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A UNIQUENESS CRITERION FOR THE SIGNORINI PROBLEM WITH COULOMB FRICTION*

YVES RENARD†

Abstract. The purpose of this paper is to study the solutions to the Signorini problem with Coulomb friction (the so-called Coulomb problem). Some optimal a priori estimates are given, and a uniqueness criterion is exhibited. Recently, nonuniqueness examples have been presented in the continuous framework. It is proved, here, that if a solution satisfies a certain hypothesis on the tangential displacement and if the friction coefficient is small enough, it is the unique solution to the problem. In particular, this result can be useful for the search of multisolutions to the Coulomb problem because it eliminates a lot of uniqueness situations.

Key words. unilateral contact, Coulomb friction, uniqueness of solution

AMS subject classifications. 35J85, 74M10

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Introduction. The so-called Signorini problem with Coulomb friction (or simply the Coulomb problem) has been introduced by Duvaut and Lions [4]. It does not exactly represent the equilibrium of a solid which encounters an obstacle because when the equilibrium is reached (or any steady state solution) the friction condition is no longer an irregular law. The aim of this problem is in fact to be very close to a time semidiscretization of an evolutionary problem by an implicit scheme. The fact that several solutions could coexist in an implicit scheme (independently of the size of the time step) may be an indication that the evolutionary problem has a dynamical bifurcation.

The first existence results for this problem were obtained by Nečas, Jarušek, and Haslinger in [15] for a two-dimensional elastic strip, assuming that the coefficient of friction is small enough and using a shifting technique previously introduced by Fichera and later applied to more general domains by Jarušek [11]. Eck and Jarušek [5] give a different proof using a penalization method. We emphasize that most results on existence for frictional problems involve a condition of smallness for the friction coefficient (and a compact support on Γ_C).

Recently, examples of nonunique solutions have been given by Hild in [7] and [8] for a large friction coefficient. As far as we know, for a fixed geometry, it is still an open question whether or not there is uniqueness of the solution for a sufficiently small friction coefficient. In the finite element approximation framework, the presence of bifurcation has been studied in [9].

The present paper gives the first (partial) result of uniqueness of a solution to the Coulomb problem. The summary is the following. Section 1 introduces strong and weak formulations of the Coulomb problem. Section 2 gives optimal estimates on the solutions. In particular, a comparison is made with the solution to the frictionless contact problem. Section 3 gives an additional estimate for the Tresca problem, i.e., the problem with a given friction threshold. And finally, section 4 gives the partial

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†MIP, CNRS UMR 5640, INSAT, Complexe scientifique de Rangueil, 31077 Toulouse, France (Yves.Renard@insa-toulouse.fr).

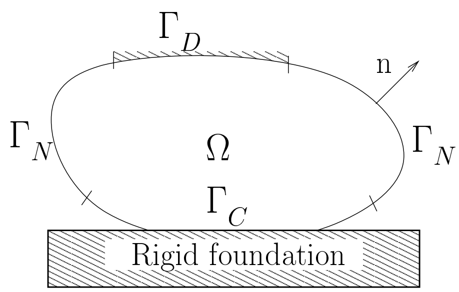

FIG. 1. *Elastic body Ω in frictional contact.*

uniqueness result. It is proved in Proposition 5 for bidimensional problems and a friction coefficient less than one that there is no multisolution with one of the solutions having a tangential displacement with a constant sign. The major result is given by Proposition 6 using the notion of a multiplier in a pair of Sobolev spaces.

1. The Signorini problem with Coulomb friction. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded Lipschitz domain representing the reference configuration of a linearly elastic body.

It is assumed that this body is submitted to a Neumann condition on a part of its boundary Γ_N, to a Dirichlet condition on another part Γ_D, and to a unilateral contact with static Coulomb friction condition on the rest of the boundary Γ_C between the body and a flat rigid foundation (see Figure 1). This latter part Γ_C is supposed to be of nonzero interior in the boundary $\partial\Omega$ of Ω. The problem consists in finding the displacement field $u(t, x)$ satisfying

$$-\operatorname{div}\sigma(u) = f \quad \text{in} \quad \Omega, \tag{1}$$

$$\sigma(u) = \mathcal{A}\varepsilon(u) \quad \text{in } \Omega, \tag{2}$$

$$\sigma(u)\mathrm{n} = F \quad \text{on } \Gamma_N, \tag{3}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{4}$$

where $\sigma(u)$ is the stress tensor, $\varepsilon(u)$ is the linearized strain tensor, n is the outward unit normal to Ω on $\partial\Omega$, F and f are the given external loads, and $\mathcal{A}$ is the elastic coefficient tensor which satisfies classical conditions of symmetry and ellipticity.

On Γ_C, it is usual to decompose the displacement and the stress vector in normal and tangential components as follows:

$$u_N = u \cdot \mathrm{n}, \quad u_T = u - u_N \mathrm{n},$$

$$\sigma_N(u) = (\sigma(u)\mathrm{n}) \cdot \mathrm{n}, \quad \sigma_T(u) = \sigma(u)\mathrm{n} - \sigma_N(u)\mathrm{n}.$$

To give a clear sense to this decomposition, we assume Γ_C to have the $\mathcal{C}^1$ regularity. The unilateral contact condition is expressed by the following complementary condition:

$$u_N \leq g, \quad \sigma_N(u) \leq 0, \quad (u_N - g)\sigma_N(u) = 0, \tag{5}$$

where g is the normal gap between the elastic solid and the rigid foundation in reference configuration (see Figure 2).

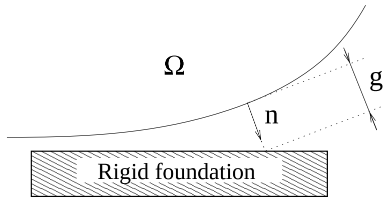

FIG. 2. *Normal gap between the elastic solid* Ω *and the rigid foundation.*

Denoting by $\mathcal{F} \geq 0$ the friction coefficient, the *static Coulomb friction* condition reads as follows:

$$\text{if } u_T = 0, \text{ then } |\sigma_T(u)| \leq -\mathcal{F}\sigma_N(u), \tag{6}$$

$$\text{if } u_T \neq 0, \text{ then } \sigma_T(u) = \mathcal{F}\sigma_N(u)\frac{u_T}{|u_T|}. \tag{7}$$

The friction force satisfies the so-called maximum dissipation principle

$$-\sigma_T(u)\cdot u_T = \sup_{\substack{\mu_T \in \mathbb{R}^{d-1} \\ |\mu_T| \leq -\mathcal{F}\sigma_N(u)}} (-\mu_T \cdot u_T). \tag{8}$$

1.1. Classical weak formulation. We present here the classical weak formulation proposed by Duvaut [3] and Duvaut and Lions [4]. Let us introduce the Hilbert spaces

$$V = \{v \in H^1(\Omega;\mathbb{R}^d), v = 0 \text{ on } \Gamma_D\},$$

$$X = \{v|_{\Gamma_C} : v \in V\} \subset H^{1/2}(\Gamma_C;\mathbb{R}^d),$$

$$X_N = \{v_N|_{\Gamma_C} : v \in V\}, \quad X_T = \{v_T|_{\Gamma_C} : v \in V\},$$

and their topological dual spaces V', X', X'_N, and X'_T. It is assumed that Γ_C is sufficiently smooth such that $X_N \subset H^{1/2}(\Gamma_C)$, $X_T \subset H^{1/2}(\Gamma_C;\mathbb{R}^{d-1})$, $X'_N \subset H^{-1/2}(\Gamma_C)$, and $X'_T \subset H^{-1/2}(\Gamma_C;\mathbb{R}^{d-1})$.

