Mass redistribution method for finite element contact problems in elastodynamics

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Abstract

This paper deals with the numerical stabilization of the semi-discretized finite element unilateral contact problem in elastodynamics. It is well known that it is ill-posed due to the nonpenetration condition on the finite element nodes lying on the contact boundary. We introduce a new method based on a redistribution of the mass matrix such that there is no inertia on the contact boundary. This leads to a mathematically well-posed and energy conserving semi-discretized problem. Finally, some numerical tests are presented.

keywords: elasticity, unilateral contact, time integration schemes, energy conservation, stability, redistributed mass matrix.

Introduction

In this paper, we are interested in the study of the numerical instabilities caused by the space semi-discretization of contact problems in elastodynamics. For the sake of simplicity, we limit ourself to the small deformations framework.

The underlying continuous elastodynamic contact problem (purely hyperbolic problem) is very difficult from a mathematical viewpoint. As far as we know, some existence results have only been established for a close but scalar and two dimensional problem in [14, 11], and in the vector case with a modified contact law in [23]. However, no uniqueness result is known in the purely hyperbolic framework.

The semi-discretized problem by finite elements is itself ill-posed, which leads to numerical instabilities of time integration schemes. Thus, many authors adapted different approaches to overcome this difficulty. To recover the uniqueness in the discretized case, one of the approaches well adapted to rigid bodies is to introduce an impact law with a restitution coefficient [20]. However, this approach seems not satisfactory for deformable bodies. On the other hand, the unilateral contact condition leads to some difficulties in the construction of energy conserving schemes [20, 13, 12, 5] because of the presence of oscillations of the displacement and of the normal stress on the contact boundary. A way to avoid a noisy behavior of the solution is to implicit the contact force [24, 3]. As a result, nodes coming to contact are stuck. The drawback of this method is that the kinetic energy of the contacting nodes is cancelled each time a new contact occurs. Another well known approach is the penalty method which introduces important oscillations that have to be reduced with a damping technique [24]. Even though it is possible to build energy conserving schemes with a penalized contact condition [2, 5], this leads to important oscillations of the normal stress. One of the key points to avoid oscillations is to try to enforce the complementarity condition with

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respect to the velocity [2, 6, 12], the so-called persistency condition. But, a compromise have to be made between the satisfaction of the latter condition and the nonpenetration condition. Such compromises are presented in [25] with additional terms in the Lagrangian of the system, in [15] with an iterative process to correct the contact stress and in [13] where a post-processing of the velocity allows to recover the energy conservation.

In this paper, we perform an analysis of the ill-posedness encountered in this kind of discretization and conclude that the main cause is that the nodes on the contact boundary have their own inertia. We propose a new method which consists in the redistribution of the mass near the contact boundary. We prove that the well-posedness of the semi-discretized problem is recovered and that the unique solution is energy conserving. Numerical simulations show that the quality of the contact stress is greatly improved by this technique.

The outline of the paper is the following. In Section 1, we give the strong and weak formulations of the continuous contact problem in elastodynamics. Then, Section 2 is devoted to the corresponding evolutionary finite element problem. Section 3 proposes an analysis of the finite element semi-discretized problem ill-posedness based on a one degree of freedom system. In section 4, we introduce our proposed method which consists in a new distribution of the body mass with conservation of the total mass, the coordinates of the center of gravity and the inertia momenta. This redistribution is done so that there is no inertia for the contact nodes (similarly to the situation in the continuous case). Using this method, we prove existence and uniqueness of the semi-discrete solution. Numerical tests are presented in a last section. Particularly, the propagation of the impact wave is exhibited. Finally, we compare the energy and normal stress evolution with and without our mass redistribution method (sometimes, we will use initials MRM thereafter in order to refer to our method).

1 The continuous contact problems in elastodynamics

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded Lipschitz domain representing the reference configuration of a linearly elastic body. This body is submitted to a Neumann condition on Γ_N, a Dirichlet condition on Γ_D and a frictionless unilateral contact condition on Γ_C between the body and a flat rigid foundation. We suppose that Γ_N, Γ_D and Γ_C form a partition of $\partial\Omega$, the boundary of Ω. Let also ρ, $\sigma(u)$, $\varepsilon(u)$ and $\mathcal{A}$ be the mass density, the stress tensor, the linearized strain tensor and the elasticity tensor, respectively. The elastodynamic problem consists in finding the displacement field $u(t, x)$ satisfying

$$\begin{cases} \rho\ddot{u} - \operatorname{div} \sigma(u) = f & \text{in } [0,T] \times \Omega, \\ \sigma(u) = \mathcal{A}\, \varepsilon(u) & \text{in } [0,T] \times \Omega, \\ u = 0 & \text{on } [0,T] \times \Gamma_D, \\ \sigma(u)\nu = g & \text{on } [0,T] \times \Gamma_N, \\ u(0) = u_0, \dot{u}(0) = u_1 & \text{in } \Omega, \end{cases} \tag{1}$$

where ν is the outward unit normal to Ω on $\partial\Omega$ and f, g are the given external loads. On Γ_C, we decompose the displacement and the stress vector in normal and tangential components as follows:

$$u_N = u.\nu, \quad u_T = u - u_N\nu,$$

$$\sigma_N(u) = (\sigma(u)\nu).\nu, \quad \sigma_T(u) = \sigma(u)\nu - \sigma_N(u)\nu.$$

To give a clear sense to this decomposition, we assume Γ_C to have the $\mathcal{C}^1$ regularity. Without real loss of generality, we assume also that there is no initial gap between the solid and the rigid foundation. The frictionless unilateral contact condition is then expressed thanks to the complementarity condition

$$u_N \leq 0,\ \sigma_N(u) \leq 0,\ u_N\sigma_N(u) = 0 \text{ and } \sigma_T(u) = 0 \quad \text{on } [0,T]\times\Gamma_C. \tag{2}$$

In order to build a finite element approximation of the problem, we need to write its weak formulation. We thus define the vector spaces

$$V = \{v \in H^1(\Omega;\mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad X_N = \{v_{N|\Gamma_C} : v \in V\},$$

their topological dual spaces V' and X'_N, the maps

$$a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v)dx, \quad l(v) = \int_\Omega f.vdx + \int_{\Gamma_N} g.vd\Gamma,$$

and the cone of admissible normal stresses

$$W_N = \left\{\mu_N \in X'_N : \langle \mu_N, v_N\rangle_{X'_N,X_N} \geq 0 \ \ \forall v \in V, v_{N|\Gamma_C} \leq 0\right\}.$$

A weak formulation of Problem (1)(2) can be written as follows:

$$\left\{\begin{array}{l} \text{find } u :\ [0,T] \longrightarrow V \text{ and } \ \lambda_N :\ [0,T] \longrightarrow W_N \text{ satisfying} \\[2mm] \langle \rho\ddot{u}(t), v\rangle_{V',V} + a(u(t),v) = l(v) + \langle \lambda_N(t), v_N\rangle_{X'_N,X_N} \quad \forall v \in V,\ a.e.\ t \in (0,T), \\[2mm] \lambda_N(t) \leq 0, \ \ \langle \mu_N - \lambda_N(t), u_N(t)\rangle_{X'_N,X_N} \geq 0 \ \ \forall \mu_N \in W_N,\ a.e.\ t \in (0,T), \\[2mm] u(0) = u^0, \quad \dot{u}(0) = u^1. \end{array}\right. \tag{3}$$

More detailed presentations of weak formulations of contact problems in elasticity can be found in [4, 8, 10]. We assume that the elasticity coefficients obey the usual symmetry and uniform ellipticity conditions and that Γ_D is of nonzero measure on $\partial\Omega$ such that $a(\cdot,\cdot)$ is itself elliptic.

