COMPACTIFYING TREES

The purpose of these notes is to present a motivating simple case of a joint project with Y. Guivarc'h and J.-Ph. Anker. The project itself concerns the compactification of Bruhat-Tits buildings, that is the geometries naturally attached to semisimple groups over (locally compact) non-Archimedean local fields. We deal with the geometric, the measure-theoretic (due to Furstenberg), the group-theoretic (due to Guivarc'h) and the polyhedral compactifications. The point is to define them, to identify them and then to use them to parametrize closed subgroups of dynamical interest. Our guideline is the case of symmetric spaces.

These notes are organized as follows. The first section states the problems we are interested in, considering at the same time symmetric spaces, Bruhat-Tits buildings and arbitrary semi-homogeneous trees. From section 2 until the end, the geometries considered are exclusively trees; §2 presents the class of locally ∞-transitive groups and their combinatorial properties. Section 3 defines the Furstenberg and Guivarc'h compactification procedures. The last section identifies all the so-obtained compact spaces, and uses them to classify maximal amenable automorphism groups of a given tree.

1. The general framework

Let us state the problems, which roughly speaking concern compactifications of geometries attached to classical groups and their generalizations.

1.A *Spaces and Groups.* — In this section, X is a (Riemannian, non-compact) symmetric space, a (locally finite) Bruhat-Tits building or an arbitrary (locally finite) semi-homogeneous tree. We assume we are given an appropriate automorphism group G, which acts by isometries on X. «Appropriate» means that G is the semisimple group over the reals (resp. a locally compact non-Archimedean field) defining X when it is a symmetric space (resp. a Bruhat-Tits building). For real groups, the symmetric space is the group G modulo one of its (all conjugate) maximal compact subgroups; for groups over non-Archimedean fields the construction of the Bruhat-Tits building is more involved [T]. In the case of trees, recall that a necessary (but not sufficient) condition for being Bruhat-Tits is to have valencies of the form «1+prime power» . We don't make this assumption, and assume that G is an arbitrary locally ∞-transitive isometry group in the sense of M. Burger and S. Mozes – see 2.A.

Examples. — 1) The group $G = \mathrm{SL}_2(\mathbf{R})$ is a real semisimple Lie group whose associated symmetric space is the so-called Poincaré hyperbolic disk. More generally, the symmetric space $\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R})$ parametrizes the scalar products on $\mathbf{R}^n$ up to homothety. Starting from the group $\mathrm{SO}(n,1)$ leads to the real hyperbolic space $\mathbb{H}^n\mathbf{R}$.
2) According to Goldman-Iwahori, the Bruhat-Tits building X of $\mathrm{SL}_n(\mathbf{Q}_p)$ parametrizes the non-Archimedean norms on $\mathbf{Q}_p^n$ up to homothety. As already said, it is *not* the quotient of $\mathrm{SL}_n(\mathbf{Q}_p)$ by a maximal compact subgroup. The maximal compact subgroups are all isomorphic to $\mathrm{SL}_n(\mathbf{Z}_p)$ but there are n conjugacy classes of such subgroups. For $n = 3$, the building X is a two-dimensional cell-complex covered by tesselation of $\mathbf{R}^2$ by regular triangles – the *apartments*. The small spheres centered at a point x of the 0-skeleton – a *vertex* – may be

seen as graphs describing the incidence relations at x. In this case, they are all isomorphic to the projective plane $\mathbb{P}^2(\mathbf{Z}/p)$.
3) Whereas for q a prime power, the semi-homogeneous tree of valencies $1+q$ and $1+q^3$ may be seen as the Bruhat-Tits building of $\mathrm{SU}_3\big(\mathbf{F}_q(\!(t)\!)\big)$, the homogeneous tree of valency 7 for instance doesn't admit any algebraic group interpretation. Nevertheless, the full automorphism group of any semi-homogeneous tree is with many respects analogous to a rank-one semisimple group over a non-Archimedean local field. As an abstract group, it is simple (Tits' theorem), and it is big enough to contain many other closed subgroups for which the analogy with rank-one groups is also relevant – see 2.A.

1.B *Problems.* — Our main problem is to study (and sometimes to give sense to) compactifications of X. For symmetric spaces, this is well-known – see [Moo1, GJT]. In fact, this classical situation will be our guideline.

As non-positively curved spaces, all the geometries we have in mind admit an asymptotic boundary $\partial_\infty X$ intrinsically defined as a set of equivalence classes of geodesic rays. It can also be seen (less intrinsically) as the set of rays emanating from a given point in X. In the case where X comes from a Lie or an algebraic group, the boundary $\partial_\infty X$ admits the structure of a spherical building – the *Tits building at infinity* of X. It describes the combinatorial structure arising from the relative Borel-Tits theory of algebraic groups over a non-necessarily algebraically closed field [BoT], [Bo, chapter V]. Then a classical procedure of gluing $\partial_\infty X$ to X makes $X \sqcup \partial_\infty X$ a compactification of X, which we call the *geometric compactification* $\overline{X}^{\mathrm{geom}}$ of X [BH, II.8].

In another direction, the *polyhedral compactification* $\overline{X}^{\mathrm{pol}}$ of X is defined by means of a gluing $\dfrac{G \times \overline{F}}{\sim}$ where $\overline{F}$ is the (simpler) compactification of a maximal isometric copy F of a Euclidean space in X. For a symmetric space such an F is called a *maximal flat*, for a Bruhat-Tits building it is an *apartment*, and for a tree it is simply a geodesic line. For all our spaces, the definition of this kind of compactification is either known or straightforward – see [AMRT], [L, §14] and section 4.A below.