Classically, $H^{1/2}(\Gamma_C)$ is the space of the restriction on Γ_C of traces on $\partial\Omega$ of functions of $H^1(\Omega)$, and $H^{-1/2}(\Gamma_C)$ is the dual space of $H^{1/2}_{00}(\Gamma_C)$, which is the space of the restrictions on Γ_C of functions of $H^{1/2}(\partial\Omega)$ vanishing outside Γ_C. We refer to [1] and [12] for a detailed presentation of trace operators.

Now, the set of admissible displacements is defined as

$$K = \{v \in V, v_N \leq g \quad \text{a.e. on } \Gamma_C\}. \tag{9}$$

The maps

$$a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v)dx,$$

$$l(v) = \int_\Omega f\cdot v dx + \int_{\Gamma_N} F\cdot v d\Gamma,$$

$$j(\mathcal{F}\lambda_N, v_T) = -\langle \mathcal{F}\lambda_N, |v_T|\rangle_{X'_N, X_N}$$

represent the virtual work of elastic forces, the external load, and the "virtual work" of friction forces, respectively. Standard hypotheses are as follows:

(10) $a(\cdot,\cdot)$ is a bilinear symmetric V-elliptic and continuous form on $V \times V$:
$\exists\ \alpha > 0,\ \exists\ M > 0,\ a(u,u) \geq \alpha \|u\|_V^2,\ a(u,v) \leq M \|u\|_V \|v\|_V\ \forall u, v \in V,$

(11) $l(\cdot)$ is a linear continuous form on V; i.e., $\exists\ L > 0, |l(v)| \leq L\|v\|_V\ \forall v \in V,$

(12) $g \in X_N,$

(13) $\mathcal{F} \in MX_N$ is a nonnegative multiplier in X_N.

The latter condition ensures that $j(\mathcal{F}\lambda_N, v_T)$ is linear continuous on λ_N and convex lower semicontinuous on v_T when λ_N is a nonpositive element of X'_N (see, for instance, [2]). To satisfy condition (10), it is necessary that Γ_D is of nonzero interior in the boundary of Ω and that the elastic coefficient tensor is uniformly elliptic (see [4]).

We refer to Maz'ya and Shaposhnikova [14] for the theory of multipliers. The set MX_N denotes the space of multipliers from X_N into X_N, i.e., the space of function $f : \Gamma_C \longrightarrow \mathbb{R}$ of finite norm

$$\|f\|_{MX_N} = \sup_{\substack{v_N \in X_N \\ v_N \neq 0}} \frac{\|f v_N\|_{X_N}}{\|v_N\|_{X_N}}.$$

This is the norm of the linear mapping $X_N \ni v \longmapsto (fv) \in X_N$. Of course, if $\mathcal{F}$ is a constant on Γ_C, one has $\|\mathcal{F}\|_{MX_N} = \mathcal{F}$. From the fact that Ω is supposed to be a bounded Lipschitz domain and Γ_C is supposed to have the $\mathcal{C}^1$ regularity, it is possible to deduce that for $d = 2$ the space $H^{1/2+\varepsilon}(\Gamma_C)$ is continuously included in MX_N for any $\varepsilon > 0$, and for $d = 3$ the space $H^1(\Gamma_C) \cap L^\infty(\Gamma_C)$ is included in MX_N, continuously for the norm $\|f\|_{H^1(\Gamma_C)} + \|f\|_{L^\infty(\Gamma_C)}$ (see [14]). In particular, the space of Lipschitz continuous functions is continuously included in MX_N.

Condition (10) implies in particular that $a(\cdot,\cdot)$ is a scalar product on V and that the associated norm

$$\|v\|_a = (a(v,v))^{1/2}$$

is equivalent to the usual norm of V:

$$\sqrt{\alpha}\|v\|_V \leq \|v\|_a \leq \sqrt{M}\|v\|_V \quad \forall v \in V.$$

The continuity constant of $l(\cdot)$ can also be given with respect to: $\|\cdot\|_a$:

$$\exists\ L_a > 0, |l(v)| \leq L_a \|v\|_a\ \forall v \in V.$$

Constants L and L_a can be chosen such that

$$\sqrt{\alpha} L_a \leq L \leq \sqrt{M} L_a.$$

The classical weak formulation of problem (1)–(7) is given by

$$(14) \qquad \begin{cases} \text{Find } u \in K \text{ satisfying} \\ \\ a(u, v-u) + j(\mathcal{F}\sigma_N(u), v_T) - j(\mathcal{F}\sigma_N(u), u_T) \geq l(v-u) \quad \forall\ v \in K. \end{cases}$$

The major difficulty about (14) is due to the coupling between the friction threshold and the contact pressure $\sigma_N(u)$. The consequence is that this problem does not represent a variational inequality, in the sense that it cannot be derived from an optimization problem.

1.2. Neumann to Dirichlet operator. In this section, the Neumann to Dirichlet operator on Γ_C is introduced together with its basic properties. This will allow to restrict the contact and friction problem to Γ_C and obtain useful estimates.

Let $\lambda = (\lambda_N, \lambda_T) \in X'$; then, under hypotheses (10) and (11), the solution u to

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ satisfying} \\ \\ a(u,v) = l(v) + \langle \lambda, v \rangle_{X',X} \quad \forall\, v \in V \end{array} \right. \tag{15}$$

is unique (see [4]). So it is possible to define the operator

$$\begin{aligned} \mathbb{E} : X' &\longrightarrow X \\ \lambda &\longmapsto u|_{\Gamma_C}. \end{aligned}$$

