TD (1) 8 AVRIL

Go back to the lecture notes

Proposition : criterion on consistency

A one step explicit method is consistent with the Cauchy problem if and only if

$$\phi(t, x, 0) = f(t, x)$$

Proposition : criterion on stability

If there exists a reel number $L>0$ such that for every $t \in [t_0, t_0+T]$, for every $x, y \in \mathbb{R}$ et for all $h<T$

$$\| \phi(t, x, h) - \phi(t, y, h) \| \leq L \| x-y \|$$

then the one step explicit method is stable, and $M = e^{LT}$

<u>Examples</u>: ① Is the Euler explicit method convergent to the solution of the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)) , & t \in [t_0, t_0+T] \\ x(t_0) = x_0 \end{cases}$$

. with f lipschitz continuous with respect to the 2^{nd} variable when $t \in [t_0, t_0+T]$ that is there exist a constant $L > 0$ such that for every $x, y \in J$ where $x : [t_0, t_0+T] \to J \subset \mathbb{R}$ and $y : [t_0, t_0+T] \to J \subset \mathbb{R}$

then $|f(t,x) - f(t,y)| < L|x-y|$

Solution The one step explicit Euler method is given by

$$x_{n+1} = x_n + h\, f(t_n, x_n) \quad , x_0 \text{ given}, \quad n = 0, \dots, N-1$$

Is it consistent? From its expression we deduce $\phi(t,x,h) = f(t,x)$ and $\phi(t,x,0) = f(t,x)$ $\rightarrow$ so the method is consistent.

Is it stable? Let's estimate

$$|\phi(t,x,h) - \phi(t,y,h)| = |f(t,x) - f(t,y)|, \quad x,y \in J$$
$$\leq L|x-y| \quad \text{(because } f \text{ is lipschitz continuous)}$$

so the explicit Euler method is stable

Conclusion: the method is STABLE and consistent, then it is convergent! (from the Lax theorem)

② Aplication

Ⓐ $\begin{cases} x'(t) = t \sin(x(t)) & t \in [0,T] \\ x(0) = \frac{\pi}{2} = x(t_0) \end{cases}$

Ⓑ $\begin{cases} x'(t) = t^2 + x + 1, & t \in [1,T] \\ x(1) = 1 = x(t_0) \end{cases}$

$h = 0,1$

Ⓐ Euler method gives:

- $\begin{cases} x_{n+1} = x_n + h \times t_n \sin(x_n) \\ x_0 = \frac{\pi}{2} \end{cases}$
- $x_1 = x_0 + h \times t_0 \sin(x_0) = \frac{\pi}{2}$ because $t_0 = 0$
- $x_2 = x_1 + h \times t_1 \sin(x_1) = \frac{\pi}{2} + 0,1 \times 0,1 \times \sin(\frac{\pi}{2})$

$= \frac{\pi}{2} + 0,1^2 = \frac{\pi}{2} + 0,01 = 1,58$

B) $\begin{cases} x_{n+1} = x_n + h \times (t^2 + x + 1) \\ x_0 = 1 \end{cases}$

$x_1 = x_0 + 0,1 \times (1^2 + 1 + 1) = 1,3 \qquad (t_0 = 1)$

$x_2 = x_1 + 0,1 \times (1,1^2 + 1,3 + 1) = 1,651 \qquad (t_0 = 1,1)$

To prove the convergence:
we know that the explicit Euler method is always consistent
<u>and</u> we showed that if f is lipschitz continuous with its 2^{nd} variable,
then it is also stable

Let's check that $f_1 : t \mapsto t\sin(x)$ and $f_2 : t \mapsto t^2 + x + 1$ are
lipschitz continuous with respect to x:

(A) <u>For</u> $f_1 : t \mapsto t\sin(x)$. we need to show that there exists
a constant $L > 0$ such that
for every x, y, $|f_1(t, x) - f_1(t, y)| \leq L|x - y|$

Reminder : if f is continuous on $[a,b]$
f is differentiable on $]a,b[$
then there exists $c \in]a,b[$ such that $|f'(c)| = \left|\frac{f(b)-f(a)}{b-a}\right|$

théorème des accroissements finis (mean value theorem)

