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A FICTITIOUS DOMAIN APPROACH FOR STRUCTURAL OPTIMIZATION WITH A COUPLING BETWEEN SHAPE AND TOPOLOGICAL GRADIENT

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Abstract

In this paper, we focus on numerical aspects of structural optimization. We combine shape and topological optimization of structure with a fictitious domain approach. After recalling some standard results of shape and topological optimization, we build an algorithm inspired from the extended finite elements method (Xfem) principles and give some numerical results. Numerical results confirm that the method is efficient and gives better result compared with the classical shape optimization techniques.

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1. Introduction

The goal of this paper is to propose a numerical scheme coupling two recent methods in shape and topological optimization of structures. Most of the known results concern structural mechanics. Classical topology optimization involves relaxed formulation and homogenization. Such methods involve many drawbacks: for example, the optimal shape is not a classical design, then some penalization method must be applied to retrieve a realistic shape; global optimization technics like genetic algorithm have been proposed but with a high computational cost.

In this paper, we recall classical results of shape and topological optimization [5, 11, 12] in order to build a numerical scheme based on the extended finite element method (Xfem).

Shape gradient is used here in order to update the level-set used in the Xfem. Shape optimization problem is a minimization problem where the unknown variable run over a class of domains; then every shape optimization problem can be written in the form

$$\min\{j(A) : A \in \mathcal{A}\},$$

where $\mathcal{A}$ is a class of admissible domains and j is the cost functional.

Topological optimization aims to write asymptotic expansion of a shape functional in the form

$$j(\Omega_\rho) = j(\Omega) = G(x_0)h(\rho) + o(h(rho)),$$

$$h(\rho) > 0, \quad \text{and} \quad \lim_{p \to 0} h(\rho) = 0,$$

where $\Omega_\rho = \Omega \backslash \overline{(x_0 + \rho\omega)}$ for $\rho > 0$, $x_0 \in \Omega \subset \mathbb{R}^n$, $n = 2, 3$, $\omega \subset \Omega$ is a reference domain. The topological sensitivity $G(x_0)$ provides an information for creating a small holes located at x_0. Hence the function G can be used like a descent direction in optimization process. Topological sensitivity analysis is used here in the algorithm in order to create new holes in fictitious initial domain.

The extended finite element method (Xfem) was introduced by Moës in [13, 19] and developed in many papers such as [7, 22, 14, 18].

In this paper, we propose a method which combines a fictitious domain approach for the approximation of the elastic displacement and the use of both shape and topological derivatives. The fictitious domain approach is the one presented in [10] which is inspired from Xfem principles [13, 19]. In the following, we describe this method adapted to our problem. It is proved in [10] that the approximation of the solution is optimal.

The paper is organized as follows: in Section 2, we introduce the model of linear elastic and some objective functionals as the compliance and the least square error. In Sections 3 and 4, we recall to classical results and give shape and topological gradient associated to our functionals. The main part of this paper is Section 5, in which we propose a numerical method for approximation of the elastic problem, to end this section, we give some numerical result in order to illustrate the efficacy of the proposed algorithm.

2. Problem Setting

Let Ω be an open and bounded set of $\mathbb{R}^n$, $n = 2$ or 3. The linearized elastic problem is the following [6]: Find u_Ω such that

$$\begin{cases} -\operatorname{div} \sigma(u_\Omega) = f, & \text{in } \Omega, \\ u_\Omega = 0, & \text{on } \Gamma_D, \\ \sigma(u_\Omega) \cdot \nu = g, & \text{on } \Gamma_N. \end{cases} \tag{1}$$

Here, ν is the outward normal to the boundary $\partial\Omega = \Gamma_N \bigcup \Gamma_D$, Γ_D and Γ_N have both nonzero Lebesgue measure in $\partial\Omega$ and $\Gamma_D \bigcap \Gamma_N = \varnothing$.

The linearized strain tensor ε and the stress tensor σ are given by

$$\sigma_{ij}(u) = \lambda(\operatorname{div} u)\delta_{ij} + 2\mu\varepsilon_{ij}(u),$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \le i, \ j \le n,$$

$\lambda > 0$ and $\mu > 0$ denote the Lamé coefficients of the material, here δ_{ij} is the Kronecker symbol.

