

# Fixed point strategies for elastostatic frictional contact problems.

Patrick LABORDE<sup>1</sup>, Yves RENARD<sup>2</sup>

## Abstract

Several fixed point strategies and Uzawa algorithms (for classical and augmented Lagrangian formulations) are presented to solve the unilateral contact problem with Coulomb friction. These methods are analyzed, without introducing any regularization, and a theoretical comparison is performed. Thanks to a formalism coming from convex analysis, some new fixed point strategies are presented and compared to known methods. The analysis is first performed on continuous Tresca problem and then on the finite dimensional Coulomb problem derived from an arbitrary finite element method.

Keywords : unilateral contact, Coulomb friction, Tresca problem, Signorini problem, bipotential, fixed point, Uzawa algorithm.

## Introduction

The main goal of this paper is to introduce a formalism to deal with contact and friction of deformable bodies, focusing on fixed point algorithms. We restrict the study to the elastostatic case, the so-called Signorini problem with Coulomb friction (or simply the Coulomb problem) introduced by Duvaut and Lions [12], whose interest is to be very close to the incremental formulation of an evolutionary friction problem.

The unilateral contact problem without friction was first considered by Signorini who shown the uniqueness of the solution. Fichera [14] proved an existence result using a quadratic minimization formulation. When friction is included, the nature of the problem changes due to the non self-adjoint character of the Coulomb friction condition. This problem no longer has a potential. Until now, only a partial uniqueness result has been obtained for the continuous (nonregularized) problem (see [26]). However, existence result have been established for a sufficiently small friction coefficient (see [25] for instance).

We introduce new fixed points formulations thanks to Moreau-Yosida resolvent and regularization using an approach similar to the proximal point algorithm. We first analyze the self-adjoint Tresca problem in which the friction threshold is assumed to be known. The properties obtained for the fixed points are independent of any spatial discretization, which is not the case for the most used algorithms in practice. As a second step, the analysis is performed on the Coulomb friction problem in finite dimension for an arbitrary finite element method. The De-Saxc bipotential for friction problem is revisited and adapted to the continuous framework in order to obtain new fixed point formulations.

The paper is outlined as follows.

- Section 1: the strong formulation of the problem is recalled and then the classical weak formulation of Duvaut and Lions is presented. The Neumann to Dirichlet operator is introduced in order to simplify the expression of friction problems.

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<sup>1</sup>MIP, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse cedex 4, France, [laborde@mip.ups-tlse.fr](mailto:laborde@mip.ups-tlse.fr)

<sup>2</sup>Corresponding author, MIP, INSAT, Complexe scientifique de Rangueil, 31077 Toulouse, France, [Yves.Renard@insa-toulouse.fr](mailto:Yves.Renard@insa-toulouse.fr)

- Section 2: a classical fixed point method for the continuous Tresca problem is analyzed. This method is deduced from the Uzawa algorithm on the classical Lagragian formulation. This is a fixed point on the contact and friction stresses. The convergence properties of this fixed point is compared to the one obtained by using an augmented Lagragian formulation.
- Section 3: we present an adaptation of this fixed point method for the continuous Coulomb problem. An equivalence result is proved.
- Section 4: the analysis is done on the Signorini problem with Coulomb friction in finite dimension, using an arbitrary finite element method and a particular discretization of contact and friction conditions allowing us to obtain uniform estimates. As in [15], but still for an arbitrary finite element method, uniqueness is obtained for a sufficiently small friction coefficient and existence for any friction coefficient.
- Section 5: a convergence analysis of the discretization method introduced in section 4 is done for the Tresca problem.
- Section 6: a new fixed point operator on the contact boundary displacement is presented. It is proved that it has the same contraction property than the classical one.
- Section 7: the De Saxcé's bipotential theory is used and a justification is presented in the continuous framework. Two new fixed points operators are derived.
- Section 8: finally, the classical fixed point on the friction threshold is compared to the previous ones.

## 1 The Coulomb problem

### 1.1 Strong formulation

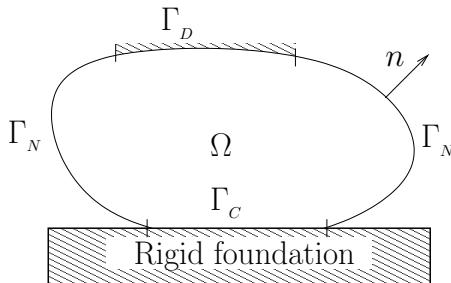


Figure 1: *Elastic body  $\Omega$  in frictional contact.*

Let  $\Omega \subset \mathbb{R}^d$  ( $d = 2$  or  $3$ ) be a bounded domain representing the reference configuration of a linearly elastic body submitted to a Neumann condition on  $\Gamma_N$ , a Dirichlet condition on  $\Gamma_D$ . On  $\Gamma_C$ , a unilateral contact with static Coulomb friction condition between the body and a flat rigid foundation is prescribed.

The problem consists in finding the displacement field  $u(x)$  satisfying:

$$-\operatorname{div} \boldsymbol{\sigma}(u) = f, \quad \text{in } \Omega, \quad (1)$$

$$\boldsymbol{\sigma}(u) = \mathcal{A}\boldsymbol{\varepsilon}(u), \quad \text{in } \Omega, \quad (2)$$

$$\boldsymbol{\sigma}(u)\mathbf{n} = g, \quad \text{on } \Gamma_N, \quad (3)$$

$$u = 0, \quad \text{on } \Gamma_D, \quad (4)$$

where  $\Gamma_N$ ,  $\Gamma_D$  and  $\Gamma_C$  are nonoverlapping open parts of  $\partial\Omega$ , the boundary of  $\Omega$ ,  $\boldsymbol{\sigma}(u)$  is the stress tensor,  $\boldsymbol{\varepsilon}(u)$  is the linearized strain tensor,  $\mathcal{A}$  is the elastic coefficient tensor which satisfies classical conditions of symmetry and ellipticity,  $\mathbf{n}$  is the outward unit normal to  $\Omega$  on  $\partial\Omega$ , and  $f$ ,  $g$  are the given external loads.

On  $\Gamma_C$ , it is usual to decompose the displacement and the stress vector in normal and tangential components:

$$u_N = u \cdot \mathbf{n}, \quad u_T = u - u_N \mathbf{n},$$

$$\boldsymbol{\sigma}_N(u) = (\boldsymbol{\sigma}(u)\mathbf{n}) \cdot \mathbf{n}, \quad \boldsymbol{\sigma}_T(u) = \boldsymbol{\sigma}(u)\mathbf{n} - \boldsymbol{\sigma}_N(u)\mathbf{n}.$$

To give a clear sense to this decomposition, we assume  $\Gamma_C$  to have the  $\mathcal{C}^1$  regularity. Prescribing also that there is no initial gap between the solid and the rigid foundation, the unilateral contact condition is expressed by the following complementary condition:

$$u_N \leq 0, \quad \boldsymbol{\sigma}_N(u) \leq 0, \quad u_N \boldsymbol{\sigma}_N(u) = 0. \quad (5)$$

Denoting by  $\mathcal{F} \geq 0$  the friction coefficient, the *static Coulomb friction* condition reads as:

$$\text{if } u_T = 0 \quad \text{then} \quad |\boldsymbol{\sigma}_T(u)| \leq -\mathcal{F}\boldsymbol{\sigma}_N(u), \quad (6)$$

$$\text{if } u_T \neq 0 \quad \text{then} \quad \boldsymbol{\sigma}_T(u) = \mathcal{F}\boldsymbol{\sigma}_N(u) \frac{u_T}{|u_T|}. \quad (7)$$

## 1.2 Classical weak formulation

Let us introduce the following Hilbert spaces

$$\begin{aligned} V &= \{v \in H^1(\Omega; \mathbb{R}^d), v = 0 \text{ on } \Gamma_D\}, \\ X &= \{v|_{\Gamma_C} : v \in V\} \subset H^{1/2}(\Gamma_C; \mathbb{R}^d), \\ X_N &= \{v_N|_{\Gamma_C} : v \in V\}, \quad X_T = \{v_T|_{\Gamma_C} : v \in V\}, \end{aligned}$$

and their topological dual spaces  $V'$ ,  $X'$ ,  $X'_N$  and  $X'_T$ . It is assumed that  $\Gamma_C$  is sufficiently smooth such that  $X_N \subset H^{1/2}(\Gamma_C)$ ,  $X_T \subset H^{1/2}(\Gamma_C; \mathbb{R}^{d-1})$ ,  $X'_N \subset H^{-1/2}(\Gamma_C)$  and  $X'_T \subset H^{-1/2}(\Gamma_C; \mathbb{R}^{d-1})$ . Classically,  $H^{1/2}(\Gamma_C)$  is the space of the restriction on  $\Gamma_C$  of traces on  $\partial\Omega$  of functions of  $H^1(\Omega)$ , and  $H^{-1/2}(\Gamma_C)$  is the dual space of  $H^{1/2}(\Gamma_C)$  which is the space of the restrictions on  $\Gamma_C$  of functions of  $H^{1/2}(\partial\Omega)$  vanishing outside  $\Gamma_C$ . We refer to [23] and [1] for a complete discussion on trace operators.

The set of admissible displacements is defined as

$$K = \{v \in V, v_N \leq 0 \text{ on } \Gamma_C\}. \quad (8)$$

The following maps

$$a(u, v) = \int_{\Omega} \mathcal{A}\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(v) dx,$$

$$l(v) = \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} g \cdot v d\Gamma,$$

$$j(s, v_T) = -\langle s, |v_T| \rangle_{X'_N, X_N}$$

represent the virtual work of elastic forces, the external load and the “virtual work” of friction forces respectively. We assume standard hypotheses:

$a(\cdot, \cdot)$  bilinear symmetric continuous coercive form on  $V \times V$  :

$$\exists \alpha > 0, \exists M > 0, a(u, u) \geq \alpha \|u\|_V^2, a(u, v) \leq M \|u\|_V \|v\|_V \quad \forall u, v \in V, \quad (9)$$

$l(\cdot)$  linear continuous form on  $V$ ,

$\mathcal{F}$  Lipschitz-continuous nonnegative function on  $\Gamma_c$ .