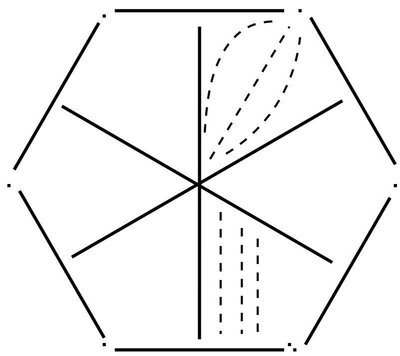
Problem A. — *Give sense to the Furstenberg and the Guivarc'h compactifications of X.*

Both compactifications rely on the simple idea which consists in defining an embedding of X into a compact metrizable space, and then taking the closure of the image. For Guivarc'h's procedure, the compact space is the set of closed subgroups of G endowed with the topology of Hausdorff convergence on compact subsets. Furstenberg's procedure involves the theory of boundaries of groups [M, VI]. The suitable compact space is that of the probability measures on a non-trivial Furstenberg boundary of G. In the case of buildings and trees, it seems that these approaches will compactify the set of vertices $\mathcal{V}_X$ of X but not the building X itself.

Problem B. — *Identify the above compactifications.*

For higher-rank symmetric spaces, the compactifications are *not* all isomorphic. The Furstenberg, the Guivarc'h and the polyhedral compactifications are G-homeomorphic [GJT, Moo1], but the geometric compactification is different from all the other ones. For the former compactifications, the boundary points of a Weyl chamber are parametrized by distances from the walls of the chamber. In the case of $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$, the boundary of a maximal flat F is the dual complex of the tesselation of F by the Weyl chambers (which may be thought of as some kind of «hexagon at infinity»). Any sequence of points in a Weyl chamber such that the distance from all walls goes to infinity, gives the same limit point : a vertex of the hexagon. On the contrary, the boundary points of a maximal flat F in the geometric compactification are parametrized by directions given by a unit vector in F.

Figure. — [Convergence of some wandering sequences in a compactified maximal flat]



PROBLEM C. — *Use the three isomorphic compactifications of* X (*all but* $\overline{X}^{\mathrm{geom}}$) *to parametrize interesting classes of closed subgroups* (*distal subgroups, some amenable subgroups...*).

In the case of symmetric spaces, points in the Guivarc'h compactification represent closed subgroups : it is a theorem by Guivarc'h that these points are precisely the maximal groups enjoying the property of *distality* – a notion of dynamical interest [G]. A subgroup is *distal* if the adjoint image of each of its elements has its spectrum contained in the unit circle. Taking the point stabilizers enables to classify a certain class of maximal amenable subgroups (Moore's theorem [Moo2]).

1.C *Results for trees.* — From now on, X is a semi-homogeneous tree and G is a closed locally ∞-transitive group of automorphisms without inversion – see 2.A. In what follows, we shall proof the following result.

THEOREM. — A) *The Furstenberg and Guivarc'h compactifications* $\overline{\mathcal{V}}_X^{\mathrm{Furs}}$ *and* $\overline{\mathcal{V}}_X^{\mathrm{Guiv}}$ *of the set of vertices* $\mathcal{V}_X$ *of* X *make sense.*
B) *The following identifications hold :* $\overline{X}^{\mathrm{geom}} \simeq \overline{X}^{\mathrm{pol}} =: \overline{X}$ *and* $\overline{\mathcal{V}}_X^{\mathrm{Furs}} \simeq \overline{\mathcal{V}}_X^{\mathrm{Guiv}} =: \overline{\mathcal{V}}_X$. *Besides, the closure of* $\mathcal{V}_X$ *in* $\overline{X}$ *identifies with* $\overline{\mathcal{V}}_X$.
C) *An amenable subgroup of* G *either fixes a vertex* $v \in X$, *either fixes a boundary point* $\xi \in \partial_\infty X$ *or stabilizes a geodesic line* $L \subset T$.

REMARKS. — 1) Point C) was proved by elementary arguments by A. Figá-Talamanca and C. Nebbia [FTN]. Here it is seen as a straightforward consequence of a measure-theoretic result (the analogue of Furstenberg's lemma for trees) due to Lubotzky-Mozes-Zimmer [LMZ].
2) The next compactifications to be considered are the ones involving potential theory (Martin compactifications) and random walks. This is another aspect of the project with J.-Ph Anker and Y. Guivarc'h, and once again for symmetric spaces and trees, this is known – see [GJT] for symmetric spaces and [W, 26.14] for trees.

2. Locally ∞-transitive groups. Combinatorics

Let X be a locally finite tree. The full automorphism group $\mathrm{Aut}X$ is endowed with the topology of uniform convergence on finite subsets [FTN, I.4]. A basis of this topology consists of the subsets $U_F(g) \subset \mathrm{Aut}X$, where

$$F \subset X \text{ is finite}, \quad g \in \mathrm{Aut}X \quad \text{and} \quad U_F(g) := \{h \in \mathrm{Aut}X : g|_F = h|_F\}.$$

This makes $\mathrm{Aut}X$ a locally compact totally disconnected topological group.

2.A *Locally ∞-transitive groups.* — Rather than studying the only group $\mathrm{Aut}X$, we have in mind a wider class of closed non-discrete subgroups [BM].

