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Semianalyticity of isoperimetric profiles

Renata Grimaldi, Stefano Nardulli, and Pierre Pansu

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Abstract

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In this talk, M is a compact real analytic Riemannian manifold, if it is not otherwise specified. We are concerned with the regularity of the *isoperimetric profile* of M.
It is shown that, in dimensions < 8, isoperimetric profiles of compact real analytic Riemannian manifolds are semi-analytic.

Definition of the isoperimetric profile function

Given $0 < v < \mathrm{vol}(M)$, consider all integral currents in M with volume v. Define $I_M(v)$ as the least upper bound of the boundary volumes of such currents. In this way, one gets a function $I_M : (0, \mathrm{vol}(M)) \to \mathbf{R}_+$ called the *isoperimetric profile* of M. In fact, for each $0 < v < \mathrm{vol}(M)$, there exist currents in M with volume v and boundary volume $I_M(v)$. Such minimizing currents will be called *bubbles*, for short.

Here is a typical example. Let S denote the circle of length 2π. Let $M = S \times S$. Then the isoperimetric profile of M is easily computed to be

$$I_M(v) = \begin{cases} \sqrt{4\pi v} & \text{for } 0 < v \leq 4\pi, \\ 4\pi & \text{for } 4\pi \leq v \leq 4\pi(\pi-1), \\ \sqrt{4\pi(4\pi^2 - v)} & \text{for } 4\pi(\pi-1) \leq v < 4\pi^2. \end{cases}$$

Sketch of proof

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This is proven as follows. In 2 dimensions, the boundaries of these bubbles are smooth, they have constant geodesic curvature, therefore they lift to disjoint unions of circles of equal radii or lines in $\mathbf{R}^2 = \tilde{M}$. It follows that bubbles are either round disks or annuli bounded by parallel geodesics, or complements of such. There remains to minimize boundary length among these three families.

First question

Question

For general real analytic manifolds, is it true that bubbles fall into finitely many analytic families, and that the profile is piecewise analytic ?

The answer is yes modulo some supplementary assumption. This has been proven in [Pan98] in dimension 2.

First, in a neighborhood of zero.

Theorem (Grimaldi-N.-Pansu, 2009)

Let M be a compact real analytic Riemannian manifold. There exists $\epsilon > 0$ such that I_M is real analytic on $(0, \epsilon)$.

The isoperimetric profile of Euclidean space $\mathbf{R}^n$ is $I_{\mathbf{R}^n}(v) = n(\omega_n)^{1/n} v^{(n-1)/n}$, where ω_n is the volume of the unit ball in $\mathbf{R}^n$. In a curved manifold, $I_M(v) \sim n(\omega_n)^{1/n} v^{(n-1)/n}$ as v tends to 0.

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Proposition

Let I be the isoperimetric profile of $\mathcal{M}$. Then

$$\limsup_{a\to 0}\frac{I(a)}{a^{\frac{n-1}{n}}}\leq c_n.$$

Demonstration: Fix a point $p\in\mathcal{M}$.

$$\limsup_{a\to 0}\frac{I(a)}{a^{\frac{n-1}{n}}}\leq \limsup_{a\to 0}\frac{Area(\partial B(p,r(a)))}{Vol(B(p,r(a)))^{\frac{n-1}{n}}}$$

with $r(a)$ such that $Vol(B(p,r(a)))=a$. Changing variables in the limits, we find

$$\limsup_{a\to 0}\frac{Area(\partial B(p,r(a)))}{Vol(B(p,r(a)))^{\frac{n-1}{n}}} = \limsup_{r\to 0}\frac{Area(\partial B(p,r))}{Vol(B(p,r))^{\frac{n-1}{n}}}$$

$$\limsup_{r\to 0}\frac{r^{n-1}Area(\mathbb{S}^{n-1})+\cdots}{[r^n Vol(\mathbb{B}^n)+\cdots]^{\frac{n-1}{n}}} = c_n.$$

Second question

Question

For a compact analytic Riemannian n-manifold, is $I_M(v)$ an analytic function of $v^{1/n}$ on $[0, \epsilon)$?

We have only a partial answer.

Theorem (Grimaldi-N.-Pansu, 2009)

Let M be a compact real analytic Riemannian manifold. Assume that the absolute maxima of scalar curvature are nondegenerate critical points. Then there exists an analytic function f defined in a neighborhood of 0 such that $I_M(v) = f(v^{1/n})$ for v small enough.

