I. Asymptotic of word metric on nilpotent groups.

II. Locally compact groups of polynomial growth.

III. Applications to equidistribution and random walks and ergodic theory.

I.1. The Bass-Guivarc'h Formula

Γ = f.g. nilpotent group

S = Finite symmetric set of generators

d_S = word metric on Γ; $B_S(n)$ = n-ball

Theorem (Bass, Guivarc'h ~1972)

$$c_1 n^d \leq |B_S(n)| \leq C_2 n^d$$

$d = d(\Gamma) \in \mathbb{N}$ integer

given by the Bass-Guivarc'h formula:

$$d(\Gamma) = \sum_{k \geq 1} k \, \mathrm{rank}\left(\gamma_k(\Gamma) / \gamma_{k+1}(\Gamma) \right)$$

where $\{\gamma_k(\Gamma)\}_k$ = central descending series

Gromov's theorem ('82): for a discrete group

Polynomial growth $\Rightarrow$ v. nilpotence

I.2. Pansu's thesis ('82)

Results:

1. $\dfrac{|B_S(n)|}{n^d}$ has a limit $c_\Gamma(S) > 0$

2. The asymptotic cone of (Γ, d_S) is unique up to isometry. It is isometric to (N_∞, ρ_∞)

the Pansu metric

"graded" Carnot group associated to $\tilde{\Gamma} = N$ = Malcev completion for Γ

The Pansu metric is the N_∞-left invariant Carnot-Caratheodory metric induced by the following polyhedral norm $\|\cdot\|_\infty$ on the horizontal tangent space m_1. ($\mathrm{Lie}\, N_\infty = m_1 \oplus [m_1, m_1] \oplus \dots$)

unitball for $\|\cdot\|_\infty$ = polyhedron in m_1

= Convex Hull $\{ \pi_1(s),\ s \in S \}$

where $\pi_1 : N_\infty \longrightarrow m_1 \simeq N_\infty / [N_\infty, N_\infty]$ linear projection.

③ Let $|\gamma|_S = d_S(e, \gamma)$, $\gamma \in \Gamma$

$|\gamma|_\infty = \rho_\infty(e, \gamma)$, $\gamma \in \Gamma$

the Pansu quasi-norm

Then $$\frac{|\gamma|_S}{|\gamma|_\infty} \longrightarrow 1 \quad \text{as } \gamma \nearrow \infty$$

Logic: ③ $\Rightarrow$ ① and ②

In words: $B_{\rho_\infty}(1)$ is the asymptotic shape of the Cayley graph balls $B_S(n)$ as $n \nearrow \infty$.

Corollary: $c_\Gamma(S) = \mathrm{vol}_{N_\infty}(B_{\rho_\infty}(1))$

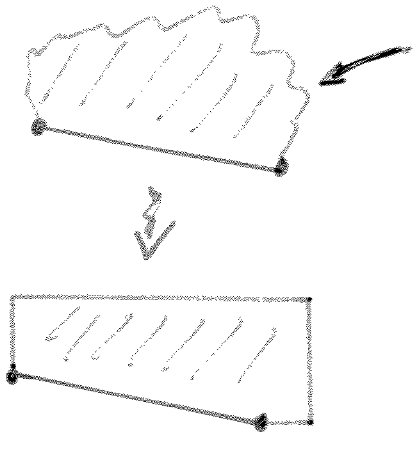
Example : Heisenberg groups

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{Z} \right\}$$

$$H_5 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & z \\ & 1 & 0 & y_2 \\ & & 1 & y_1 \\ & & & 1 \end{pmatrix}, \quad x_i, y_i, z \in \mathbb{Z} \right\}$$