2 The finite element approximation of contact problems in elastodynamics

We consider a Lagrange finite element method to approximate the contact problem in elastodynamics (3). Let $a_1, ..., a_n$ be the finite element nodes, $\varphi_1, ..., \varphi_{n_d}$ the (vector) basis functions of the finite element displacement space and $I_C = \{i : a_i \in \Gamma_C\}$. We denote by m the number of nodes on Γ_C and n_d the number of degrees of freedom. Let U be the vector of degrees of freedom of the finite element displacement field $u_h(x)$ such that

$$u_h(x) = \sum_{1\leq i\leq n_d} u_i\varphi_i \quad \text{and} \quad U = (u_i) \in \mathbb{R}^{n_d}.$$

By approximating the contact condition with a nodal condition, the space semi-discretization of Problem (3) can be written as follows:

$$\begin{cases} \text{Find } U : [0,T] \mapsto \mathbb{R}^{n_d} \text{ and } \Lambda_{_N} : [0,T] \mapsto \mathbb{R}^m \text{ satisfying on } (0,T) \\ M\,\ddot{U} + KU = L + \displaystyle\sum_{i \in I_C} \lambda^i_{_N} N_i, \\ N_i^T U \leq 0, \quad \lambda^i_{_N} \leq 0, \quad (N_i^T U)\lambda^i_{_N} = 0 \quad \forall\, i \in I_C, \\ U(0) = U^0, \quad \dot{U}(0) = U^1, \end{cases} \tag{4}$$

where

$$K_{ij} = a(\varphi_i, \varphi_j) \quad \text{and} \quad M_{ij} = \int_\Omega \rho\, \varphi_i.\varphi_j\, dx \qquad (1 \leq i, j \leq n_d)$$

are the components of the stiffness matrix K and the components of the mass matrix M, respectively. The components of the external loads vector $L = (L_i)$ are given by

$$L_i = \int_\Omega f.\varphi_i dx + \int_{\Gamma_C} g.\varphi_i dx.$$

The vectors $N_i \in \mathbb{R}^n$ are chosen such that the normal displacement on the contact surface reads as

$$u^h_{_N}(a_i) = N_i^T U \qquad \forall i \in I_C. \tag{5}$$

The multipliers $\lambda^i_{_N}$ define the nodal equivalent contact forces and $\Lambda_{_N}$ stands for the vector

$$\Lambda_{_N} = \left(\lambda^i_{_N}\right) \in \mathbb{R}^m.$$

Problem (4) represents a differential inclusion with measure solution (see [19], [20]). More details on the discretization of contact with friction problems can be found for instance in [10, 8].

3 On the ill-posedness of elastodynamic frictionless contact problem

It is known that Problem (4) is ill-posed [17, 18, 21, 22]. In this section, we illustrate this ill-posedness by exhibiting an infinite number of solutions to the one degree of freedom system represented in Fig. 1.

The vertical motion $U \in \mathbb{R}$ of this simple mechanical system is governed by the following set of equations

$$\begin{cases} m\ddot{U} + kU = \Lambda_{_N}, \\ U \geq 0, \ \Lambda_{_N} \geq 0, \ \Lambda_{_N}\, U = 0, \\ U(0) \text{ and } \dot{U}(0) \text{ given,} \end{cases} \tag{6}$$

where k is the stiffness coefficient of the spring, m is the mass placed in its extremity and $\Lambda_{_N}$ is the reaction force of the rigid foundation. With the particular initial data $U(0) = -1$ and $\dot{U}(0) = 0$, and for any $\alpha \geq 0$, a solution to Problem (6) is given by

$$U(t) = cos\left(t\sqrt{\frac{k}{m}}\right), \; 0 \leq t < \frac{\pi}{2}\sqrt{\frac{m}{k}},$$

$$U(t) = -\alpha \; cos\left(t\sqrt{\frac{k}{m}}\right), \; \frac{\pi}{2}\sqrt{\frac{m}{k}} < t < \pi\sqrt{\frac{m}{k}}.$$

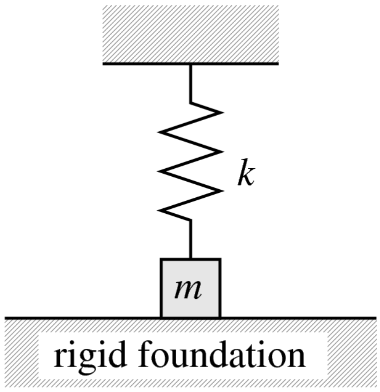


Figure 1: *A one degree of freedom contact problem.*

Despite its simplicity, this system reflects well the difficulty caused by the unilateral contact condition in a dynamic discrete problem. Actually, it appears on the normal component at each contact node in Problem (4) with a supplementary right hand side corresponding to the remaining terms. Hence, the space semi-discretized elastodynamic frictionless contact problem (4) admits also an infinite number of solutions and is ill-posed in that sense.

4 Mass redistribution method

In order to recover the uniqueness of the solution to the semi-discretized problem, one of the approaches well adapted to rigid bodies is to introduce an impact law with a restitution coefficient [17, 18, 21, 22]. This seems not to be completely satisfactory for deformable bodies because, whatever is the restitution coefficient, the system tends to a global restitution of energy when the mesh parameter goes to zero. Moreover, if an impact law with a nonzero restitution coefficient is applied on a contact node, the velocity on this node cannot vanish, which means that this node oscillates without the possibility to remain in a persistent contact.

The aim now is to present a new method which permits to recover the uniqueness for the finite element semi-discretized elastodynamic contact problem, to recover the energy conservation of the solution and which avoids the oscillatory behavior on the contact nodes. Some of the results presented below were announced in [9].

4.1 Construction of the redistributed mass matrix

The ill-posedness of Problem (4) comes from the fact that the nodes on the contact boundary have their own inertia (Problem (6) with $m = 0$ has the unique solution $U = \Lambda_N = 0$). This leads to instabilities even for energy conserving schemes. An explanation of those instabilities is that if a node is stopped on the contact boundary, its kinetic energy is definitively lost. Thus, energy conserving schemes make the node on the contact boundary oscillate in order to keep this kinetic energy.