The weak form of the problem is classically written

$$\text{Find } u_\Omega \in V : a(u_\Omega, v) = l(v) \quad \forall v \in V, \tag{2}$$

with

$$V = \{v \in H^1(\Omega; \mathbb{R}^n) : v|_{\Gamma_D} = 0\},$$

$$a(u, v) = \int_\Omega \sigma(u) : \varepsilon(v)\,dx,$$

$$l(v) = \int_\Omega f \cdot v dx + \int_{\Gamma_N} g \cdot v ds.$$

We suppose that all required hypotheses for uniqueness and existence of a solution are satisfied, in particular we may assume that $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, Ω sufficiently regular and Γ_D of nonzero measure on $\partial\Omega$. In such case, it is well know that (2) admits a unique solution in V due to Korn inequality.

A shape optimization problem is a minimization problem where the unknown variable run over a class of domains, then every shape optimization problem can be written in the form

$$\min\{F(A) : A \in \mathcal{U}_{ad}\},$$

where $\mathcal{U}_{ad}$ is the class of admissible domains and F is the cost function that we have to minimize over $\mathcal{U}_{ad}$. More details can be found in [4].

The problem consists in calculating the shape derivative and the topological one of a shape functional $J(u_\Omega)$, where u_Ω is the solution of (1) in order to propose an implementation.

We focus on the two following functional, but more general functional can be used

$$J_1(\Omega, u) = \int_\Omega f \cdot u dx + \int_{\Gamma_N} g \cdot u ds$$
$$= \int_\Omega \sigma(u) : \varepsilon(u)\,dx \tag{3}$$

and

$$J_2(\Omega, u) = \int_\Omega | u - u_d |^2 dx. \tag{4}$$

It is well known that the maximization of (3) or the minimization of (4) is not well posed. In order to obtain the existence of optimal shape, some smoothness, geometrical or topological constraints will be prescribed in the class of admissible domains $\mathcal{U}_{ad}$.

Let us recall the Lagrangian method [5, 11] in order to compute shape and topological derivatives. The Lagrangian operator is defined by

$$\mathfrak{L}(u, v) = a(u, v) - l(v) + J(u).$$

Writing that the variation of the cost functional is equal to the one of the Lagrangian, it follows the results which give the topological and the shape gradient.

3. Shape Derivative

In order to compute shape derivative, we use the approach of Murat and Simon in [15], recalled by Henrot and Pierre in [11] and Allaire et al. in [1]. We consider a perturbation of the domain Ω in the following sense: for $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $\Omega_\theta = (Id + \theta)(\Omega)$. It is well known that for θ sufficiently small $(Id + \theta)$ is a diffeomorphism in $\mathbb{R}^n$.

Definition 3.1. The shape derivative of $J(\Omega)$ at Ω is defined as the *Frechet derivative* in $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ at 0 of the application $\theta \to J((Id + \theta)(\Omega))$, i.e.,

$$J((Id + \theta)(\Omega)) = J(\Omega) + J'(\Omega)(\theta) + o(\theta),$$

where $J'(\Omega)$ is a continuous and linear form on $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

It follows from the definition and some symbolic calculus, the following result which is somewhat standards (see [1, 5, 11]).

Theorem 3.2. *Let Ω be a smooth bounded open set and $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$. We assume that the data f, g and the solution u_Ω of* (1) *are sufficiently smooth. Then the shape derivative of* (3) *is given by*

$$J_1'(\Omega)(\theta) = \int_{\Gamma_N} \left(2\left[\frac{\partial(g \cdot u_\Omega)}{\partial \nu} + Hg \cdot u_\Omega + f \cdot u_\Omega \right] - \sigma(u_\Omega) : \varepsilon(u_\Omega) \right) \theta \cdot \nu ds$$
$$+ \int_{\Gamma_D} (\sigma(u_\Omega) : \varepsilon(u_\Omega) - f \cdot u_\Omega) \theta \cdot \nu ds, \tag{5}$$

and the shape derivative of (4) *is*

$$J_2'(\Omega)(\theta)$$
$$= \int_{\Gamma_N} \left(C| u_\Omega - u_d |^2 + \sigma(p_\Omega) : \varepsilon(u_\Omega) - \frac{\partial(g \cdot p_\Omega)}{\partial \nu} - Hg \cdot p_\Omega - f \cdot p_\Omega \right) \theta \cdot \nu ds$$
$$+ \int_{\Gamma_D} (C| u_\Omega - u_d |^2 + \sigma(u_\Omega) : \varepsilon(p_\Omega) - f \cdot u_\Omega) \theta \cdot \nu ds, \tag{6}$$

where p is the solution of the adjoint equation, assumed also to be smooth and defined by

$$\begin{cases} -\mathrm{div}\, \sigma(p_\Omega) = -2(u_\Omega - u_d), & in \quad \Omega, \\ p_\Omega = 0, & on \quad \Gamma_D, \\ \sigma(p_\Omega)\nu = 0, & on \quad \Gamma_N, \end{cases} \tag{7}$$

and H is the mean curvature defined on Γ_N *and C is a positive constant which depend on the space dimension*.