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Away from 0, our result also requires an extra assumption.

Theorem (Grimaldi-N.-Pansu, 2009)

Let M be a compact real analytic Riemannian manifold. Let $0 < v_0 < \mathrm{vol}(M)$. Assume that all bubbles of volume v_0 are smooth. Then I_M is semi-analytic on a neighborhood of v_0.

Since bubbles are known to be smooth in dimensions < 8, [Alm76], it follows that

Corollary

If the dimension of M is less than 8, I_M is semianalytic on $[0, \mathrm{vol}(M)]$.

Third question

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Question

Our method of proof relies on the regularity of bubbles. Can this be circumvented ?

I would guess yes. But I have not yet a written proof at disposal.

Sketch of the proof of Theorem 1

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It relies on results from [Nar08]. There, it is shown that the small bubbles form a subset of a smooth finite dimensional family of domains called *pseudo-bubbles*. We just need to show that if the metric is real analytic, pseudo-bubbles form a compact, finite dimensional real analytic set, on which the volume and boundary volume functions are real analytic.

Definition of Pseudo-bubbles

Definition

We call pseudo-bubble a hypersurface N embedded in M such that there exists a point $p \in M$ and a function $u \in C^{2,\alpha}(T_p^1 M \simeq \mathbb{S}^{n-1}, \mathbb{R})$, such that N is the graph of u in normal polar coordinates centered at p, i.e., $N = \left\{ exp_p(u(\theta)\theta),\ \theta \in T_p^1 M \right\}$ and

$$Q(H(u)) = const. \in \mathbb{R},$$

where H is the mean curvature operator, $Q = id - P$, P is the orthogonal projector of $L^2(T_p^1 M)$ on the first eigenspace of the Laplacian on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, and $T_p^1 M$ is the fiber over p of the unit tangent bundle over the Riemannian manifold M.

Why to introduce pseudo-bubbles

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We can observe that pseudo-bubbles are solutions of a differential equation which is weaker than constancy of mean curvature, but to which the implicit function theorem can be applied. Specificly, for $k \geq 0$, consider the bundle $\mathcal{F}^{k,\alpha} \to M$ whose fiber at $p \in M$ consists of $C^{k,\alpha}$ functions on the unit sphere of the tangent space T_pM (for the details on the construction of such an infinite dimensional vector bundle whose fiber is a Banach space, the reader can see the article [AF03]). There is a smooth map Φ with the following properties.

Main properties of Φ

1. Let $r > 0$, $p \in M$ and $x \in \mathcal{F}_p^2$. If the graph, in polar coordinates, of $r(1+x)$ has constant mean curvature, then $\Phi(r, p, x) = 0$.
2. For all $p \in M$, $\Phi(0, p, 0) = 0$.
3. The differential of Φ restricted to the fibers is an isomorphism.

More explicitly

We let

$$\Psi : \begin{cases} \mathbb{R} \times \Gamma(\mathcal{F}^{2,\alpha}) & \rightarrow & \Gamma(\mathcal{F}^{0,\alpha}) \\ (r, y) & \mapsto & r\left(H(p, r(1+x)) - \frac{n-1}{r}\right). \end{cases}$$

In other words, if $p \in M$, $r \in \mathbb{R}$ and x is a $C^{2,\alpha}$ function on $T^1_p M$, then $\Psi(p, r, x)$ is the $C^{0,\alpha}$ function on $T^1_p M$ defined by

$$\Psi(p, r, x) := r\left(H(p, r(1+x)) - \frac{n-1}{r}\right).$$

The center of mass

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Lemma

There exists an analytic map

$$c : \begin{cases} \mathbb{R} \times \Gamma(\mathcal{F}^{2,\alpha}) & \rightarrow & M \\ (r, p, x) & \mapsto & c(r, p, x) \end{cases}$$

defined implicitly by the equation

$$\int_{\mathcal{N}_{p,r,x}} exp_c^{-1} z dVol(z) = 0 \qquad (1)$$

in $T_c\mathcal{M}$ *where*

$$\mathcal{N}_{p,r,x} = \{exp_p(r(1 + x(\theta))\theta) | \theta \in \mathbb{S}^{n-1}\},$$

and $dVol$ *is the Riemannian volume form on* M*.*

$c(r, p, x)$ is the center of mass of $\mathcal{N}_{r,p,x}$ and $c(0, p, x) = p$.