DEFINITION. — *A subgroup of tree automorphisms $G < \mathrm{Aut}X$ is* locally ∞-transitive *if for every vertex $v \in X$ the fixator $\mathrm{Fix}_G(v)$ is transitive on all spheres $S(v,n)$ centered at v.*

We denote by $S(v,n)$ (resp. by $B(v,n)$) the sphere (resp. the ball) of radius n centered at v. From now on, we assume that the tree X admits a closed locally ∞-transitive group of automorphism G. Using the G-action on spheres of radius 1, we see immediately that X must be semi-homogeneous. We attach to each vertex v a *type* $\tau_v \in \{0;1\}$ in such a way that two adjacent vertices don't have the same type. We also assume that the group G is type-preserving, which of course can always be done after passing to a subgroup of index at most 2. The maximal compact subgroups in G are the vertex fixators; they are open. We will mainly use the following transitivity properties [BM, 3.1.1].

PROPOSITION. — *Let X be a locally finite tree, and $G < \mathrm{Aut}X$ be a closed subgroup of tree automorphisms. Then the following are equivalent.*

(i) *The group G is locally ∞-transitive.*
(ii) *For every vertex $v \in X$, the group K_v is transitive on $\partial_\infty X$.*
(iii) *The group G is non-compact and transitive on $\partial_\infty X$.*
(iv) *The group G is 2-transitive on $\partial_\infty X$.* □

REMARK. — The equivalence (i) $\iff$ (ii) is due to C. Nebbia – see [FTN, I.10].

2.B *Tits systems.* — These transitivity properties have deep combinatorial consequences, involving the structure of Tits systems [Bou2, IV.2].

LEMMA. — *Let G be a closed locally ∞-transitive subgroup of $\mathrm{Aut}X$. We make the following choices of subsets in $X \sqcup \partial_\infty X$:*

– *a geodesic line $L := (\xi\eta)$ in X, which defines two boundary points ξ and η in $\partial_\infty X$;*
– *an edge $E := [v;v']$ in L, which defines two adjacent vertices v and v' in X.*

We set $P = P_\xi := \mathrm{Fix}_G(\xi)$, $N = N_L := \mathrm{Stab}_G(L)$ and $B = B_E := \mathrm{Fix}_G(E)$. Then, there exists in G an autmorphism s_v (resp. $s_{v'}$), stabilizing L, fixing v (resp. v') and permuting ξ and η so that :

(i) *The quadruple (G, P_ξ, N_L, s_v) is a spherical Tits system with Weyl group $\mathbf{Z}/2$;*
(ii) *The quadruple $(G, B_E, N_L, \{s_v; s_{v'}\})$ is an affine Tits system with Weyl group D_∞, the infinite dihedral group.*

Proof. Existence of symmetries. Let us prove the existence of a symmetry s_v w.r.t. the vertex v. For any radius $n \in \mathbf{N}$, by local ∞-transitivity there exists an automorphism $g_n \in G$ whose restriction to $B(v,n)$ is a symmetry s_v around v stabilizing the diameter $[x;x'] := L \cap B(v,n)$. Indeed, first use transitivity on $S(v,n)$ to get $g'_n \in K_v$ sending x on x'; then use the sphere centered at x' and of radius twice bigger to get $g''_n \in K_v$ sending $g'_n x'$ on x. The automorphism $g_n := g''_n g'_n$ is an approximation of s_v on $B(v,n)$. All the elements g_n are in the compact subgroup $\mathrm{Fix}_G(v)$, hence the sequence $\{g_n\}_{n\in\mathbf{N}}$ admits a cluster value. Such an automorphism can be chosen as s_v, and the same argument applies for the vertex $s_{v'}$.

Point (i). We show that the axioms (T1)-(T4) of a Tits system as stated in [Bou2, IV.2] are satisfied. By the previous point, we know that there exists an element $s \in N$ which permutes ξ and η; indeed, we can choose s_v or $s_{v'}$ as above for instance. (We could also have used 2-transitivity of G on $\partial_\infty X$.) Let $g \in G$. Assume that $g.\xi \neq \xi$. Then, by 2-transitivity of G on $\partial_\infty X$, there exists $p \in P_\xi$ such that $p^{-1}g.\xi = \eta$, which implies $sp^{-1}g.\xi = \xi$. This argument proves the Bruhat decomposition of G :

$$G = P_\xi \sqcup P_\xi s P_\xi,$$

which implies in particular the first half of (T1) : $G = \langle P_\xi, s \rangle$, and axiom (T3). Axiom (T4) is clear since $sP_\xi s^{-1} = P_\eta := \mathrm{Fix}_G(\eta) \neq P_\xi$. The group $P_\xi \cap N_L$ is nothing else than the pointwise fixator of $\{\xi; \eta\}$: it is clearly normal in the global stabilizer N_L of this pair of boundary points. This remark implies the second half of axiom (T1), and axiom (T2).

Point (ii). Recall that the tree X is a building with Weyl group D_∞. The apartments are the geodesic lines and the chambers are the edges in X. First, since the group N_L contains the reflections s_v and $s_{v'}$, it is transitive on the edges contained in L. Combined with the 2-transitivity of G on $\partial_\infty X$, this implies the transitivity of G on pairs of edges at given distance. Hence, in the terminology of [Ron, §5], we proved that G is strongly transitive on X w.r.t. L, which implies (ii), according to [Ron, Theorem 5.2]. □

Remarks. — 1) From now on, we will use the notation K_v to denote the group $\mathrm{Fix}_G(v)$. This subgroup K_v is a parabolic subgroup – in the abstract sense of Tits systems – of G :

$$K_v = B_E \sqcup B_E s_v B_E.$$

2) By irreducibility of the Coxeter systems of both Weyl groups, P_ξ and K_v are maximal subgroups. This property is formally derived from the Tits system structure [Bou2, IV.2].