4. Topological Derivative

For a given $x_0 \in \Omega,$ consider the perforated open set $\Omega_\rho = \Omega \backslash \overline{\omega_\rho},$ $\omega_\rho = x_0 + \rho\omega,$ $\omega \in \mathbb{R}^n$ is a fixed reference domain. We recall here to the general adjoint method and domain truncation [12] in order to get topological derivative.

Let u_{Ω_ρ} be the solution of the equation in the perturbed domain:

$$\begin{cases} -\mathrm{div}\, \sigma(u_{\Omega_\rho}) = f & \text{in} \quad \Omega_\rho, \\ u_{\Omega_\rho} = 0 & \text{on} \quad \Gamma_D, \\ \sigma(u_{\Omega_\rho}) \cdot \nu = g & \text{on} \quad \Gamma_N, \\ \sigma(u_{\Omega_\rho}) \cdot \nu = 0 & \text{on} \quad \partial\omega_\rho. \end{cases} \tag{8}$$

The aim of the topological optimization is to compute the difference $J(\Omega_\rho) - J(\Omega).$ For many cases, the asymptotic expansion of the function *J* can be obtained in the following form:

$$J(\Omega_\rho) = J(\Omega) + h(\rho)G(x_0) + o(h(\rho)),$$

$$\lim_{\rho \to 0} h(\rho) = 0, \quad h(\rho) > 0. \tag{9}$$

The function $h(\rho)$ depends of the space dimension and the boundary conditions on $\partial\omega.$ See [9], for example.

The function $G(x_0)$ is called *topological derivative* (or *topological sensitivity*) and provides an information for creating a small hole located at x_0. Hence the function *G* can be used like a descent direction in optimization process.

Possibly changing the coordinate system, we can suppose for convenience $x_0 = 0$. Let v_ω be the solution of the exterior problem

$$\begin{cases} -\operatorname{div} \sigma(v_\omega) = 0, & \text{in} \quad \mathbb{R}^n \backslash \overline{\omega}, \\ v_\omega = 0, & \text{at} \quad \infty, \\ \sigma(v_\omega) \cdot \nu = \sigma(u_\omega) \cdot \nu & \text{in} \quad \partial\omega. \end{cases} \tag{10}$$

This function can be expressed by a single layer potential on $\partial\omega$

$$v_\omega(y) = \int_{\partial\omega} E(y - x)\, p(x)\, d\gamma(x), \quad \forall y \in \mathbb{R}^n \backslash \overline{\omega},$$

where $p \in H^{-1/2}(\partial\omega)$ is the solution of the boundary integral equation

$$\frac{p(y)}{2} + \int_{\partial\omega} \sigma_y(E(y - x))\, p(x)\, d\gamma(x) = \sigma(u_\Omega(x_0))\nu(y), \quad \forall y \in \partial\omega, \tag{11}$$

and E is the fundamental solution for the elasticity problem in $\mathbb{R}^n$.

Using Taylor expansion of E, it follows that v_ω can be written as $v_\omega(y) = P_\omega(y) + W(y)$ with

$$P_\omega(y) = \int_{\partial\omega} DE(x)\, x p(x)\, d\gamma(x), \quad W_\omega(y) = O\left(\frac{1}{\| y \|^n}\right).$$

Let $D_\rho = B(x_0, R) \backslash \omega_\rho$ (R is choosing such that $\omega_\rho \subset B(x_0, R)$), and Q_ω be the solution of the interior problem

$$\begin{cases} -\operatorname{div} \sigma(Q_\omega) = 0, & \text{in} \quad D_0, \\ Q_\omega = P_\omega, & \text{on} \quad \partial B(x_0, R) = \Gamma_R. \end{cases} \tag{12}$$

Using topological optimization result, cf. [12], the variation of the bilinear form associated to equation (8) can be written

$$\delta_a(u_\Omega, p_\Omega) = \int_{\Gamma_R} \sigma(Q_\omega - P_\omega) \cdot \nu p_\Omega\, d\gamma(x),$$

where p_Ω is the solution of the adjoint problem.

