April 30, 2024

– Exercises – **Runge-Kutta method**

Ex 5. (d)

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}
$$

Is developed as a R-K scheme as follows:

Remark:
$$
\begin{cases}
X_i = x_n + h \sum_{j=1}^{i-1} a_{ij} f(t_n + c_j h, X_j) & i = 1, \dots, s \\
x_{n+1} = x_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, X_i) & \text{with } \sum_{i=1}^{0} \dots = 0
\end{cases}
$$

We have $S = 4$.

- As usual: $\boxed{X_1 = x_n}$
- $X_2 = x_n + h \sum_{j=1}^{1} a_{2j} f(t_n + c_j h, X_j)$

  $\boxed{X_2 = x_n + \frac{1}{2} h f(t_n, x_n)}$
-

- $X_3 = x_n + h \sum_{j=1}^{2} a_{3j} f(t_n + c_j h, X_j)$

$$\boxed{X_3 = x_n + \frac{1}{2} h \left(f(t_n + 1/2 h, x_n + 1/2 h f(t_n, x_n)) \right)}$$

- $X_4 = x_n + h \sum_{j=1}^{3} a_{4j} f(t_n + c_j h, X_j)$

$$\boxed{X_4 = x_n + h f(t_n + 1/2 h, x_n + 1/2 h (f(t_n + 1/2 h, x_n + 1/2 h f(t_n, x_n))))}$$

- $X_{n+1} = x_n + h \sum_{i=1}^{4} b_i f(t_n + c_i h, X_i)$

$$\boxed{\begin{aligned} X_{n+1} = x_n &+ 1/6 h f(t_n, x_n) \\ &+ 1/3 h f(t_n + 1/2 h, X_2) \\ &+ 1/3 h f(t_n + 1/2 h, X_3) \\ &+ 1/6 h f(t_n + h, X_4) \end{aligned}}$$

<u>Question</u> : is this scheme of order . at least 1 ?

. at least 2 ?

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

. <u>at least 1</u> : $\sum_{i=1}^{4} b_i = \frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = \frac{3}{3} = \boxed{1}$ <u>ok</u>

. <u>at least 2</u>

It is of order at least 2 if: $\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$ <u>and</u> $\sum_{i=1}^{s} b_i \left(\sum_{j=1}^{i-1} a_{ij} \right) = \frac{1}{2}$

i) $\sum_{i=1}^{4} b_i c_i = 0 \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} = \frac{1}{2}$ (ok)

ii) $\sum_{i=1}^{4} b_i \sum_{j=1}^{i-1} a_{ij} = b_2 a_{21} + b_3 \cancel{a_{31}} + b_3 a_{32} + b_4 \cancel{a_{41}} + b_4 \cancel{a_{42}} + b_4 a_{43}$

$= \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{6} \cdot 1 = \frac{1}{2}$ (ok)

<u>Note</u> : this scheme is in fact of order $\underline{\underline{4}}$, and is called RK4 (it is the one used in the matlab solver ODE45

RK4 (it is the one used in the matlab solver ODE45

2\. (a)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & \alpha & \alpha \end{array}$$

(a). Develop this scheme:

(b) Find α in order to get order at least 1

Is then possible to get order 2?

(a) $X_1 = x_n + h \sum_{j=1}^{0} \dots = x_n$

$X_2 = x_n + h \sum_{j=1}^{1} a_{ij} f(t_n + c_j h, X_j) = x_n + h\frac{2}{3} f(t_n + 0h, x_n) = x_n + h\frac{2}{3} f(t_n, x_n)$

$x_{n+1} = x_n + h \sum_{i=1}^{2} b_i f(t_n + c_i h, X_i)$

$= x_n + h(\alpha f(t_n + 0h, X_1) + \alpha f(t_n + \frac{2}{3}h, X_2))$

$x_{n+1} = x_n + h\alpha f(t_n, X_1) + h\alpha f(t_n + \frac{2}{3}h, X_2)$

(b) we want $\sum b_i = 1 \Rightarrow \alpha + \alpha = 1 \Rightarrow 2\alpha = 1 \Rightarrow \alpha = \frac{1}{2}$

order at least 2 :

$\sum b_i c_i = \alpha \times 0 + \alpha \times \frac{2}{3} = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \neq \frac{1}{2}$ it doesn't work

$$\begin{array}{c|cc} 0 & 0 & 0 \\ & \gamma & \end{array}$$

we want $\sum_{i=1}^{2} b_i c_i = \frac{1}{2} \Rightarrow \frac{1}{2} \times 0 + \frac{1}{2} \times \beta = \frac{1}{2} \Rightarrow \beta = 1$

$$\begin{array}{c|cc} \beta & \gamma & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

we want $\sum_{i=1}^{2} b_i c_i = \frac{1}{2} \Rightarrow \frac{1}{2}\times 0 + \frac{1}{2}\times\beta = \frac{1}{2} \Rightarrow \beta = 1$