The latter condition ensures that  $j(\mathcal{F}\lambda_N, v_T)$  is linear continuous on  $\lambda_N$  and also convex lower semi-continuous on  $v_T$  when  $\lambda_N$  is a nonpositive element of  $X'_N$  (see for instance [3]). Problem (1) – (7) is then formally equivalent to the following inequality formulation (Duvaut and Lions [12]):

$$\begin{cases} \text{Find } u \in K \text{ satisfying} \\ a(u, v - u) + j(\mathcal{F}\sigma_N(u), v_T) - j(\mathcal{F}\sigma_N(u), u_T) \geq l(v - u), \quad \forall v \in K. \end{cases} \quad (12)$$

Existence results for this problem can be found in Nečas, Jaruček and Haslinger [25] for a two-dimensional elastic strip, assuming that the coefficient of friction is small enough and using a shifting technique, previously introduced by Fichera, and later applied to more general domains by Jaruček [20] [21]. Recently, Eck and Jaruček [13] have given a different proof using a penalization method. We emphasize that most results on existence for frictional problems involve a condition of smallness for the friction coefficient (and a compact support on  $\Gamma_c$ ). As far as we know, it does not exist a global uniqueness result for the continuous problem. A partial uniqueness result is presented in [26] and some multi-solutions for a large friction coefficient are presented by P. Hild in [17, 18].

The major difficulty about (12) is due to the coupling between the friction threshold and the contact pressure  $\sigma_N(u)$ . The consequence is that this problem does not represent a variational inequality, in the sense that there does not exist a potential for the Coulomb friction force.

### 1.3 Neumann to Dirichlet operator

Now, we introduce the Neumann to Dirichlet operator on  $\Gamma_c$  which allows to restrict the problem on  $\Gamma_c$ . Let  $\lambda = (\lambda_N, \lambda_T) \in X'$  then, there exists a unique solution  $u$  to

$$\begin{cases} \text{Find } u \in V \text{ satisfying} \\ a(u, v) = l(v) + \langle \lambda, v \rangle_{X', X} \quad \forall v \in V, \end{cases} \quad (13)$$

under hypotheses (9) and (10) (see [12]). So, it is possible to define the operator

$$\mathbb{E} : X' \longrightarrow X$$

$$\lambda \longmapsto u|_{\Gamma_c}$$

This operator is affine and continuous. Moreover, it is invertible and its inverse is continuous. It is possible to express  $\mathbb{E}^{-1}$  as follows: for  $w \in X$ , let  $u$  be the solution to the Dirichlet problem

$$\begin{cases} \text{Find } u \in V \text{ satisfying } u|_{\Gamma_c} = w \text{ and} \\ a(u, v) = l(v), \quad \forall v \in V, v|_{\Gamma_c} = 0, \end{cases} \quad (14)$$

then  $\mathbb{E}^{-1}(w)$  is equal to  $\lambda \in X'$  defined by

$$\langle \lambda, v \rangle_{X', X} = a(u, v) - l(v), \quad \forall v \in V.$$

In a weak sense, one has the relation  $\mathbb{E}^{-1}(u) = \sigma(u)n$  on  $\Gamma_c$ . The continuity of  $\mathbb{E}$  and  $\mathbb{E}^{-1}$  is given by the following lemma.

**Lemma 1** *Under hypotheses (9) and (10), the following estimates hold:*

$$\|\mathbb{E}(\lambda^1) - \mathbb{E}(\lambda^2)\|_X \leq \frac{C_1^2}{\alpha} \|\lambda^1 - \lambda^2\|_{X'} \quad (15)$$

$$\|\mathbb{E}^{-1}(u^1) - \mathbb{E}^{-1}(u^2)\|_{X'} \leq MC_2^2 \|u^1 - u^2\|_X \quad (16)$$

where  $C_1$  is the continuity constant of the trace operator on  $\Gamma_c$ ,  $\alpha$  the coercivity constant of the bilinear form  $a(\cdot, \cdot)$ ,  $M$  is the continuity constant of  $a(\cdot, \cdot)$  and  $C_2 > 0$  is the continuity constant of the homogeneous Dirichlet problem corresponding to (14) (i.e. with  $l(v) \equiv 0$ ).

**Proof.** Let  $\lambda^1$  and  $\lambda^2$  be given in  $X'_t$  and  $u^1, u^2$  the corresponding solutions to (13), then

$$\begin{aligned} \|u^1 - u^2\|_v^2 &\leq \frac{1}{\alpha} a(u^1 - u^2, u^1 - u^2) = \frac{1}{\alpha} \langle \lambda^1 - \lambda^2, u^1 - u^2 \rangle_{X', X} \\ &\leq \frac{C_1}{\alpha} \|\lambda^1 - \lambda^2\|_{X'} \|u^1 - u^2\|_v \end{aligned}$$

and consequently

$$\|u^1 - u^2\|_v \leq \frac{C_1}{\alpha} \|\lambda^1 - \lambda^2\|_{X'} \quad (17)$$

which gives the first estimate using again the continuity of the trace operator on  $\Gamma_c$ . The second estimate can be performed as follows:

$$\begin{aligned} \|\mathbb{E}^{-1}(u^1) - \mathbb{E}^{-1}(u^2)\|_{X'} &= \sup_{\substack{w \in X \\ w \neq 0}} \frac{\langle \mathbb{E}^{-1}(u^1) - \mathbb{E}^{-1}(u^2), w \rangle_{X', X}}{\|w\|_X} \\ &= \sup_{\substack{w \in X \\ w \neq 0}} \left( \inf_{\substack{\{v \in V : v|_{\Gamma_c} = w\}} \frac{a(u^1 - u^2, v)}{\|w\|_X} \right) \\ &\leq M \|u^1 - u^2\|_v \sup_{\substack{w \in X \\ w \neq 0}} \left( \inf_{\substack{\{v \in V : v|_{\Gamma_c} = w\}} \frac{\|v\|_v}{\|w\|_X} \right) \\ &\leq M \gamma \|u^1 - u^2\|_v \end{aligned} \quad (18)$$

where  $\gamma = \sup_{\substack{w \in X \\ w \neq 0}} \inf_{\{v \in V : v|_{\Gamma_c} = w\}} \frac{\|v\|_v}{\|w\|_X}$ . Since  $\gamma \leq C_2$ , this gives (16). ■

## 2 A classical fixed point method for the Tresca problem

### 2.1 The Tresca problem

Let us introduce the so-called Tresca problem, which is a static friction problem with a prescribed friction threshold  $-s$  defined on  $\Gamma_c$  satisfying

$$s \in X'_N, \quad s \text{ nonpositive in the weak sense: } \langle s, v \rangle_{X'_N, X_N} \geq 0, \quad \forall v \in K_N.$$

The Tresca problem can be written as follows:

$$\begin{cases} \text{Find } u \in K \text{ satisfying} \\ a(u, v - u) + j(s, v_T) - j(s, u_T) \geq l(v - u), \quad \forall v \in K. \end{cases} \quad (19)$$

Of course, finding a solution to the Coulomb friction problem is finding  $s \in X'_N$  and a solution to (19) such that  $s = \mathcal{F}\sigma_N(u)$ . The Tresca problem corresponds to a variational problem. Denoting

$$J(u) = \frac{1}{2}a(u, u) - l(u) + j(s, u_T) + I_K(u),$$

where  $I_K$  is the indicator function of  $K$  (if  $u \in K$  then  $I_K(u) = 0$ , else  $I_K(u) = +\infty$ ), Problem (19) is equivalent to

$$\begin{cases} \text{Find } u \in V \text{ satisfying} \\ J(u) = \inf_{v \in V} J(v). \end{cases} \quad (20)$$

Under classical assumptions (9) (10) (11) the functional  $J$  is strictly convex, coercive and lower semicontinuous. Thus,  $J$  admits a unique minimizer (see [23] for instance) in  $V$ .

### 2.2 Classical Lagrangian for Tresca problem

The set of admissible normal stresses on  $\Gamma_c$  can be defined as

$$\Lambda_N = \{f_N \in X'_N : \langle f_N, v_N \rangle_{X'_N, X_N} \geq 0, \quad \forall v_N \in K_N\}.$$

This is the opposite of  $K_N^*$  the polar cone to  $K_N$ . Let us also introduce the set of admissible tangential stresses on  $\Gamma_c$ :

$$\Lambda_T(s) = \{f_T \in X'_T : -\langle f_T, w_T \rangle_{X'_T, X_T} + \langle s, |w_T| \rangle_{X'_N, X_N} \leq 0, \quad \forall w_T \in X_T\}.$$

**Remark 1** When  $s \in L^2(\Gamma_c)$  then  $s \leq 0$  a.e. on  $\Gamma_c$  and  $\Lambda_T(s) = \{\lambda_T \in L^2(\Gamma_c, \mathbb{R}^{n-1}) : |\lambda_T| \leq -s \text{ a.e. on } \Gamma_c\}$ .

Using these definitions, it is classical to consider the following Lagrangian for the Tresca Problem (see [23], [2] for instance)

$$\mathcal{L}(u, \lambda) = \frac{1}{2}a(u, u) - l(u) - \langle \lambda, u \rangle_{X', X} - I_{\Lambda_T(s)}(\lambda_T) - I_{\Lambda_N}(\lambda_N).$$

The following saddle point problem is then equivalent to Problem (19):

$$\begin{cases} \text{Find } u \in V \text{ and } \lambda \in X' \text{ satisfying} \\ \mathcal{L}(u, \lambda) = \inf_{v \in V} \sup_{\mu \in X'} \mathcal{L}(v, \mu). \end{cases} \quad (21)$$

We choose here to express the constraints on  $\mathcal{L}(u, \lambda)$  thanks to indicator functions. This Lagrangian problem corresponds to a dualization of the indicator function in the expression of  $J(u)$ , in the sense of Rockafellar [28]. Optimality conditions of Problem (21) are

$$\begin{cases} a(u, v) = l(v) + \langle \lambda_N, w_N \rangle_{X'_N, X_N} + \langle \lambda_T, w_T \rangle_{X'_T, X_T} \quad \forall v \in V, \\ u_N + N_{\Lambda_N}(\lambda_N) \ni 0, \\ u_T + N_{\Lambda_T(s)}(\lambda_T) \ni 0, \end{cases} \quad (22)$$

which is classically equivalent to Problem (19).