Let

$$A_{ik}(\sigma(u_\Omega)(x_0)) = \int_{\partial\omega} p_i(x)\, x_k\, d\gamma(x), \quad p = (p_i)_{1 \le i \le n}, \quad x = (x_k)_{1 \le k \le n}$$

be the mass matrix, if $-\mathrm{div}\,(\sigma(p_\Omega)) = 0$ in D_0, then

$$a_\rho(u_\Omega,\ p_\Omega) - a_0(u_\Omega,\ p_\Omega) = \delta_a(u_\Omega,\ p_\Omega) = A(\sigma(u_\Omega(x_0) : \varepsilon(p_\Omega(x_0)))).$$

Using Saint-Venant principle, a good approximation of p can be obtained by computing u_{ext} in $\Omega\backslash\omega$ and u_{int} on ω, thus p is approximatively equal to the jump $\sigma(u_{ext})\nu - \sigma(u_{\mathrm{int}})\nu$.

When ω be the unit ball of $\mathbb{R}^n$, P and A can be explicitly computed.

Since the topological gradient is in the form $\delta_{\mathcal{L}}(u_\Omega,\ p_\Omega) = \delta_a(u_\Omega,\ p_\Omega) + \delta_J(u_\Omega)$.

The topological derivative of (3) and (4) follows, cf. [9], for the proof.

Theorem 4.1. *Let Ω be a smooth bounded open set and Ω_ρ defined as below and $\omega = B(0, 1)$, (the unit ball of $\mathbb{R}^n$, $n = 2, 3$). Then the topological derivative is given by*:

For $J_1(\Omega)$,

$$G(x_0) = -\frac{\pi(\lambda+2\mu)}{2\mu(\lambda+\mu)}[4\mu\sigma(u_\Omega):\varepsilon(u_\Omega)+(\lambda-\mu)tr\sigma(u_\Omega)tr\varepsilon(u_\Omega)],\ \ in\ \ \mathbb{R}^2, \tag{13}$$

$$G(x_0) = -\frac{\pi(\lambda+2\mu)}{\mu(9\lambda+14\mu)}[20\mu\sigma(u_\Omega):\varepsilon(u_\Omega)+(3\lambda-2\mu)tr\sigma(u_\Omega)tr\varepsilon(u_\Omega)],\ \ in\ \ \mathbb{R}^3. \tag{14}$$

For $J_2(\Omega)$,

$$G(x_0) = -\frac{\pi(\lambda+2\mu)}{2\mu(\lambda+\mu)}[4\mu\sigma(u_\Omega):\varepsilon(p_\Omega)+(\lambda-\mu)tr\sigma(u_\Omega)tr\varepsilon(p_\Omega)],\ \ in\ \ \mathbb{R}^2, \tag{15}$$

$$G(x_0) = -\frac{\pi(\lambda+2\mu)}{\mu(9\lambda+14\mu)}[20\mu\sigma(u_\Omega):\varepsilon(p_\Omega)+(3\lambda-2\mu)tr\sigma(u_\Omega)tr\varepsilon(p_\Omega)],\ \ in\ \ \mathbb{R}^3. \tag{16}$$

where p_Ω is the solution of the adjoint equation, assumed to be smooth, as u_Ω, defined by (7).

5. The Proposed Numerical Method

5.1. Numerical approximation

In this section, we propose a method which combines a fictitious domain approach for the approximation of the elastic displacement and the use of both shape and topological derivatives. The fictitious domain approach is the one presented in [10] which is inspired from Xfem principles [13, 19]. In the following, we describe this method adapted to our problem. It is proven in [10] that the approximation of the solution is optimal.

The boundary of the domain being an unknown of the problem, we introduce $\widetilde{\Omega}$ a fixed (in general, rectangular or parallelepiped) domain which includes all the potential domains Ω. This fictitious domain approach requires the introduction of two finite element spaces $\widetilde{V}^h \subset H^1(\widetilde{\Omega}; \mathbb{R}^n)$ and $\widetilde{W}^h \subset L^2(\widetilde{\Omega}; \mathbb{R}^n)$ on the fictitious domain $\widetilde{\Omega}$. As $\widetilde{\Omega}$ can be a rectangular or parallelepiped domain, the ones can be defined on the same structured mesh $\mathcal{T}^h$ (see, Figure 1). Next, we shall suppose that

$$\widetilde{V}^h = \{v^h \in \mathcal{C}(\overline{\widetilde{\Omega}}; \mathbb{R}^n) : v^h|_T \in (P(T))^n \ \forall T \in \mathcal{T}^h\}, \tag{17}$$

where $P(T)$ is a finite dimensional space of regular functions such that $P(T) \supseteq P_k(T)$ for some $k \geq 1$, integer. The mesh parameter h stands for $h = \max_{T \in \mathcal{T}^h} h_T$, where h_T is the diameter of T.

