Ricci curvature in metric spaces: the case of the Heisenberg group

Nicolas JUILLET

Université de Strasbourg

Bures-sur-Yvette, september 2009

Outline

Geodesic space

In a metric space (X,d), $\gamma : [0,1] \to X$ is a geodesic if for every $s, s' \in [0,1]$

$$d(\gamma(s),\gamma(s')) = |s-s'|\, d(\gamma(0),\gamma(1)).$$

A metric space (X,d) is geodesic if for all (p,q) there is a geodesic γ from p to q.

If γ is unique, we define $\mathcal{M}^s(p,q) = \gamma(s)$, the interpolation map.

Rényi Entropy

The N-entropy of a probability measure μ of density ρ is given by

$$\mathrm{Ent}_N(\rho\nu \mid \nu) = -\int \rho^{1-1/N} d\nu(x).$$

If μ is singular $\mathrm{Ent}(\mu) = 0$.

Big entropy: μ concentrated on a small space.
Small entropy: μ fills a lot of space.

Curvature-dimension $CD(0,N)$

A space (X,d,ν) satisfies $CD(0,N)$ if

for every absolutely continuous $\mu_0,\mu_1 \in \mathcal{P}_2(X)$, there is a geodesic $(\mu_s)_{s\in[0,1]}$ such that $s \in [0,1] \to \mathrm{Ent}_N(\mu_s \mid \nu) \in \mathbb{R}$ is convex.

(The exact statement is: there exists a geodesic $(\mu_s)_{s\in[0,1]}$ such that for any

$s \in [0,1]$, $\mathrm{Ent}_N(\mu_s \mid \nu) \leq (1-s)\,\mathrm{Ent}_N(\mu_0 \mid \nu) + s\,\mathrm{Ent}_N(\mu_1 \mid \nu)$)

Synthetic Ricci curvature

New definitions of positive Ricci curvature for metric measure spaces (X, d, ν):

- Measure Contraction Property $MCP(K, N)$ (Sturm; Ohta 2006)
- Curvature-Dimension $CD(K, N)$ (Lott-Villani; Sturm 2006)

$$CD(0, N) \Rightarrow \text{Brunn-Minkowski}(0, N) \Rightarrow MCP(0, N).$$

(The second implications is actually only known in the case the number of geodesics between p and q of X is almost surely 1).

Synthetic Ricci curvature

New definitions of positive Ricci curvature for metric measure spaces (X, d, ν):

- Measure Contraction Property $MCP(K, N)$ (Sturm; Ohta 2006)
- Curvature-Dimension $CD(K, N)$ (Lott-Villani; Sturm 2006)

$$CD(0, N) \Rightarrow \text{Brunn-Minkowski}(0, N) \Rightarrow MCP(0, N).$$

(The second implications is actually only known in the case the number of geodesics between p and q of X is almost surely 1).

The Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with the multiplicative structure:

$$(x,y,t)\cdot(x',y',t') = (z,t)\cdot(z',t') = \left(z+z', t+t' - \frac{\mathrm{Im}(z\overline{z'})}{2}\right).$$

The Lebesgue measure $\mathcal{L}^3$ is left-invariant.

The left invariant vector fields

$$\mathbf{X} = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t} \quad \text{and} \quad \mathbf{Y} = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t}$$

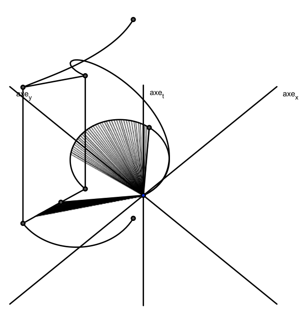
and $\mathbf{T} = [\mathbf{X},\mathbf{Y}] = \frac{\partial}{\partial \mathbf{t}}$ span the tangent space in any point.