2.C *Decompositions.* — We can proceed further in the decomposition of the group G and of some of its subgroups, after defining some additional subgroups.

Definition. — (i) *Let $U = U_\xi < P_\xi$ be the subgroup consisting of the elements of G fixing ξ and stabilizing each horocycle centered at ξ.*
(ii) *For any hyperbolic translation τ of step 2, with attracting (resp. repelling) point ξ (resp. η), we denote by $T = T_\tau$ the cyclic subgroup of G generated by τ, and by $\overline{T}^+ = \overline{T}^+_\tau$ its subsemigroup of non-negative powers $\tau^n, n \geq 0$.*

Remarks. — 1) The next proposition will justify that there do exist such hyperbolic translations τ in G.
2) For Bruhat-Tits buildings, there is a dictionary between apartments and maximal k-split tori. We note here that the definition of the subgroup T – the analogue of a maximal k-split torus – not only depends on the choice of the geodesic line L, but also on that of τ. Nevertheless, we may often use the notation $T_{\xi,\eta}$ for a subgroup T_τ as above, the choice of τ among other hyperbolic translations along $(\xi\eta)$ being usually harmless.

In spite of the slight differences in the expected definitions, we can prove analogues of the well-known decompositions concerning Lie groups. We denote by $\partial^2_\infty T$ the product $\partial_\infty T \times \partial_\infty T$ minus its diagonal.

Proposition. — (i) *For every couple of distinct points $(\xi', \eta') \in \partial^2_\infty T$, the group G contains a hyperbolic translation $\tau_{\xi',\eta'}$ of step 2, with attracting (resp. repelling) point ξ' (resp. η').*

(ii) *The group* U_ξ *is transitive on every horosphere centered at* ξ *and the group* G *admits an* Iwasawa decomposition : $G = K_v \cdot T_\tau \cdot U_\xi$, *for any hyperbolic translation* τ *with attracting (resp. repelling) point* ξ *(resp.* η*).*
(iii) *The group* P_ξ *is amenable and admits a semidirect product decomposition :* $P_\xi = T_\tau \ltimes U_\xi$.
(iv) *The group* G *also admits a* Cartan decomposition : $G = K_v \cdot \overline{T}_\tau^+ \cdot K_v$.

REMARKS. — 1) Points (iii) and (iv) are used in [LM] in order to prove the vanishing at ∞ of matrix coefficients of unitary representations.
2) The quotient group $N_{(\xi\eta)}/\mathrm{Fix}_G\big(\{\xi;\eta\}\big)$ is isomorphic to $\mathbf{Z}/2$, generated by the permutation of ξ and η. The subgroup $\mathrm{Fix}_G\big(\{\xi;\eta\}\big)$ admits the semidirect product decomposition $\mathrm{Fix}_G\big(\{\xi;\eta\}\big) = T_\tau \ltimes \mathrm{Fix}_G\big((\xi\eta)\big)$.

EXAMPLE. — The group $\mathrm{SL}_2(\mathbf{Q}_p)$ is locally ∞-transitive on the homogeneous tree T_{p+1} of valency $p+1$. The diagonal matrices $\mathrm{Diag}(k,k^{-1})$ $(k \in \mathbf{Q}_p^\times)$ form a maximal split torus $T = \mathbf{T}(\mathbf{Q}_p)$ to which is associated a geodesic line L. The line is stabilized by T and the element $\tau := \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \in T$ is a step 2 hyperbolic translation along L. The (pointwise) fixator of L is the compact group $T = \mathbf{T}(\mathbf{Z}_p)$ made of the matrices $\mathrm{Diag}(k,k^{-1})$ $(k \in \mathbf{Z}_p^\times)$. We can see now why our notations may lead to confusion since for a boundary point ξ of L, the group U_ξ is the extension $\mathbf{T}(\mathbf{Z}_p) \ltimes \begin{pmatrix} 1 & \mathbf{Q}_p \\ 0 & 1 \end{pmatrix}$, though we could think about it as the unipotent group $\begin{pmatrix} 1 & \mathbf{Q}_p \\ 0 & 1 \end{pmatrix}$. Our notations are more convenient to define subgroups geometrically. At last, the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix}$ lift symmetries w.r.t. adjacent vertices.

Proof. Point (i). The automorphism $s_v \circ s_{v'}$ or its inverse is a required hyperbolic translation along the geodesic line L. The case of an arbitrary geodesic line L' follows by conjugation since G is 2-transitive on $\partial_\infty X$.

Point (ii). Let v and v' be two points on the same horosphere centered at ξ. Denote by $\{v_n\}_{n \geq 0}$ the set of vertices of $[v\xi) \cap [v'\xi)$, with $\beta_{v_0,\xi}(v_n) = -n$ where $\beta_{v_0,\xi}$ is the Busemann function associated to the ray $[v_0\xi)$. For each $n \geq 0$, v and v' are on the same sphere centered at v_n. Hence by local ∞-transitivity, there exists an element $g_n \in G$ mapping $[v_n;v]$ onto $[v_n;v']$. Each element g_n fixes $[v_0;v_n]$; consequently, lies in the compact subgroup $\mathrm{Fix}_G(v_0)$. Any cluster value of the sequence $\{g_n\}_{n\geq 0}$ is an element of U_ξ sending v to v'. This implies that U_ξ is transitive on every horosphere centered at ξ. This also leads to the Iwasawa decomposition. Indeed, let $g \in G$; by the previous result, gv can be sent by an element $u \in U_\xi$ to a point of $(\xi\eta)$. Then, by type-preservation, a suitable power of τ sends $(ug).v$ to v.