We will now formulate the classical Uzawa algorithm on Problem (21) (see [24] for instance) in the continuous framework. It corresponds to a gradient with projection algorithm on  $\lambda$ . In order to define the projection step, we introduce the following duality map from  $X'_N$  to  $X_N$ :

$$\begin{aligned} i_N : X'_N &\longrightarrow X_N, \\ \lambda_N &\longmapsto v_N \text{ defined by } \langle \lambda_N, w_N \rangle_{X'_N, X_N} = (v_N, w_N)_{X_N} \quad \forall w_N \in X_N. \end{aligned}$$

and the duality map from  $X'_T$  to  $X_T$ :

$$\begin{aligned} i_T : X'_T &\longrightarrow X_T, \\ \lambda_T &\longmapsto v_T \text{ defined by } \langle \lambda_T, w_T \rangle_{X'_T, X_T} = (v_T, w_T)_{X_T} \quad \forall w_T \in X_T, \end{aligned}$$

where  $(\cdot, \cdot)_{X_N}$  and  $(\cdot, \cdot)_{X_T}$  are the inner products of  $X_N$  and  $X_T$  respectively. These two duality maps are isometries. For the sake of convenience  $i(\lambda)$  will stand for the pair  $(i_N(\lambda_N), i_T(\lambda_T))$ .

Denoting  $\tilde{\lambda}_N = i_N(\lambda_N)$ ,  $\tilde{\lambda}_T = i_T(\lambda_T)$ ,  $\tilde{\Lambda}_N = i_N(\Lambda_N)$  and  $\tilde{\Lambda}_T(s) = i_T(\Lambda_T(s))$ , the Uzawa algorithm can be written as follows:

$$\begin{aligned} &\bullet \text{ Step 0 : } \lambda^0 = (\lambda_N^0, \lambda_T^0) \text{ with } \lambda_N^0 \in \Lambda_N \text{ and } \lambda_T^0 \in \Lambda_T(s) \text{ arbitrary chosen.} \\ &\bullet \text{ Step 1 : } \lambda^n = (\lambda_N^n, \lambda_T^n) \text{ fixed, find } u^{n+1} \in V \text{ solution to} \\ &\quad \mathcal{L}(u^{n+1}, \lambda^n) = \inf_{v \in V} \mathcal{L}(v, \lambda^n). \\ &\bullet \text{ Step 2 : } \begin{aligned} &\text{Update multipliers by} \\ &\tilde{\lambda}_N^{n+1} = P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N^n - ru_N^{n+1}), \\ &\tilde{\lambda}_T^{n+1} = P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T^n - ru_T^{n+1}). \end{aligned} \end{aligned} \quad (23)$$

Loop to step 1 until a "stop criterion" is reached.

In this algorithm,  $P_{\tilde{\Lambda}_N}$  and  $P_{\tilde{\Lambda}_T(s)}$  denote the projection operator onto  $\tilde{\Lambda}_N$  in  $X_N$  and the projection operator onto  $\tilde{\Lambda}_T(s)$  in  $X_T$  respectively. The parameter  $r$  (which may be variable from an iteration to another) is the "descent" step of the gradient method.

The Uzawa algorithm corresponds to the iterations of the following fixed point operator:

$$\begin{aligned} T^1 : \quad X &\longrightarrow X \\ (\tilde{\lambda}_N, \tilde{\lambda}_T) &\longmapsto \left( P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N - ru_N), P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T - ru_T) \right), \\ &\text{where } (u_N, u_T) = \tilde{\mathbb{E}}(\tilde{\lambda}_N, \tilde{\lambda}_T). \end{aligned}$$

In this definition,  $\tilde{\mathbb{E}}(\tilde{\lambda}_N, \tilde{\lambda}_T)$  is the trace on  $\Gamma_c$  of the solution  $u$  to the following problem (compare to (13)):

$$u \in V, \quad a(u, v) = l(v) + (\tilde{\lambda}, v)_X \quad \forall v \in V.$$

### 2.3 Contraction property of the fixed point operator

**Theorem 1** *Provided that hypotheses (9), (10), (11) are satisfied, the mapping  $T^1$  is a strict contraction for  $r > 0$  sufficiently small.*

**Proof.** Since projection operators in  $X$  are contractions, one has

$$\begin{aligned} \|T^1(\tilde{\lambda}^1) - T^1(\tilde{\lambda}^2)\|_X^2 &\leq \|\delta\tilde{\lambda}_N - r\delta u_N\|_{X_N}^2 + \|\delta\tilde{\lambda}_T - r\delta u_T\|_{X_T}^2 \\ &\leq \|\delta\tilde{\lambda}\|_X^2 - 2r(\delta\tilde{\lambda}, \delta u)_X + r^2\|\delta u\|_X^2 \\ &\leq \|\delta\tilde{\lambda}\|_X^2 - 2r a(\delta u, \delta u) + r^2\|\delta u\|_X^2 \\ &\leq \|\delta\tilde{\lambda}\|_X^2 - 2r\alpha\|\delta u\|_V^2 + r^2C_1^2\|\delta u\|_V^2, \end{aligned}$$

for all  $\tilde{\lambda}^1$  and  $\tilde{\lambda}^2$  in  $X$  and using the notation  $\delta\lambda = \tilde{\lambda}^1 - \tilde{\lambda}^2$ . Now, denoting  $\beta = \frac{\|\delta u\|_V}{\|\delta\tilde{\lambda}\|_X}$ , it follows from (18) and (17) that  $\frac{1}{M\gamma} \leq \beta \leq \frac{C_1}{\alpha}$ , and

$$\|T^1(\tilde{\lambda}^1) - T^1(\tilde{\lambda}^2)\|_X^2 \leq \|\delta\tilde{\lambda}\|_X^2(1 - 2r\alpha\beta^2 + r^2C_1^2\beta^2)$$

which means that  $T^1$  is a strict contraction, at least for  $0 < r < 2r_1$  with  $r_1 = \frac{\alpha}{C_1^2}$ . The minimum value of  $p_1(r) = 1 - 2r\alpha\beta^2 + r^2C_1^2\beta^2$  is  $p_1(r_1) = 1 - \frac{\alpha^2\beta^2}{C_1^2} \leq 1 - \frac{\alpha^2}{M^2C_1^2\gamma^2}$ . ■

### 2.4 Augmented Lagrangian for Tresca problem

The following augmented Lagrangian is the proximal Lagrangian in the sense of Rockafellar (see [28] for instance). It was introduced for the friction problems by P. Alart and A. Curnier (see [2]):

$$\begin{aligned} \mathcal{L}_\rho(u, \lambda) &= \frac{1}{2}a(u, u) - l(u) - \langle \lambda, u \rangle_{X', X} - \frac{1}{2\rho}\|\tilde{\lambda}_N - \rho u_N - P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N - \rho u_N)\|_{X_N}^2 \\ &\quad - \frac{1}{2\rho}\|\tilde{\lambda}_T - \rho u_T - P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T - \rho u_T)\|_{X_T}^2 + \frac{\rho}{2}\|u\|_X^2, \end{aligned}$$

where  $\rho > 0$  is the given augmentation parameter. The following saddle point problem is then also equivalent to Tresca problem (19)

$$\begin{cases} \text{Find } u \in V \text{ and } \lambda \in X' \text{ satisfying:} \\ \mathcal{L}_\rho(u, \lambda) = \inf_{v \in V} \sup_{\mu \in X'} \mathcal{L}_\rho(v, \mu). \end{cases} \quad (24)$$

The optimality conditions for Problem (24) are

$$\left\{ \begin{array}{l} u \in V \text{ and } \lambda \in X' \text{ such that} \\ a(u, v) - l(v) - (P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N - \rho u_N), v_N)_{X_N} - (P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T - \rho u_T), v_T)_{X_T} = 0, \forall v \in V, \\ \frac{1}{\rho}(P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N - \rho u_N) - \tilde{\lambda}_N) = 0, \\ \frac{1}{\rho}(P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T - \rho u_T) - \tilde{\lambda}_T) = 0. \end{array} \right. \quad (25)$$

The saddle point problem (24) has no constraint. An Uzawa algorithm for this problem, corresponding now to a simple gradient iteration on  $\lambda$ , can be written as follows:

$$\left| \begin{array}{l} \bullet \text{ Step 0 : } \lambda^0 = (\lambda_N^0, \lambda_T^0) \text{ arbitrary chosen in } X'. \\ \bullet \text{ Step 1 : } \lambda^n = (\lambda_N^n, \lambda_T^n) \text{ fixed, find } u^{n+1} \in V \text{ solution to} \\ \quad \mathcal{L}_\rho(u^{n+1}, \lambda^n) = \inf_{v \in V} \mathcal{L}_\rho(v, \lambda^n). \\ \bullet \text{ Step 2 : } \text{Update multipliers with} \\ \quad \tilde{\lambda}_N^{n+1} = \tilde{\lambda}_N^n + \frac{r}{\rho}(P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N^n - \rho u_N^{n+1}) - \tilde{\lambda}_N^n), \\ \quad \tilde{\lambda}_T^{n+1} = \tilde{\lambda}_T^n + \frac{r}{\rho}(P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T^n - \rho u_T^{n+1}) - \tilde{\lambda}_T^n). \\ \quad \text{Loop to step 1 until a "stop criterion" is reached.} \end{array} \right. \quad (26)$$

We have denoted  $r > 0$  the descent step in the update of  $\lambda$ . There are two parameters,  $\rho$  is the augmentation parameter of the augmented Lagrangian and  $r$  is the descent step of the Uzawa algorithm. When  $r = \rho$ , step 2 is the same as the one for classical Lagrangian (23). Indeed, step 2 in (26) can be written in the general case

$$\begin{aligned} \tilde{\lambda}_N^{n+1} &= (1 - \frac{r}{\rho})\tilde{\lambda}_N^n + \frac{r}{\rho}P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N^n - \rho u_N^{n+1}), \\ \tilde{\lambda}_T^{n+1} &= (1 - \frac{r}{\rho})\tilde{\lambda}_T^n + \frac{r}{\rho}P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T^n - \rho u_T^{n+1}), \end{aligned}$$

which can be viewed as a relaxation for  $r < \rho$  (and an over-relaxation for  $r > \rho$ ) of step 2 in (23). An important difference here is the nonlinearity of step 1 in (26). Of course, such a difference is less important in nonlinear elasticity.

