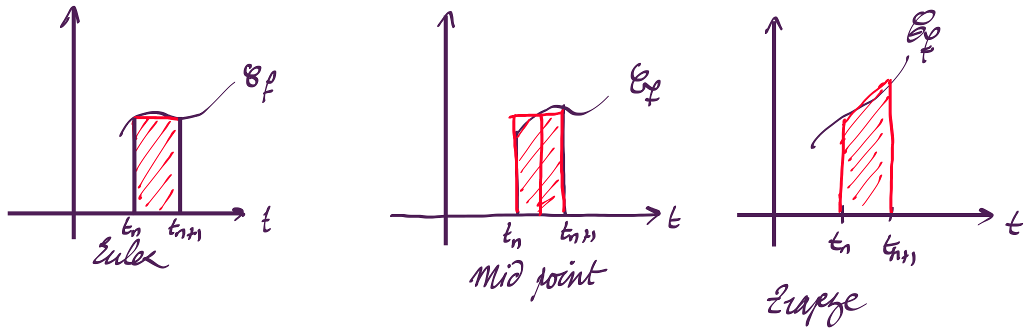
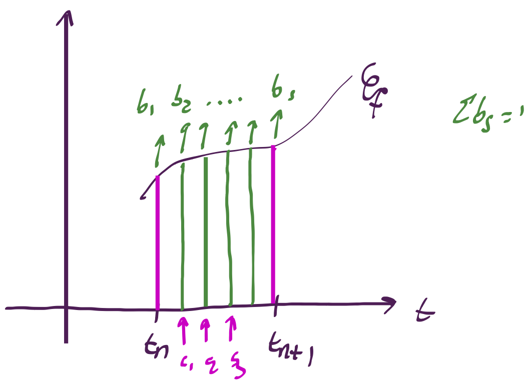
April 29, 2024

4. The RUNGE-KUTTA methods

Remind:



idea



<u>Definition</u>: The general structure of an explicit s-stages Runge-Kutta method is given by the following system:

$$(RK)\quad \begin{cases} X_i = x_n + h \sum_{j=1}^{i-1} a_{ij}\, f(t_n + c_j h, X_j) & i = 1, \dots, s \\ x_{n+1} = x_n + h \sum_{i=1}^{s} b_i\, f(t_n + c_i h, X_i) \end{cases}$$

with $n = 0, \dots, N-1$, x_0 given and we assume that $\sum_{i=1}^{0} \dots = 0$

<u>Remarks</u>: 1. s: number of time steps between t_n and t_{n+1} defined by $t_n + c_i h$ where $c_i \in [0,1]$ for $i = 1, \dots, s$

2. there are s weights, denoted by b_i, $i = 1, \dots, s$ where $\sum_{i=1}^{s} b_i = 1$

3. a_{ij}: intermediate weights to adjust the intermediate estimates

Usually we represent the Runge-Kutta method with the following BUTCHER table:

$$\begin{array}{c|cccccc}
c_1 & 0 & & & & & \\
c_2 & a_{21} & 0 & & & & \\
c_3 & a_{31} & a_{32} & 0 & & 0 & \\
\vdots & \vdots & & & \ddots & & \\
c_{s-1} & a_{s-1,1} & a_{s-1,2} & \cdots & & \ddots & \\
c_s & a_{s,1} & a_{s,2} & \cdots & & a_{s,s-1} & 0 \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_{s-1} & b_s
\end{array}$$

<u>example</u>: Give the R-K. method for the following Butcher tables:

① $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$ ② $\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$ ③ $\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$

① $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$ $\qquad s = 1,\quad b_1 = 1,\quad c_1 = 0$

The R-K system is then:

$$X_1 = x_n + h \sum_{j=1}^{1-1=0} \underbrace{\dots}_{=0} \qquad i=1$$

$$x_{n+1} = x_n + h \sum_{i=1}^{1} b_i\, f(t_n + c_i h, X_i)$$

$$= x_n + h \,.\, b_1\, f(t_n + c_1 h, X_1)$$

$$\boxed{x_{n+1} = x_n + h \,.\, f(t_n, x_n)}$$ Euler

② $\begin{array}{cc|cc} & c_1 & & a_{21} \\ & 0 & 0 & 0 \\ c_2 & 1/2 & 1/2 & 0 \\ \hline & & 0 & 1 \\ & & b_1 & b_2 \end{array}$ $\qquad s = 2$

$$X_1 = x_n$$

$$X_2 = x_n + h \sum_{j=1}^{2-1=1} a_{2j}\, f(t_n + c_j h, X_j)$$

$$x_{n+1} = x_n + h \sum_{i=1}^{2} b_i\, f(t_n + c_i h, X_i)$$

so $X_2 = x_n + h\, a_{21}\, f(t_n + c_1 h, X_1)$

$$= x_n + \frac{h}{2} f(t_n, x_n)$$

and $x_{n+1} = x_n + h \left(0 + f\left(t_n + \frac{1}{2} h,\ x_n + \frac{h}{2} f(t_n, x_n)\right)\right)$

$$\boxed{x_{n+1} = x_n + h\, f\left(t_n + \frac{1}{2} h,\ x_n + \frac{h}{2} f(t_n, x_n)\right)}$$ Mid point

③

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

c_1, c_2, a_{21}, b_1, b_2

$s=2$

$$X_1 = x_n$$

$$X_2 = x_n + h\, a_{21} f(t_n + c_1 h, X_1)$$

$$= x_n + h . f(t_n, x_n)$$

and $x_{n+1} = x_n + h\left(b_1 f(t_n + c_1 h, X_1) + b_2 f(t_n + c_2 h, X_2)\right)$

$$\boxed{x_{n+1} = x_n + h\left(\frac{1}{2} f(t_n, x_n) + \frac{1}{2} f(t_n + h, x_n + h f(t_n, x_n))\right)}$$

Trapeze method

__Question__ : how can we guess the order of convergence of a R-K method just by looking at its coefficients: $b_1, .., b_s$, $c_1, ..., c_s$, $a_{ij} ...$?