This operator is affine and continuous. Moreover, it is invertible and its inverse is continuous. It is possible to express $\mathbb{E}^{-1}$ as follows: For $w \in X$, let u be the solution to the Dirichlet problem

$$\left\{ \begin{array}{l} \text{Find } u \in V \text{ satisfying } u|_{\Gamma_C} = w \text{ and} \\ \\ a(u,v) = l(v) \quad \forall\, v \in V,\ v|_{\Gamma_C} = 0; \end{array} \right. \tag{16}$$

then $\mathbb{E}^{-1}(w)$ is equal to $\lambda \in X'$ defined by

$$\langle \lambda, v \rangle_{X',X} = a(u,v) - l(v) \quad \forall\, v \in V.$$

In a weak sense, one has the relation $\mathbb{E}^{-1}(u) = \sigma(u)\mathrm{n}$ on Γ_C. Now, under hypotheses (10) and (11) one has

$$\|\mathbb{E}(\lambda^1) - \mathbb{E}(\lambda^2)\|_X \le \frac{C_1^2}{\alpha} \|\lambda^1 - \lambda^2\|_{X'}, \tag{17}$$

where C_1 is the continuity constant of the trace operator on Γ_C and α is the coercivity constant of the bilinear form $a(\cdot,\cdot)$. One can verify it as follows. Let λ^1 and λ^2 be given in X'_T and let u^1, u^2 be the corresponding solutions to (15); then

$$\begin{aligned} \alpha \|u^1 - u^2\|_V^2 \ &\le\ a(u^1 - u^2, u^1 - u^2) \ = \ \langle \lambda^1 - \lambda^2, u^1 - u^2 \rangle_{X',X} \\ &\le\ C_1 \|\lambda^1 - \lambda^2\|_{X'} \|u^1 - u^2\|_V \end{aligned} \tag{18}$$

and, consequently,

$$\|u^1 - u^2\|_V \le \frac{C_1}{\alpha} \|\lambda^1 - \lambda^2\|_{X'}. \tag{19}$$

Conversely, one has

$$\|\mathbb{E}^{-1}(u^1) - \mathbb{E}^{-1}(u^2)\|_{X'} \le M C_2^2 \|u^1 - u^2\|_X, \tag{20}$$

where M is the continuity constant of $a(\cdot,\cdot)$ and $C_2 > 0$ is the continuity constant of the homogeneous Dirichlet problem corresponding to (16) (i.e., with $l(v) \equiv 0$ and

$C_2 = \sup_{\substack{v\in X\\ v\neq 0}} \frac{\|w\|_V}{\|v\|_X}$, where $w_{|_{\Gamma_C}} = v$ and $a(w,z) = 0 \ \ \forall z \in V$). This latter estimate can be performed as follows:

$$\begin{aligned}\|\mathbb{E}^{-1}(u^1) - \mathbb{E}^{-1}(u^2)\|_{X'} &= \sup_{\substack{v\in X\\ v\neq 0}} \frac{\langle \mathbb{E}^{-1}(u^1) - \mathbb{E}^{-1}(u^2), v\rangle_{X',X}}{\|v\|_X} \\ &= \sup_{\substack{v\in X\\ v\neq 0}} \left(\inf_{\{w\in V : w_{|_{\Gamma_C}} = v\}} \frac{a(u^1-u^2, w)}{\|v\|_X} \right) \\ &\leq M\gamma \|u^1 - u^2\|_V, \end{aligned} \tag{21}$$

where $\gamma = \sup_{\substack{v\in X\\ v\neq 0}} \inf_{\{w\in V : w_{|_{\Gamma_C}} = v\}} \frac{\|w\|_V}{\|v\|_X}$ is the continuity constant of the homogeneous Poisson problem with respect to a Dirichlet condition on Γ_C. Using $\gamma \leq C_2$, this gives (20).

It is also possible to define the following norms on Γ_C relative to $a(\cdot,\cdot)$:

$$\|v\|_{a,\Gamma_C} = \inf_{w\in V,\ w_{|_{\Gamma_C}} = v} \|w\|_a,$$

$$\|\lambda\|_{-a,\Gamma_C} = \sup_{\substack{v\in X\\ v\neq 0}} \frac{\langle \lambda, v\rangle_{X',X}}{\|v\|_{a,\Gamma_C}} = \sup_{\substack{v\in V\\ v\neq 0}} \frac{\langle \lambda, v\rangle_{X',X}}{\|v\|_a}.$$

These are equivalent, respectively, to the norms in X and X':

$$\frac{\sqrt{\alpha}}{C_1}\|v\|_X \leq \|v\|_{a,\Gamma_C} \leq \sqrt{M}\gamma \|v\|_X,$$

$$\frac{1}{\sqrt{M}\gamma}\|\lambda\|_{X'} \leq \|\lambda\|_{-a,\Gamma_C} \leq \frac{C_1}{\sqrt{\alpha}}\|\lambda\|_{X'}.$$

With these norms, the estimates are straightforward since the following lemma holds.

LEMMA 1. *Let λ^1 and λ^2 be two elements of X' and let $u^1 = \mathbb{E}(\lambda^1)$, $u^2 = \mathbb{E}(\lambda^2)$; then under hypotheses* (10) *and* (11) *one has*

$$\|u^1 - u^2\|_{a,\Gamma_C} = \|u^1 - u^2\|_a = \|\lambda^1 - \lambda^2\|_{-a,\Gamma_C}.$$

Proof. On the one hand, one has

$$\|u^1 - u^2\|^2_{a,\Gamma_C} = \inf_{\substack{w\in V\\ w_{|_{\Gamma_c}} = u^1-u^2}} \|w\|^2_a = \|u^1 - u^2\|^2_a,$$

because $u^1 - u^2$ is the minimum of $\frac{1}{2}\|w\|^2_a$ under the constraint $w_{|_{\Gamma_C}} = u^1 - u^2$. This implies

$$\begin{aligned}\|u^1 - u^2\|^2_{a,\Gamma_C} &= a(u^1-u^2, u^1-u^2) = \langle \lambda^1 - \lambda^2, u^1 - u^2\rangle_{X',X} \\ &\leq \|\lambda^1 - \lambda^2\|_{-a,\Gamma_C} \|u^1 - u^2\|_{a,\Gamma_C},\end{aligned}$$

and finally

$$\|u^1 - u^2\|_{a,\Gamma_C} \leq \|\lambda^1 - \lambda^2\|_{-a,\Gamma_C}.$$

On the other hand, one has

$$\|\lambda^1-\lambda^2\|_{-a,\Gamma_C} = \sup_{\substack{v\in X\\ v\neq 0}} \frac{\langle \lambda^1-\lambda^2, v\rangle_{X',X}}{\|v\|_{a,\Gamma_C}} = \sup_{\substack{v\in X\\ v\neq 0}} \inf_{\substack{w\in V\\ w|_{\Gamma_C}=v}} \frac{a(u^1-u^2,w)}{\|v\|_{a,\Gamma_C}},$$

$$\leq \sup_{\substack{v\in X\\ v\neq 0}} \inf_{\substack{w\in V\\ w|_{\Gamma_C}=v}} \frac{\|u^1-u^2\|_a\|w\|_a}{\|v\|_{a,\Gamma_C}} = \|u^1-u^2\|_a = \|u^1-u^2\|_{a,\Gamma_C},$$

which ends the proof of the lemma. □

1.3. Direct weak inclusion formulation. Let

$$K_N = \{v_N \in X_N : v_N \leq 0 \text{ a.e. on } \Gamma_C\}$$

be the (translated) set of admissible normal displacements on Γ_C. The normal cone in X'_N to K_N at $v_N \in X_N$ is defined as

$$N_{K_N}(v_N) = \{\mu_N \in X'_N : \langle \mu_N, w_N - v_N\rangle_{X'_N,X_N} \leq 0 \ \forall w_N \in K_N\}.$$