Conversely to what happens in the continuous case where no mass is attached to the contact zone, the mass matrix has nonzero components corresponding to the nodes on Γ_C. We propose here to introduce a new distribution of the mass which conserves the total mass, the center of gravity and the inertia momenta but is built so that there is no inertia for the contact nodes. This restore, in a sense, the continuous framework situation.

Let us denote by M_r the redistributed mass matrix. The fact that the mass on the contact boundary is eliminated leads to the following constraints:

$$N_i^T M_r N_j = 0, \forall\ i, j \in I_C, \tag{7}$$

where N_i is still defined by (5).

The total mass can be expressed from M as follows (since a Lagrange finite element method is used):

$$\int_\Omega \rho\ dx = X^T M\ X,$$

where $X = 1/\sqrt{d}\ \ (1...1)^T \in \mathbb{R}^n$ $(d = 2, 3)$. The k^{th}-coordinate of the center of gravity is defined by

$$\int_\Omega \rho\ x_k\ dx = X^T M\ Y_k \quad (1 \le k \le d),$$

denoting $Y_k = (y_i) \in \mathbb{R}^n$ the vector such that

$$1/\sqrt{d}\ \sum_{i,j} y_i\ \varphi_i\ .\varphi_j = x_k.$$

Finally, the moment of inertia matrix is derived from the quantities

$$\int_\Omega \rho\ x_k\ x_l\ dx = Y_k^T M\ Y_l \quad (1 \le k, l \le d).$$

The matrix M_r will be said to be *equivalent* to M if the following equality constraints are satisfied:

$$\begin{cases} X^T(M_r - M)\ X = 0, \\ X^T(M_r - M)\ Y_k = 0 \ \ (1 \le k \le d), \\ Y_k^T(M_r - M)\ Y_l = 0 \ \ (1 \le k, l \le d). \end{cases} \tag{8}$$

Moreover, for a reason of computational cost, the construction of the matrix M_r is done with the same sparsity than M (*i.e* without adding nonzero elements) and keeping the symmetry.

Finally, the new mass matrix M_r is subjected to the above-mentioned constraints and minimizes the distance to the standard finite element matrix M with respect to the Fröbenius norm. This choice leads to a very simple system (6×6 in 2D and 10×10 in 3D) to be solved whose unknowns are the Lagrange multipliers associated to the constraints.

4.2 Semi-discretized problem with redistributed mass matrix

If we number the degrees of freedom such that the last ones correspond to the normal components on the finite element nodes on the contact boundary, hypothesis (7) leads to a new mass matrix having the following pattern

$$M_r = \begin{pmatrix} \overline{M_r} & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

We can also split each matrix and vector into interior part and contact boundary part as follows:

$$K = \begin{pmatrix} \overline{K} & C^T \\ C & \widetilde{K} \end{pmatrix}, \quad N_i = \begin{pmatrix} 0 \\ \widetilde{N}_i \end{pmatrix}, \quad L = \begin{pmatrix} \overline{L} \\ \widetilde{L} \end{pmatrix} \quad \text{and } U = \begin{pmatrix} \overline{U} \\ \widetilde{U} \end{pmatrix}.$$

Replacing M by M_r, Problem (4) becomes

$$\begin{cases} \text{Find } U : [0,T] \mapsto \mathbb{R}^{n_d} \text{ and } \Lambda_{_N} : [0,T] \mapsto \mathbb{R}^m \text{ satisfying on } (0,T) \\[2mm] \begin{pmatrix} \overline{M_r} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\overline{U}} \\ \ddot{\widetilde{U}} \end{pmatrix} + \begin{pmatrix} \overline{K} & C^T \\ C & \widetilde{K} \end{pmatrix} \begin{pmatrix} \overline{U} \\ \widetilde{U} \end{pmatrix} = \begin{pmatrix} \overline{L} \\ \widetilde{L} \end{pmatrix} + \sum_{i \in I_C} \lambda_{_N}^i \begin{pmatrix} 0 \\ \widetilde{N}_i \end{pmatrix}, \\[2mm] \widetilde{N}_i^T \widetilde{U} \le 0, \quad \lambda_{_N}^i \le 0, \quad \lambda_{_N}^i (\widetilde{N}_i^T \widetilde{U}) = 0 \quad \forall\, i \in I_{_C}, \\[2mm] U(0) = U^0, \dot{U}(0) = U^1. \end{cases} \tag{10}$$

4.3 Stability analysis

In this section, we analyze the properties of Problem (10). We first state the uniqueness of the solution to this problem. Then, we prove that the persistency condition in verified on the finite element nodes. Finally, this permits us to prove that the solution is energy conserving.

Theorem 1 *Let us assume that the load vector L is a Lipschitz continuous function on $[0,T]$. Then, there exists one and only one Lipschitz continuous function $t \longrightarrow (U(t), \Lambda_{_N}(t))$ solution to the discretized Problem (10).*

Proof. Problem (10) is equivalent to:

$$\begin{cases} \overline{M_r}\, \ddot{\overline{U}} + \overline{K}\, \overline{U} = \overline{L} - C^T \widetilde{U}, \\[2mm] C\overline{U} + \widetilde{K}\widetilde{U} = \widetilde{L} + \sum_{i \in I_C} \lambda_{_N}^i \widetilde{N}_i, \\[2mm] \widetilde{N}_i^T \widetilde{U} \le 0, \quad \lambda_{_N}^i \le 0, \quad \lambda_{_N}^i (\widetilde{N}_i^T \widetilde{U}) = 0 \quad \forall\, i \in I_{_C}, \\[2mm] U(0) = U^0, \dot{U}(0) = U^1. \end{cases} \tag{11}$$

The following sub-system of (11):

$$\begin{cases} \widetilde{K}\widetilde{U} + C\overline{U} = \widetilde{L} + \sum_{i \in I_C} \lambda_{_N}^i \widetilde{N}_i, \\[2mm] \widetilde{N}_i^T \widetilde{U} \le 0, \quad \lambda_{_N}^i \le 0, \quad \lambda_{_N}^i (\widetilde{N}_i^T \widetilde{U}) = 0 \quad \forall\, i \in I_{_C} \end{cases} \tag{12}$$

can be expressed as follows :

$$a(\widetilde{U}, \widetilde{V} - \widetilde{U}) \geq l_{\overline{U}}(\widetilde{V} - \widetilde{U}) \quad \forall\, \widetilde{V} \in Q, \tag{13}$$

where $\quad a(\widetilde{U}, \widetilde{V}) = \widetilde{V}^T \widetilde{K} \widetilde{U}, \;\; l_{\overline{U}}(\widetilde{V}) = \widetilde{V}^T \widetilde{L} - \widetilde{V}^T C \overline{U} \quad$ and $\quad Q = \{V : \widetilde{N}_i^T \widetilde{V} \leq 0, \; i \in I_C\}$.