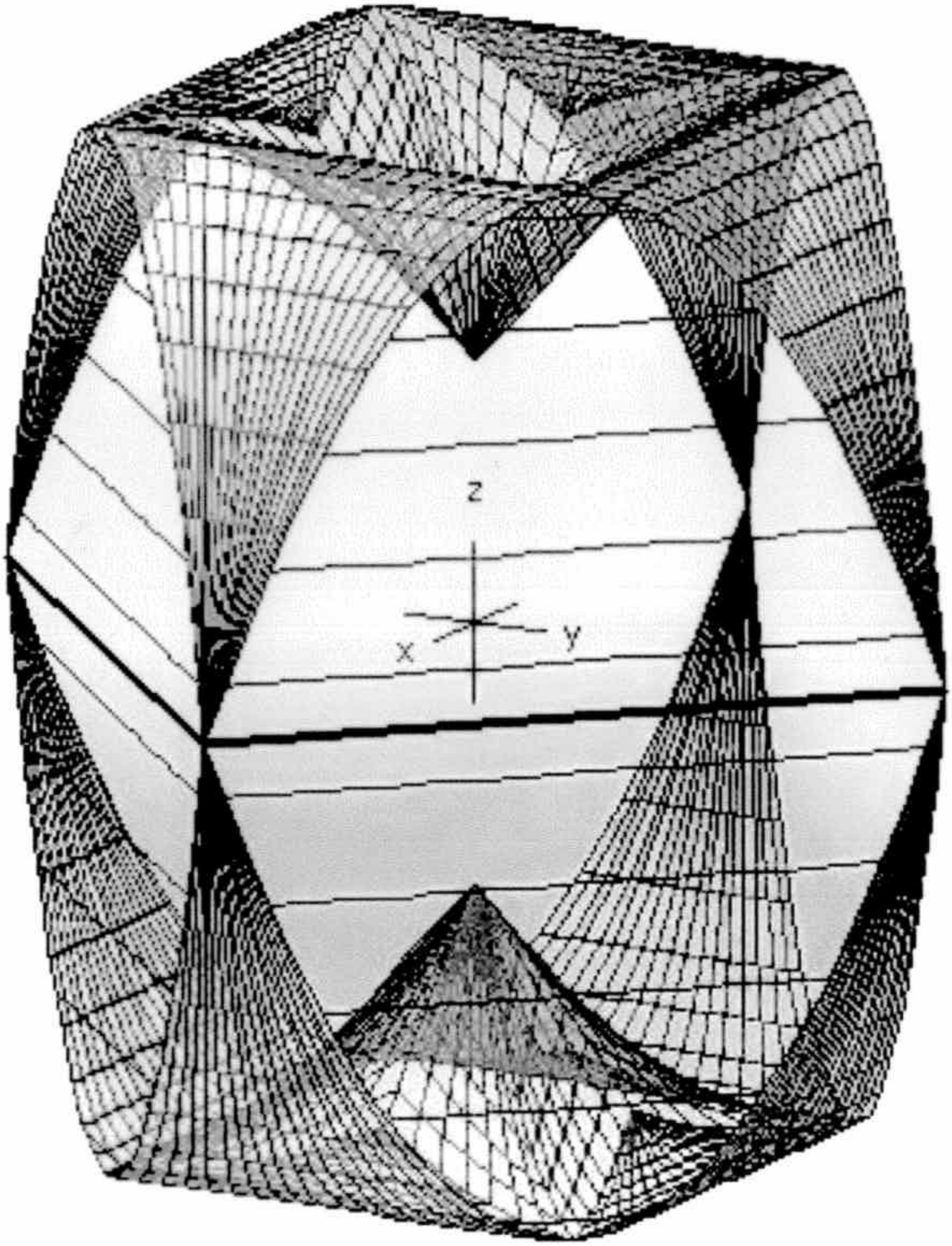
In those cases $B_{\ell_\infty}(1)$ (i.e. the limit shape) can be explicitly computed ; also CC-geodesics.

$\rightsquigarrow$ need to solve Dido's isoperimetric problem.



fixed perimeter length

need to maximize enclosed area.



$$\mathrm{Vol}_3 = \frac{31}{72} \qquad , \quad \mathrm{vol}_5 = \frac{2009}{21870} + \frac{\log 2}{32805}$$

I.2. Error terms and the effectivization of Pansu's proof.

Questions: . Can one compute $B_{\rho_\infty}(1)$?

a. . Is $B_{\rho_\infty}(1)$ semi-algebraic set ?

b. . Is there a PL-geodesic btw any 2 points?

In 2006 Burago and Margulis conjectured that

$$\big|\, |\gamma|_S - |\gamma|_\infty \big| \ll O(1)$$

counterexample: $\Gamma = \mathbb{Z} \times H_3(\mathbb{Z})$

$$\big|\, |\gamma_n|_S - |\gamma_n|_\infty \big| \geqslant \sqrt{|\gamma_n|_\infty} \quad \text{for some } \gamma_n = (n, 0, 0, n)$$

True for $H_3(\mathbb{Z})$! (Krat '03)

Principle: good error term $\Leftrightarrow$ every ρ_∞-geodesic g is well approximated by PL-paths

$\Leftrightarrow$ $g = \text{Lift}(c(s))$
$c : [0,1] \to m_1$
c is more than 1-Lipschitz

Condition (*): $\|$ $\exists$ piecewise C^1-geodesic btw every two points in N_∞

Thm[4] (B '08) Under (*) we have: (*) 7

- $\exists$ PL geodesic btw any 2 points
- $\left| |\gamma|_S - |\gamma|_\infty \right| \ll |\gamma|_\infty^{1 - 1/6r^2}$ $\quad r =$ nilp. class
- $|B_S(n)| = c \cdot n^d + O(n^{d - 1/6r^2})$

Rk Stoll 96: (*) holds for 2-step grps.