$\sum_{i=1}^{2} b_i \sum_{j=1} a_{ij} = \frac{1}{2} \Rightarrow \frac{1}{2}\times\sum_{j=1}^{0} \dots + \frac{1}{2}\times\sum_{j=1}^{1} a_{ij} = \frac{1}{2}$

$\Rightarrow \frac{1}{2}\times\gamma = \frac{1}{2} \Rightarrow \gamma = 1$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Trapeze method

Implicit methods

1. Is the explicit Euler method A-stable?
2. Is the implicit Euler method A-stable?

1. To answer the question, we need to apply the

1. To answer the question, we need to apply the STANDARD LINEAR TEST equation to the explicit Euler method:

. Remember: the explicit Euler method is

$$\begin{cases} x_{n+1} = x_n + hf(t_n, x_n) & n=0,\dots,N-1 \\ x_0 \text{ given} \end{cases} \longrightarrow \begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

Applied to the (SLT): $\begin{cases} x' = -Lx \\ x(t_0) = x_0 \text{ given} \end{cases}$, with $L>0$

we set $f(t,x) = -Lx$

Applied to the explicit Euler method: (1) $x_{n+1} = x_n - h.L x_n$ where $L>0$, $0<h<T$

(1) $\Leftrightarrow \begin{cases} x_{n+1} = (1-hL)x_n \\ x_0 \text{ given} \end{cases}$, $n \in \mathbb{N}$ (SLT) $\to$ we need $\underline{n \to +\infty}$ $h = \frac{T}{N}$

this is a geometric sequence of the form $x_{n+1} = q \cdot x_n$

so $x_n = x_0 \cdot q^n$, for every $n \in \mathbb{N}$

here $x_n = x_0 \cdot (1-hL)^n$, $n \in \mathbb{N}$

And to satisfy A-stability we need to get $x_n \xrightarrow[n\to+\infty]{} 0$ for any $L>0$ and $0<h<T$

is it the case here?

Here $x_n \xrightarrow[n\to+\infty]{} 0$ if and only if $|1-hL|<1$

$\Leftrightarrow -1 < 1-hL < 1$

$\Leftrightarrow -2 < -hL < 0$

$\Leftrightarrow 0 < hL < 2 \Leftrightarrow \boxed{L < \frac{2}{h}}$ or $\boxed{h < \frac{2}{L}}$

2)

$$\begin{cases} x_{n+1} = x_n - hLx_{n+1} \\ x_0 \text{ given} \end{cases}$$

$$x_{n+1}(1 + hL) = x_n$$

$$x_{n+1} = \frac{1}{1+hL} x_n \Rightarrow \quad x_n = \Big(\underbrace{\frac{1}{1+hL}}_{q}\Big)^n x_0$$

$0 < q < 1$

because : $h > 0$, $L > 0$

$q^n \xrightarrow[n\to+\infty]{} 0$

so $x_n \xrightarrow[n\to+\infty]{} 0$ for any h and $L > 0$ unconditionnally!

Conclusion: the implicit Euler method is A-stable

Exercice 6. $(S_1)\begin{cases} u'(t) = -100\,u(t) + 25 \\ u(0) = 1 \end{cases}$

$x'' + x' + 3x = g(t)$
order 2

1. PART I : Exact solution

a. order of the equation: 1. (highest derivative) here: $\underline{\underline{x'}}$

linearity: yes, because it is of the form $a(t)u'(t) + b(t)u(t) = g(t)$

with $a(t) = 1$, $b(t) = 100$ and $g(t) = 25$

, autonomous. (it is not explicitely dependent)

b. yes it is globally unique because the problem is linear

c. Remember : $\begin{cases} x' + a(t)x = f(t) \\ x(t_0) = x_0 \end{cases}$, $t \in I$

multiply the ODE by $e^{\int_{t_0}^t a(s)ds}$ and you get :

$$\left(x(t) e^{\int_{t_0}^t a(s)ds} \right)' = e^{\int_{t_0}^t a(s)ds} \cdot f(t)$$

we integrate between t_0 and t: $\displaystyle\int_{t_0}^t \left(x(z) e^{\int_{t_0}^z a(s)ds} \right)' dz = \int_{t_0}^t e^{\int_{t_0}^z a(s)ds} f(z)dz$

$$\Leftrightarrow x(t) e^{\int_{t_0}^t a(s)ds} - \underbrace{x(t_0)}_{x_0} \underbrace{e^{\int_{t_0}^{t_0} a(s)ds}}_{=1} = \text{''}$$

$$\Leftrightarrow x(t) e^{\int_{t_0}^t a(s)ds} - x_0 = \dots$$

$$\Leftrightarrow x(t) = e^{-\int_{t_0}^t a(s)ds} \left(x_0 + \int_{t_0}^t e^{\int_{t_0}^z a(s)ds} f(z)dz \right)$$

