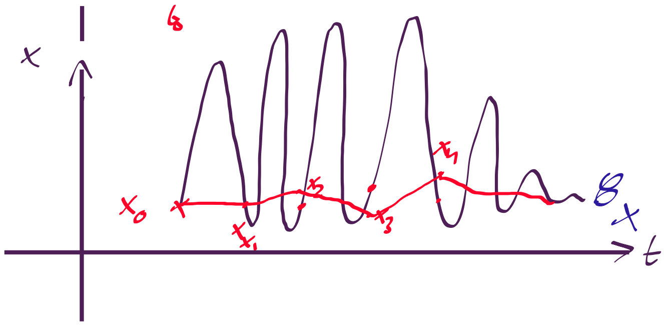
$$\begin{cases} x_{n+1} = x_n + h\,\phi(t_n, x_n, h) & , n = 0, \dots, N-1 \\ x(t_0) = x_0 \text{ given} \end{cases}$$

where ϕ depends on the chosen method.

Example: for **Euler**: $\phi(t_n, x_n, h) = f(t_n, x_n)$

Question: How do we know if these methods are efficient or not?



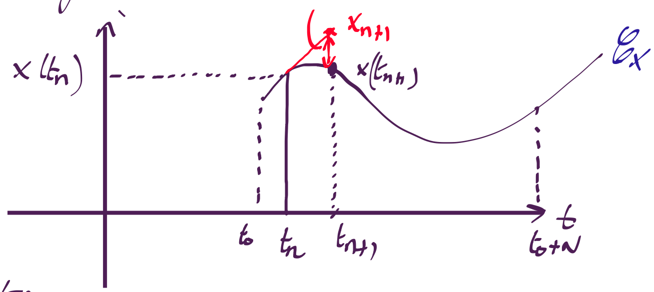


3. Consistency, stability and convergence

a. Consistency

↗ consistency error

Consistency



Definition

we call the consistency error, and denote it by τ_n the reel number defined by:

$$\tau_{n+1}(h) = x(t_{n+1}) - x_{n+1}$$

$$= x(t_{n+1}) - x(t_n) - h\phi(t_n, x(t_n), h) \quad , n=0,\dots,N-1$$

attention we assume for the consistency error that the previous point is the exact value!

in other words:

$$x(t_{n+1}) = \underbrace{x(t_n) + h\phi(t_n, x(t_n), h)}_{x_{n+1}} + \tau_{n+1}(h) \quad , n=0,\dots,N-1$$

Definition: consistent method

we say that a one step explicit scheme

$$x_{n+1} = x_n + h\,\phi(t_n, x_n, h)$$

is consistent with the Cauchy problem $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$

if and only if

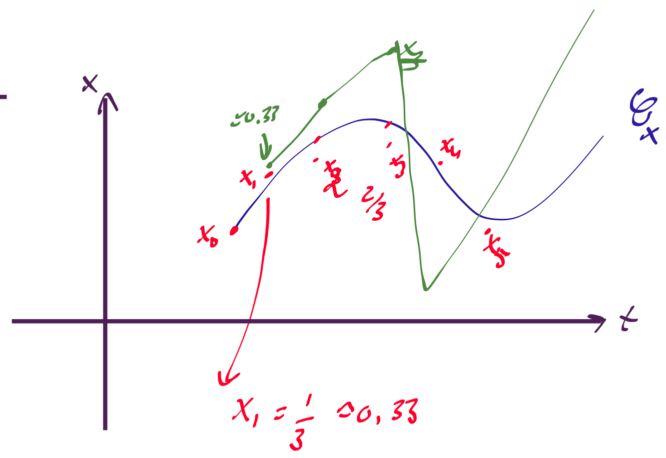
$$\lim_{h \to 0} \sum_{n=1}^{N} |\tau_n(h)| = 0$$

$h \to 0 \Rightarrow N \to +\infty$

$T = N \cdot h$

$N = \frac{T}{h}$

6. Stability



Definition: Stability

We say that a one step explicit method

$$\begin{cases} x_{n+1} = x_n + h\phi(t_n, x_n, h) & , n = 0, \dots, N-1 \\ x_0 \text{ given} \end{cases}$$

is stable if there exists a reel number $M \geq 0$ such that given $\varepsilon_1, \dots, \varepsilon_N$ a perturbation of this scheme noted

$$y_{n+1} = y_n + h \cdot \phi(t_n, y_n, h) + \varepsilon_{n+1}$$

same ϕ

with $y_0 = x_0$ ← important

then
$$\max_{0 \leq n \leq N} \|y_n - x_n\| \leq M \sum_{n=1}^{N} |\varepsilon_n|$$

c. Convergence :

Definition : we say that an explicit one step method is convergent to the exact solution of the Cauchy problem is $\lim_{h \to 0} \max_{1 \leq n \leq N} \| x(t_n) - x_n \| = 0$

Theorem : LAX theorem

If a one step explicit method is stable and consistent then it is convergent

Proof: Consider the explicit method

$$\begin{cases} x_{n+1} = x_n + h.\phi(t_n, x_n, h) \\ x_0 \text{ given} \end{cases}$$

since the method is STABLE then, for any y_n given and for $\varepsilon_1, .., \varepsilon_n \in R$

since the method is STABLE then, for any y_n given and for $\varepsilon_1, .., \varepsilon_n \in \mathbb{R}$

$$\begin{cases} y_{n+1} = y_n + \phi(t_n, y_n, h) + \varepsilon_{n+1} \\ y_0 = x_0 \end{cases}$$, there exists $M > 0$ such that

we have $\max\limits_{1 \leq n \leq N} \| y_n - x_n \| \leq M \sum_{n=1}^{N} |\varepsilon_n|$

in particular, this formula works if $y_n = x(t_n)$

that is $x(t_{n+1}) = x(t_n) + h\phi(t_n, x(t_n), h) + \varepsilon_{n+1}$

in that case, $\varepsilon_{n+1} = \tau_{n+1}$ (consistency error)

and since the method is consistent: $\sum_{n=1}^{N} |\varepsilon_n| \xrightarrow[N \to +\infty]{} 0$

Combining the 2 results:

$$\lim_{\substack{h \to 0 \\ (\Leftrightarrow N \to +\infty)}} \max_{1 \leq n \leq N} \| x(t_n) - x_n \| \leq M \lim_{\substack{h \to 0 \\ (\Leftrightarrow N \to +\infty)}} \sum_{n=1}^{N} |\underbrace{\varepsilon_n}_{\tau_n}| \longrightarrow 0$$

$\longrightarrow 0$

conclusion: the method is convergent