Here : $|f(t,x) - f(t,y)| = |t \sin(x) - t \sin(y)|$

$= |t| \cdot |\sin(x) - \sin(y)| \qquad t \in [0,T]$

$\leq T \cdot |\sin(x) - \sin(y)|$

From the mean value theorem, on the interval $[x,y]$: sin is continuous, and differentiable on $]x,y[$, then there exists a $c \in]x,y[$ such that $\left|\frac{\sin(x)-\sin(y)}{x-y}\right| = |\sin'(c)| = |\cos(c)|$

But $|\cos(c)| \leq 1$ for every $c \in \mathbb{R}$, in particular for $c \in]x,y[$

thus $\left|\frac{\sin(x)-\sin(y)}{x-y}\right| \leq 1$ that is $|\sin(x)-\sin(y)| \leq |x-y|$

Finally, for every $x,y \in \mathbb{R}$, $|f(t,x) - f(t,y)| \leq T \cdot |x-y|$

so f is lipschitz continuous on $\mathbb{R}$, and so the method is stable so f is lipschitz continuous on $\mathbb{R}$, on so the method is stable

Conclusion: in that case, the Euler method is convergent.

(B) Is the Euler method convergent for B.

We know it is always consistent.
We need to prove that f_2 is lipschitz continuous with respect to its 2^{nd} variable:

For every X and Y in $\mathbb{R}$ (and also for every $t \in [1,T]$):

$$|f_2(t,X) - f_2(t,Y)| = |t^2 + X + 1 - (t^2 + Y + 1)| = |t^2 - t^2 + X - Y + 1 - 1|$$
$$= |X - Y| = 1 \times |X - Y| \leq L\,|X - Y| \quad \text{with } L = 1$$

Conclusion: (1) f_2 is Lipschitz continuous $\Rightarrow$ B scheme is stable
(2) B. scheme is consistent
$\Rightarrow$ B. scheme is convergent

Let's go back to the lecture:

We've seen that if f is lipschitz continuous with respect to its 2^{nd} variable then the one step explicit Euler method is convergent.

But we do not know if it converges to the exact solution of the Cauchy problem "rapidly" or not.

One criterion to specify this, is called "convergence of order p"

<u>Definition</u>: order of consistency

A one step explicit method is consistent of order p with the Cauchy problem if there exists a $K>0$ such that

$$\sum_{n=1}^{N} |\tau_n(h)| \leq K \cdot h^p$$

Proposition If a one step explicit method is <u>stable</u> and if it is consistent of order p with the Cauchy problem

then $\max\limits_{1 \le n \le N} \|x(t_n) - x_n\| \le M.K.h^p$ (that is if $h \to 0$ $x(t_n) - x_n \to 0$ as fast as h^p)

we say then that the method is convergent of order p.

Proposition: For any function f of class $\mathcal{C}^p$ (its p differentiable, and $f^{(p)}$ is continuous), the one step explicit method is consistent of order p with the Cauchy problem if:

order 1 $\quad \cdot\ \phi(t,x,0) = f(t,x)$

order 2 $\quad \cdot\ \frac{\partial}{\partial h} \phi(t,x,0) = \frac{1}{2} D f(t,x) = \frac{1}{2}\left(\frac{\partial}{\partial t} f(t,x) + f(t,x) . \frac{\partial}{\partial x} f(t,x)\right)$

$\vdots \qquad \vdots$

order p $\quad \cdot\ \frac{\partial^{p-1}}{\partial h^{p-1}} \phi(t,x,0) = \frac{1}{p} D^{p-1} f(t,x)$

and $\quad \frac{\partial^p}{\partial h^p}\phi(t,x,0) \neq \frac{1}{p+1} D^p f(t,x)$

with $\quad D^p f(t,x) = D\left(D^{p-1} f(t,x)\right)$

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explanation on how to compute $Df(t,x)$

"tree diagram"

$$f(t,x(t)) = f(u(t),v(t))$$

where $u(t)=t$ and $v(t)=x(t)$

$u(t)=t$ $\qquad \frac{\partial}{\partial u}f \qquad \frac{\partial}{\partial v}f \qquad x'(t)=f(t,x)$

$u'(t)=\frac{dt}{dt}=1 \qquad \frac{d}{dt}u(t)=1 \qquad \frac{d}{dt}v(t)=\frac{d}{dt}x(t)=x'(t)=f(t,x)$

$$Df(t,x) = 1\cdot\frac{\partial}{\partial u}f(t,x) + f(t,x)\cdot\frac{\partial}{\partial v}f(t,x) \qquad u=t \text{ and } v=x$$

$$= \frac{\partial}{\partial t}f(t,x) + f(t,x)\frac{\partial}{\partial x}f(t,x)$$