Point (iii). The amenability assertion is justified for instance in [LM]. Let $g \in P_\xi$ and S be a horosphere centered at ξ. Then, by type-preservation, a suitable power τ^n sends $g.S \cap (\eta\xi)$ onto $S \cap (\eta\xi)$. Hence, $\tau^n g$ is in U_ξ, since once an element of P_ξ stabilizes a horosphere centered at ξ, it fixes all of them. The argument also shows that $\langle\tau\rangle$ normalizes U_ξ. This proves the semidirect product assertion, since the only power τ^n stabilizing S is the trivial one.

Point (iv). This is the same argument of successive rectifications of an automorphism in order to send the point v back onto itself. The idea here is that K_v is transitive on all spheres centered at v. □

3. Measure- and group-theoretic compactifications

We keep our closed locally ∞-transitive type-preserving group G of automorphisms of the (semi-homogeneous, labelled) tree X. We first define the measure-theoretic compactification of the tree, and then use it as a tool to define the group-theoretic compactification.

3.A *Distinguished sequences of vertices.* — We use the notations of 2.B and consider the ray $R = [v\xi) \subset L$. The *fundamental sequences* will converge in the various compactifications under consideration.

Definition. — (ii) *A* wandering sequence *is a sequence* $\{v_n\}_{n\geq 1}$ *of vertices in the tree* X *which eventually leaves every finite subset of* X.
(ii) *A* canonical sequence (w.r.t. the ray R) *is a sequence* $\{v_n\}_{n\geq 1}$ *of vertices in the ray* R *which eventually leaves every finite subset of* R.
(iii) *A* fundamental sequence (w.r.t. the ray R) *is a sequence* $\{v_n\}_{n\geq 1}$ *of vertices in the tree* X *for which there exists a converging sequence* $\{k_n\}_{n\geq 1}$ *of elements of* K_v *such that* $\{k_n^{-1}.v_n\}_{n\geq 1}$ *is a canonical sequence* (*w.r.t.* R).

Remark. — The transitivity of the compact vertex fixator K_v on spheres centered at v shows that any wandering sequence admits a subsequence which is fundamental w.r.t. the ray R. Combined with convergence computations, this easy fact will be used in 3.C and 3.E to justify discreteness assertions.

3.B *Geometric and measure-theoretic boundaries.* — As a consequence of the transitivity properties, of the Iwasawa decomposition of G and of the amenability of P_ξ, we have :

Lemma. — *The geometric boundary* $\partial_\infty X$ *is the maximal Furstenberg boundary of* G.

Proof. The space $\partial_\infty X$ is homogeneous – hence minimal (any orbit is dense) – under G. In view of the beginning of 3.C below and of the existence of enough hyperbolic translations in G, the G-orbit of any probability measure μ on $\partial_\infty X$ contains a Dirac measure in its closure. This proves the strong proximality of $\partial_\infty X$ under the G-action. Thus, $\partial_\infty X$ is a Furstenberg boundary of G. At last, since G writes $G = KTU$ with K compact and TU amenable (Iwasawa – see 2.C), it follows that every Furstenberg boundary of G is an equivariant image of $G/TU \cong \partial_\infty X$ – see [F, 4.4]. Hence the maximality of $\partial_\infty X$. □

It follows from a general fact [Bou1, III §1, exercice 14.a] that the space $\mathcal{M}^1(\partial_\infty X)$ of probability measures on the asymptotic boundary $\partial_\infty X$ is compact and metrizable.

3.C *Convergence and discreteness for measures.* — In view of the dynamics of hyperbolic translations on $\partial_\infty X$ [GH, II.8.16], the Lebesgue dominated convergence theorem shows that :

$$\lim_{n\to\infty} \tau_*^n \mu = \delta_\xi,$$

for any probability measure μ on $\partial_\infty X$ and any hyperbolic translation τ with attracting point ξ, provided that the reppelling point of τ is not an atom of μ.

Lemma. — (i) *To each vertex* $v \in X$ *is associated a probability measure* μ_v *whose fixator is precisely the maximal compact subgroup* K_v.
(ii) *The assignment* $\phi^{\text{Furs}} : v \mapsto \mu_v$ *defines an embedding of the discrete set of vertices of* X *into the space of probability measures* $\mathcal{M}^1(\partial_\infty X)$.

Proof. Point (i). According to the local ∞-transitivity assumption on G, the compact group K_v is transitive on $\partial_\infty X$ for every vertex v. Hence, there exists a unique probability measure

$\mu_v \in \mathcal{M}^1(\partial_\infty X)$ which is fixed under K_v. According to the ordered structure of parabolic subgroups (in the abstract sense) of Tits systems, if $\mathrm{Fix}_G(\mu_v)$ were strictly bigger than K_v, it would be the whole group G. This is impossible since G contains hyperbolic translations, and any such τ satisfies : $\lim_{n\to\infty} \tau_*^n \mu_v = \delta_\xi$, where ξ is the attracting point of τ.