The standard assumptions on the elasticity problem imply on the one hand that $\widetilde{U}$ is uniquely defined from the variational inequality (13) for given $\overline{U}$ and $\widetilde{L}$, and on the other hand that $\widetilde{U}$ and Λ_N are Lipschitz continuous functions with respect to $\overline{U}$ and $\widetilde{L}$. It follows that the first equation in System (11) is a second order Lipschitz ordinary differential equation with respect to the unknown $\overline{U}$. Such an equation, with the initial conditions, has a unique solution $\overline{U}$ with a Lipschitz continuous derivative. Since $\overline{U}$ and $\widetilde{L}$ are Lipschitz continuous functions in time, Λ_N is Lipschitz in time too.

□

Proposition 1 *The solution (U, Λ_N) to Problem (10) satisfies the following persistency condition at each node on Γ_C:*

$$\forall i \in I_C, \quad \lambda_N^i (N_i^T \dot{U}) = 0 \quad \textit{a.e. on } [0, T].$$

Proof. Thanks to the fact that the solution (U, Λ_N) to Problem (10) is Lipschitz continuous, we have:

$$\lambda_N^i = 0 \;\; \text{on} \;\; \text{Supp}(N_i^T U) = \omega_i \subset [0, T] \quad (i \in \Gamma_C),$$

where $\text{Supp}(\psi)$ denotes the support of the function $\psi(t)$. The continuity of λ_N^i on $[0, T]$ implies

$$\lambda_N^i = 0 \;\; \text{on} \;\; \overline{\omega_i}.$$

On the other hand,

$$N_i^T \dot{U} = 0 \quad \text{a.e. on} \;\; \theta_i,$$

where θ_i is the complementary part in $[0, T]$ of the interior of ω_i. Hence

$$\lambda_N^i (N_i^T \dot{U}) = 0, \;\; \text{a.e. on} \;\; [0, T].$$

□

Theorem 2 *If we assume that the load vector L is constant in time then the solution to finite element elastodynamic system with unilateral contact (10) is energy conserving.*

Proof. The discrete energy of system (10) is given by:

$$E(t) = \frac{1}{2}\, \dot{U}^T M_r \dot{U} + \frac{1}{2}\, U^T K U - U^T L.$$

The first equation in (10) implies:

$$\dot{U}^T M_r \ddot{U} + \dot{U}^T K U = \dot{U}^T L + \sum_{i \in I_C} \lambda_N^i \dot{U}^T N_i.$$

Integrating from 0 to t, it follows:

$$\frac{1}{2}\,\dot{U}^T M_r \dot{U} + \frac{1}{2}\,U^T K U - U^T L = \sum_{i \in I_C} \int_0^t \lambda_{_N}^i\ \dot{U}^T N_i\ dt + E(0).$$

In others words, one has

$$E(t) = \sum_{i \in I_C} \int_0^t \lambda_{_N}^i\ \dot{U}^T N_i\ dt + E(0) \qquad \forall t \in [0, T].$$

Thanks to Proposition 1, we finally obtain

$$E(t) = E(0) \qquad \forall t \in [0, T].$$

□

Note that the same result can be obtained for a more general load vector whenever it admits a potential.

5 Numerical results

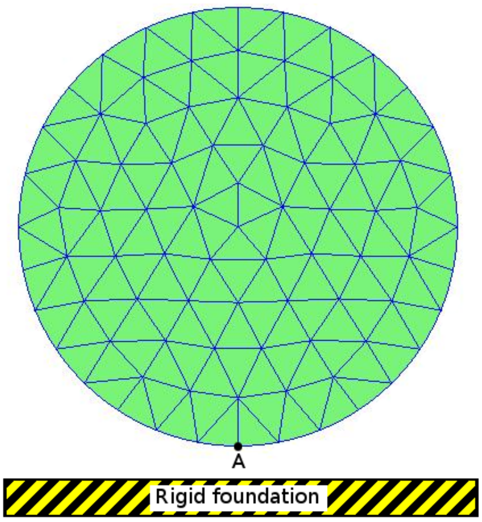


Figure 2: *Mesh of the disc (isoparametric P_2 elements).*

Here, we present some numerical experiments of the dynamic contact of an elastic disc on a rigid foundation (see Fig. 2). The properties of the elastic disc are summarized in Table 1. We denote by A the lowest point of the disc (the first point which comes into contact with the foundation). The numerical tests are performed thanks to the finite element library Getfem++ [26] with isoparametric P_2 elements. The test program is itself available on the Getfem++ website.

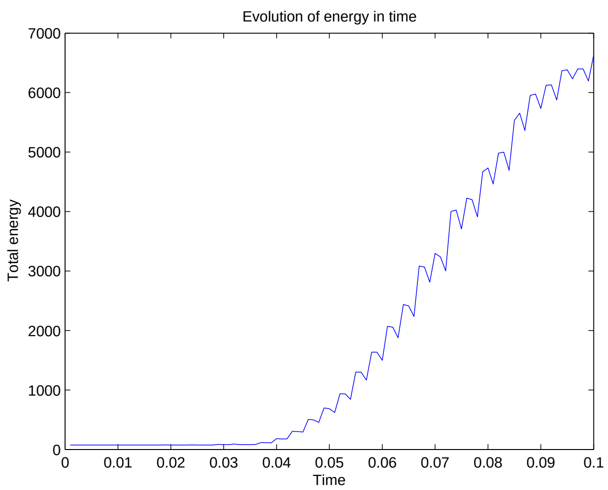
Disc property	Values	Property of the resolution method	Values
ρ, diameter	$6\ 10^3 kg/m^3$, $0.2\ m$	Time step	$10^{-3}s$
Lamé coefficients	$\lambda = 10^6\ P$, $\mu = 5\ 10^5\ P$	Simulation time	$0.3\ s$
u^0, v^0	$0.01\ m$, $-0.1\ m/s$	Mesh parameter	$\simeq 0.02\ cm$

Table 1: *Characteristics of the elastic disc and the resolution method.*

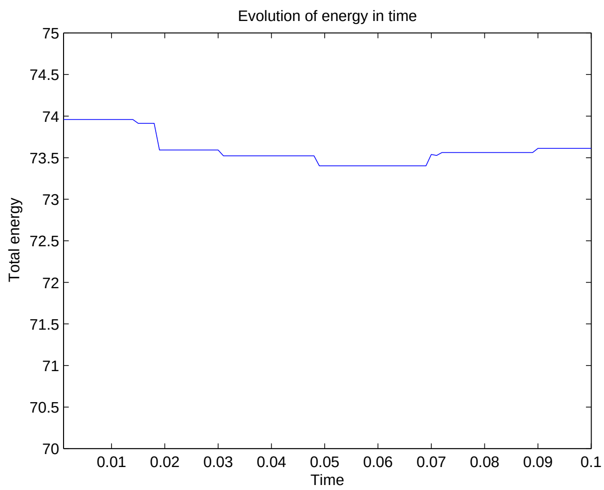
The time integration scheme we use is the popular Newmark scheme. This scheme applied to Problem (10) can be written as follows:

$$\begin{cases} U^{n+1} = U^n + \Delta t\ V^n + \dfrac{\Delta t}{2}\left((1-2\beta)A^n + 2\beta A^{n+1}\right), \\ V^{n+1} = V^n + \Delta t\ \left((1-\gamma)A^n + \gamma A^{n+1}\right), \\ M_r A^{n+1} + KU^{n+1} = L + \displaystyle\sum_{i \in I_C} (\lambda_N^i)^{n+1} N_i, \\ N_i^T U^{n+1} \le 0, \quad (\lambda_N^i)^{n+1} \le 0, \quad (N_i^T U^{n+1})(\lambda_N^i)^{n+1} = 0 \quad \forall\, i \in I_C, \\ U(0) = U^0, \dot{U}(0) = U^1, \end{cases} \tag{14}$$

where U^n, V^n and A^n approximate $U(t_n), \dot{U}(t_n)$ and $\ddot{U}(t_n)$ respectively and $2\beta \in [0,1]$, $\gamma \in [1/2, 1]$ are the two classical parameters of the scheme.



With the standard mass matrix

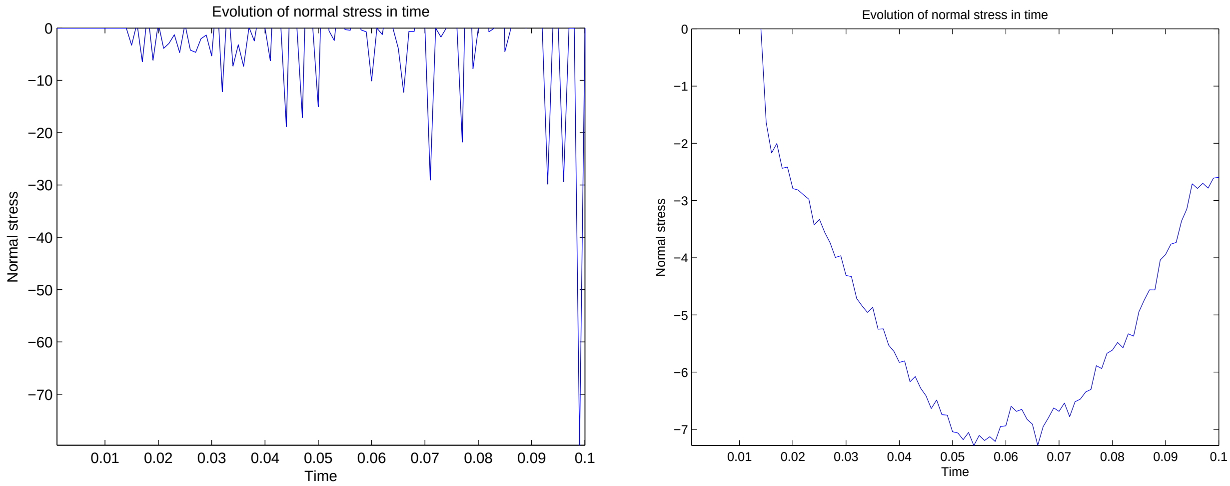


With the redistributed mass matrix

Figure 3: *Energy evolution for the Crank-Nicholson scheme ($\Delta t\ = 10^{-3}$).*

The first numerical test is done for $2\beta = \gamma = 1/2$ which corresponds to the Crank-Nicholson scheme. The energy evolution of the discrete solution with and without mass redistribution method is shown on Fig. 3. Even though the Crank-Nicholson scheme is energy conserving for the elastodynamic part, the energy is blowing up very rapidly with the standard mass matrix due to the contact condition. The use of the mass redistribution method clearly stabilizes the scheme. The remaining energy fluctuations are rather small.

The evolution of the numerical normal stress at point A is presented on Fig. 4. The normal stress is quite smooth with the mass redistribution method (MRM) unlike with the standard finite element mass matrix where it is not exploitable.



With the standard mass matrix　　　With the redistributed mass matrix

Figure 4: *Normal stress evolution of the lowest point of the disc for the Crank-Nicholson scheme* ($\Delta t = 10^{-3}$).

The fact that Problem (10) corresponds to a Lipschitz ordinary differential equation in time allows the convergence of any classical scheme when the time step goes to zero. It is illustrated with the use of a time step equal to 10^{-4} on Fig. 5 which shows that the energy tends to be conserved when the time step decreases.

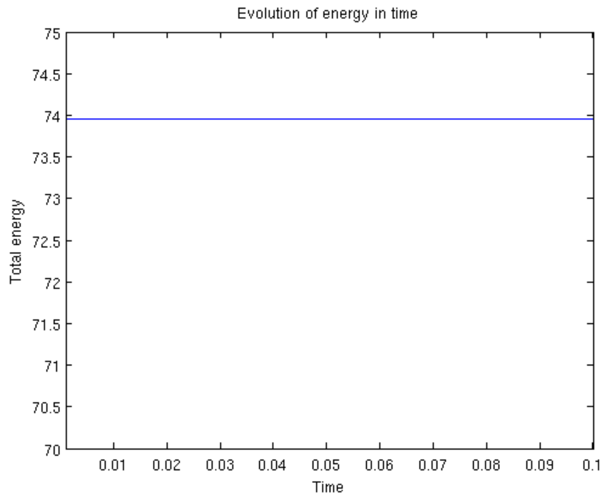
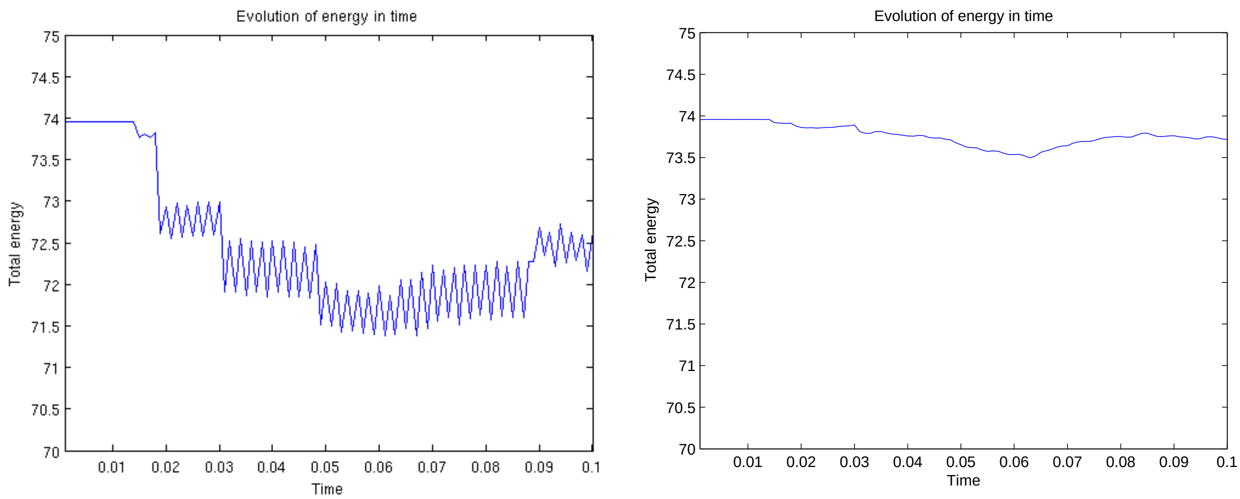