In particular, $N_{K_N}(v_N) = \emptyset$ if $v_N \notin K_N$. The subgradient of $j(\mathcal{F}\lambda_N, u_T)$ with respect to the second variable is given by

$$\partial_2 j(\mathcal{F}\lambda_N, u_T) = \{\mu_T \in X'_T : j(\mathcal{F}\lambda_N, v_T) \geq j(\mathcal{F}\lambda_N, u_T) + \langle \mu_T, v_T - u_T\rangle_{X'_T,X_T} \ \forall v_T \in X_T\}.$$

With this notation, problem (14) is equivalent to the following problem:

$$\text{(22)} \qquad \begin{cases} \text{Find } u \in V,\ \lambda_N \in X'_N, \text{ and } \lambda_T \in X'_T \text{ satisfying} \\ (u_N, u_T) = \mathbb{E}(\lambda_N, \lambda_T), \\ -\lambda_N \in N_{K_N}(u_N - g) \quad \text{in } X'_N, \\ -\lambda_T \in \partial_2 j(\mathcal{F}\lambda_N, u_T) \quad \text{in } X'_T. \end{cases}$$

More details on this equivalence can be found in [13].

Remark 1. Inclusion $-\lambda_N \in N_{K_N}(u_N - g)$ is equivalent to the complementarity relations

$$u_N \leq g, \quad \langle \lambda_N, v_N\rangle_{X'_N,X_N} \geq 0 \ \ \forall v_N \in K_N, \quad \langle \lambda_N, u_N - g\rangle_{X'_N,X_N} = 0,$$

which is the weak formulation of the strong complementarity relations (5) for the contact conditions. Similarly, the second inclusion $-\lambda_T \in \partial_2 j(\mathcal{F}\lambda_N, u_T)$ represents the friction condition.

1.4. Hybrid weak inclusion formulation. We will now consider the sets of admissible stresses. The set of admissible normal stresses on Γ_C can be defined as

$$\Lambda_N = \{\lambda_N \in X'_N : \langle \lambda_N, v_N\rangle_{X'_N,X_N} \geq 0 \ \ \forall v_N \in K_N\}.$$

This is the opposite of K_N^*, the polar cone to K_N. The set of admissible tangential stresses on Γ_C can be defined as

$$\Lambda_T(\mathcal{F}\lambda_N) = \{\lambda_T \in X_T' : -\langle \lambda_T, w_T\rangle_{X_T', X_T} + \langle \mathcal{F}\lambda_N, |w_T|\rangle_{X_N', X_N} \leq 0 \ \ \forall w_T \in X_T\}.$$

With this, problem (14) is equivalent to the following problem:

$$(23) \qquad \begin{cases} \text{Find } u \in V,\ \lambda_N \in X_N',\ \text{and } \lambda_T \in X_T' \text{ satisfying} \\ (u_N, u_T) = \mathbb{E}(\lambda_N, \lambda_T), \\ -(u_N - g) \in N_{\Lambda_N}(\lambda_N) \quad \text{in } X_N, \\ -u_T \in N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T) \quad \text{in } X_T, \end{cases}$$

where the two inclusions can be replaced by inequalities as follows:

$$(24) \qquad \begin{cases} \text{Find } u \in V,\ \lambda_N \in X_N',\ \text{and } \lambda_T \in X_T' \text{ satisfying} \\ (u_N, u_T) = \mathbb{E}(\lambda_N, \lambda_T), \\ \lambda_N \in \Lambda_N, \ \ \langle \mu_N - \lambda_N, u_N - g\rangle_{X_N', X_N} \geq 0 \ \ \forall \mu_N \in \Lambda_N, \\ \lambda_T \in \Lambda_T(\mathcal{F}\lambda_N), \ \ \langle \mu_T - \lambda_T, u_T\rangle_{X_T', X_T} \geq 0 \ \ \forall \mu_T \in \Lambda_T(\mathcal{F}\lambda_N). \end{cases}$$

Remark 2. The inclusion $-u_T \in N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T)$ implies the complementarity relation

$$\langle \lambda_T, u_T\rangle_{X_T', X_T} = \langle \mathcal{F}\lambda_N, |u_T|\rangle_{X_N', X_N}$$

and the weak maximum dissipation principle

$$-\langle \lambda_T, u_T\rangle_{X_T', X_T} = \sup_{\mu_T \in \Lambda_T(\mathcal{F}\lambda_N)} \langle -\mu_T, u_T\rangle_{X_T', X_T},$$

which is the weak formulation of (8).

2. Optimal a priori estimates on the solutions to the Coulomb problem. For the sake of simplicity, a vanishing contact gap ($g \equiv 0$) will be considered in the following.

Remark 3. In the case of a nonvanishing gap, it is possible to find $u_g \in V$ such that $u_g|_{\Gamma_C} = g\mathrm{n}$, and then $w = u - u_g$ is solution to the problem

$$(25) \qquad \begin{cases} \text{Find } w \in V,\ \lambda_N \in X_N',\ \text{and } \lambda_T \in X_T' \text{ satisfying} \\ a(w, v) = l(v) - a(u_g, v) + \langle \lambda_N, w_N\rangle_{X_N', X_N} + \langle \lambda_T, w_T\rangle_{X_T', X_T}, \\ -w_N \in N_{\Lambda_N}(\lambda_N) \quad \text{in } X_N, \\ -w_T \in N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T) \quad \text{in } X_T, \end{cases}$$

i.e., a contact problem without gap but with a modified source term.

Following Remarks 1 and 2, a solution (u, λ) to problem (22) (i.e., a solution u to problem (14)) satisfies the complementarity relations

$$\langle \lambda_N, u_N \rangle_{X'_N, X_N} = 0,$$

$$\langle \lambda_T, u_T \rangle_{X'_T, X_T} = \langle \mathcal{F}\lambda_N, |u_T| \rangle_{X'_N, X_N}.$$

This implies

$$\langle \lambda, u \rangle_{X', X} \leq 0,$$

which expresses the dissipativity of contact and friction conditions. The first consequence of this is that solutions to problem (14) can be bounded independently of the friction coefficient.