__Exercise__ : compute $D^2 f$. $\qquad f(t,x)$

$$D^2 f = D(Df) = D\left(\frac{\partial}{\partial t} f(t,x) + f(t,x)\frac{\partial}{\partial x} f(t,x)\right)$$

$$= D\left(\frac{\partial}{\partial t} f(t,x)\right) + D\left(f(t,x)\frac{\partial}{\partial x} f(t,x)\right)$$

$$= \underbrace{D\left(\frac{\partial}{\partial t} f(t,x)\right)}_{1} + \underbrace{D(f(t,x)) \cdot \frac{\partial}{\partial x} f(t,x)}_{2} + \underbrace{f(t,x) \cdot D\left(\frac{\partial}{\partial x} f(t,x)\right)}_{3}$$

1 $\qquad \frac{\partial}{\partial t} f(t,x)$

$$\frac{\partial}{\partial t} f(t,x)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} f(t,x)\right) \qquad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} f(t,x)\right)$$

$$\frac{dt}{dt} = 1 \qquad \frac{dx}{dt} = x'(t) = f(t,x)$$

$$D\frac{\partial}{\partial t} f(t,x) = 1 \cdot \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} f(t,x)\right) + f(t,x) \cdot \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} f(t,x)\right)$$

$$= \boxed{\frac{\partial^2}{\partial t^2} f(t,x) + f(t,x) \cdot \frac{\partial^2}{\partial x \partial t} f(t,x)}$$

② $$D(f(t,x)) \cdot \frac{\partial}{\partial x} f(t,x) = \left(\frac{\partial}{\partial t} f(t,x) + f(t,x) \cdot \frac{\partial}{\partial x} f(t,x)\right) \cdot \frac{\partial}{\partial x} f(t,x)$$

$$= \boxed{\frac{\partial}{\partial t} f(t,x) \cdot \frac{\partial}{\partial x} f(t,x) + f(t,x) \cdot \left(\frac{\partial}{\partial x} f(t,x)\right)^2}$$

③ $f(t,x) \cdot D\left(\frac{\partial}{\partial x} f(t,x)\right)$ but $D\frac{\partial}{\partial x} f(t,x) = \ldots$

$$\frac{\partial}{\partial x} f(t,x)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x} f\right) \qquad \frac{\partial}{\partial x} \cdot \left(\frac{\partial}{\partial x} f\right)$$

$$\frac{dt}{dt} = 1 \qquad \frac{dx}{dt} = x'(t) = f(t,x)$$

$$D\frac{\partial}{\partial x} f(t,x) = 1 \cdot \frac{\partial^2}{\partial t \partial x} f(t,x) + f(t,x) \cdot \frac{\partial^2}{\partial x^2} f(t,x)$$

$$f(t,x) \cdot D\left(\frac{\partial}{\partial x} f(t,x)\right) = f(t,x) \cdot \left(\frac{\partial^2}{\partial t \partial x} f(t,x) + f(t,x) \cdot \frac{\partial^2}{\partial x^2} f(t,x)\right)$$

$$= \boxed{f(t,x) \cdot \frac{\partial^2}{\partial t \partial x} f(t,x) + f^2(t,x) \cdot \frac{\partial^2}{\partial x^2} f(t,x)}$$

Conclusioni:

$$D^2 f(t,x) = \frac{\partial^2}{\partial t^2} f(t,x) + f(t,x) \frac{\partial^2}{\partial x \partial t} f(t,x) + \frac{\partial}{\partial t} f(t,x) \cdot \frac{\partial}{\partial x} f(t,x) + f(t,x)\left(\frac{\partial}{\partial x} f(t,x)\right)^2$$

$$+ f(t,x) \cdot \frac{\partial^2}{\partial t \partial x} f(t,x) + f^2(t,x) \cdot \frac{\partial^2}{\partial x^2} f(t,x)$$