Point (ii). Point (i) establishes a one-to-one correspondence between the measures μ_v and the maximal compact subgroups K_v, hence a one-to-one correspondence between the measures μ_v and the vertices. By uniqueness of the measure attached to a vertex, we have $\mu_{g.v} = g_*\mu_v$. Assume now that μ is a cluster point in the subset $\{\mu_v\}_{v\in\mathcal{V}_X}$ of $\mathcal{M}^1(\partial_\infty X)$. Consequently, μ is the limit of an injective sequence $\{\mu_{v_n}\}_{n\geq 1}$. At the level of vertices, we get an injective hence wandering sequence of vertices $\{v_n\}_{n\geq 1}$. Then, as seen in 3.A, this sequence of vertices admits a fundamental subsequence $\{v_{n_j}\}_{j\geq 1}$ w.r.t. to a fixed ray $[v\xi)$, with $\{k_j\}_{j\geq 1}$ a sequence in K_v converging to $k \in K_v$ and such that $\lim_{j\geq 1} k_j^{-1} v_{n_j} = \xi$. This implies $\mu = \delta_{k.\xi}$, which is not fixed by a maximal compact subgroup : contradiction. □

Remark. — The measure μ_v can be described more explicitly as the weak limit (in the space of probability measures on $X \sqcup \partial_\infty X$) of the equidistributed probability measures on spheres centered at v, as the radius goes to ∞.

Since $\partial_\infty X$ is the maximal measure-theoretic boundary of G, we are led to the following natural analogue of the Furstenberg compactification for trees.

Definition. — *The closure $\overline{\phi^{\mathrm{Furs}}(\mathcal{V}_X)}$ of the image of the above map is the* Furstenberg compactification *of the set of vertices $\mathcal{V}_X$ of X. It is denoted by $\overline{\mathcal{V}}_X^{\mathrm{Furs}}$.*

We turn now to the other compactification procedure, which uses the space of closed subgroups. To this end, we first need to define distances on the group $\mathrm{Aut}X$.

3.D *Metrics on automorphism groups.* — We choose once and for all a real number $q > 1$. Let us fix a vertex $v\in X$. Given $h, h' \in \mathrm{Aut}X$, two cases may occur :

– either $h.v \neq h'.v$, in which case we set $d_v(h,h') := q$;
– or $h.v = h'.v$, in which case we set $d_v(h,h') := q^{-\omega_v(h,h')}$, where $\omega_v(h,h')$ is the maximal radius of the sphere centered at v on which h and h' coincide.

We have either $d_v(h,h') \leq 1$ or $d_v(h,h') > 1$ (*i.e.*, $d_v(h,h') = q$) according to whether v has the same image under h and h', or not. For $h \in G$ and $r \in \mathbf{R}_+$, we denote by $\overline{B}_G(h,r)$ (resp. $B_G(h,r)$) the set of tree automorphisms in G at distance $\leq r$ (resp. $< r$) from h. The «closed ball» of radius 1 centered at the identity is by definition the fixator of the vertex v : $B_G(\mathrm{id}, \leq 1) = K_v$. Conversely, any ball of radius q is the whole automorphism group. Elementary geometric arguments prove the following.

Lemma. — (i) *For any vertex v, the application d_v is an ultrametric distance on* $\mathrm{Aut}X$.
(ii) *All metrics $\{d_v\}_{v\in\mathcal{V}_X}$ are in the same Lipschitz class and induce the above defined topology.*
(iii) *The metrics d_v are left invariant under* $\mathrm{Aut}X$. *Right multiplication by any $g\in \mathrm{Aut}X$ is a bi-Lipschitz map for any d_v.* □

We can now introduce the second compact space into which $\mathcal{V}_X$ can be embedded.

Definition. — *Let $\mathcal{C}(G)$ be the set of closed subsets of G; let $\mathcal{S}(G)$ be that of closed subgroups.*

We choose a metric d_v as above, in order to proceed as in [GJT, 9.4]. On every compact subset of G, this defines a Hausdorff distance which we always denote by D_H. We define a topology on

$\mathcal{C}(G)$ via the following notion of convergence. A sequence of closed subsets $\{F_n\}_{n\geq 1}$ converges to F in $\mathcal{C}(G)$ if for any compact subset C, the intersections $C\cap F_n$ converge to $C\cap F$ for the Hausdorff distance D_H. More precisely, for every $\varepsilon>0$, every non-empty compact subset C and every closed subset F, let us define

$$P(C,\varepsilon;F) := \{F'\in\mathcal{C}(G) : D_H(C\cap F, C\cap F')<\varepsilon\}.$$

Definition. — *The $P(C,\varepsilon;F)$'s are fundamental neighborhoods of F for a metrizable topology which does not depend on the choice of d_v : the* topology of Hausdorff convergence on compact subsets on $\mathcal{C}(G)$. □

The definition of the topology implies that if a sequence of closed subsets F_n converges to F, then F is the set of limits of converging sequences $\{x_n\}_{n\geq 1}$ with $x_n\in F_n$; hence, the subspace $\mathcal{S}(G)$ is closed in $\mathcal{C}(G)$. We turn now to the fundamental compactness result, a standard consequence of the diagonal process.

Proposition. — *The space of closed subgroups $\mathcal{S}(G)$ is compact for the topology of Hausdorff convergence on compact subsets.* □

3.E *Convergence and discreteness for subgroups.* — As already mentioned, we use now probability measures as tools, for instance thanks to the following lemma [GJT, 9.7], another consequence of the Lebesgue dominated convergence theorem.