PROPOSITION 1. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied and* $g \equiv 0$, *let* (u, λ) *be a solution to problem* (22), *which means that* u *is a solution to problem* (14); *then*

$$\|u\|_a \leq L_a, \quad \|\lambda\|_{-a,\Gamma_C} \leq L_a,$$

$$\|u\|_V \leq \frac{L}{\alpha}, \quad \|\lambda\|_{X'} \leq L\gamma\sqrt{\frac{M}{\alpha}}.$$

Proof. One has

$$\|u\|_a^2 = a(u, u) = l(u) + \langle \lambda, u \rangle_{X', X} \leq L_a \|u\|_a,$$

which states the first estimates. The estimate on $\|\lambda\|_{-a,\Gamma_C}$ can be performed using the intermediary solution $u^{\mathcal{N}}$ to the following problem with a homogeneous Neumann condition on Γ_C:

$$a(u^{\mathcal{N}}, v) = l(v) \quad \forall v \in V. \tag{26}$$

Since $\|u^{\mathcal{N}}\|_a \leq L_a$ for the same reason as for u, and using Lemma 1, one has

$$\|\lambda - 0\|_{-a,\Gamma_C}^2 = a(u - u^{\mathcal{N}}, u - u^{\mathcal{N}}) = \langle \lambda, u - u^{\mathcal{N}} \rangle_{X', X} \leq -\langle \lambda, u^{\mathcal{N}} \rangle_{X', X} \leq L_a \|\lambda\|_{-a,\Gamma_C}.$$

The two last estimates can be stated thanks to equivalence of norms introduced in section 1.1. $\square$

It is possible to compare $\|u\|_a$ to the corresponding norm of the solution u^c to the Signorini problem without friction defined as follows:

$$\left\{ \begin{array}{l} \text{Find } u^c \in K \text{ satisfying} \\ \\ a(u^c, v - u^c) \geq l(v - u^c) \quad \forall\, v \in K. \end{array} \right. \tag{27}$$

It is well known that under hypotheses (10) and (11), this problem has a unique solution (see [12]).

PROPOSITION 2. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied and* $g \equiv 0$, *let* u *be a solution to problem* (14), *let* u^c *be the unique solution to problem* (27), *and let* $u^{\mathcal{N}}$ *be the solution to problem* (26); *then*

$$\|u\|_a \leq \|u^c\|_a \leq \|u^{\mathcal{N}}\|_a.$$

Proof. One has

$$a(u^{\mathcal{N}}, u^{\mathcal{N}}) = l(u^{\mathcal{N}}), \quad a(u^c, u^c) = l(u^c), \quad a(u, u) = l(u) + \langle \lambda_T, u_T \rangle_{X'_T, X_T}.$$

Since u^c is the solution to the Signorini problem without friction, it minimizes over K the energy functional $\frac{1}{2}a(v,v) - l(v)$. The solution $u^{\mathcal{N}}$ minimizes this energy functional over V. Thus, since $u \in K$, one has

$$\frac{1}{2}a(u^{\mathcal{N}}, u^{\mathcal{N}}) - l(u^{\mathcal{N}}) \leq \frac{1}{2}a(u^c, u^c) - l(u^c) \leq \frac{1}{2}a(u, u) - l(u),$$

and the following relations allow one to conclude that

$$a(u^c, u^c) - a(u^{\mathcal{N}}, u^{\mathcal{N}}) = l(u^c - u^{\mathcal{N}}),$$
$$a(u, u) - a(u^c, u^c) = l(u - u^c) + \langle \lambda_T, u_T \rangle_{X'_T, X_T}$$

because then

$$\frac{1}{2}a(u^c, u^c) - \frac{1}{2}a(u^{\mathcal{N}}, u^{\mathcal{N}}) \leq 0,$$
$$\frac{1}{2}a(u, u) \leq \frac{1}{2}a(u^c, u^c) + \langle \mathcal{F}\lambda_N, |u_T| \rangle_{X'_N, X_N} \leq \frac{1}{2}a(u^c, u^c). \qquad \square$$

It is also possible to estimate how far from u^c is a solution u to problem (14). Let us introduce the following norms on Γ_C. For $v \in X$ let us define

$$\|v_T\|_{a,\Gamma_C} = \inf_{\substack{w \in V \\ w_T = v_T}} \|w\|_a = \inf_{\substack{z \in X \\ z_T = v_T}} \|z\|_{a,\Gamma_C},$$
$$\|v_N\|_{a,\Gamma_C} = \inf_{\substack{w \in V \\ w_N = v_N}} \|w\|_a = \inf_{\substack{z \in X \\ z_N = v_N}} \|v\|_{a,\Gamma_C}.$$

One has

$$\|v_T\|_{a,\Gamma_C} \leq \|v\|_{a,\Gamma_C}, \quad \|v_N\|_{a,\Gamma_C} \leq \|v\|_{a,\Gamma_C}.$$

Now, for $\lambda \in X'$, let us define

$$\|\lambda_T\|_{-a,\Gamma_C} = \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\langle \lambda_T, v_T \rangle_{X'_T, X_T}}{\|v_T\|_{a,\Gamma_C}} = \sup_{\substack{v \in X \\ v \neq 0}} \frac{\langle \lambda_T, v_T \rangle_{X'_T, X_T}}{\|v\|_{a,\Gamma_C}},$$
$$\|\lambda_N\|_{-a,\Gamma_C} = \sup_{\substack{v_N \in X_N \\ v_N \neq 0}} \frac{\langle \lambda_N, v_N \rangle_{X'_N, X_N}}{\|v_N\|_{a,\Gamma_C}} = \sup_{\substack{v \in X \\ v \neq 0}} \frac{\langle \lambda_N, v_N \rangle_{X'_N, X_N}}{\|v\|_{a,\Gamma_C}}.$$

Then, the following equivalences of norms are immediate:

$$\frac{\sqrt{\alpha}}{C_1}\|v_N\|_{X_N} \leq \|v_N\|_{a,\Gamma_C} \leq \gamma\sqrt{M}\|v_N\|_{X_N},$$
$$\frac{\sqrt{\alpha}}{C_1}\|v_T\|_{X_T} \leq \|v_T\|_{a,\Gamma_C} \leq \gamma\sqrt{M}\|v_T\|_{X_T},$$

$$\frac{1}{\gamma\sqrt{M}}\|\lambda_N\|_{X'_N} \leq \|\lambda_N\|_{-a,\Gamma_C} \leq \frac{C_1}{\sqrt{\alpha}}\|\lambda_N\|_{X'_N},$$

$$\frac{1}{\gamma\sqrt{M}}\|\lambda_T\|_{X'_T} \leq \|\lambda_T\|_{-a,\Gamma_C} \leq \frac{C_1}{\sqrt{\alpha}}\|\lambda_T\|_{X'_T}.$$

And the following result can be easily deduced.

LEMMA 2. *There exists $C_3 > 0$ such that for all $\lambda \in X'$*

$$\|\lambda_T\|_{-a,\Gamma_C} \leq C_3\|\lambda\|_{-a,\Gamma_C}, \quad \|\lambda_N\|_{-a,\Gamma_C} \leq C_3\|\lambda\|_{-a,\Gamma_C}.$$

This also allows us to define an equivalent norm on MX_N given for $\mathcal{F} \in MX_N$ by

$$\|\mathcal{F}\|_a = \sup_{\substack{v_N \in X_N \\ v_N \neq 0}} \frac{\|\mathcal{F}v_N\|_{a,\Gamma_C}}{\|v_N\|_{a,\Gamma_C}},$$

which satisfies

$$\frac{\sqrt{\alpha}}{C_1\gamma\sqrt{M}}\|\mathcal{F}\|_a \leq \|\mathcal{F}\|_{MX_N} \leq \frac{C_1\gamma\sqrt{M}}{\sqrt{\alpha}}\|\mathcal{F}\|_a.$$

With these definitions, the following result holds.