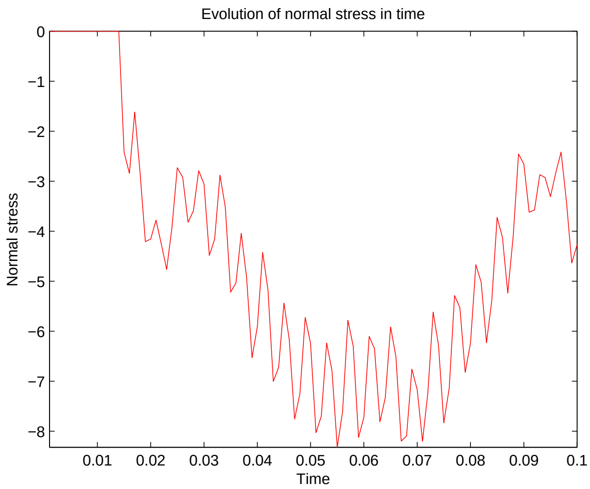
Figure 5: *Energy evolution for the Crank-Nicholson scheme and a smaller time step* ($\Delta t = 10^{-4}$).



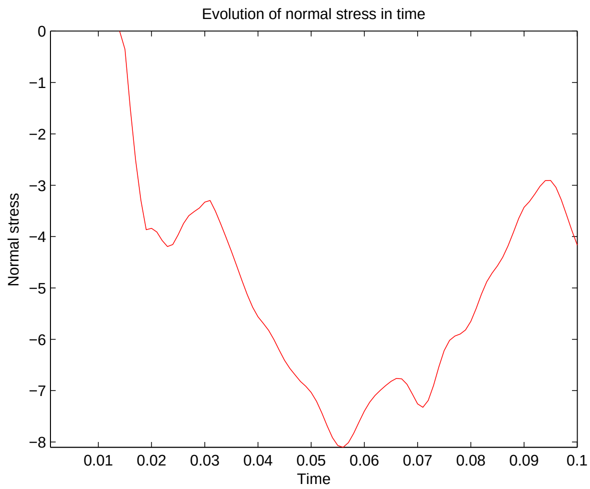
With the standard mass matrix

With the redistributed mass matrix

Figure 6: *Energy evolution for the elastodynamic contact problem for* $\beta = \gamma = 1/2$ ($\Delta t = 10^{-3}$).

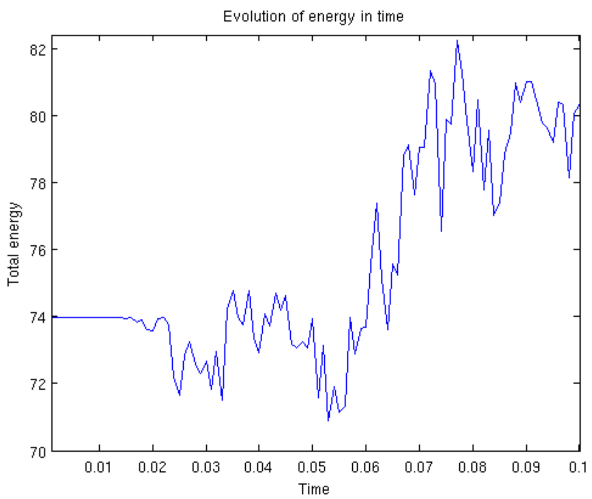


With the standard mass matrix

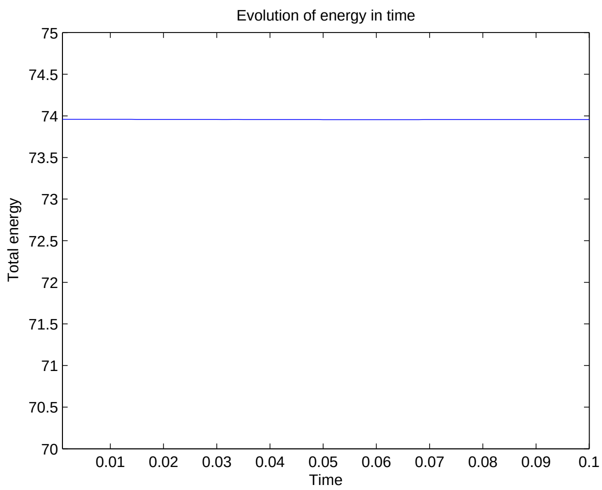


With the redistributed mass matrix

Figure 7: *Normal stress evolution on the lowest point of the disc for $\beta = \gamma = 1/2$ ($\Delta t \; = 10^{-3}$).*



With the standard mass matrix



With the redistributed mass matrix

Figure 8: *Energy evolution for $\beta = \gamma = 1/2$ and a smaller time step ($\Delta t \; = 10^{-4}$).*

Figures 6, 7 and 8 corresponds to a numerical test with the Newmark scheme with $\beta = \gamma = 1/2$. This method have a better behavior without the redistributed mass matrix compared to the Crank-Nicholson scheme. However, when the time step decreases the method becomes more and more unstable. Here again, the MRM improve the stability and the approximation of the normal stress.

Propagation of an impact wave. We represent on Fig. 9 the evolution of the Von Mises stress during the first impact for the simulation with a Crank-Nicholson scheme and the MRM. We remark a return journey of the shear wave. We conclude that the MRM does not affect to much the behavior of the impact wave. Note that Lamé coefficients have been chosen (see Table 1) in order to have visible deformations (even though it is rather out of the scope of linear elasticity).