The center of mass on the tangent space

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Lemma

There exists an analytic map $\mathcal{A}$ such that

$$exp_p^{-1}(c(r,y)) = r\mathcal{A}(r,y) \tag{2}$$

and

$$\mathcal{A}(0,y) = \frac{\int_{\mathbb{S}^{n-1}}(1+x(\theta))^{n-1}\theta\sqrt{||dx||^2+(1+x)^2}}{\int_{\mathbb{S}^{n-1}}(1+x(\theta))^{n-2}\sqrt{||dx||^2+(1+x)^2}}. \tag{3}$$

Implicit function theorem for Φ

We define $\Phi(r,p,x) = (\mathcal{A}(r,p,x), Q \circ \Psi(r,p,x))$.

Lemma

Φ is a real analytic map.

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Implicit function theorem for Φ

Lemme

Let

$$\Phi : \begin{cases} \mathbb{R} \times \Gamma(\mathcal{F}^{2,\alpha}) & \rightarrow \quad T\mathcal{M} \times \left(C^{0,\alpha}(\mathbb{S}^{n-1}) \cap (ker(L))^{\perp}\right) \\ (r, p, x) & \mapsto \quad (A(r, y), Q \circ \Psi(r, p, x)) \end{cases}$$

Then, for r sufficiently small, there exists a unique $x(p, r)$ in $C^{2,\alpha}(\mathbb{S}^{n-1})$ of small $C^{2,\alpha}$ norm which is a solution of the equation

$$\Phi(r, p, x(p, r)) = \Phi(r, y_r) = (y_0, 0).$$

Here y_0 is the zero section of $T\mathcal{M}$. Furthermore, x depends analytically on p and r.

Small volume bubbles are pseudo-bubbles

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From the implicit function theorem, it follows that for all $p \in M$ and for r small enough, the equation $\Phi(r, p, x) = 0$ has a unique small solution $x = x(r, p)$ which depends analytically on $(r, p) \in \mathbf{R} \times M$. Theorem 6 of [Nar08] asserts that there exist $r_0 > 0$ and $v_0 > 0$ such that every bubble of volume less than v_0 coincides with the domain $\mathcal{N}^+(p, r)$ bounded by the graph, in polar coordinates, of the function $r(1 + x(r, p))$, for some $r \leq r_0$ and some $p \in M$.

Proof of theorem 1

From the implicit function theorem, it follows that for all $p \in M$ and for r small enough, the equation $\Phi(r,p,x)=0$ has a unique small solution $x=x(r,p)$ which depends analytically on $(r,p) \in \mathbf{R} \times M$. Theorem 6 of [Nar08] asserts that there exist $r_0>0$ and $v_0>0$ such that every bubble of volume less than v_0 coincides with the domain $\mathcal{N}^+(p,r)$ bounded by the graph, in polar coordinates, of the function $r(1+x(r,p))$, for some $r \leq r_0$ and some $p \in M$.

Proof of theorem 1

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Therefore, for $v < v_0$,

$$I_M(v) = \min\{\mathrm{vol}(\partial\mathcal{N}^+(p,r))\,|\,p \in M,\ 0 \leq r \leq r_0,\ \mathrm{vol}(\mathcal{N}^+(p,r)) = v\}.$$

Define the *lower contour* $c(A)$ of a subset $A \subset \mathbf{R}^2$ as the function $v \mapsto \inf\{w \in \mathbf{R}\,|\,(v,w) \in A\}$.

Proof of theorem 1

Then the restriction of I_M to $[0, v_0]$ coincides with the lower contour of the image of the real analytic map

$$\Omega : [0, r_0] \times M \to \mathbf{R}^2, \quad (r, p) \mapsto (\mathrm{vol}(\mathcal{N}^+(p, r)), \mathrm{vol}(\partial \mathcal{N}^+(p, r))).$$

End of proof of theorem 1

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Since $[0, r_0] \times M$ is semi-analytic and compact, $\Omega([0, r_0] \times M)$ is a compact subanalytic set. Its lower contour is a subanalytic function. According to S. Lojasiewicz (Theorem 6.1 in [BM88]), subanalytic functions on the real line are semi-analytic. Semi-analytic functions are piecewise analytic, thus there exists $\epsilon > 0$ such that I_M is analytic on $(0, \epsilon)$. This completes the proof of Theorem 1.