Lemma. — *Let $\{F_n\}_{n\geq 1}$ be a sequence of closed subgroups of a locally compact metrizable group G, converging to $F\in\mathcal{S}(G)$. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of probability measures on a locally compact metrizable G-space Y, converging to $\mu\in\mathcal{M}^1(Y)$. Assume that for each $n\geq 1$, the measure μ_n is fixed under F_n. Then μ is fixed under F.* □

The latter result enables to justify that fundamental sequences converge in the space $\mathcal{S}(G)$.

Proposition. — (i) *Let $\{v_n\}_{n\geq 1}$ be a canonical sequence of vertices w.r.t. $R=[v\xi)$. Then the corresponding sequence of maximal compact subgroups $\{K_n := \mathrm{Fix}_G(v_n)\}_{n\geq 1}$ converges in $\mathcal{S}(G)$ to the subgroup U_ξ.*
(ii) *The set $\mathcal{M}(G)$ of maximal compact subgroups of G is discrete in $\mathcal{S}(G)$. The assignment $\phi^{\mathrm{Guiv}} : v\mapsto K_v$ thus defines an embedding of the discrete set of vertices $\mathcal{V}_X$ into $\mathcal{S}(G)$.*

Proof. Point (i). By compactness of $\mathcal{S}(G)$, it suffices to show that any cluster value of the sequence $\{K_n\}_{n\geq 1}$ equals U_ξ. Let $D=\lim\limits_{j\to\infty} K_{n_j}<G$ be such a subgroup. Choose a geodesic line $(\xi\eta)$ extending R and a hyperbolic translation of step 2 along $(\xi\eta)$, with attracting point ξ. At last, fix v' a vertex of $(\xi\eta)$ adjacent to v. After passing to a subsequence, we may assume that $K_{n_j}=\tau^{n_j}K_{v''}\tau^{-n_j}$ for a sequence of positive exponents $\{n_j\}_{j\geq 1}$ satisfying $\lim\limits_{j\to\infty} n_j=\infty$ and for $v''=v$ or v'. Then by the above lemma, $\lim\limits_{n\to\infty}\tau^n_*\mu_{v''}=\delta_\xi$ implies that D fixes δ_ξ hence ξ : $D<P_\xi$.

Let g be an automorphism of D. By 2.C (iii), it writes $g=u\tau^N$, with $N\in\mathbf{Z}$ and $u\in U_\xi$. As an element of a limit group, it also writes $g=\lim\limits_{j\to\infty}\tau^{n_j}k_j\tau^{-n_j}$, for a sequence $\{k_j\}_{j\geq 1}$ of elements of $K_{v''}$. Hence, there exists $J\geq 1$ such that for any $j\geq J$, we have $(u\tau^N).v''=(\tau^{n_j}k_j\tau^{-n_j}).v''$. Since u stabilizes all the horospheres centered at ξ, there is a vertex z in $(\xi\eta)$ such that $(u\tau^N).v''$ and $\tau^N.v''$ are on the same sphere centered at z. Hence, we may choose j large enough to have $d\big(\tau^{n_j}.v'',(u\tau^N).v''\big)=2n_j-2N$. But, the group $\tau^{n_j}K_{v''}\tau^{-n_j}$ stabilizes the spheres centered at $\tau^{n_j}.v''$, which implies that $d\big(\tau^{n_j}.v'', (\tau^{n_j}k_j\tau^{-n_j}).v''\big) = 2n_j$. Thus in order to have $(u\tau^N).v'' = (\tau^{n_j}k_j\tau^{-n_j}).v''$, we must have $N = 0$, hence $g = u$. This shows that $D = U_\xi$.

Point (ii). Once we have the convergence result of the previous point (with non-compact limit groups), this is the same argument as for measures – see 3.C. □

At last, we can define the group-theoretic compactification.

Definition. — *The closure* $\overline{\phi^{\rm Guiv}(\mathcal{V}_X)}$ *of the image of the above map is the* Guivarc'h compactification *of the set of vertices* $\mathcal{V}_X$ *of* X. *It is denoted by* $\overline{\mathcal{V}}_X^{\rm Guiv}$.

Remark. — It would be interesting to have an intrinsic description of the closed subgroups that appear on the boundary of the compactification. For symmetric spaces, they are the non-compact maximal distal subgroups [1.B, problem C]. Since there is no adjoint representation here, the notion of distality is not defined.

4. Polyhedral compactification. Identifications. Amenable subgroups

The last compactification to be defined is the polyhedral one. As for the case of Bruhat-Tits buildings, we will take into account the whole tree, and extend an equivalence relation defining the tree by gluing. This leads to a compactification of the tree as a metric space. Taking the closure of the set of vertices gives a compact space for which the comparisons with the previous compactifications make sense. The last subsection uses «the» compactification to parametrize maximal amenable subgroups.

4.A *Polyhedral compactification.* — In order to define the polyhedral compactification of the tree X, we mimick the construction done in [L, §14]. Since X is not necessarily Bruhat-Tits and since the arguments in the tree case are simple, we write them explicitly. With the notations of 2.B, we consider the closure $\overline{L} \subset X^{\rm geom}$ of the geodesic $L = (\eta\xi)$ containing the standard edge $E = [v; v']$. The subspace $\overline{L}$ admits an action by D_∞ via the restriction map $N \twoheadrightarrow N|_{\overline{L}}$.