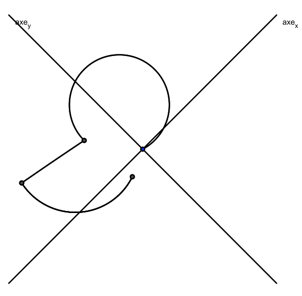
$$d_c(p,q) = \inf_{\gamma} \int_0^1 \sqrt{a^2(s)+b^2(s)}ds$$

where γ is horizontal:

$$\dot\gamma(s) = a(s)\mathbf{X}(\gamma(s)) + b(s)\mathbf{Y}(\gamma(s)).$$

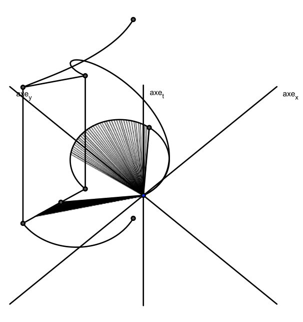
Geodesics of $\mathbb{H}$.

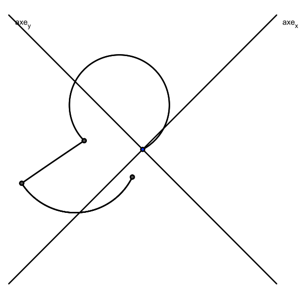




- A curve is horizontal if and only if the third coordinate evolves like the algebraic area swept by the complex projection.
- The length of the horizontal curves is exactly the length of the projection in $\mathbb{C}$.
- The geodesics of $\mathbb{H}_1$ are the horizontal curves whose projection is a circle arc or a line.

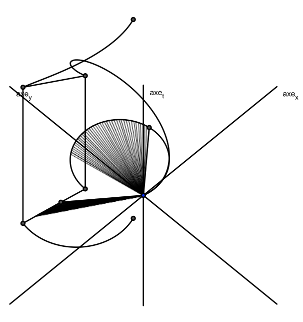
Geodesics of $\mathbb{H}$.

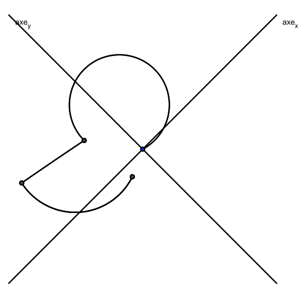




- A curve is horizontal if and only if the third coordinate evolves like the algebraic area swept by the complex projection.
- The length of the horizontal curves is exactly the length of the projection in $\mathbb{C}$.
- The geodesics of $\mathbb{H}_1$ are the horizontal curves whose projection is a circle arc or a line.

Geodesics of $\mathbb{H}$.





- A curve is horizontal if and only if the third coordinate evolves like the algebraic area swept by the complex projection.
- The length of the horizontal curves is exactly the length of the projection in $\mathbb{C}$.
- The geodesics of $\mathbb{H}_1$ are the horizontal curves whose projection is a circle arc or a line.

The key estimate

Key estimate

For any $e \in \mathbb{H}$. The contraction map $\mathcal{M}_e^s : f \to \mathcal{M}^s(e,f)$ is differentiable with

$$\mathrm{Jac}(\mathcal{M}_e^s)(f) \geq s^5.$$

Equality case : e and f are on a line.

As a consequence $(\mathbb{H}, d_c, \mathcal{L}^3)$ satisfies $MCP(0,5)$:

(Rough) definition of the Measure Contraction Property $MCP(0,N)$ for (X,d,ν):

for every point $e \in X$, for every $F \subset X$ and for all $s \in [0,1]$,

$$\nu(\mathcal{M}^s(e,F)) \geq s^N \nu(F).$$

First result

Theorem (Ambrosio, Rigot, 2004)

Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H})$ such that μ_0 is absolutely continuous. Then there is a **unique** *optimal coupling π. It is $\pi = (\mathrm{Id} \otimes T)_\# \mu_0$ for some map T. Moreover there is a* **unique** *geodesic between p and $T(p)$ (μ_0-almost surely).*

Let $T_s(p) = \mathcal{M}^s(p, T(p))$. There is a unique geodesic $(\mu_s)_{s\in[0,1]}$ between μ_0 and μ_1.

It is defined by $\mu_s = \mathcal{M}^s(\mu_0, \mu_1) = (T_s)_\# \mu_0$.