**Remark 2** When the solution  $(u, \lambda)$  is such that  $\lambda$  belongs to  $L^2(\Gamma_C)$ , it is possible to use projection operators in  $L^2(\Gamma_C)$  instead of  $X$ . The norms are taken in  $L^2(\Gamma_C)$  instead of  $X$  and there is no need of  $i(\cdot)$  due to the classical identification between  $L^2(\Gamma_C)$  and its dual space.

The following statement is an adaptation of a result established by G. Stadler [29] [30] in the case  $r = \rho$  and  $\lambda \in L^2(\Gamma_C)$ :

**Theorem 2** Provided that hypotheses (9), (10), (11) are satisfied, the Uzawa algorithm for the augmented Lagrangian (26) converges for all  $\rho > 0$  and for  $0 < r \leq \rho$ .

**Proof.** Let  $(u, \lambda)$  be the unique solution to the Tresca problem. We use the following notations:

$$\begin{aligned} \tilde{u}_N^{n+1} &= P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N^n - \rho u_N^{n+1}), \quad \tilde{u}_T^{n+1} = P_{\tilde{\Lambda}_T(s)}(\tilde{\lambda}_T^n - \rho u_T^{n+1}), \\ \tilde{\theta}_N^{n+1} &= \tilde{u}_N^{n+1} - \tilde{\lambda}_N, \quad \tilde{\theta}_T^{n+1} = \tilde{u}_T^{n+1} - \tilde{\lambda}_T, \quad \tilde{\delta}_N^{n+1} = \tilde{\lambda}_N^{n+1} - \tilde{\lambda}_N, \quad \tilde{\delta}_T^{n+1} = \tilde{\lambda}_T^{n+1} - \tilde{\lambda}_T, \quad \delta_u^{n+1} = u^{n+1} - u. \end{aligned}$$

From (25) and (26), one deduces the following equalities:

$$\begin{aligned} a(u, \delta_u^{n+1}) - l(\delta_u^{n+1}) - (\tilde{\lambda}_N, (\delta_u^{n+1})_N)_{X_N} - (\tilde{\lambda}_T, (\delta_u^{n+1})_T)_{X_T} &= 0, \\ a(u^{n+1}, \delta_u^{n+1}) - l(\delta_u^{n+1}) - (\tilde{\mu}_N^{n+1}, (\delta_u^{n+1})_N)_{X_N} - (\tilde{\mu}_T^{n+1}, (\delta_u^{n+1})_T)_{X_T} &= 0, \end{aligned}$$

thus

$$a(\delta_u^{n+1}, \delta_u^{n+1}) = (\tilde{\theta}_N^{n+1}, (\delta_u^{n+1})_N)_{X_N} + (\tilde{\theta}_T^{n+1}, (\delta_u^{n+1})_T)_{X_T}. \quad (27)$$

Now, since

$$\tilde{\lambda}_N = P_{\tilde{\lambda}_N}(\tilde{\lambda}_N - \rho u_N), \quad \tilde{\lambda}_T = P_{\tilde{\lambda}_T(s)}(\tilde{\lambda}_T - \rho u_T),$$

and due to the monotonicity property of the normal cone, one has

$$(\tilde{\mu}_N^{n+1} - \tilde{\lambda}_N, [\tilde{\lambda}_N^n - \rho u_N^{n+1} - \tilde{\mu}_N^{n+1}]) - [\tilde{\lambda}_N^n - \rho u_N^n - \tilde{\lambda}_N^n] \geq 0. \quad (28)$$

But

$$(\tilde{\theta}_N^{n+1}, (\delta_u^{n+1})_N)_{X_N} = -\frac{1}{\rho} (\tilde{\theta}_N^{n+1}, (\tilde{\lambda}_N^n - \rho u_N^{n+1}) - (\tilde{\lambda}_N - \rho u_N))_{X_N} + \frac{1}{\rho} (\tilde{\theta}_N^{n+1}, \tilde{\delta}_N^n)_{X_N}.$$

Thus thank to (28) one has

$$\begin{aligned} (\tilde{\theta}_N^{n+1}, (\delta_u^{n+1})_N)_{X_N} &\leq -\frac{1}{\rho} \|\tilde{\theta}_N^{n+1}\|_{X_N}^2 + \frac{1}{\rho} (\tilde{\theta}_N^{n+1}, \tilde{\delta}_N^n)_{X_N} \\ &\leq \frac{1}{2\rho} (\|\tilde{\delta}_N^n\|_{X_N}^2 - \|\tilde{\theta}_N^{n+1}\|_{X_N}^2). \end{aligned}$$

For the friction part, the same calculus gives

$$(\tilde{\theta}_T^{n+1}, (\delta_u^{n+1})_T)_{X_T} \leq \frac{1}{2\rho} (\|\tilde{\delta}_T^n\|_{X_T}^2 - \|\tilde{\theta}_T^{n+1}\|_{X_T}^2).$$

Finally, together with (27), the two last inequalities yield to

$$a(\delta_u^{n+1}, \delta_u^{n+1}) \leq \frac{1}{2\rho} (\|\tilde{\delta}_N^n\|_{X_N}^2 + \|\tilde{\delta}_T^n\|_{X_T}^2 - \|\tilde{\theta}_N^{n+1}\|_{X_N}^2 - \|\tilde{\theta}_T^{n+1}\|_{X_T}^2). \quad (29)$$

This implies

$$\|\tilde{\theta}_N^{n+1}\|_{X_N}^2 + \|\tilde{\theta}_T^{n+1}\|_{X_T}^2 \leq \|\tilde{\delta}_N^n\|_{X_N}^2 + \|\tilde{\delta}_T^n\|_{X_T}^2.$$

Using the fact that  $\tilde{\lambda}^{n+1} = (1 - \frac{r}{\rho})\tilde{\lambda}^n + \frac{r}{\rho}\tilde{\mu}^{n+1}$  and consequently that

$$\tilde{\delta}^{n+1} = (1 - \frac{r}{\rho})\tilde{\delta}^n + \frac{r}{\rho}\tilde{\theta}^{n+1}, \quad (30)$$

one has

$$\begin{aligned} (\|\tilde{\delta}_N^{n+1}\|_{X_N}^2 + \|\tilde{\delta}_T^{n+1}\|_{X_T}^2)^{1/2} &\leq (1 - \frac{r}{\rho})(\|\tilde{\delta}_N^n\|_{X_N}^2 + \|\tilde{\delta}_T^n\|_{X_T}^2)^{1/2} \\ &\quad + \frac{r}{\rho} (\|\tilde{\theta}_N^{n+1}\|_{X_N}^2 + \|\tilde{\theta}_T^{n+1}\|_{X_T}^2)^{1/2} \\ &\leq (\|\tilde{\delta}_N^n\|_{X_N}^2 + \|\tilde{\delta}_T^n\|_{X_T}^2)^{1/2}. \end{aligned}$$

This is sufficient to conclude that  $\|\tilde{\delta}_N^n\|_{X_N}^2 + \|\tilde{\delta}_T^n\|_{X_T}^2$  converges, thus thanks to (30) the quantity  $\|\tilde{\theta}_N^n\|_{X_N}^2 + \|\tilde{\theta}_T^n\|_{X_T}^2$  converges also towards the same limit and finally (29) implies  $\lim_{n \rightarrow \infty} a(\delta_u^{n+1}, \delta_u^{n+1}) = 0$ . ■

### 3 Generalization to the Coulomb problem

#### 3.1 Definition of the fixed point operator

The Coulomb problem (see section 1.2) is not a variational problem and cannot be expressed in terms of a saddle point problem. However, the optimality system for the Tresca problem (22) is close to the following hybrid formulation of the Coulomb problem:

$$\begin{cases} \text{Find } u \in V, \lambda_N \in X'_N \text{ and } \lambda_T \in X'_T \text{ satisfying} \\ \mathbb{E}(\lambda_N, \lambda_T) = (u_N, u_T), \\ u_N + N_{\Lambda_N}(\lambda_N) \ni 0 \text{ in } X_N, \\ u_T + N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T) \ni 0 \text{ in } X_T. \end{cases} \quad (31)$$

This formulation is equivalent to Problem (12) (see [22]). (The terminology hybrid comes from the fact that the contact force is considered as a multiplier in this formulation). The fixed point operator  $T^1$  can be adapted to the Coulomb problem as follows:

$$\begin{aligned} T^1 : \quad X &\longrightarrow X \\ (\tilde{\lambda}_N, \tilde{\lambda}_T) &\longmapsto \left( P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N - ru_N), P_{\tilde{\Lambda}_T(\mathcal{F}\lambda_N)}(\tilde{\lambda}_T - ru_T) \right), \\ &\quad \text{where } (u_N, u_T) = \tilde{\mathbb{E}}(\tilde{\lambda}_N, \tilde{\lambda}_T). \end{aligned}$$

#### 3.2 Moreau-Yosida transformations and equivalence with the hybrid formulation

In order to verify that the fixed point problem associated to  $T^1$  is equivalent to the Coulomb problem (31), let us consider the general inclusion

$$a \in F(b), \quad (32)$$

where  $F : H \longrightarrow \mathcal{P}(H)$  is a maximal monotone multivalued map and  $H$  an Hilbert space. This equation is equivalent to

$$b = (I + rF)^{-1}(b + ra),$$

where  $r > 0$  and  $I$  is the identity operator in  $H$ . The term  $(I + rF)^{-1}$  is known as the (Moreau-Yosida) resolvent  $J_r^F$  of  $F$ . Since  $F$  is a maximal monotone map,  $J_r^F$  is a single-valued map and a contraction (see [9] for instance). Inclusion (32) is then equivalent to

$$b = J_r^F(b + ra). \quad (33)$$

This approach is quite similar to the one which gives the proximal algorithm (see [27]).

Since the resolvent of a normal cone to a convex set in a Hilbert space is the projection operator onto this convex set, the equivalence between (32) and (33) implies

$$\begin{aligned} u_N + N_{\Lambda_N}(\lambda_N) \ni 0 &\iff \tilde{\lambda}_N = P_{\tilde{\Lambda}_N}(\tilde{\lambda}_N - ru_N), \\ u_T + N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T) \ni 0 &\iff \tilde{\lambda}_T = P_{\tilde{\Lambda}_T(\mathcal{F}\lambda_N)}(\tilde{\lambda}_T - ru_T). \end{aligned}$$

Hence, the fixed point problem associated to  $T^1$  is equivalent to the hybrid formulation (31). However, the convergence of the fixed point iterations of  $T^1$  is an open problem (and would provide an existence result for the Coulomb problem). In the next session, the finite dimensional framework is investigated.