Application of Pansu's theorem to ergodic theorems

A. Nevo: if $B_S(n)$ Folner, then the ergodic theorem holds for Γ-actions with averages over $B_S(n)$.

Pansu: $B_S(n)$ is Folner: indeed $|B_S(n)| \sim c\, n^d$

$$\Downarrow$$

$$|B_S(n+1) \setminus B_S(n)| = o(B_S(n))$$

By Thm 3: also holds in general $\forall$ G locally cpct with pol. growth

2 other argument due to Colding - Minicozzi gives $\frac{|S(n)|}{|B(n)|} \leq n^{-\varepsilon}$

8

II. Locally compact groups with polynomial growth

$G :=$ l.c. + cptly gen. + pol. growth

Losert (generalizing Gromov) :

$\exists K \triangleleft G$ compact subgrp s.t. G/K is a Lie group

(not connected in g^al)

Thm 2 (B. '06) : G is weakly commensurable to a connected and simply connected solvable Lie group of pol. growth ($\Leftrightarrow$ type(R)), which we call S, the Lie shadow of G.

weak comm. : (up to co-cpct subgrp
G/K embeds in S as a co-cpct sub grp

exple : $G = \mathbb{Z} \ltimes_\alpha \mathbb{R}^2$ $\quad \alpha =$ irrat. rotation

$S = \mathbb{R} \ltimes_\alpha \mathbb{R}^2$

$S_{Nil} = \mathbb{R}^3$

Philosophy : large scale geometry of G is governed by a nilpotent group : the nilshadow of G

(Auslander-Green '69) : idea : Get a new nilpotent Lie product on S by "removing the semisimp part of $Ad(g)$"

get a natural map $G \to S_{Nil} \to S_{Nil,\infty} : x \mapsto \bar{x}$

(9)

$\uparrow$ graded nilshadow

<u>Thm 3</u> (B'06) If ρ is any (coarsely geodesic) periodic distance on G then

ⓐ $\exists\, \rho_{Nil}$ periodic distance on S_{Nil} s.t.

$$\frac{\rho(e,x)}{\rho_{Nil}(e,\bar{x})} \longrightarrow 1 \quad \text{as } x \nearrow \infty$$

ⓑ $$\frac{\rho(e,x)}{|\bar{x}|_\infty} \longrightarrow 1 \qquad |\bar{x}|_\infty := \text{Pansu's metric on } S_{Nil,\infty}.$$

<u>Corollary</u>: 1. $\left(B_\rho(n), \frac{1}{n}\right) \xrightarrow{G-H} B_{\rho_\infty}(1)$

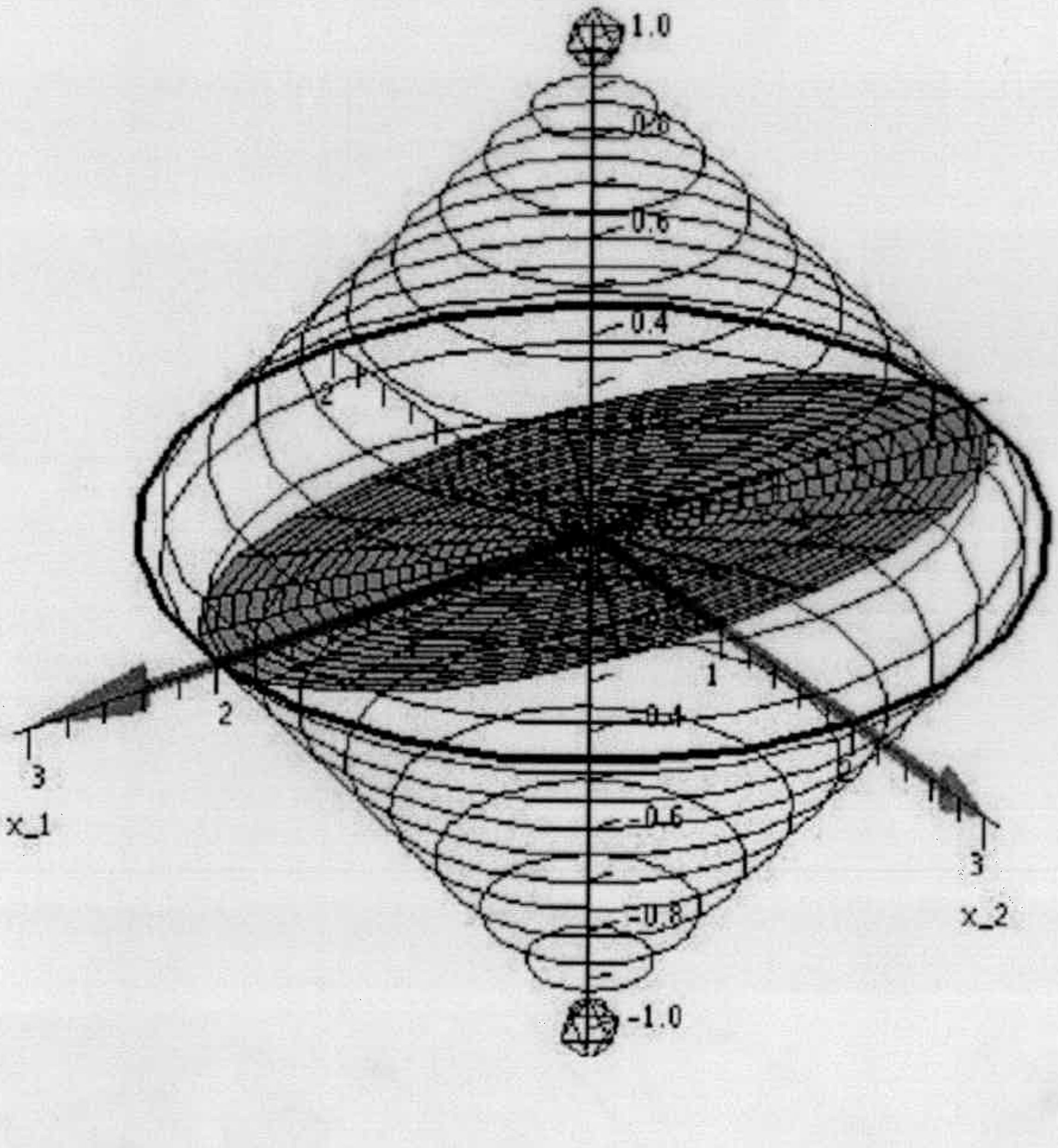
(i.e. $B_{\rho_\infty}(1)$ is the asymptotic shape of $B_\rho(n)$)