Open question (Ambrosio, Rigot)

Let μ_0 be absolutely continuous and $s < 1$. Is μ_s absolutely continuous as well?

Theorem (Figalli, J.)

Let $(\mu_s)_{s\in[0,1]}$ be a geodesic of $\mathcal{P}_2(\mathbb{H})$ and μ_0 absolutely continuous with respect to $\mathcal{L}^3$. Then for all $s \in [0,1)$, μ_s is absolutely continuous too.

First result

Theorem (Ambrosio, Rigot, 2004)

Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H})$ such that μ_0 is absolutely continuous. Then there is a **unique** *optimal coupling π. It is $\pi = (\mathrm{Id} \otimes T)_{\#}\mu_0$ for some map T. Moreover there is a* **unique** *geodesic between p and $T(p)$ (μ_0-almost surely).*

Let $T_s(p) = \mathcal{M}^s(p, T(p))$. There is a unique geodesic $(\mu_s)_{s\in[0,1]}$ between μ_0 and μ_1.

It is defined by $\mu_s = \mathcal{M}^s(\mu_0, \mu_1) = (T_s)_{\#}\mu_0$.

Open question (Ambrosio, Rigot)

Let μ_0 be absolutely continuous and $s < 1$. Is μ_s absolutely continuous as well?

Theorem (Figalli, J.)

Let $(\mu_s)_{s\in[0,1]}$ be a geodesic of $\mathcal{P}_2(\mathbb{H})$ and μ_0 absolutely continuous with respect to $\mathcal{L}^3$. Then for all $s \in [0,1)$, μ_s is absolutely continuous too.

First result

Theorem (Ambrosio, Rigot, 2004)

Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H})$ such that μ_0 is absolutely continuous. Then there is a **unique** *optimal coupling π. It is $\pi = (\mathrm{Id} \otimes T)_\# \mu_0$ for some map T. Moreover there is a* **unique** *geodesic between p and $T(p)$ (μ_0-almost surely).*

Let $T_s(p) = \mathcal{M}^s(p, T(p))$. There is a unique geodesic $(\mu_s)_{s\in[0,1]}$ between μ_0 and μ_1.

It is defined by $\mu_s = \mathcal{M}^s(\mu_0, \mu_1) = (T_s)_\# \mu_0$.

Open question (Ambrosio, Rigot)

Let μ_0 be absolutely continuous and $s < 1$. Is μ_s absolutely continuous as well?

Theorem (Figalli, J.)

Let $(\mu_s)_{s\in[0,1]}$ be a geodesic of $\mathcal{P}_2(\mathbb{H})$ and μ_0 absolutely continuous with respect to $\mathcal{L}^3$. Then for all $s \in [0,1)$, μ_s is absolutely continuous too.

Second result

Theorem (J.)

In $(\mathbb{H}, \mathcal{L}^3, d_c)$, the Heisenberg group with the Lebesgue measure and the Carnot-Carathéodory distance

- *$MCP(0, N)$ is true if and only if $N \geq 5$,*
- *$CD(0, N)$ and $BM(0, N)$ are false for every N.*

Theorem (J.)

In $(\mathbb{H}_n, \mathcal{L}^{2n+1}, d_c)$, the nth-Heisenberg group with the Lebesgue measure and the Carnot-Carathéodory distance

- *$MCP(K, N)$ is true if and only if $N \geq 2n+3$ and $K \leq 0$,*
- *$CD(K, N)$ and $BM(K, N)$ are false for every (K, N).*

Brunn-Minkowski inequality $BM(0,N)$:

A space (X,d,ν) satisfies the Brunn-Minkowski inequality $BM(0,N)$ if

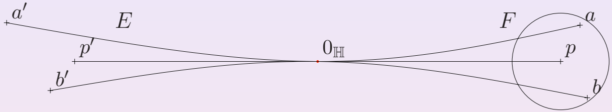
For every $E,F \subset X$ and for all $s \in [0,1]$,

$$\nu(\mathcal{M}^s(E,F))^{1/N} \geq (1-s)\nu(E)^{1/N} + s\nu(F)^{1/N}.$$

In particular if $\nu(E) = \nu(F)$,

$$\nu(\mathcal{M}^{1/2}(E,F))^{1/N} \geq \frac{\nu(E)^{1/N} + \nu(F)^{1/N}}{2} = \nu(E)^{1/N} = \nu(F)^{1/N}.$$