## 4 The finite element Coulomb problem

For finite dimensional problems, the results in section 2 on the Tresca problem are still valid, and estimates of convergence rate are independent of the discretization. But it is necessary to use projection operators with respect to the  $H^{1/2}$ -inner product, which could be expansive from a numerical viewpoint. Let us now consider the Coulomb friction model.

### 4.1 Finite element framework

In this section, a discretization of the fixed points is made using arbitrary finite element method. Estimates of the contraction constant of the fixed point operators are given, which depend on the constant of equivalence between  $H^{1/2}$  norm and  $L^2$  norm on  $\Gamma_c$ . This generalizes some results given in [15].

Classically, let  $V^h \subset V$  be a family of finite dimensional sub-vector spaces defined from a regular finite element discretization of the domain  $\Omega$ , supposed now to be polygonal ( $h$  represents the radius of the largest element). Let us define

$$\begin{aligned} X_N^h &= \{v_N^h|_{\Gamma_c} : v^h \in V^h\}, \\ X_T^h &= \{v_T^h|_{\Gamma_c} : v^h \in V^h\}, \\ X^h &= \{v^h|_{\Gamma_c} : v^h \in V^h\} = X_N^h \times X_T^h. \end{aligned}$$

Now, in order to approximate the dual space  $X'$ , we make the choice  $X'^h = X^h$  (through the identification between  $L^2(\Gamma_c)$  and its dual space). We refer also to [15, 4, 5, 6, 16, 19, 22] for the discretization of Signorini problems. Let  $\mathbb{E}^h$  be the finite element approximation of  $\mathbb{E}$ :

$$\begin{aligned} \mathbb{E}^h : X^h &\longrightarrow X^h, \\ \lambda^h &\longmapsto u^h|_{\Gamma_c}, \end{aligned}$$

where  $u^h \in V^h$  is solution to the problem

$$a(u^h, v^h) = l(v^h) + \int_{\Gamma_c} \lambda^h \cdot v^h d\Gamma, \quad \forall v^h \in V^h. \quad (34)$$

We assume that the finite element discretization satisfies the following assumptions:

- there exists  $C > 0$  independent of  $h$  such that  $\|P_{X^h}(v)\|_X \leq C\|v\|_X \quad \forall v \in X$ ; (35)

- there exists a linear lifting operator  $L_h : X^h \longrightarrow V^h$ ,  
and  $C > 0$  independent of  $h$  with  $\|L_h(v)\|_V \leq C\|v\|_X, \quad \forall v \in X^h$ , (36)

where  $P_{X^h}$  represents the  $L^2$  projection operator on  $X^h$ . These conditions are obtained for classical finite element methods under condition on the regularity of the mesh (see [4], [7] for instance). Moreover, for such methods, the so-called inverse inequality holds with  $C > 0$  a constant independent of  $h$  (see [10] for instance):

$$\|v\|_X \leq Ch^{-1/2}\|v\|_{L^2(\Gamma_c)}, \quad \forall v \in X^h.$$

Classically, this allows to settle that there exists  $C_3 > 0$ , independent of  $h$ , such that

$$\|\delta\lambda^h\|_{L^2(\Gamma_c)} \leq MC_3h^{-1/2}\|\delta u^h\|_V. \quad (37)$$

For discrete problems, this estimate will play the role of (18) used for continuous problems.

## 4.2 Discrete Coulomb problems

The fixed point operator  $T^1$  can be adapted for the finite dimension, with  $P$  designing now projection operators with respect to the  $L^2(\Gamma_c)$  inner product as follows:

$$\begin{aligned} T^{1h} : X^h &\longrightarrow X^h \\ (\lambda_N^h, \lambda_T^h) &\longmapsto \left( P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h)), P_{X_T^h}(P_{\Lambda_T(\mathcal{F}(\lambda_N^h)_-)}(\lambda_T^h - ru_T^h)) \right), \\ \text{where } (u_N^h, u_T^h) &= \mathbb{E}^h(\lambda_N^h, \lambda_T^h), \end{aligned}$$

and  $(x)_- = \min\{x; 0\}$ . Compared to the continuous case, two projection operators  $P_{X_N^h}$  and  $P_{X_T^h}$  are introduced in order to have the range of operator  $T^{1h}$  in  $X^h$  (note, that if the projection operators are not added,  $T^{1h}$  is an operator with values in  $X$  and the convergence on the displacement will not be modified, but the convergence on the contact stress should be perturbed). The fixed point of operator  $T^{1h}$  defines a discrete Coulomb problem which depends on the parameter  $r$  and can be expressed

$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h, \lambda_N^h \in X_N^h \text{ and } \lambda_T^h \in X_T^h \text{ satisfying} \\ \mathbb{E}^h(\lambda_N^h, \lambda_T^h) = (u_N^h, u_T^h), \\ \lambda_N^h = P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h)), \\ \lambda_T^h = P_{X_T^h}(P_{\Lambda_T(\mathcal{F}(\lambda_N^h)_-)}(\lambda_T^h - ru_T^h)). \end{array} \right. \quad (38)$$

This fixed point formulation give implicitly an algorithm to solve numerically the corresponding discrete problems.

**Theorem 3** *Let  $h > 0$  be given, under hypotheses (9), (10), (11), (35) and (36) and for  $\|\mathcal{F}\|_{L^\infty}$  sufficiently small, there exists  $r > 0$  such that the operator  $T^{1h}$  is a strict contraction.*

Let us first give the following lemma which allows to obtain more optimal estimates.

**Lemma 2** *Under the conditions of Theorem 3, for  $\lambda_N^1, \lambda_N^2 \in X_N^h$  and  $\lambda_T^1, \lambda_T^2 \in X_T^h$  one has*

$$\|P_{X_T^h}(P_{\Lambda_T(\mathcal{F}(\lambda_N^1)_-)}(\lambda_T^1)) - P_{X_T^h}(P_{\Lambda_T(\mathcal{F}(\lambda_N^2)_-)}(\lambda_T^2))\|_{L^2(\Gamma_C)}^2 \leq \|\lambda_T^1 - \lambda_T^2\|_{L^2(\Gamma_C)}^2 + \|\mathcal{F}\|_{L^\infty}^2 \|\lambda_N^1 - \lambda_N^2\|_{L^2(\Gamma_C)}^2.$$

**Proof of the lemma.** As the projection is in  $L^2(\Gamma_c)$ , one has

$$P_{\Lambda_T(\mathcal{F}(\lambda_N)_-)}(\lambda_T)(x) = \lambda_T(x) \min \left( 1, -\frac{\mathcal{F}(x)(\lambda_N(x))_-}{|\lambda_T(x)|} \right), \quad \text{a.e. on } \Gamma_c,$$

where the minimum is assumed to be 0 when  $\lambda_T(x) = 0$ . In particular, this means that the estimate can be obtained comparing the pointwise projection onto discs of different sizes. A simple enumeration of the different possible situations allows to conclude. ■

### Proof of the theorem.

Using Lemma 2, one can state the following estimate

$$\begin{aligned} \|T^{1h}(\lambda^1) - T^{1h}(\lambda^2)\|_{L^2(\Gamma_C)}^2 &\leq \|\delta\lambda_N - r\delta u_N\|_{L^2(\Gamma_C)}^2 + \|\delta\lambda_T - r\delta u_T\|_{L^2(\Gamma_C)}^2 + \|\mathcal{F}\|_{L^\infty}^2 \|\delta\lambda\|_{L^2(\Gamma_C)}^2 \\ &\leq \|\delta\lambda - r\delta u\|_{L^2(\Gamma_C)}^2 + \|\mathcal{F}\|_{L^\infty}^2 \|\delta\lambda\|_{L^2(\Gamma_C)}^2. \end{aligned}$$

Using the same method as in the proof of Theorem 1 one obtains

$$\|\delta\lambda - r\delta u\|_{L^2(\Gamma_C)}^2 \leq \|\delta\lambda\|_{L^2(\Gamma_C)}^2 (1 - 2r\alpha\beta^2 + r^2 C_1^2 \beta^2), \quad (39)$$

with  $\beta = \frac{\|\delta u\|_v}{\|\delta\lambda\|_{L^2(\Gamma_C)}} \geq \frac{1}{MC_3 h^{-1/2}}$  from (37). The minimum of the contraction constant is  $\left(1 - \frac{\alpha^2}{M^2 C_1^2 C_3^2 h^{-1}} + \|\mathcal{F}\|_{L^\infty}^2\right)^{1/2}$ .

It is less than one when

$$\|\mathcal{F}\|_{L^\infty} < \frac{\alpha\sqrt{h}}{MC_1 C_3}.$$

■

### 4.3 Existence result for an arbitrary $\mathcal{F}$

An existence result can be obtained for an arbitrary  $\|\mathcal{F}\|_{L^\infty}$  in the finite dimensional framework.