2. $(S_{Nil,\infty}, \rho_\infty)$ is the asymptotic cone of (G,ρ)

3. $\mathrm{vol}_G(B_\rho(n)) \sim n^d \cdot \mathrm{vol}_{S_{Nil,\infty}}(B_{\rho_\infty}(1))$

? no error term in general: take $\mathbb{Z} \ltimes_\alpha \mathbb{R}^2$, α Liouville

Key point in thm 3: ρ is asymptotically invariant under the semisimple parts of $Ad(g)$.


1.0
0.8
0.6
0.4
2
1
2
2
-0.4
2
3
x_1
-0.6
3
x_2
-0.8
-1.0

III. Applications to equidistribution and RWs

Weyl's equidistribution:

$$\alpha \notin \mathbb{Q} \iff \{n\alpha \mod 1\}_{n\in\mathbb{Z}} \text{ is equidistributed in } \mathbb{R}/\mathbb{Z}$$

$\iff$ the n-ball $[-n,n] \subseteq \mathbb{Z}^2$ becomes equidistributed in $\mathbb{R}$ under the dense embedding $\mathbb{Z}^2 \hookrightarrow \mathbb{Z} + \alpha\mathbb{Z} \subseteq \mathbb{R}$

<u>Thm 4</u> (B'07) N = connected nilpotent Lie group
Γ = f.g. dense subgroup, $\Gamma = \langle S \rangle$

then $|B_S(n) \cap \mathcal{U}| \sim c(S) \cdot \mathrm{vol}_N(\mathcal{U}) \cdot n^{d(\Gamma) - d(N)}$

[$\mathcal{U}$ = open set in N]

<u>Rk</u>: $c(S) = \mathrm{vol}(\mathrm{Ker}\,\varphi \cap B_{f_\infty}(1))$, where φ is given by Malcev Rigidity: $\widetilde{\Gamma} \xrightarrow{\varphi} N$

and f_∞ is Pansu's metric in $\widetilde{\Gamma}$ = Malcev closure.

key ingredients:

1. Pansu's thesis : comparison btw d_S and f_∞ .

2. Unique ergodicity of nilflows:

Γ dense in N $\iff$ $\ker\varphi \curvearrowright \tilde{\Gamma}/_{\Gamma}$ minimally and uniquely ergodically

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ = compact nilmanifold.

Applying deep work of Alexopoulos, one gets:

Cor: Let $\mu = \sum_{g \in S} \mu(g)\, \delta_g$ symmetric proba. measure on S.

Then $\mu^{*n}(\mathcal{U}) \sim c(\mu) \cdot \mathrm{vol}_N(\mathcal{U}) \cdot n^{d(N)/2}$

in words: the local limit theorem on N folds for finitely supported symmetric measures.