Schwarz theorem: $\frac{\partial^2}{\partial t \partial x} = \frac{\partial^2}{\partial x \partial t}$ if f is continuously differentiable (here f is C^2 so ok)

$$D^2 f(t,x) = \frac{\partial^2}{\partial t^2} f(t,x) + 2 f(t,x) \frac{\partial^2}{\partial x \partial t} f(t,x) + \frac{\partial}{\partial t} f(t,x) \cdot \frac{\partial}{\partial x} f(t,x) + f(t,x)\left(\frac{\partial}{\partial x} f(t,x)\right)^2 + f^2(t,x) \frac{\partial^2}{\partial x^2} f(t,x)$$

Application. We know that for the one step explicit Euler method

$$\phi(t,x,h) = f(t,x)$$

so $\phi(t,x,0) = f(t,x)$ $\rightarrow$ consistency of order at least 1 so $\phi(t,x,0) = f(t,x)$ $\longrightarrow$ consistency of order at least 1

. for the order 2: do we have

$$\frac{\partial}{\partial h}\phi(t,x,0) = \frac{1}{2} D f(t,x) = \frac{1}{2}\left(\frac{\partial}{\partial t} f(t,x) + f(t,x)\frac{\partial}{\partial x} f(t,x)\right) ?$$

But $\phi(t,x,h) = f(t,x)$ independant of h

thus $\frac{\partial}{\partial h}\phi(t,x,h) = \frac{\partial}{\partial h} f(t,x) = 0$

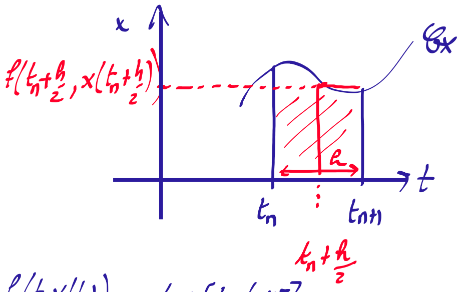
And in general $0 \neq \frac{1}{2}\left(\frac{\partial}{\partial t} f(t,x) + f(t,x)\frac{\partial}{\partial x} f(t,x)\right)$

except if $f(t,x)$ is a constant $\Rightarrow$ but this is not a general case

Besides if $f(t,x) = k$ (constant)

solving $x'(t) = k$ is OBVIOUS $\Rightarrow$ no numerical scheme needed.

Exercice 2 : 1) Mid point scheme



Start with the Cauchy problem $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \text{ given} \end{cases}$, $t \in [t_0, t_0+T]$

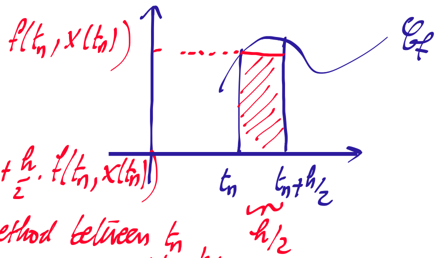
Consider $n = 0, \dots, N-1$

integrate between n and $n+1$: $\displaystyle\int_{t_n}^{t_{n+1}} x'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, x(s))\,ds$

$$x(t_{n+1}) - x(t_n) = \quad ''$$

with the mid-point rectangle $\displaystyle\int_{t_n}^{t_{n+1}} f(s, x(s))\,ds \simeq h \,.\, f\left(t_n + \frac{h}{2}, x\left(t_n + \frac{h}{2}\right)\right)$

which gives: $x_{n+1} - x_n = h\, f\left(t_n + \frac{h}{2}, \underbrace{x\left(t_n + \frac{h}{2}\right)}_{?\text{ how to estimate } x(t_n+\frac{h}{2})}\right)$



$x\left(t_n + \frac{h}{2}\right) \simeq x(t_n) + \frac{h}{2} . f(t_n, x(t_n))$

(Euler Method between t_n and $t_n + h/2$

thus $\begin{cases} x_{n+1} = x_n + h\, f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2} f(t_n, x_n)\right) \\ x_0 \text{ given} \end{cases}$ Mid point method $n = 0, \dots, N-1$