LEMMA 3. *There exists $C_4 > 0$ such that*

$$\|\mathcal{F}|v_T|\,\|_{a,\Gamma_C} \leq C_4\|\mathcal{F}\|_a\|v_T\|_{a,\Gamma_C} \quad \forall v_T \in X_T.$$

Proof. One has

$$\|\mathcal{F}|v_T|\,\|_{a,\Gamma_C} \leq \|\mathcal{F}\|_a\|\,|v_T|\,\|_{a,\Gamma_C}.$$

Moreover, it is known (see [1]) that the norm $\|\cdot\|_{X_N}$ is equivalent to the norm

$$\|v_N\|^2_{1/2,\Gamma_C} = \|v_N\|^2_{L^2(\Gamma_C)} + \int_{\Gamma_C}\int_{\Gamma_C} \frac{|v_N(x) - v_N(y)|^2}{|x-y|^d}dxdy,$$

and it is easy to verify that $\|\,|v_T|\,\|_{1/2,\Gamma_C} \leq \|v_T\|_{1/2,\Gamma_C}$ for any $v_T \in X_T$. Thus, the result can be deduced from the previously presented equivalences of norms. ∎

Of course the tangential stress on Γ_C corresponding to u^c is vanishing. The tangential stress corresponding to u can be estimated as follows. As $\lambda_T \in \Lambda_T(\mathcal{F}\lambda_N)$, one has

$$\|\lambda_T\|_{-a,\Gamma_C} = \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\langle \lambda_T, v_T\rangle_{X'_T,X_T}}{\|v_T\|_{a,\Gamma_C}} \leq \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{-\langle \mathcal{F}\lambda_N, |v_T|\rangle_{X'_N,X_N}}{\|v_T\|_{a,\Gamma_C}}$$
$$\leq C_4\|\mathcal{F}\|_a\|\lambda_N\|_{-a,\Gamma_C}.$$

Now, with the result of Proposition 1 this means that

$$\|\lambda_T\|_{-a,\Gamma_C} \leq L_a C_3 C_4\|\mathcal{F}\|_a, \tag{28}$$

and the following result holds.

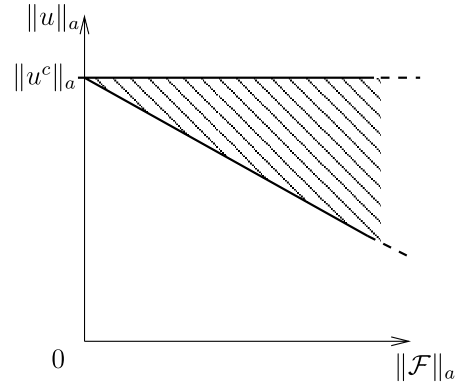

FIG. 3. *Admissibility zone for* $\|u\|_a$.

PROPOSITION 3. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied and* $g \equiv 0$, *let* u *be a solution to problem* (14) *and let* u^c *be the solution to problem* (27); *then*

$$\|u^c - u\|_a \leq L_a C_3 C_4 \|\mathcal{F}\|_a,$$

$$\|u^c - u\|_V \leq \frac{L C_3 C_4}{\alpha} \|\mathcal{F}\|_a.$$

Proof. With $\lambda \in X'$ and $\lambda^c \in X'$ the corresponding stresses on Γ_C, because $-\lambda^c_N \in N_{K_N}(u^c_N)$ and $-\lambda_N \in N_{K_N}(u_N - g)$ and because of the fact that N_{K_N} is a monotone set-valued map, one has

$$\langle \lambda^c_N - \lambda_N, u^c_N - u_N \rangle_{X'_N, X_N} \leq 0.$$

Now, $\|u^c - u\|_a$ can be estimated as follows:

$$\|u^c - u\|^2_a = a(u^c - u, u^c - u) = \langle \lambda^c - \lambda, u^c - u \rangle_{X', X} \leq \|\lambda_T\|_{-a, \Gamma_C} \|u^c - u\|_a,$$

which gives the result taking into account (28). ◻

The latter result implies that if problem (14) has several solutions, then they are in a ball of radius $L_a C_3 C_4 \|\mathcal{F}\|_a$ centered around u^c. In particular, if u^1 and u^2 are two solutions to problem (14), one has $\|u^1 - u^2\|_a \leq 2 L_a C_3 C_4 \|\mathcal{F}\|_a$. This is illustrated by Figure 3.

Remark 4. For a friction coefficient $\mathcal{F}$ constant on Γ_C, the graph in Figure 3 can be more precise for $\mathcal{F} = \|\mathcal{F}\|_a$ small, since, from the proof of Proposition 2 and the continuity result given by the latter proposition, one can deduce $\|u\|^2_a \leq \|u^c\|^2_a + \mathcal{F}\langle \lambda^c_N, |u^c_T| \rangle_{X'_N, X_N}$ at least if $\langle \lambda^c_N, |u^c_T| \rangle_{X'_N, X_N} < 0$. Of course, if $\langle \lambda^c_N, |u^c_T| \rangle_{X'_N, X_N} = 0$, the solution u^c to the Signorini problem without friction is also a solution to the Coulomb problem for any friction coefficient.

3. Elementary estimates on the Tresca problem. What is usually called the Tresca problem is the friction problem with a given friction threshold. Let $\theta \in X'_N$ be given. Then it can be formulated as follows:

$$(29) \qquad \begin{cases} \text{Find } u \in K \text{ satisfying} \\ a(u, v - u) + j(\theta, v_T) - j(\theta, u_T) \geq l(v - u) \quad \forall\, v \in K. \end{cases}$$

It is well known that under standard hypotheses (10), (11), and (13), this problem has a unique solution (see [12]) which minimizes the functional $\frac{1}{2}a(u,u)+j(\theta,u)-l(u)$.

In fact, it is not difficult to verify that all the estimates given in the previous section for the solutions to the Coulomb problem are still valid for the solution to the Tresca problem. Moreover, the solution to the Tresca problem continuously depends on the friction threshold θ. This result is stated in the following lemma.

