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Bio-informatics and modeling
3rd year
Numerical analysis of ODEs
Spring 2023

Exam on Numerical Analysis of ODEs-3BIM
June 21, 2023

This is a 45 minutes exam. Document : only one sheet (hand written) is allowed. No calculator, no phone are allowed. It is necessary to justify and carefully present the answers.

Exercise 1 (15 minutes) (6 points)

1. (1 point) Remind the s-stage Runge-Kutta general explicit method as well as the Butcher table
2. (1 point) Remind the consistency of order at least 2 for a one-step explicit method. Explain which equality should be verified to get this method exactly of order 2.
3. (1 point) Give the expression of function ϕ for the general s-stage Runge-Kutta method and prove that it is of order at least 1 if and only if
$$\sum_{i=1}^{s} b_i = 1.$$
4. (1 point) Write the expression of $\dfrac{\partial \phi}{\partial h}(t, x, h)$ then $\dfrac{\partial \phi}{\partial h}(t, x, h = 0)$.
5. (2 points) Deduce that the general s-stage Runge-Kutta method is of order at least 2 if and only if we have also
$$\sum_{i=1}^{s} b_i c_i = \frac{1}{2} \quad \text{et} \quad \sum_{i=1}^{s} b_i \sum_{j=1}^{i-1} a_{ij} = \frac{1}{2}.$$

Exercise 2 (30 minutes) (14 points)

We consider the following Cauchy problem : find$u \in \mathscr{C}^1([0,1], \mathbb{R})$ such that

$$(\mathscr{S}_1) \left\{ \begin{array}{lcl} u'(t) & = & -150u(t) + 30, \\ u(0) & = & 1. \end{array} \right.$$

1. **Part 1 : exact solution of $(\mathscr{S}_1)$ (4 points)**
   (a) (1.5 point) Give the order of the equation of $(\mathscr{S}_1)$. Specify if it is linear or not, autonomous or not ? Justify your answer.
   (b) (1.5 point) Compute the exact solution of $(\mathscr{S}_1)$.
   (c) (1 point) What is the limit of $u(t)$ when t tends to $+\infty$?
2. **Part 2 : explicit Euler method (5 points)**
   (a) (1 point) Write the explicit one step Euler method for $(\mathscr{S}_1)$.
   (b) (1 point) Recognize a classic type of sequence for $(u_n)_{n\in\mathbb{N}}$.
   (c) (1 point) Compute an expression of u_n in function of n and h. Give a detailed computation.
   (d) (1 point) Assume that $h = 1/25$. Compute u_n, then $\lim\limits_{n\to+\infty} |u_n|$.

(e) (1 point) What can we conclude about the A-stability of the method ? Explain.

(f) (1 point) Give the maximal value $h > 0$ such that $\lim_{n \to +\infty} |u_n|$ corresponds to the one of question (c) or Part 1.

3. **Part 3 : implicit Euler method 5 points)**
   Answer questions (a) to (e) of part 2, replacing explicit method by implicit method. Does question f make sense in this part 3 ? Justify your answers.