**Theorem 4** *Under hypotheses (9), (10), (11), (35) and (36), in particular for  $\mathcal{F}$  Lipschitz continuous on  $\Gamma_C$ , the mapping  $T^{1h}$  has at least one fixed point for  $r > 0$  sufficiently small.*

**Proof.** First, let us establish that for a sufficiently small  $r > 0$  and a sufficiently large  $\lambda^h$ , one has  $\|T^{1h}(\lambda^h)\|_{L^2(\Gamma_C)} < \|\lambda^h\|_{L^2(\Gamma_C)}$ . The following estimate can be performed using the fact that projection operators are contractions and that the concerned convex sets contain the origin:

$$\begin{aligned} \|T^{1h}(\lambda^h)\|_{L^2(\Gamma_C)}^2 &= \|P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h))\|_{L^2(\Gamma_C)}^2 + \|P_{X_T^h}(P_{\Lambda_T}(\mathcal{F}(\lambda_N^h)_-)(\lambda_T^h - ru_T^h))\|_{L^2(\Gamma_C)}^2 \\ &\leq \|\lambda^h - ru^h\|_{L^2(\Gamma_C)}^2 \\ &\leq \|\lambda^h\|_{L^2(\Gamma_C)}^2 - 2r \int_{\Gamma_C} \lambda^h \cdot u^h d\Gamma + r^2 \|u^h\|_{L^2(\Gamma_C)}^2. \end{aligned}$$

But,

$$\int_{\Gamma_C} \lambda^h \cdot u^h d\Gamma = a(u^h, u^h) - l(u^h) \geq \alpha \|u^h\|_v^2 - L \|u^h\|_v,$$

and also

$$\|u^h\|_{L^2(\Gamma_C)} \leq \frac{C_1}{\alpha} (L + C_1 \|\lambda^h\|_{L^2(\Gamma_C)}), \quad (40)$$

where  $L$  is the norm of the linear mapping  $l(\cdot)$ . Now, using (35) and (36), one obtains

$$\|\lambda^h\|_{L^2(\Gamma_C)} \leq C_3 h^{-1/2} (M \|u^h\|_v + L),$$

and

$$\|u^h\|_v \geq \frac{1}{C_3 h^{-1/2} M} \|\lambda^h\|_{L^2(\Gamma_C)} - \frac{L}{M}.$$

Finally, one has

$$\begin{aligned} \|T^{1h}(\lambda^h)\|_{L^2(\Gamma_C)}^2 &\leq \|\lambda^h\|_{L^2(\Gamma_C)}^2 - 2r\alpha \left( \frac{\sqrt{h}}{C_3 M} \|\lambda^h\|_{L^2(\Gamma_C)} - \frac{L}{M} \right)^2 \\ &\quad + 2r \frac{L}{\alpha} \left( L + C_1 \|\lambda^h\|_{L^2(\Gamma_C)} \right) + r^2 \frac{C_1^2}{\alpha^2} \left( L + C_1 \|\lambda^h\|_{L^2(\Gamma_C)} \right)^2. \end{aligned}$$

Thus, there exists  $C^h > 0$  such that, for  $\|\lambda^h\|_{L^2(\Gamma_C)} > C^h$ , the term in factor of  $2r\alpha$  is always strictly negative and there will be a  $r_0$  such that

$$\|T^{1h}(\lambda^h)\|_{L^2(\Gamma_C)} < \|\lambda^h\|_{L^2(\Gamma_C)},$$

for  $\|\lambda^h\|_{L^2(\Gamma_C)} > C^h$  and  $0 < r < r_0$ .

Now, by definition of  $T^{1h}$  and using (40), there exists  $\bar{C}^h > 0$  and  $\bar{L}^h > 0$  such that

$$\|T^{1h}(\lambda^h)\|_{L^2(\Gamma_C)} \leq \|\lambda^h\|_{L^2(\Gamma_C)} + r\|u^h\|_{L^2(\Gamma_C)} \leq \bar{C}^h \|\lambda^h\|_{L^2(\Gamma_C)} + \bar{L}^h,$$

and thus

$$\|T^{1h}(\lambda^h)\|_{L^2(\Gamma_C)} \leq C^h \bar{C}^h + \bar{L}^h, \quad \text{when } \|\lambda^h\|_{L^2(\Gamma_C)} \leq C^h \bar{C}^h + \bar{L}^h.$$

This means that  $T^{1h}$  is a continuous map from the ball of radius  $C^h \bar{C}^h + \bar{L}^h$  of  $X^h$  into itself. Then one can conclude using Brouwer's fixed point theorem.  $\blacksquare$

Of course, each fixed point satisfies  $\|\lambda^h\|_{L^2(\Gamma_C)} \leq C^h$ , but this estimate does not use dissipativity properties of contact and friction conditions. It is possible to obtain an estimate which is independent of the discretization. This is the aim of the following proposition.

**Proposition 1** *Under hypotheses (9), (10), (11), (35) and (36) each solution to Problem (38) satisfies*

$$\|u^h\|_V \leq \frac{L}{\alpha}.$$

**Proof.** Let  $\bar{\lambda}_N^h$  and  $\bar{\lambda}_T^h$  be defined as

$$\bar{\lambda}_N^h = P_{\Lambda_N}(\lambda_N^h - ru_N^h), \quad \bar{\lambda}_T^h = P_{\Lambda_T(\mathcal{F}\lambda_N^h)}(\lambda_T^h - ru_T^h),$$

which is equivalent to

$$\lambda_N^h - ru_N^h - \bar{\lambda}_N^h \in N_{\Lambda_N}(\bar{\lambda}_N^h), \quad \lambda_T^h - ru_T^h - \bar{\lambda}_T^h \in N_{\Lambda_T(\mathcal{F}(\lambda_N^h)_-)}(\bar{\lambda}_T^h).$$

Thus, due to the definition of normal cones

$$\int_{\Gamma_C} (\lambda_N^h - ru_N^h - \bar{\lambda}_N^h)(\mu_N - \bar{\lambda}_N^h) d\Gamma \leq 0, \quad \forall \mu_N \in \Lambda_N,$$

$$\int_{\Gamma_C} (\lambda_T^h - ru_T^h - \bar{\lambda}_T^h) \cdot (\mu_T - \bar{\lambda}_T^h) d\Gamma \leq 0, \quad \forall \mu_T \in \Lambda_T(\mathcal{F}(\lambda_N^h)_-).$$

But one has  $\lambda_N^h = P_{X_N^h}(\bar{\lambda}_N^h)$ ,  $\lambda_T^h = P_{X_T^h}(\bar{\lambda}_T^h)$ , thus  $\|\lambda_N^h\|_{L^2(\Gamma_C)} \leq \|\bar{\lambda}_N^h\|_{L^2(\Gamma_C)}$  and  $\|\lambda_T^h\|_{L^2(\Gamma_C)} \leq \|\bar{\lambda}_T^h\|_{L^2(\Gamma_C)}$  due to the contraction property of projection operators. It follows  $\int_{\Gamma_C} (\lambda_N^h - \bar{\lambda}_N^h) \bar{\lambda}_N^h d\Gamma \leq 0$  and  $\int_{\Gamma_C} (\lambda_T^h - \bar{\lambda}_T^h) \bar{\lambda}_T^h d\Gamma \leq 0$ . Taking now  $\mu_N = 0$  and  $\mu_T = 0$ , one obtains

$$\int_{\Gamma_C} u_N^h \bar{\lambda}_N^h d\Gamma \leq 0, \quad \int_{\Gamma_C} u_T^h \bar{\lambda}_T^h d\Gamma \leq 0,$$

and, because  $u_N^h \in X_N^h$ ,  $u_T^h \in X_T^h$ , one has  $\int_{\Gamma_C} u_N^h \bar{\lambda}_N^h d\Gamma = \int_{\Gamma_C} u_N^h \lambda_N^h d\Gamma$  and  $\int_{\Gamma_C} u_T^h \bar{\lambda}_T^h d\Gamma = \int_{\Gamma_C} u_T^h \lambda_T^h d\Gamma$ . Thus

$$\int_{\Gamma_C} u_N^h \lambda_N^h d\Gamma \leq 0, \quad \int_{\Gamma_C} u_T^h \lambda_T^h d\Gamma \leq 0. \quad (41)$$

This result allows to conclude, because one has

$$\alpha \|u^h\|_v^2 \leq a(u^h, u^h) = l(u^h) + \int_{\Gamma_C} u_N^h \lambda_N^h d\Gamma + \int_{\Gamma_C} u_T^h \lambda_T^h d\Gamma \leq L \|u^h\|_v. \quad \blacksquare$$

**Remark 3** Relations (41) mean that the numerical scheme respects the dissipativity of contact and friction condition.

## 5 A convergence result for the Tresca problem

In order to justify the discretization of the Coulomb problem presented in the previous section, we prove here a convergence result for the Tresca problem.

The analogous of (38) for the discrete Tresca problem is

$$\begin{cases} \text{Find } u^h \in V^h, \lambda_N^h \in X_N^h \text{ and } \lambda_T^h \in X_T^h \text{ satisfying} \\ \mathbb{E}^h(\lambda_N^h, \lambda_T^h) = (u_N^h, u_T^h), \\ \lambda_N^h = P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h)), \\ \lambda_T^h = P_{X_T^h}(P_{\Lambda_T(s)}(\lambda_T^h - ru_T^h)), \end{cases} \quad (42)$$

where the prescribed friction threshold defined on  $\Gamma_C$  is  $s$  (see section 2) with

$$s \in L^2(\Gamma_C), \quad s \leq 0.$$

Let now  $(u, \lambda)$  be the solution to Problem (19),  $(u^h, \lambda^h)$  be the solution to Problem (42) and  $u_0^h$  be the solution to the problem

$$u_0^h \in V^h, \quad a(u_0^h, v^h) = l(v^h) + \int_{\Gamma_C} \lambda \cdot v^h d\Gamma, \quad \forall v^h \in V^h.$$

We assume that there exists  $v > 0$  and  $C > 0$  independent of  $h$  such that

$$\|u - u_0^h\|_v \leq Ch^v \|u\|_{H^{1+v}(\Omega)} \quad (43)$$

$$\|\lambda - P_{X_N^h}(\lambda)\|_{L^2(\Gamma_C)} \leq Ch^v \|\lambda\|_{H^v(\Gamma_C)} \quad (44)$$

$$\inf_{\mu^h \in X^h} \|\mu^h - v\|_{X'} \leq Ch^v \|v\|_{H^{v-1/2}(\Gamma_C)}, \quad \forall v \in H^{v-1/2}(\Gamma_C). \quad (45)$$

Again, these estimates are obtained for classical finite element methods under condition on the regularity of the mesh and  $v$  generally depends on the degree of the finite element method.