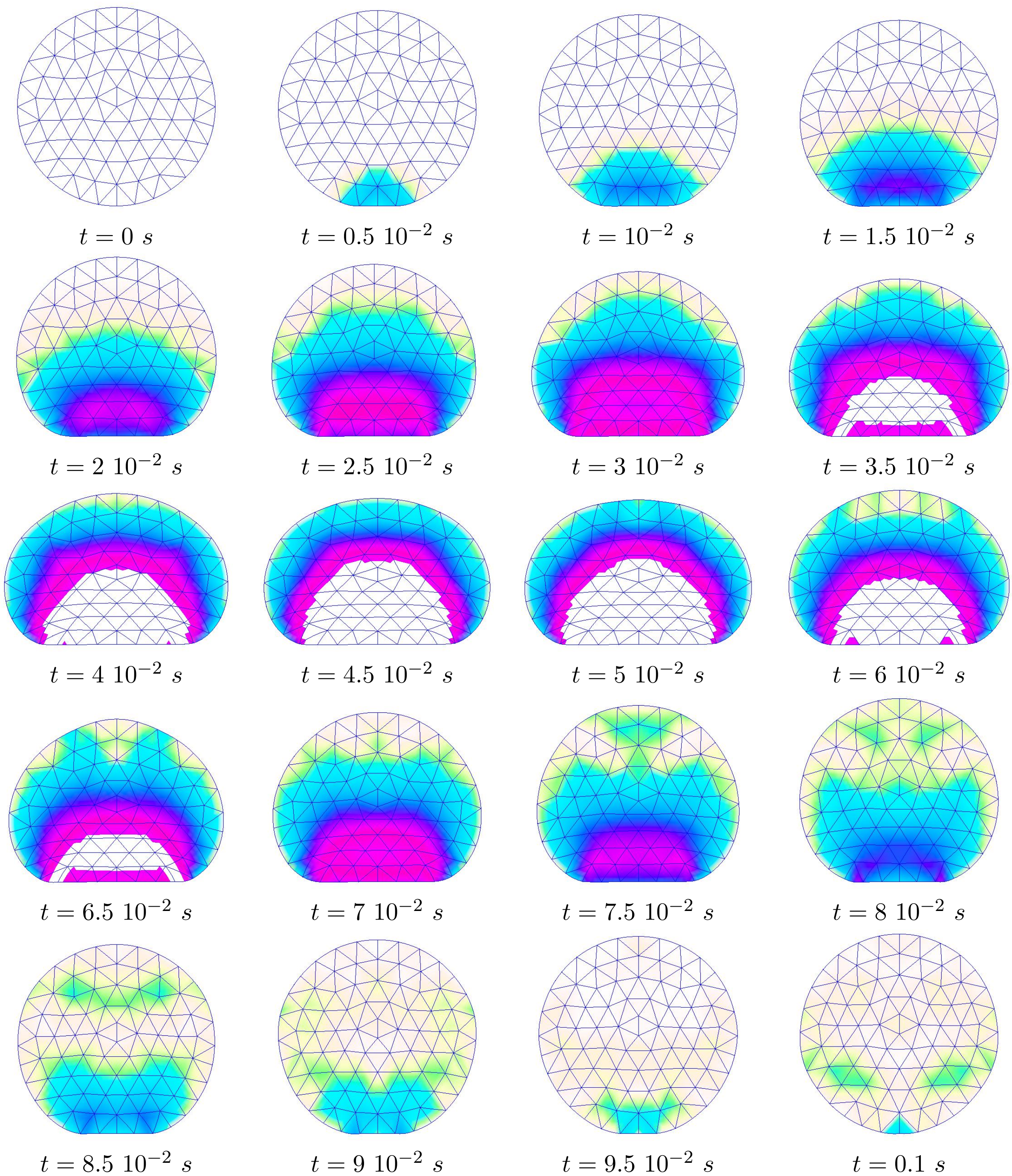


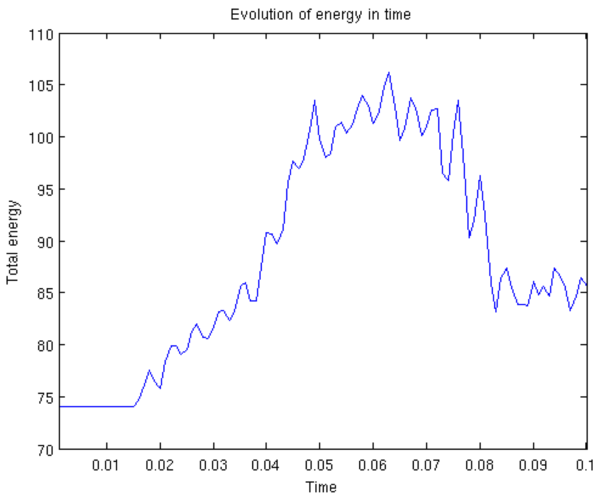
Figure 9: *Von Mises stress evolution during the first impact (Crank-Nicholson scheme with MRM).*

5.1 Comparison with Paoli-Schatzman scheme

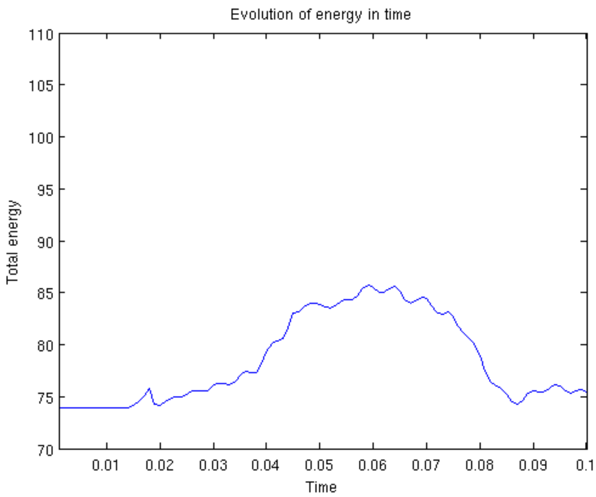
The Paoli-Schatzman scheme is based on a centered difference scheme concerning the approximation of the elastodynamic part and on a contact condition defined on an intermediary time step relatively to a restitution coefficient (see [20]). It can be expressed as follows:

$$\left\{\begin{array}{l} U^0 \text{ and } V^0 \text{ given}, U^1 = U^0 + \Delta t\ V^0 + \Delta t\ z(\Delta t\ ),\ \text{avec} \lim\limits_{\Delta t\ \longrightarrow 0} z(\Delta t\ ) = 0, \\ \\ M\dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} + K\dfrac{U^{n+1} + 2U^n + U^{n-1}}{4} = L + B_N^T \Lambda_N^n + B_T^T \Lambda_T^n, \quad n \geq 2, \\ \\ U^{n,e} = \dfrac{B_N U^{n+1} + eB_N U^{n-1}}{1+e}, \\ \\ N_i^T U^{n,e} \leq 0, \quad (\lambda_N^i)^n \leq 0, \quad (\lambda_N^i)^n (N_i^T U^{n,e}) = 0 \qquad \forall\, i \in I_C, \end{array}\right. \tag{15}$$

where e is the restitution coefficient ($e \in [0,1]$). In [20], it is proved that for a diagonal mass matrix, the solution to (15) tends to a contact condition with a restitution coefficient e on each contact node when the time step goes to zero. The result is less obvious for a more general mass matrix. Fig. 10 and 11 shows the result of numerical tests for $e = 1/2$. They have to be compared with Fig. 6, 7 and 8. The Paoli-Schatzman scheme has a better behavior than the Newmark scheme without MRM since it converges when the time step decreases. However, the energy and the normal stress on the contact boundary still have some small oscillations.