We note $\phi(t,x,h) = f\left(t+\frac{h}{2}, x+\frac{h}{2} f(t,x)\right)$

consistency of order at least 1:

$$\phi(t,x,0) = f(t,x) \qquad \text{ok}$$

stability:

for every $x, y \in \mathbb{R}$ and every $t \in [t_0, t_0+T]$

$$\begin{aligned}
|\phi(t,x,h) - \phi(t,y,h)| &= \left|f\left(t+\frac{h}{2}, x+\frac{h}{2} f(t,x)\right) - f\left(t+\frac{h}{2}, y+\frac{h}{2} f(t,y)\right)\right| \\
&\leq L\left|x+\frac{h}{2} f(t,x) - y - \frac{h}{2} f(t,y)\right| \\
&= L\left|x-y+\frac{h}{2}\left(f(t,x)-f(t,y)\right)\right| \\
&\leq L\left(|x-y| + \frac{h}{2}|f(t,x)-f(t,y)|\right) \\
&\leq L\left(|x-y| + L\frac{h}{2}|x-y|\right) \\
&= \left(L + L^2\frac{h}{2}\right)|x-y|
\end{aligned}$$

$\Rightarrow$ stable with constant $= L + L^2 \frac{h}{2}$

Conclusion: the mid-point method is consistent of order at least 1
and stable
so it is convergent of order at least 1.

<u>Question</u>: is it convergent of order at least 2?

Let us compute: $\frac{\partial}{\partial h} \phi(t,x,h)$

then $\frac{\partial}{\partial h} \phi(t,x,0) \overset{?}{=} \frac{1}{2}\left(\frac{\partial}{\partial t} f(t,x) + f(t,x) \frac{\partial}{\partial x} f(t,x)\right)$?

$$\frac{\partial}{\partial h}\phi(t,x,0) \overset{?}{=} \frac{1}{2}\cdot\left(\frac{\partial}{\partial t} f(t,x) + f(t,x)\frac{\partial}{\partial x} f(t,x)\right)?$$

Here $\phi(t,x,h) = f\left(t+\frac{h}{2},\ x+\frac{h}{2} f(t,x)\right)$

$= f(u(h),\ v(h))$ with: $u(h) = t+\frac{h}{2} \Rightarrow u(0)=t$

$v(h) = x+\frac{h}{2} f(t,x) \Rightarrow v(0)=x$

$\frac{\partial}{\partial u} f$ $\qquad$ $\frac{\partial}{\partial v} f$

$\frac{1}{2} = \frac{d}{dh} u(h)$ $\qquad$ $\frac{d}{dh} v(h) = \frac{1}{2} f(t,x)$

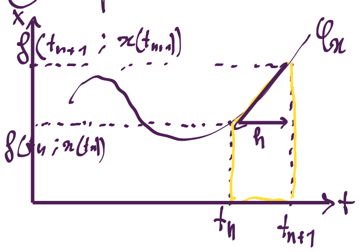
so $\frac{\partial}{\partial h}\phi(t,x,h) = \frac{1}{2}\cdot\frac{\partial}{\partial u} f(u(h),v(h)) + \frac{1}{2} f(t,x)\cdot\frac{\partial}{\partial v} f(u(h),v(h))$

and for $h=0$ then $u(0)=t$, $v(0)=x$

$$\frac{\partial}{\partial h}\phi(t,x,0) = \frac{1}{2}\frac{\partial}{\underset{u(0)}{\partial t}} f(t,x) + \frac{1}{2} f(t,x)\cdot\frac{\partial}{\underset{v(0)}{\partial x}} f(t,x) = \frac{1}{2}\left(\frac{\partial f}{\partial t}(t,x) + f(t,x)\frac{\partial}{\partial x} f(t,x)\right)$$

$= \frac{1}{2} Df(t,x)$ OK

② Trapeze Method:



$$\text{Area} = \frac{(\text{small base} + \text{large base}).\text{height}}{2}$$

We have : $t_{n+1} = t_n + h$

We approximate :

$$\int_{t_n}^{t_{n+1}} x'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s; x(s))\,ds$$

$$\Leftrightarrow x(t_{n+1}) - x(t_n) \simeq \frac{h}{2}\Big(f(t_n; x(t_n)) + f(t_n + h; x(t_{n+1}))\Big)$$

we approximate $x(t_{n+1})$ by explicit Euler method.

We obtain $x_{n+1} = x_n + \frac{h}{2}\Big[f(t_n; x_n) + f(t_n + h; x_n + h f(t_n; x_n))\Big]$

Consistency: $\Phi(t; x; 0) = \frac{1}{2}\big(f(t; x) + f(t; x)\big) = f(t; x)$ OK ✓