LEMMA 4. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied, if* u^1, u^2 *are the solutions to problem* (29) *for a friction threshold* $\theta^1 \in \Lambda_N$ *and* $\theta^2 \in \Lambda_N$, *respectively, then there exists a constant* $C_5 > 0$ *independent of* θ^1 *and* θ^2 *such that the following estimate holds:*

$$\|u^1 - u^2\|_a^2 \leq C_5 \|\theta^1 - \theta^2\|_{-a,\Gamma_C}.$$

Proof. One has

$$a(u^1, u^2 - u^1) - l(u^2 - u^1) + j(\theta^1, u^2) - j(\theta^1, u^1) \geq 0,$$
$$a(u^2, u^1 - u^2) - l(u^1 - u^2) + j(\theta^2, u^1) - j(\theta^2, u^2) \geq 0,$$

which implies

$$\|u^1 - u^2\|_a^2 \leq \langle \theta^1 - \theta^2, |u_T^1| - |u_T^2| \rangle_{X_N', X_N},$$

which gives the estimate using Proposition 1 (in fact, $C_5 \leq 2C_4 L_a$). ∎

Remark 5. It does not seem possible to establish a Lipschitz continuity with respect to the friction threshold θ. Such a result would automatically imply the uniqueness of the solution to the Coulomb problem for a sufficiently small friction coefficient.

4. A uniqueness criterion. Hild in [7, 8] exhibits some multisolutions for the Coulomb problem on triangular domains. These solutions have been obtained for a large friction coefficient ($\mathcal{F} > 1$) and for a tangential displacement having a constant sign. For the moment, it seems that no multisolution has been exhibited for an arbitrary small friction coefficient in the continuous case, although such a result exists for finite element approximation in [6], albeit for a variable geometry. As far as we know, no uniqueness result has been proved even for a sufficiently small friction coefficient. The result presented here is a partial uniqueness result, which determines some cases where it is possible to say that a particular solution of the Coulomb problem is in fact the unique solution. A contrario, this result can be used to search multisolutions for an arbitrary small friction coefficient, by the fact that it eliminates a lot of situations. The partial uniqueness results we present in this section are deduced from the estimate given by the following lemma.

LEMMA 5. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied and* $g \equiv 0$, *if* u^1 *and* u^2 *are two solutions to problem* (14) *and* λ^1 *and* λ^2 *are the corresponding contact stresses on* Γ_C, *then one has the following estimate:*

$$\|u^1 - u^2\|_a^2 = \|\lambda^1 - \lambda^2\|_{-a,\Gamma_C}^2 \leq \langle \zeta - \lambda_T^2, u_T^1 - u_T^2 \rangle_{X_T', X_T} \qquad \forall \zeta \in -\partial_2 j(\mathcal{F}\lambda_N^1, u_T^2).$$

Proof. One has

$$\|u^1 - u^2\|_a^2 = \|\lambda^1 - \lambda^2\|_{-a,\Gamma_C}^2 = \langle \lambda_N^1 - \lambda_N^2, u_N^1 - u_N^2 \rangle_{X_N', X_N} + \langle \lambda_T^1 - \lambda_T^2, u_T^1 - u_T^2 \rangle_{X_T', X_T}.$$

Because N_{K_N} is a monotone set-valued map, one has $\langle \lambda^1_N - \lambda^2_N, u^1_N - u^2_N \rangle_{X'_N, X_N} \leq 0$. Thus

$$\|u^1 - u^2\|^2_a \leq \langle (\lambda^1_T - \zeta) + (\zeta - \lambda^2_T), u^1_T - u^2_T \rangle_{X'_T, X_T} \qquad \forall \zeta \in -\partial_2 j(\mathcal{F}\lambda^1_N, u^2_T).$$

But $\partial_2 j(\mathcal{F}\lambda_N, u_T)$ is also a monotone set-valued map with respect to its second variable, which implies the result (and also the fact that $\|u^1 - u^2\|^2_a \leq \inf_{\zeta \in -\partial_2 j(\mathcal{F}\lambda^1_N, u^2_T)} \|\zeta - \lambda^2_T\|_{-a,\Gamma_C}$). □

An immediate consequence of this lemma is the following result for a vanishing tangential displacement.

PROPOSITION 4. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied and* $g \equiv 0$*, if* u *is a solution to problem* (14) *such that* $u_T = 0$ *a.e. on* Γ_C *and if* $C_3 C_4 \|\mathcal{F}\|_a < 1$*, then* u *is the unique solution to problem* (14).

Proof. Let us assume that $\overline{u}$ is another solution to problem (14). Then from Lemma 5 one has

$$\|u - \overline{u}\|^2_a \leq \langle \zeta - \overline{\lambda}_T, u_T - \overline{u}_T \rangle_{X'_T, X_T} \qquad \forall \zeta \in -\partial_2 j(\mathcal{F}\lambda_N, \overline{u}_T),$$

but, because $u_T = 0$ and due to the complementarity relations $\langle \overline{\lambda}_T, \overline{u}_T \rangle_{X'_T, X_T} = \langle \mathcal{F}\overline{\lambda}_N, |\overline{u}_T| \rangle_{X'_N, X_N}$ and $\langle \zeta, \overline{u}_T \rangle_{X'_T, X_T} = \langle \mathcal{F}\lambda_N, |\overline{u}_T| \rangle_{X'_N, X_N}$, it implies using Lemma 3 that

$$\begin{aligned} \|u - \overline{u}\|^2_a &\leq \langle \mathcal{F}(\overline{\lambda}_N - \lambda_N), |\overline{u}_T| \rangle_{X'_N, X_N} \leq C_3 C_4 \|\mathcal{F}\|_a \|\lambda - \overline{\lambda}\|_{-a,\Gamma_C} \|u_T - \overline{u}_T\|^2_{a,\Gamma_C} \\ &\leq C_3 C_4 \|\mathcal{F}\|_a \|u - \overline{u}\|^2_a, \end{aligned}$$

which concludes the proof. □

In the case $d = 2$, it is possible to give a result to a solution having a tangential displacement with a constant sign on Γ_C. We will say that a tangential displacement $u_T \in X_T$ is strictly positive if $\langle \mu_T, u_T \rangle_{X'_T, X_T} > 0$ for all $\mu_T \in X'_T$ such that $\mu_T \geq 0$ (i.e., $\langle \mu_T, v_T \rangle_{X'_T, X_T} \geq 0$ for all $v_T \in X_T$, $v_T \geq 0$, a.e. on Γ_C) and $\mu_T \neq 0$.