Definition of analytic and semi-analytic sets

A subset A of a real analytic manifold M is called *semi-analytic iff, for every $x \in M$, we can find a neighbourhood U of x in M and $2pq$ real analytic functions g_{ij} and h_{ij} $(1 \leq i \leq p$ and $1 < j \leq q)$ such that* $A \cap U = \bigcup_{1=i}^{p} \{y \in U : g_{ij}(y) = 0, h_{ij}(y) > 0, j = 1, ..., q\}$. *We let $SEM(M)$ denote the family of semi-analytic subsets of M. This definition generalizes the notion of real analytic set (take all h_{ij} constant).*

Definition of sub-analytic sets

A subset A of M is called *subanalytic iff, for every* x *in* M, *we can find a neighbourhood* U *of* x *in* M *and* $2p$ *pairs* $(\varphi_i^\delta, A_i^\delta)$ $(1 \leq i \leq p$ *and* $\delta = 1, 2)$, *where* $A_i^\delta \in SEM(M_i^\delta)$ *for some real analytic manifolds* M_i^δ, *and where the maps* $\varphi_i^\delta : M_i^\delta \longrightarrow M$ *are proper analytic, such that* $A \cap U = \bigcup_{1=i}^{p}(\varphi_i^1(A_i^1) - \varphi_i^2(A_i^2))$. *We let* $SUB(M)$ *denote the family of subanalytic subsets of* M. *Clearly every semianalytic set is also subanalytic.*

Definition of sub-analytic functions

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A continuous map $\varphi : N \to M$ is said to be subanalytic iff its graph $G_\varphi \subset N \times M$ belongs to $SUB(N \times M)$. We denote the family of subanalytic maps from N to M by $\mathcal{SUB}(N, M)$. $\mathcal{SUB}(N, M)$ is a natural set of morphisms to study in connection with subanalytic sets.

Proof of Theorem 2

For $(p, \rho) \in M \times \mathbf{R}_+$, let $\beta(p, \rho)$ denote the pseudo-bubble defined by

$$\beta(p, \mathrm{vol}(\mathcal{N}^+(p, r))^{1/n}) = \mathcal{N}^+(p, r).$$

Since $\mathrm{vol}(\mathcal{N}^+(p, r))^{1/n} \sim \omega_n^{1/n} r$ is a 1 to 1 analytic function of r, the notation is unambiguous.

Proof of Theorem 2

Let

$$f(p,\rho) = \mathrm{vol}(\partial\beta(p,\rho)) = f_\rho(p).$$

Then f is real analytic. Furthermore, among pseudo-bubbles of volume $v = \rho^n$, bubbles are characterized as minima of f_ρ.

Proof of Theorem 2

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The following expansion

$$f_\rho(p) = c_n \rho^{n-1}\left(1 - \frac{1}{2n(n+2)} \omega_n^{-2/n} Sc(p)\rho^2 + \mathcal{O}(\rho^4)\right)$$

is computed in [Nar08], Lemma 3.6, compare [Ye91].

Proof of Theorem 2

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If the absolute maxima $p_1, \dots, p_k$ of the scalar curvature function Sc are non degenerate critical points, then each of them deforms into a critical point $p_i(\rho)$ of f_ρ that depends analytically on ρ. Therefore (Theorem 8 in [Nar08]),

$$I_M(\rho^n) = \min_{i=1,\dots,k} f_\rho(p_i(\rho)).$$

End of the proof of Theorem 2

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There exists $\epsilon > 0$ and i such that the minimum is equal to $f_\rho(p_i(\rho))$ for all $\rho \in [0, \epsilon)$. Indeed, otherwise, some function $f_\rho(p_i(\rho)) - f_\rho(p_j(\rho))$ would change sign infinitely many times near 0, contradiction. Thus the right hand side is analytic on $[0, \epsilon)$. This completes the proof of Theorem 2.

We follow M. Tamm's strategy, [Tam81]. We aim at including bubbles in a parametrized analytic variety. We shall first do this in a neighborhood of a smooth bubble B with volume v_0. Our first candidate is the set of domains whose boundary is a graph in normal exponential coordinates to ∂B and has constant mean curvature. To decide wether this set is a submanifold in some function space, let us examine the mean curvature operator and its linearization.