Here : we have $\begin{cases} u' + 100u = 25 \; (*) \\ u(0) = 1 \end{cases}$

we multiply $(*)$ by $e^{\int_0^t 100 ds} = e^{100t}$

we get $\left(u(t) e^{100t} \right)' = 25 e^{100t}$

we integrate between 0 et t: and we get $u(t) e^{100t} - \underbrace{u(0)}_{=1} = \underbrace{\int_0^t 25 e^{100s} ds}$

$$= \frac{25}{100} \cdot \left[e^{100s} \right]_0^t = \frac{1}{4} \cdot \left[e^{100t} - 1 \right]$$

Finally, $u(t) = e^{-100t} \left[1 + \frac{1}{4} e^{100t} - \frac{1}{4} \right]$

$$= \frac{3}{4} e^{-100t} + \frac{1}{4} = \frac{1}{4} \left(3e^{-100t} + 1 \right)$$

d. $\displaystyle\lim_{t \to +\infty} u(t) = \lim_{t \to +\infty} \frac{1}{4} \left(3 \underbrace{e^{-100t}}_{\downarrow t \to +\infty \; 0} + 1 \right) = \boxed{\frac{1}{4}}$

$$\Big\downarrow {\scriptstyle t \to +\infty}$$

$$0$$

Part 2:

a. Explicit Euler method:

$$\begin{cases} u_{n+1} = u_n + h f(t_n, u_n) \\ u_0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} u_{n+1} = u_n + h(-100 u_n + 25) \\ u_0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} u_{n+1} = u_n(1 - 100h) + 25h \\ u_0 = 1 \end{cases}$$

b. $u_{n+1} = u_n(1-100h) + 25h$

c. We have: $f : x \longmapsto x(1-100h) + 25h$

we need to find $f(\alpha) = \alpha$

$$\alpha(1-100h) + 25h = \alpha$$

$$\Leftrightarrow \alpha(1-100h) - \alpha + 25h = 0$$

$$\Leftrightarrow -100h\,\alpha = -25h$$

$$\Leftrightarrow \alpha = \frac{1}{4}$$

We note $v_n = u_n - \frac{1}{4}$ $\Leftrightarrow$ $u_n = v_n + \frac{1}{4}$ and $v_0 = \frac{3}{4}$

$$\begin{aligned} v_{n+1} &= u_{n+1} - \frac{1}{4} \\ &= u_n(1-100h) + 25h - \frac{1}{4} \\ &= \left(v_n + \frac{1}{4}\right)(1-100h) + 25h - \frac{1}{4} \\ &= v_n - 100h\,v_n + \cancel{\frac{1}{4}} - \cancel{25h} + \cancel{25h} - \cancel{\frac{1}{4}} \\ &= v_n(1-100h) \end{aligned}$$

so $(v_n)_{n\in\mathbb{N}}$ is a geometric sequence

$$\begin{aligned} v_n &= v_0(1-100h)^n \\ &= \frac{3}{4}(1-100h)^n \end{aligned}$$

$$u_n = \frac{3}{4}(1-100h)^n + \frac{1}{4}$$

d. $h = \frac{1}{25}$ then $u_n = \frac{3}{4}\left(1 - \frac{100}{25}\right)^n + \frac{1}{4}$

$$= \frac{3}{4}(-3)^n + \frac{1}{4}$$

$$|u_n| \xrightarrow[n\to+\infty]{} +\infty$$

e. $u_n \not\longrightarrow \frac{1}{4}$ so the problem isn't A-stable

f. $u_n \xrightarrow[n \to +\infty]{} \frac{1}{4}$ if $|1-100h| < 1$

$\Leftrightarrow -1 < 1-100h < 1$

$\Leftrightarrow -2 < -100h < 0$

$\Leftrightarrow 0 < 100h < 2$

$\Leftrightarrow h < \frac{1}{50}$

Part 3 : with the implicit Euler method

a. Remember : Implicit Euler method:

$$X_{n+1} = X_n + h f(t_{n+1}, X_{n+1})$$

Here, applied to our problem :

$$\begin{cases} U_{n+1} = U_n + h f(t_{n+1}, U_{n+1}) \\ U_0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} U_{n+1} = U_n + h(-100 U_{n+1} + 25) \\ U_0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} U_{n+1} = U_n - 100h U_{n+1} + 25h \\ U_0 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} U_{n+1} = \frac{1}{1+100h}(U_n + 25h) \\ U_0 = 1 \end{cases}$$