Classically, along with the fact that an inf-sup condition is satisfied for our discretization (since  $X^h = X'$ , see [4]), this allows to conclude that (see [23] or [4] for instance)

$$\|\lambda - \lambda^h\|_{X'} \leq C \left( \|u - u^h\|_v + h^v \|\lambda\|_{H^{v-1/2}(\Gamma_C)} \right). \quad (46)$$

**Theorem 5** Under hypotheses (9), (10), (11), (35), (36), (43), (44) and (45), let  $(u, \lambda)$  be the solution to the continuous Tresca problem (19) and  $(u^h, \lambda^h)$  be the solution to the discrete Tresca problem (42). Then, if  $u \in H^{1+v}(\Omega)$ ,  $\lambda \in H^v(\Gamma_C)$  and for  $r > 0$  sufficiently small, there exists a constant  $C > 0$  independent of  $h$  such that

$$\|u - u^h\|_v \leq Ch^v \left( \|u\|_{H^{1+v}(\Omega)} + \|\lambda\|_{H^v(\Gamma_C)} \right).$$

**Proof.** One as

$$\begin{aligned} \|\lambda - \lambda^h\|_{L^2(\Gamma_C)}^2 &= \|P_{\Lambda_N}(\lambda_N - ru_N) - P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h))\|_{L^2(\Gamma_C)}^2 \\ &\quad + \|P_{\Lambda_T(s)}(\lambda_T - ru_T) - P_{X_T^h}(P_{\Lambda_T(s)}(\lambda_T^h - ru_T^h))\|_{L^2(\Gamma_C)}^2. \end{aligned}$$

Now, thanks to the properties of projection operators, it follows successively:

$$\begin{aligned} \|\lambda - \lambda^h\|_{L^2(\Gamma_C)}^2 &= \|P_{\Lambda_N}(\lambda_N - ru_N)\|_{L^2(\Gamma_C)}^2 - \|P_{X_N^h}(P_{\Lambda_N}(\lambda_N - ru_N))\|_{L^2(\Gamma_C)}^2 \\ &\quad + \|P_{X_N^h}(P_{\Lambda_N}(\lambda_N - ru_N)) - P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h))\|_{L^2(\Gamma_C)}^2 \\ &\quad + \|P_{\Lambda_T(s)}(\lambda_T - ru_T)\|_{L^2(\Gamma_C)}^2 - \|P_{X_T^h}(P_{\Lambda_T(s)}(\lambda_T - ru_T))\|_{L^2(\Gamma_C)}^2 \\ &\quad + \|P_{X_T^h}(P_{\Lambda_T(s)}(\lambda_T - ru_T)) - P_{X_T^h}(P_{\Lambda_T(s)}(\lambda_T^h - ru_T^h))\|_{L^2(\Gamma_C)}^2 \\ &\leq \|\lambda_N - P_{X_N^h}(\lambda_N)\|_{L^2(\Gamma_C)}^2 + \|\lambda_N - ru_N - \lambda_N^h + ru_N^h\|_{L^2(\Gamma_C)}^2 \\ &\quad + \|\lambda_T - P_{X_T^h}(\lambda_T)\|_{L^2(\Gamma_C)}^2 + \|\lambda_T - ru_T - \lambda_T^h + ru_T^h\|_{L^2(\Gamma_C)}^2. \end{aligned}$$

Then

$$0 \leq \|\lambda - P_{X^h}(\lambda)\|_{L^2(\Gamma_C)}^2 - 2r \int_{\Gamma_C} (\lambda - \lambda^h) \cdot (u - u^h) d\Gamma + r^2 \|u - u^h\|_{L^2(\Gamma_C)}^2.$$

Now, inserting  $u_0^h$ , one has

$$\begin{aligned} 0 &\leq \|\lambda - P_{X^h}(\lambda)\|_{L^2(\Gamma_C)}^2 - 2r \int_{\Gamma_C} (\lambda - \lambda^h) \cdot (u - u_0^h) d\Gamma - 2r \int_{\Gamma_C} (\lambda - \lambda^h) \cdot (u_0^h - u^h) d\Gamma \\ &\quad + 2C_1^2 r^2 \|u - u_0^h\|_v^2 + 2C_1^2 r^2 \|u_0^h - u^h\|_v^2. \end{aligned}$$

And thus

$$(2r\alpha - 2r^2 C_1^2) \|u_0^h - u^h\|_v^2 \leq \|\lambda - P_{X^h}(\lambda)\|_{L^2(\Gamma_C)}^2 + 2rC_1 \|\lambda - \lambda^h\|_{X'} \|u - u_0^h\|_v + 2C_1^2 r^2 \|u - u_0^h\|_v^2.$$

This allows to conclude, for  $r$  small enough, using (43), (44) and (46).  $\blacksquare$

**Remark 4** This result is not optimal, since it is assumed for  $\lambda$  to be in  $H^v(\Gamma_C)$ . The interest of this estimate is to be independent of the finite element method. Quasi optimal results can be found in [4] for the Signorini problem using linear elements and in [19] using quadratic elements.

## 6 A fixed point on the contact boundary displacement

In the continuation of section 3.2, for an inclusion of the form  $a \in F(b)$  with  $F : H \longrightarrow \mathcal{P}(H)$  a maximal monotone multivalued map, one defines the Moreau-Yosida approximation of  $F$  as

$$F_r = \frac{1}{r}(I - J_r^F) = (F^{-1} + rI)^{-1}. \quad (47)$$

The previous inclusion is also equivalent to

$$a = F_r(b + ra). \quad (48)$$

Since  $F$  is maximal monotone, the Yosida approximation  $F_r$  is single-valued and  $\frac{1}{r}$ -Lipschitz continuous. From (47), one can note that the Moreau-Yosida approximation of  $F$  and the resolvent of  $F^{-1}$  are linked by

$$F_r(x) = J_{1/r}^{F^{-1}}(x/r). \quad (49)$$

The computation of the Moreau-Yosida regularization of normal cones  $N_{\Lambda_N}(\lambda_N)$  and  $N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T)$  leads to the following equivalence for  $r > 0$ :

$$\begin{aligned} u_N + N_{\Lambda_N}(\lambda_N) &\ni 0 \iff u_N = P_{K_N}(u_N - r\tilde{\lambda}_N), \\ u_T + N_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T) &\ni 0 \iff u_T = u_T - r\tilde{\lambda}_T - rP_{\tilde{\Lambda}_T(\lambda_N)}\left(\frac{1}{r}u_T - \tilde{\lambda}_T\right). \end{aligned}$$

We can deduce the following fixed point operator:

$$\begin{aligned} T^2 : X &\longrightarrow X \\ (u_N, u_T) &\longmapsto \left( P_{K_N}(u_N - r\tilde{\lambda}_N), u_T - r\tilde{\lambda}_T - rP_{\tilde{\Lambda}_T(\lambda_N)}\left(\frac{1}{r}u_T - \tilde{\lambda}_T\right) \right), \\ \text{where } (\lambda_N, \lambda_T) &= \mathbb{E}^{-1}(u_N, u_T). \end{aligned}$$

The associated fixed point problem is equivalent to the Coulomb problem (31). An adaptation to the finite element discretization of the Coulomb problem can be written as follows:

$$\begin{aligned} T^{2h} : X^h &\longrightarrow X^h \\ (u_N^h, u_T^h) &\longmapsto \left( P_{X_N^h}(P_{K_N}(u_N^h - r\lambda_N^h)), u_T^h - r\lambda_T^h - rP_{X_T^h}(P_{\Lambda_T(\mathcal{F}(\lambda_N^h))}(u_T^h - \lambda_T^h)) \right), \\ \text{where } (\lambda_N^h, \lambda_T^h) &= (\mathbb{E}^h)^{-1}(u_N^h, u_T^h). \end{aligned}$$

The following result holds:

**Theorem 6** *Let  $h > 0$  be given, under hypotheses (9), (10), (11), (35) and (36) and for  $\|\mathcal{F}\|_{L^\infty}$  sufficiently small, there exists  $r > 0$  such that the operator  $T^{2h}$  is a strict contraction.*

The proof of this theorem is analogous to the one of Theorem 3. In particular  $T^{2h}$  is a strict contraction for  $\|\mathcal{F}\|_{L^\infty} < \frac{\alpha\sqrt{h}}{MC_1C_3}$ .

## 7 A new weak inclusion formulation using De Saxcé's bipotential theory

One of the difficulties about (31) is that the two inclusions are linked by the fact that the set  $\Lambda_T$  of admissible tangential stresses depends on  $\lambda_N$ . In a discrete framework, De Saxcé [11] (see also [8]) gives a new formulation of the contact and friction conditions allowing to write them using a unique inclusion.

## 7.1 The bipotential of the Coulomb friction law

Let  $H$  be an Hilbert space. Following De Saxcé, a *bipotential* is a map  $b : H' \times H \longrightarrow \overline{\mathbb{R}}$  which is convex, lower semi-continuous with respect to each of its variables satisfying additionally the generalized Fenchel inequality

$$b(\xi, y) \geq \langle \xi, y \rangle_{H', H}, \quad \forall \xi \in H', \forall y \in H. \quad (50)$$

We prefer here to give a slightly more restrictive definition, and we will prescribe for the bipotential to satisfy the two following relations (better corresponding to an internal conjugacy property):

$$\inf_{y \in H} (b(\zeta, y) - \langle \zeta, y \rangle_{H', H}) \in \{0, +\infty\}, \quad \forall \zeta \in H', \quad (51)$$

$$\inf_{\xi \in H'} (b(\xi, x) - \langle \xi, x \rangle_{H', H}) \in \{0, +\infty\}, \quad \forall x \in H. \quad (52)$$

Of course, (51) or (52) implies (50). The value  $+\infty$  cannot be avoided, since the bipotential may contain some indicator functions. We will see in the following that these conditions are naturally satisfied by the bipotential representing the Coulomb friction law.

Now, a pair  $(\zeta, x)$  is said to be *extremal* if it satisfies the following relation

$$b(\zeta, x) = \langle \zeta, x \rangle_{H', H}. \quad (53)$$

Subtracting (53) from (50), this means that

$$b(\zeta, y) - b(\zeta, x) \geq \langle \zeta, y - x \rangle_{H', H} \quad \forall y \in H,$$

which is equivalent to

$$\zeta \in \partial_x b(\zeta, x). \quad (54)$$

A similar reasoning leads to

$$x \in \partial_\zeta b(\zeta, x). \quad (55)$$

Moreover, due to (51), inclusion (54) is clearly equivalent to (53) and due to (52) inclusion (55) is also equivalent to (53). Thus (54) and (55) are equivalent one to each other. Inequality (50) is not sufficient to conclude to this equivalence, this is the reason why we introduced (51) and (52).