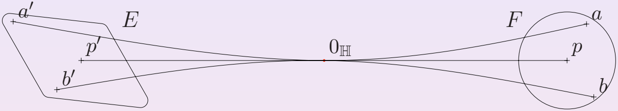
Sketch of proof



Let F be a small ball such that $0_{\mathbb{H}}$ and the center of the ball are on a “bad" geodesic.

For E we take the “geodesic inverse" of F.

Sketch of proof

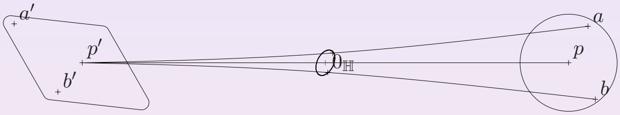


Let F be a small ball such that $0_{\mathbb{H}}$ and the center of the ball are on a “bad" geodesic.

For E we take the “geodesic inverse" of F. It turns out that $\mathcal{L}^3(E) = \mathcal{L}^3(F)$.

We want to prove $\mathcal{L}^3\left(\mathcal{M}^{1/2}(E, F)\right) < \mathcal{L}^3(F)$ because it is a contradiction to the $BM(0, N)$.

Sketch of proof

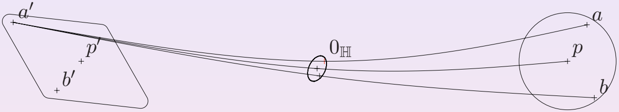


For each $e \in E$, the contracted set $\mathcal{M}^{1/2}(e, F)$ is a sort of ellipsoid that contains $0_{\mathbb{H}}$.

The volume of such an ellipsoid is

$$2^{-5}\mathcal{L}^3(F).$$

Sketch of proof

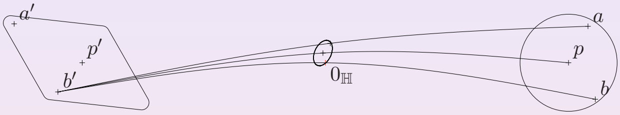


For each $e \in E$, the contracted set $\mathcal{M}^{1/2}(e, F)$ is a sort of ellipsoid that contains $0_{\mathbb{H}}$.

The volume of such an ellipsoid is

$$2^{-5}\mathcal{L}^3(F).$$

Sketch of proof

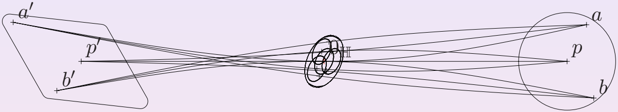


For each $e \in E$, the contracted set $\mathcal{M}^{1/2}(e, F)$ is a sort of ellipsoid that contains $0_{\mathbb{H}}$.

The volume of such an ellipsoid is

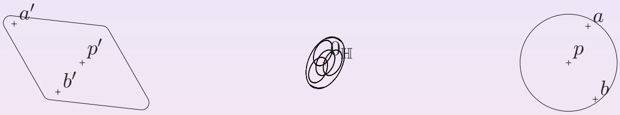
$$2^{-5}\mathcal{L}^3(F).$$

Sketch of proof



The midset $\mathcal{M}^{1/2}(E, F)$ is made of the union of these ellipsoids.

Sketch of proof



All of them contain $0_{\mathbb{H}}$. Then $\mathcal{M}^{1/2}(E,F)$ is an ellipsoid of size 2. Its volume is

$$2^3\left(\cdot 2^{-5}\mathcal{L}^3(F)\right) = \frac{\mathcal{L}^3(F)}{4}.$$

Then $\mathcal{L}^3(\mathcal{M}^{1/2}(E,F)) < \mathcal{L}^3(F) = \mathcal{L}^3(E)$.