Let B be a smooth bubble with volume v_0. Let $H_B : C^{2,\alpha}(\partial B) \to C^{0,\alpha}(\partial B)$ denote the operator which to a function u on ∂B associates the mean curvature of the graph of u in normal exponential coordinates to ∂B. In particular, $H_B(0) = H(\partial B) = h_B$ is the constant mean curvature of ∂B. Let $L_B : C^{2,\alpha}(\partial B) \to C^{0,\alpha}(\partial B)$ denote its linearization at 0 (sometimes called the *Jacobi operator*).

Lemma

For all $v \in C^{2,\alpha}(\partial B)$

$$L_B(v) = -\Delta_{\partial B} v - (||II_{\partial B}||^2 + Ric(\nu))v, \tag{4}$$

where $\Delta_{\partial B} v = div(\nabla v)$ is the Laplace operator on ∂B (with negative spectrum when taken on the round sphere), $||II_{\partial B}||^2$ is the Hilbert-Schmidt squared norm ($tr(A^t A)$ for a square matrix A) of the second fundamental form of ∂B and $Ric(\nu)$ is the Ricci curvature of the ambient manifold M in the direction ν of the unit outward normal vector to ∂B evaluated at a point of ∂B.

We recall here formula (8) of 3.3 of [Nar07]

$$
\begin{aligned}
H(u) &= -div_{(\mathbb{S}^{n-1},g_u)}\left(\frac{\overrightarrow{\nabla}_{g_u} u}{W_u}\right) - \frac{1}{W_u^2} < \nabla_{\overrightarrow{\nabla}_{g_u} u}\left(\frac{u\overrightarrow{\nabla}_{g_u} u}{W_u}\right), \overrightarrow{\nabla}_{g_u} u > \\
&+ \frac{u^2}{W_u^3} II_\theta^u(\overrightarrow{\nabla}_{g_u} u, \overrightarrow{\nabla}_{g_u} u) \\
&- \frac{1}{W_u} H_\theta^u(u) + \frac{1}{W_u} < \overrightarrow{\nabla}_{g_u}\left(\frac{1}{W_u}\right), u\frac{\overrightarrow{\nabla}_{g_u} u}{W_u}\right) >_{g_u} .
\end{aligned}
$$

Here θ denotes the gradient of the signed distance function to ∂B. For the meaning of the other terms involved in (5), see [Nar07].

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The operator L_B satisfies then $H(tv) = H_{\partial B} + tL_B + \mathcal{O}(t^2)$. At first observe that only the first and fourth term of (5) contribute to L_B.

Denoting by $U(r)=\nabla\theta$ the shape operator of the equidistant hypersurfaces to ∂B at distance r, we have $U(r)=U_0+U_1 r+\cdots$, hence by using the Riccati equation satisfied by U (see [Cha06]) we can compute $U_1=-U_0^2-R$ where R is the curvature tensor.

Now taking traces we get
$H_{\theta}^{tv} = tr(-U) = H_{\partial B,ext} + (tr(U_0^2) + Ric)t + \cdots$ where $H_{\partial B,ext}$ is the outward mean curvature of the boundary of B.
Finally (4) follows easily.

L_B is a selfadjoint elliptic operator, which has a discrete spectrum. Let $K_B = \mathrm{kernel}(L_B)$, $m(B) = \dim(K_B)$.
If L_B were invertible (i.e. $m(B) = 0$), the implicit function theorem would imply that nearby domains with constant mean curvature boundary form an analytic family parametrized by the value of mean curvature.

Unfortunately, L_B is not always invertible. Therefore, instead of solving $H(u) = h$, $h \in \mathbf{R}$, we shall solve

$$\Phi_B(u,h) = P_B(H_B(u) - h) = 0,$$

where P_B is the orthogonal projection onto the L^2-orthogonal complement $K_B^\perp$ of K_B in $C^{0,\alpha}(\partial B)$.

Then

$$\Phi_B : C^{2,\alpha}(\partial B) \times \mathbf{R} \to K_B^{\perp}$$

is a real analytic map, whose linearization at 0 is $P_B \circ L_B$. By construction, it is onto. In fact, the restriction of $P_B \circ L_B$ to $K_B^{\perp}$ is an isomorphism (see for example [Bes87], page 464). Note that $\Phi_B(0, h_B) = 0$.