PROPOSITION 5. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied,* $g \equiv 0$*, and* $d = 2$*, if* u *is a solution to problem* (14) *such that* $u_T > 0$ *and* $C_3 \|\mathcal{F}\|_a < 1$*, then* u *is the unique solution to problem* (14) *(when* $\mathcal{F}$ *is constant over* Γ_C*, the condition reduces to* $C_3 \mathcal{F} < 1$*).*

Proof. Let us assume that $\overline{u}$ is another solution to problem (14), with $\overline{\lambda}_N$ and $\overline{\lambda}_T$ the corresponding contact stresses on Γ_C. Then from Lemma 5 one has

$$\|\overline{u} - u\|^2_a \leq \langle \zeta - \lambda_T, \overline{u}_T - u_T \rangle_{X'_T, X_T} \qquad \forall \zeta \in -\partial_2 j(\mathcal{F}\overline{\lambda}_N, u_T).$$

Because $u_T > 0$, one has $\lambda_T = \mathcal{F}\lambda_N$ and $-\partial_2 j(\mathcal{F}\overline{\lambda}_N, u_T)$ contains $\mathcal{F}\overline{\lambda}_N$. Thus, taking $\zeta = \mathcal{F}\overline{\lambda}_N$, one obtains

$$\|\overline{u} - u\|^2_a \leq \langle \mathcal{F}(\overline{\lambda}_N - \lambda_N), \overline{u}_T - u_T \rangle_{X'_T, X_T} \leq \|\overline{\lambda} - \lambda\|_{-a,\Gamma_C} \|\mathcal{F}(\overline{u} - u)\|_a \leq \|\mathcal{F}\|_a \|\overline{u} - u\|^2_a,$$

which implies $\overline{u} = u$ when $\|\mathcal{F}\|_a < 1$. □

Of course, the same reasoning is valid for $u_T < 0$.

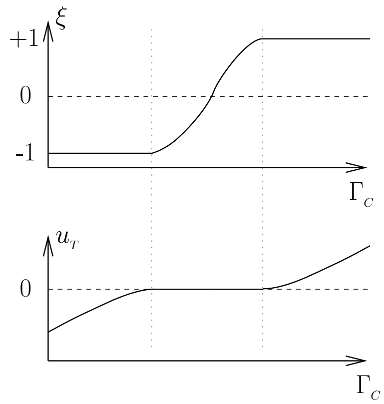

FIG. 4. *Example of tangential displacement u_T and a possible corresponding multiplier ξ for $d = 2$.*

Let us now define the space of multipliers $M(X_T \to X_N)$ of the functions $\xi : \Gamma_C \to \mathbb{R}^d$ such that $\xi \cdot \mathrm{n} = 0$ a.e. on Γ_C and such that the following two equivalent norms are finite:

$$\|\xi\|_{M(X_T \to X_N)} = \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\|\xi \cdot v_T\|_{X_N}}{\|v_T\|_{X_T}} \quad \text{and} \quad \|\xi\|_a = \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\|\xi \cdot v_T\|_{a,\Gamma_C}}{\|v_T\|_{a,\Gamma_C}}.$$

Because Γ_C is assumed to have the $\mathcal{C}^1$ regularity, $M(X_T \to X_N)$ is isomorphic to $(MX_N)^{d-1}$.

It is possible to give a more general result assuming that $\lambda_T = \mathcal{F}\lambda_N \xi$, with $\xi \in M(X_T \to X_N)$. It is easy to see that this implies that $|\xi| \leq 1$ a.e. on the support of λ_N and, more precisely, that $\xi \in \mathrm{Dir}_T(u_T)$ a.e. on the support of λ_N, where $\mathrm{Dir}_T(.)$ is the subderivative of the convex map $\mathbb{R}^d \ni x \longmapsto |x_T|$. This means that it is reasonable to assume that $\xi \in \mathrm{Dir}_T(u_T)$ a.e. on Γ_C.

PROPOSITION 6. *Assuming hypotheses* (10), (11), *and* (13) *are satisfied and* $g \equiv 0$*, if u is a solution to problem* (14) *such that* $\lambda_T = \mathcal{F}\lambda_N \xi$*, with* $\xi \in M(X_T \to X_N)$*,* $\xi \in \mathrm{Dir}_T(u_T)$ *a.e. on* Γ_C*, and* $C_3 \|\mathcal{F}\|_a \|\xi\|_a < 1$*, then u is the unique solution to problem* (14).

Proof. Let us assume that $\overline{u}$ is another solution to problem (14), with $\overline{\lambda}_N$ and $\overline{\lambda}_T$ the corresponding contact stresses on Γ_C. Then from Lemma 5 one has

$$\|\overline{u} - u\|_a^2 \leq \langle \zeta - \lambda_T, \overline{u}_T - u_T \rangle_{X'_T, X_T} \qquad \forall \zeta \in -\partial_2 j(\mathcal{F}\overline{\lambda}_N, u_T).$$

Then, a possible choice is $\zeta = \mathcal{F}\overline{\lambda}_N \xi$, which, together with the fact that $\|\mathcal{F}\xi\|_a \leq \|\mathcal{F}\|_a \|\xi\|_a$, gives

$$\begin{aligned} \|\overline{u} - u\|_a^2 \leq \langle \mathcal{F}\xi(\overline{\lambda}_N - \lambda_N), \overline{u}_T - u_T \rangle_{X'_T, X_T} &\leq C_3 \|\mathcal{F}\|_a \|\xi\|_a \|\overline{\lambda} - \lambda\|_{-a,\Gamma_C} \|\overline{u} - u\|_a \\ &\leq C_3 \|\mathcal{F}\|_a \|\xi\|_a \|\overline{u} - u\|_a^2, \end{aligned}$$

which implies $\overline{u} = u$ when $C_3 \|\mathcal{F}\|_a \|\xi\|_a < 1$. ◻

Remark 6. Using equivalences of norms, one can deduce that a more restrictive condition than $C_3 \|\mathcal{F}\|_a \|\xi\|_a < 1$ is the condition $\|\mathcal{F}\|_{MX_N} \|\xi\|_{M(X_T \to X_N)} < \frac{\sqrt{\alpha}}{C_1 C_3 \gamma \sqrt{M}}$.

As illustrated in Figure 4, for $d = 2$, the multiplier ξ has to vary from -1 to $+1$ each time the sign of the tangential displacement changes from negative to positive.

The set $M(X_T \to X_N)$ does not contain any multiplier having a discontinuity of the first kind. This implies that in order to satisfy the assumptions of Proposition 6 the tangential displacement of the solution u cannot pass from a negative value to a positive value, being zero on only a single point of Γ_C.

Perspectives. As far as we know, the result given by Propositions 4, 5, and 6 are the first results dealing with the uniqueness of the solution to the Coulomb problem without considering a regularization of the contact or the friction law. In the future, it may be interesting to investigate the following open problems.

Is it possible to prove that, for a sufficiently regular domain and a sufficiently regular loading, a solution of the Coulomb problem is necessarily such that $\lambda_T = \mathcal{F}\lambda_N\xi$ with $\xi \in M(X_T \to X_N)$? This could be a way to prove a uniqueness result for a sufficiently small friction coefficient and regular loadings.

The more the tangential displacement u_T oscillates around 0 (i.e., the more u_T changes its sign for $d = 2$), the more the multiplier ξ varies and thus the greater $\|\xi\|_{M(X_T \to X_N)}$ is. Does it mean that a multisolution for an arbitrary small friction coefficient and a fixed geometry has to be searched with very oscillating tangential displacement (necessarily for all the solutions)?

Finally, the convergence of finite element methods in the uniqueness framework given by Proposition 6 will be presented in [10].

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