b) $U_{n+1} = \frac{1}{1+100h}(U_n + 25h)$

c) As before, we're looking for an α such as $f(\alpha) = \alpha$,
with $f : X \longmapsto \frac{1}{1+100h}(X + 25h)$

$$\frac{1}{1+100h}(\alpha + 25h) = \alpha$$

$$\Leftrightarrow \left(\frac{1}{1+100h} - 1\right)\alpha + \frac{25h}{1+100h} = 0$$

$$\Leftrightarrow \left(\frac{-100h}{1+100h}\right)\alpha = \frac{-25h}{1+100h}$$

$$\Leftrightarrow \alpha = \frac{1}{4}$$

Then we use an auxilary sequence : $V_n = U_n - \alpha$

auxilary sequence : $V_n = U_n - \alpha$

So $V_{n+1} = U_{n+1} - \alpha = U_{n+1} - \frac{1}{4}$

$$= \frac{1}{1+100h}(U_n + 25h) - \frac{1}{4}$$

$$= \frac{1}{1+100h}\left(V_n + \frac{1}{4} + 25h\right) - \frac{1}{4}$$

$$= V_n \times \frac{1}{1+100h} + \frac{1}{4+400h} + \frac{25h}{1+100h} - \frac{1}{4}$$

$$= V_n \times \frac{1}{1+100h} + \frac{1}{4}\left(\frac{1}{1+100h} - 1\right) + \frac{25h}{1+100h}$$

$$= V_n \times \frac{1}{1+100h} + \frac{1}{4}\left(\frac{-100h}{1+100h}\right) + \frac{25h}{1+100h}$$

$$= V_n \times \frac{1}{1+100h} + 0$$

$$V_{n+1} = V_n \times \frac{1}{1+100h}$$

So $(V_n)_{n\in\mathbb{N}}$ is a geometric sequence.

$V_0 = U_0 - \alpha$
$= \frac{3}{4}$

$$V_n = V_0 \left(\frac{1}{1+100h}\right)^n$$

$$= \frac{3}{4}\left(\frac{1}{1+100h}\right)^n$$

So : $U_n = V_n + \alpha$

$$= \frac{3}{4}\left(\frac{1}{1+100h}\right)^n + \frac{1}{4}$$

$0 < \frac{1}{1+100h} < 1$, which means that :

for every $h > 0$, $U_n \xrightarrow[n\to+\infty]{} \frac{1}{4}$

This confirme that the implicit Euler method is A-stable.

The question f is useless.

Exercise sheet 2

II 1. Let us prove that a R·K method is order 1 if and only if $\sum_{i=1}^{s} b_i = 1$

Remember: the R-K scheme:

$$X_i = x_n + h \sum_{j=1}^{i-1} a_{ij} \, f(t+c_j h, X_j) \quad i=1,\dots,s$$

$$x_{n+1} = x_n + h \underbrace{\sum_{i=1}^{s} b_i \, f(t_n + c_i h, X_i)}_{\phi(t_n, x_n, h)} \qquad x_{n+1} = x_n + h\,\phi(t_n, x_n, h)$$

. an explicit method is of order at least one if and only if $\phi(t,x,0) = f(t,x)$

here, $\sum_{i=1}^{s} b_i \, f(t+0, X_i) = f(t,x)$ with $X_i = x + 0 \cdot \Sigma \dots = x$

$$\phi(t,x,0) = \sum^{s} b_i \, f(t,x) = f(t,x) \Leftrightarrow f(t,x) \dots \sum^{s}$$

so $\phi(t,x,0) = \sum_{i=1}^{s} b_i \underbrace{f(t,x)}_{\text{independent of } i} = f(t,x) \Leftrightarrow f(t,x) \cdot \sum_{i=1}^{s} b_i = f(t,x)$

$\Leftrightarrow \sum_{i=1}^{s} b_i = 1$

2. for order at least 2 we need to prove that $\frac{\partial}{\partial h}\phi(t,x,0) = \frac{1}{2} Df = \frac{1}{2}\left(\frac{\partial}{\partial t}f + f\frac{\partial}{\partial x}f\right)$

$= \frac{1}{2}\frac{\partial}{\partial t}f + \frac{1}{2}f \cdot \frac{\partial}{\partial x}f$

<u>hint:</u> we can find that

$$\frac{\partial}{\partial h}\phi(t,x,0) = \left(\sum_{i=1}^{s} b_i c_i\right)\frac{\partial f}{\partial t}(t,x) + \left(\sum_{i=1}^{s} b_i \sum_{j=1}^{i-1} a_{ij}\right)\frac{\partial}{\partial x}f(t,x) \cdot f(t,x)$$