The following variant of the implicit function theorem provides us with an open neighborhood U_B of $(0, h_B)$ in $C^{2,\alpha}(\partial B) \times \mathbf{R}$ in which the solutions of $\Phi_B(u, h) = 0$ form a real analytic submanifold. We shall call such solutions *B-pseudo-bubbles*.

Lemma

Let E, P and F be real analytic Banach manifolds, let $e_0 \in E$, $f_0 \in B$, $p_0 \in P$ be such that $\Phi(e_0, p_0) = f_0$. Let $\Phi : E \times P \to B$ be a real analytic map. Assume that the differential $d\Phi$ of Φ at (e_0, p_0) in the direction of E has a finite dimensional kernel $K \subset T_{e_0}E$, which admits a closed complement $K^\perp$. Assume that the restriction of $d\Phi$ to $K^\perp$ is invertible. Then, in a neighborhood of (e_0, p_0) the solutions of equation $\Phi(e, p) = f_0$ form a real analytic submanifold parametrized by a neighborhood of $(0, p_0)$ in $K \times P$.

Proof: Apply the implicit function theorem to $\Psi : E \times P \to F \times K$ defined by $\Psi(e,p) = (\Phi(e,p), \pi_K(e))$ where π_K is a local submersion onto K. ∎

Let us define the $C^{2,\alpha}$-topology on the space of domains with smooth boundary as follows : as neighborhoods of a smooth domain β, take all domains S whose boundary is the graph, in normal exponential coordinates, of a $C^{2,\alpha}$-small function on $\partial\beta$. Using a result from [Nar07], we show that on smooth bubbles with volume close to v_0, the topologies induced by the $C^{2,\alpha}$-topology on smooth domains and the flat topology on currents coincide.

Lemma

Let B be a bubble of volume v_0. For all $\delta > 0$, there exist $\epsilon > 0$ such that if β is a bubble of volume $\in [v_0 - \epsilon, v_0 + \epsilon]$ with $\mathrm{vol}(\beta \Delta B) < \epsilon$, then there exists a smooth function u on ∂B with $\| u \|_{C^{2,\alpha}} < \delta$ such that $\partial\beta$ is the graph in normal exponential coordinates of u. Conversely, the graph of a $C^{2,\alpha}$-small function on ∂B bounds a current which is close to B volumewise.

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Proof: By contradiction. Otherwise, there exists a sequence β_j of bubbles with $\mathrm{vol}(\beta_j) \to v_0$ and $\mathrm{vol}(\beta_j \Delta B) \to 0$ such that $\partial\beta_j$ is not the normal exponential graph of a $C^{2,\alpha}$-small function on ∂B. Theorem 1 of [Nar07] asserts that for j large enough, $\partial\beta_j$ is the graph in normal exponential coordinates of a function u_j on ∂B whose $C^{2,\alpha}$-norm tends to zero, contradiction. The converse statement is obvious. ■

Lemma

Assume that all volume v_0 bubbles in M are smooth. Then there exists $\epsilon > 0$ such that the set $\mathcal{B}$ of pseudo-bubbles with volumes $\in [v_0 - \epsilon, v_0 + \epsilon]$ is contained in a finite union of compact semi-analytic pieces of finite dimensional real analytic manifolds, on which the volume and boundary volume functions are real analytic.

Proof: It was just proven that the set $\mathcal{B}$ of bubbles with volumes $\in [v_0 - \epsilon, v_0 + \epsilon]$ is compact in flat topology and the set $\mathcal{BH} = \{(B, h_B) \mid B \in \mathcal{B}\}$ is covered by the sets $U(B)$. According to Lemma 6, it is compact in $C^{2,\alpha}$-topology as well. Therefore, $\mathcal{BH}$ can be covered with finitely many open sets $U(B_1),\dots,U(B_N)$. The set ΨB_i of B_i-pseudo-bubbles in $U(B_i)$ is an analytic submanifold. There exist compact semi-analytic subsets $W_i \subset \Psi B_i$ which suffice to cover $\mathcal{BH}$. ∎

The proof of Theorem 3 is completed in the same manner as the proof of Theorem 1. In dimensions less than 8, F. Almgren has shown that all bubbles are smooth. Therefore the profile is semi-analytic in a neighborhood of every point of the closed interval $[0, \mathrm{vol}(M)]$. It follows that it is semi-analytic on this interval. This proves Corollary 4.

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