Definition. — *The* polyhedral compactification $\overline{X}^{\rm pol}$ *is the quotient space* $\dfrac{G \times \overline{L}}{\sim}$, *where* $(g, x) \sim (h, x)$ *if and only if there exists* $n \in N$ *such that* $y = n.x$ *and* $g^{-1}hn \in K_x := {\rm Fix}_G(x)$. *We denote by* $[g, x]$ *the class of* (g, x) *for* $\sim$, *and by* $\pi : G \times \overline{L} \twoheadrightarrow \overline{X}^{\rm pol}$ *the natural projection.*

By definition, we have $[g, x] = [gk, x]$ for all $x \in K_x$ and $[n, x] = [1, n.x]$ for all $n \in N$. The group G obvioulsy acts on $\overline{X}^{\rm pol}$ by : $h[g, x] := [hg, x]$. We can also define the map

$$\begin{array}{rccc} \phi : & G \times \overline{L} & \longrightarrow & \overline{X}^{\rm geom} \\ & (g, x) & \mapsto & g.x \end{array}$$

If $(g, x) \sim (h, x)$, then $y = n.x$ and $g^{-1}hn \in K_x$, which leads to $h.y = hn.x = gk_x.x = g.x$. Conversely, if $h.y = g.x$ then in view of the action of G on the tree X, the equality $x = g^{-1}h.y$ implies that $y = n.x$ for some $n \in N$. Then $h.y = g.x$ writes $g.x = hn.y$, so that $g^{-1}hn.x = x$. Since ϕ is surjective, we get by factorization a bijection $\bar{\phi} : \overline{X}^{\rm geom} \cong \overline{X}^{\rm pol}$, which is clearly G-equivariant.

Lemma. — *The space* $\overline{X}^{\rm pol}$ *is compact and the factorization map* $\bar{\phi}$ *is a* G*-homeomorphism. Hence,* $\overline{X}^{\rm pol}$ *is a compactification of the tree* X.

Proof. Let us denote by $\overline{R} = [v\xi]$ the (compact) closure of the ray $R = [v\xi)$ in $\overline{X}^{\rm geom}$. By local ∞-transitivity of G, it is a fundamental domain for the action of K_v on $\overline{X}^{\rm geom}$. Since $\bar{\phi}$

is a G-equivariant bijection, this shows that the restricted projection map $\pi : K_v \times \overline{R} \to \overline{X}^{\rm pol}$ is surjective. Hence, in order to conclude that $\overline{X}^{\rm pol}$ is compact, we need to show that it is Hausdorff. This amounts to proving that the graph of $\sim$ restricted to $(K_v \times \overline{R}) \times (K_v \times \overline{R})$ is closed. Since $(k, x) \sim (k', x')$ is equivalent to $k.x = k'.x'$ in $\overline{X}^{\rm geom}$, this comes from the continuity of the G-action on the geometric compactification. □

4.B *Identifications.* — We can now state the comparison result.

Proposition. — *Let X be a semi-homogeneous tree, whose set of vertices is denoted by $\mathcal{V}_X$. Then, the following compactifications of $\mathcal{V}_X$ are* Aut$X$*-homeomorphic.*

(i) *The geometric compactification $\mathcal{V}_X \sqcup \partial_\infty X$.*
(ii) *The polyhedral compactification $\overline{\mathcal{V}}_X^{\rm pol}$, constructed by gluing.*
(iii) *The Guivarc'h compactification $\overline{\mathcal{V}}_X^{\rm Guiv} = \{K_v\}_{v \in \mathcal{V}_X} \sqcup \{U_\xi\}_{\xi \in \partial_\infty X}$.*
iv) *The Furstenberg compactification $\overline{\mathcal{V}}_X^{\rm Furs} = \{\mu_v\}_{v \in \mathcal{V}_X} \sqcup \{\delta_\xi\}_{\xi \in \partial_\infty X}$.*

Besides, for every locally ∞-transitive group $G <$ Aut$X$*, the fixator of any boundary point is a parabolic subgroup of the spherical Tits system of G.*

Proof. Note first that in any compactification above, the stabilizer of the boundary point indexed by $\xi \in \partial_\infty X$ is precisely P_ξ, since it is a maximal proper subgroup [2.B, remark 2]. Then the result follows form the domination criterion [GJT, 3.28]. □

4.C *Amenable subgroups.* — The determination of maximal amenable subgroups of tree automorphism groups was done in [FTN]. It is proved there by elementary geometric arguments. In our context, it is more natural to prove it by some kind of Furstenberg lemma about supports of limit measures – see [LMZ, 4.3].

Lemma. — *Let $\{g_n\}_{n \geq 1}$ be an unbounded sequence of tree automorphisms. Assume there are two probability measures μ, ν on $\partial_\infty X$ such that $\lim_{n \to \infty} g_{n*}\mu = \nu$. Then the support of the limit measure ν contains at most two points.* □

Now we can state the result about amenable groups of tree automorphisms [FTN, I.8.1].

Proposition. — *Let H be an amenable subgroup of* Aut$X$*. Then, either H fixes a vertex $v \in X$, either it fixes a boundary point $\xi \in \partial_\infty X$ or it stabilizes a geodesic line $L \subset T$.*

Proof. Being amenable, the subgroup H fixes a probability measure $\mu \in \mathcal{M}^1(\partial_\infty X)$. If H is compact, it fixes a vertex v and we are done. If not, then by the Furstenberg lemma, the support of μ contains at most two points. The support supp(μ) is stabilized by H, and we obtain the last two possibilities according to whether supp$(\mu) = \{\xi\}$ or supp$(\mu) = \{\xi; \eta\}$. □

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