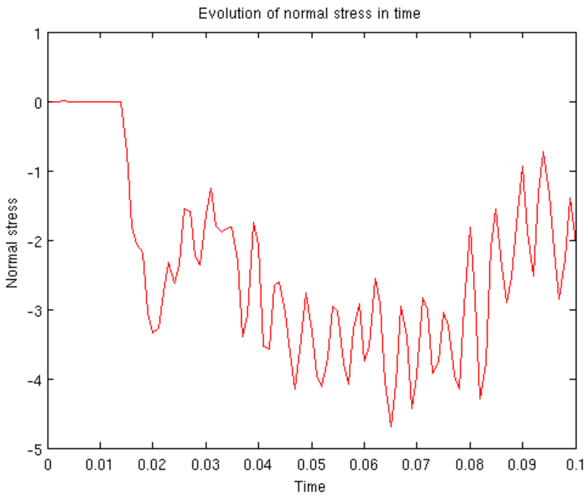


$\Delta t = 10^{-3}$

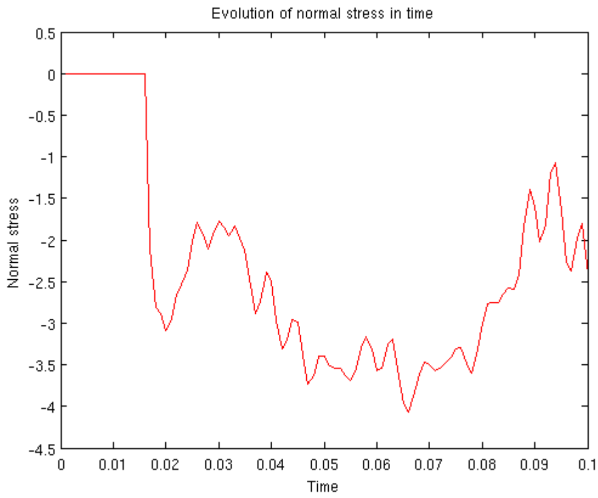


$\Delta t = 10^{-4}$

Figure 10: *Energy evolution for Paoli-Schatzman scheme with* $e = 1/2$.



$\Delta t = 10^{-3}$



$\Delta t = 10^{-4}$

Figure 11: *Normal stress evolution on the lowest point of the disc for Paoli-Schatzman scheme with $e = 1/2$.*

More detailed numerical tests on this method can also be found in [7].

5.2 Comparison with Laursen-Chawla scheme

The Laursen-Chawla scheme [12], which is energy conserving, is based on a midpoint rule for the elastodynamic part (modified for large deformations) and a particular discretization of the contact condition both in term of normal displacement and in normal velocity allowing the persistency condition to be satisfied. However, a small interpenetration is allowed.

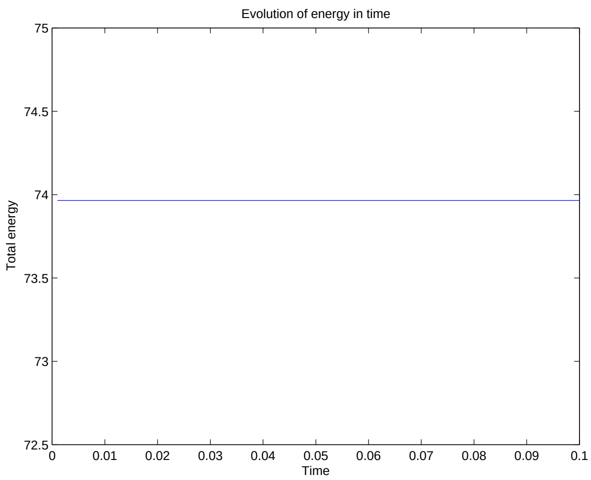
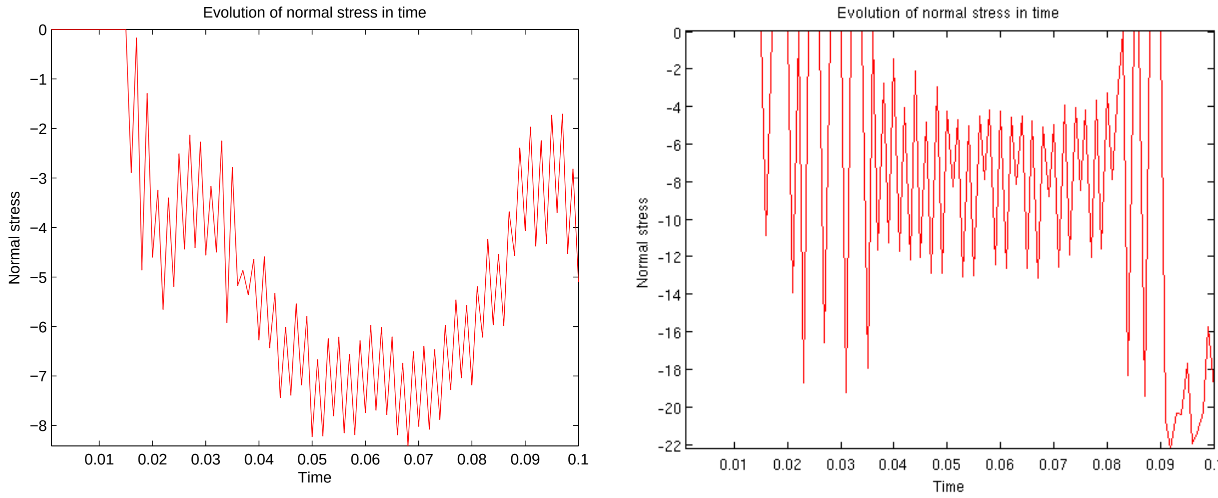


Figure 12: *Energy evolution for Laursen-Chawla scheme ($\Delta t = 10^{-3}$).*

The result of the numerical test is presented on Fig. 12 and 13. Despite the fact that the conservation of energy is verified, the normal stress on the contact boundary is oscillating and does not seem to converge.



$\Delta t = 10^{-3}$ $\qquad$ $\Delta t = 10^{-4}$

Figure 13: *Normal stress evolution on the lowest point of the disc for the Laursen-Chawla scheme.*

Concluding remarks

The main advantage of the presented mass redistribution method compared to other methods is that it leads to a well-posed semi-discretized finite element problem whose solution is energy conserving. This allows the use of a large set of time integration schemes for the approximation of elastodynamic contact problems. The numerical tests shows that the quality of the normal stress on the contact surface is greatly improved.

For the sake of simplicity, we presented the method in the framework of small deformations and with a frictionless contact condition. In fact, adding a Coulomb friction condition is not a difficulty from a stability viewpoint due to its dissipative characteristic. The extension to large deformations is quite straightforward. Note also that the mass redistribution method is consistent in the sense that when the mesh parameter goes to zero the amount of redistributed mass goes to zero as well.

Aknowledgments

We would like to thank Julien Pommier[4] for insightful discussions and help in numerical improvements of the mass redistribution method.

This work is supported by "l'Agence Nationale de la Recherche", project ANR-05-JCJC-0182-01.

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