Proposition : Consider a RK method (explicit)

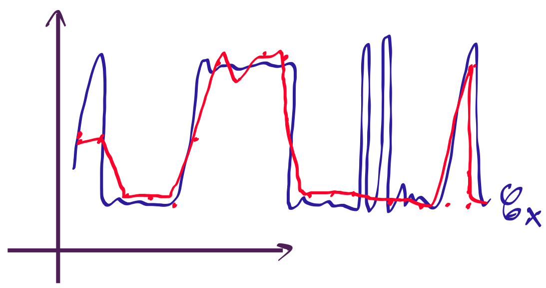
1. This scheme is consistent of order at least 1 with the Cauchy problem if and only if $\sum_{i=1}^{s} b_i = 1$

2. This scheme is consistent of order at least 2 with the Cauchy problem if and only if $\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$ $\sum_{i=1}^{s} b_i \sum_{j=1}^{i-1} a_{ij} = \frac{1}{2}$

Proposition All the explicit RK methods of stable as soon as f is continuous and globally lipschitz continuous with to its variable x on the interval $[t_0, t_0+T]$.

<u>Remark</u> : ODE solvers in MATLAB : ODE23
ODE45

II <u>Stiff problems and implicit schemes</u>



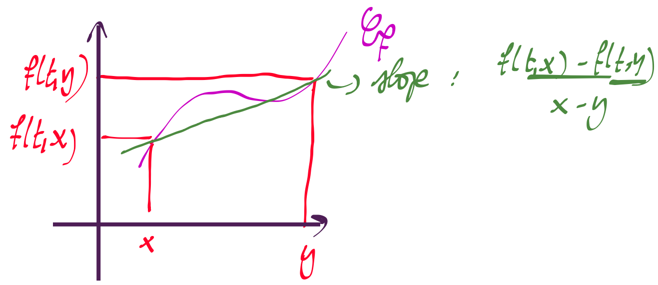
$x' = f(t, x)$

To see if a problem is stiff or not we need to observe the variations of f: and the variations of f depend on its Lipschitz constant value

$$|f(t,x) - f(t,y)| \leq L\,|x-y| \quad \text{for all } t \in [t_0, t_0+T],\ x, y \in J \subset \mathbb{R}$$

$| f(t,x - f(t,y$

$$\Leftrightarrow \quad \left| \frac{f(t,x) - f(t,y)}{x - y} \right| \leq L = \max \left| \frac{\partial}{\partial x} f(t,x) \right|$$



To investigate the resistance of a method to a stiff problem, we need to test the method with the simplest problem which can manage the stiffness.

This test is called the STANDARD LINEAR TEST (SLT)

we test any explicit or implicit method with this test:

$$(SLT) \quad \begin{cases} x'(t) = -Lx(t) & , t \in [0,T] \quad , L > 0 \\ x(0) = x_0 \end{cases}$$

The solution of this problem is known: $x(t) = x_0 e^{-Lt} \xrightarrow[t \to +\infty]{L>0} 0$

and the larger L is, the stiffer the problem is!!

Definition: we say that a method is A-stable (asymptotically) if and only if this method applied to the (SLT) gives a solution x_n such that $\lim_{n \to +\infty} x_n = 0$ REGARDLESS the value of $L > 0$ and the time step h!!

we also say that this method is unconditionally stable

Remark: example: Euler implicit method implicit

$$\begin{cases} x_{n+1} = x_n + h\, f(t_n, x_{n+1}) & n=0,\dots,N-1 \\ x_0 = x(t_0) \end{cases}$$

explicit

$$\begin{cases} x_{n+1} = x_n + h\, f(t_n, x_n) \\ x_0 = x(t_0) \end{cases}$$

Proposition: good news: the implicit Euler method is A-stable

what about the Runge-Kutta implicit methods?

BUTCHER TABLE:

$$
\begin{array}{c|ccccc}
c_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & a_{23} & \cdots & a_{2s} \\
c_3 & \vdots & & & & \vdots \\
\vdots & \vdots & & & & \vdots \\
c_s & a_{s1} & \cdots & & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s
\end{array}
$$

and the scheme is given by:

$$
(RK_{imp}) \begin{cases} X_i = x_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, X_j) & i = 1, \dots, s \\ x_{n+1} = x_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, X_i) \end{cases}
$$

$n = 0, \dots, N-1$ $\quad x_0$ given

Example: $\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$ implicit Euler method.

stability, consistency, convergence of order p: work also with the same criteria for the implicit methods

Now: $x_{n+1} = x_n + h\phi(t_n, x_n, x_{n+1}, h)$ How to solve this problem?

$x_{n+1} = g(x_{n+1})$ → solve a fixed point problem

→ use numerical methods for the fixed point problems:

for example NEWTON-RAPHSON method