De Saxcé defined the so-called bipotential of the Coulomb friction law, which can be written in a continuous version as follows:

$$b(-\lambda, u) = \langle -\lambda_N, \mathcal{F}|u_T| \rangle_{X'_N, X_N} + I_{\Lambda_{\mathcal{F}}}(\lambda) + I_{K_N}(u_N), \quad (56)$$

where  $\Lambda_{\mathcal{F}}$  is the weak friction cone given by

$$\Lambda_{\mathcal{F}} = \{F = (\lambda_N, \lambda_T) \in X' : -\langle \lambda_T, w_T \rangle_{X'_T, X_T} + \langle \mathcal{F}\lambda_N, |w_T| \rangle_{X'_N, X_N} \leq 0, \forall w_T \in X_T\},$$

(the minus before  $\lambda$  comes from the convention taken for the multiplier  $\lambda$ , see Remark 4. The inclusion  $-\lambda \in \partial_u b(-\lambda, u)$  gives exactly Problem (12) (see [22]). Thus, if  $b(-\lambda, u)$  is a bipotential, then Problem (12) is equivalent to  $u \in \partial_\lambda b(-\lambda, u)$  which gives

$$-(u_N - \mathcal{F}|u_T|, u_T) \in N_{\Lambda_{\mathcal{F}}}(\lambda_N, \lambda_T). \quad (57)$$

**Lemma 3**  $b(., .)$  defined by (56) is a bipotential.

**Proof.**  $b(.,.)$  is clearly convex and lower semi-continuous of each of its variables. Now, if  $u_N \notin K_N$  or  $\lambda \notin \Lambda_{\mathcal{F}}$  relation (50) is satisfied since  $b(-\lambda, u) = +\infty$ . If  $u_N \in K_N$  and  $\lambda \in \Lambda_{\mathcal{F}}$  then

$$\begin{aligned} b(-\lambda, u) &= \langle -\lambda_N, \mathcal{F}|u_T| \rangle_{X'_N, X_N} \\ &\geq \langle -\lambda_T, u_T \rangle_{X'_T, X_T} \\ &\geq \langle -\lambda, u \rangle_{X', X} \end{aligned}$$

and relation (50) is also satisfied. Now, relation (51) is satisfied for  $\lambda \notin \Lambda_{\mathcal{F}}$ , and for  $\lambda \in \Lambda_{\mathcal{F}}$ , the relation is satisfied with  $y = 0$ . Similarly, relation (52) is satisfied for  $u_N \notin K_N$ , and also for  $u_N \in K_N$  with  $\xi = 0$ .  $\blacksquare$

Using inclusion (57), the expression of the Signorini problem with Coulomb friction (31) is equivalent to

$$\begin{cases} \text{Find } u \in V, \lambda_N \in X'_N \text{ and } \lambda_T \in X'_T \text{ satisfying} \\ \mathbb{E}(\lambda_N, \lambda_T) = (u_N, u_T), \\ -(u_N - \mathcal{F}|u_T|, u_T) \in N_{\Lambda_{\mathcal{F}}}(\lambda_N, \lambda_T) \text{ in } X. \end{cases} \quad (58)$$

## 7.2 Fixed point formulations associated to De-Saxcé inclusion formulation

Applying again the same transformations to Problem (58) and defining

$$\tilde{\Lambda}_{\mathcal{F}} = \{x = (x_N, x_T) \in X : -(x_T, w_T)_{X_T} + (\mathcal{F}x_N, |w_T|)_{X_N} \leq 0, \forall w_T \in X_T\},$$

one will obtain using (33):

$$\begin{cases} \text{Find } u \in V, \tilde{\lambda} \in X \text{ satisfying} \\ \tilde{\mathbb{E}}(\tilde{\lambda}_N, \tilde{\lambda}_T) = (u_N, u_T), \\ \tilde{\lambda} = P_{\tilde{\Lambda}_{\mathcal{F}}}(\tilde{\lambda} - r(u_N - \mathcal{F}|u_T|, u_T)), \end{cases} \quad (59)$$

and using (48):

$$\begin{cases} \text{Find } u \in V \text{ and } \tilde{\lambda} \in X \text{ satisfying} \\ \tilde{\mathbb{E}}(\tilde{\lambda}_N, \tilde{\lambda}_T) = (u_N, u_T), \\ (u_N - \mathcal{F}|u_T|, u_T) = \frac{1}{r} \left( r(u_N - \mathcal{F}|u_T|, u_T) - \tilde{\lambda} + P_{\tilde{\Lambda}_{\mathcal{F}}}(\tilde{\lambda} - r(u_N - \mathcal{F}|u_T|, u_T)) \right). \end{cases} \quad (60)$$

The mappings defining the corresponding fixed points from (59) and (60) are respectively:

$$\begin{aligned} T^3 : X &\longrightarrow X \\ \tilde{\lambda} &\longmapsto P_{\tilde{\Lambda}_{\mathcal{F}}}(\tilde{\lambda} - r(u_N - \mathcal{F}|u_T|, u_T)), \\ \text{where } (u_N, u_T) &= \tilde{\mathbb{E}}(\tilde{\lambda}_N, \tilde{\lambda}_T), \end{aligned}$$

and replacing  $r$  by  $1/r$  for commodity:

$$\begin{aligned} T^4 : X &\longrightarrow X \\ (u_N, u_T) &\longmapsto (q_N + \mathcal{F}|q_T|, q_T), \\ \text{where } (q_N, q_T) &= \left( (u_N - \mathcal{F}|u_T|, u_T) - r\tilde{\lambda} + rP_{\tilde{\Lambda}_{\mathcal{F}}}(\tilde{\lambda} - \frac{1}{r}(u_N - \mathcal{F}|u_T|, u_T)) \right), \\ \text{and } \tilde{\lambda} &= \tilde{\mathbb{E}}^{-1}(u_N, u_T). \end{aligned}$$

These two fixed point operators can also be adapted to finite element discretization of the Coulomb problem and an analogous result to Theorem 3 can be proved.

## 8 Fixed point on the friction threshold

Another classical possibility is to make a fixed point on the friction threshold, which corresponds to a sequence of Tresca problems. Let us define for  $r > 0$

$$\begin{aligned} T^{5h} : X_N^h &\longrightarrow X_N^h \\ s^h &\longmapsto (\lambda_N^h)_-, \\ \text{where } (u_N^h, u_T^h) &= \mathbb{E}^h(\lambda_N^h, \lambda_T^h), \\ \lambda_N^h &= P_{X_N^h}(P_{\Lambda_N}(\lambda_N^h - ru_N^h)), \\ \lambda_T^h &= P_{X_T^h}(P_{\Lambda_T}(s^h)(\lambda_T^h - ru_T^h)). \end{aligned}$$

The computation of  $T^{5h}(s^h)$  is equivalent to solve a Tresca problem (see section 2.2). With the same kind of analysis as in Theorem 3, it can be proved that, for  $r$  sufficiently small,  $T^{5h}$  defines a unique fixed point  $\lambda_N^h$ .

**Theorem 7** *Under hypotheses (9), (10), (11), (35) and (36) and for  $\|\mathcal{F}\|_{L^\infty}$  sufficiently small, there exists  $r > 0$  such that  $T^{5h}$  is a strict contraction.*

**Proof.** For  $s^1 < 0, s^2 < 0$  in  $X^h$ , let  $u^1, u^2 \in X^h$ , and  $\lambda^1, \lambda^2 \in X^h$  be the corresponding displacements and stresses on the contact boundary coming from the computation of  $T^{5h}(s_1)$  and  $T^{5h}(s_2)$  respectively. From (37) One has

$$\begin{aligned} \|\delta(\lambda_N)_-\|_{L^2(\Gamma_C)}^2 &\leq \|\delta\lambda_N\|_{L^2(\Gamma_C)}^2 \\ &\leq \|\delta\lambda\|_{L^2(\Gamma_C)}^2 \\ &\leq M^2 C_3^2 h^{-1} \|\delta u\|_v^2. \end{aligned}$$

Now, using lemma 2

$$\begin{aligned} \|\delta\lambda\|_{L^2(\Gamma_C)}^2 &\leq \|\delta\lambda - r\delta u\|_{L^2(\Gamma_C)}^2 + \|\mathcal{F}\|_{L^\infty}^2 \|\delta s\|^2 \\ &\leq \|\delta\lambda\|_{L^2(\Gamma_C)}^2 - 2ra(\delta u, \delta u) + r^2 \|\delta u\|_{L^2(\Gamma_C)}^2 + \|\mathcal{F}\|_{L^\infty}^2 \|\delta s\|^2. \end{aligned}$$

Thus

$$(2r\alpha - r^2 C_1^2) \|\delta u\|_v^2 \leq \|\mathcal{F}\|_{L^\infty}^2 \|\delta s\|^2,$$

consequently for  $r = \frac{\alpha}{C_1^2}$

$$\|\delta(\lambda_N)_-\|_{L^2(\Gamma_C)}^2 \leq \frac{M^2 C_1^2 C_3^2 h^{-1}}{\alpha} \|\mathcal{F}\|_{L^\infty}^2 \|\delta s\|^2.$$

This means that  $T^{5h}$  is a strict contraction for  $r = \frac{\alpha}{C_1^2}$  and

$$\|\mathcal{F}\|_{L^\infty} < \frac{\alpha \sqrt{h}}{MC_1 C_3}.$$

■

**Remark 5** *An interesting property of this fixed point operator is that  $\|\mathcal{F}\|_{L^\infty}$  is in factor of the contraction constant, which means that for a small  $\|\mathcal{F}\|_{L^\infty}$  the contraction property should be better than for the other fixed point operators. Of course each iteration needs to solve a Tresca problem.*

## Concluding remarks

In this paper, we presented a new formalism to deal with contact and friction problems. It turns out it is well adapted for the analysis of this kind of problems and allows to present very concise proofs.

Among the fixed points presented, the more used in practical computations are the fixed point on the contact and friction stresses ( $T^{1h}$  operator) and the fixed point on the friction threshold ( $T^{5h}$  operator). The operator  $T^{5h}$  has a better contraction constant, but has the drawback to need the resolution of a nonlinear problem at each iteration. The same drawback exists for the Uzawa algorithm applied to the augmented Lagrangian formulation. Operators  $T^{1h}$ ,  $T^{2h}$  and  $T^{3h}$  have theoretically the same contraction constant and need only to solve a linear problem at each iteration. The operator  $T^{3h}$  is defined thanks to De Saxcé's bipotential theory. An advantage of the last formulation is that only one projection is required (compared to two, for the others) which simplifies the analysis.

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