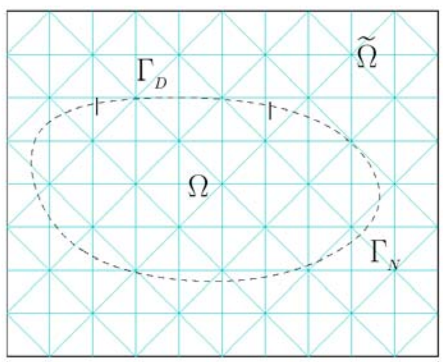

Figure 1. Example of a structured mesh.

Thus we can build

$$V^h := \widetilde{V}^h|_{\Omega} \quad \text{and} \quad W^h := \widetilde{W}^h|_{\Gamma_D}$$

which are natural discretizations of V and $W = L^2(\Gamma_D; \mathbb{R}^n)$, respectively. A mixed approximation of Problem (2) is defined as follows:

$$\begin{cases} \text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that} \\ a(u^h, v^h) + \int_{\Gamma_D} \lambda^h \cdot v^h ds = l(v^h) \quad \forall v^h \in V^h, \\ \int_{\Gamma_D} \mu^h \cdot u^h ds = 0 \quad \forall \mu^h \in W^h. \end{cases} \tag{18}$$

Similarly to Xfem, where the shape functions of the finite element space is multiplied with an Heaviside function, this corresponds here to the multiplication of the shape functions with the characteristic function of Ω.

Unfortunately, such a simple formulation leads to a potentially poor approximation of the solution (in $O(\sqrt{h})$ generally, see [10]) due to a possible locking phenomenon on the Dirichlet boundary. This is why it is necessary to consider an additional stabilization. Here, we adapt a stabilization technique presented by Barbosa and Hughes in [2, 3] in order to recover an optimal rate of convergence. Note that this stabilization technique can be viewed as a generalization of the former Nitsche's method [16], where the multipliers are eliminated (see [23] for the link between the two stabilization techniques). We present its symmetric version although the nonsymmetric one can be considered in the same way. This technique is based on the addition of a supplementary term involving the normal derivative on Γ_D. Let us suppose that we have at our disposal an operator

$$R^h : V^h \to L^2(\Gamma_D)$$

which approximates the normal stress on Γ_D (i.e., for $v^h \in V^h$ converging to a sufficiently smooth function v, $R^h(v^h)$ tends to $\sigma(v) \cdot \nu$ in an appropriate sense). A first straightforward choice is given by

$$R^h(v^h) = \sigma(v^h)\nu = (\lambda(\operatorname{div} v^h)\delta_{ij} + 2\mu\varepsilon_{ij}(v^h))\nu_j.$$

In [10], we can see that this gives some satisfactory numerical results in most of the cases except where there is some very small intersection of an element with the real domain Ω. In the latter case, it is proven that a good approximation can be recovered using the extrapolation of $\sigma(v^h)\nu$ on a neighbor element having a sufficiently large intersection with Ω.

Now, we obtain the stabilized problem by considering the following Lagrangian for $v^h \in V^h$ and $\mu^h \in W^h$:

$$\mathcal{L}_h(v^h, \mu^h) = a(v^h, v^h) - l(v^h) + \int_{\Gamma_D} \mu^h \cdot v^h ds - \frac{\gamma}{2}\int_{\Gamma_D} \| \mu^h + R^h(v^h) \|^2 ds,$$

where, for the sake of simplicity, $\gamma := h\gamma_0$ is chosen to be a positive constant over Ω (for non-uniform meshes, an element dependent parameter $\gamma = h_T\gamma_0$ is a better choice).

The corresponding discrete problem reads as follows:

$$\begin{cases} \text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that} \\ a(u^h, v^h) + \int_{\Gamma_D} \lambda^h \cdot v^h ds - \gamma \int_{\Gamma_D} (\lambda^h + R^h(u^h)) \cdot R^h(v^h) ds = l(v^h) \quad \forall v^h \in V^h, \\ \int_{\Gamma_D} \mu^h \cdot u^h ds - \gamma \int_{\Gamma_D} (\lambda^h + R^h(u^h)) \cdot \mu^h ds = 0 \quad \forall \mu^h \in W^h. \end{cases} \tag{19}$$

This formulation is consistent in the sense that the additional term $\| \mu^h + R^h(v^h) \|^2$ should vanish when h goes to zero since it is well known that in Problem (18), the multiplier μ^h is an approximation of the opposite of the normal stress.

The parameter γ_0 has to be taken sufficiently small such that the coerciveness of the bilinear form is kept. The quality of the approximation is not very sensitive to the parameter γ_0 which can be chosen in a wide range of values.

Now, the shape optimization process needs the description of the boundary of Ω. As in [1] or also in the framework of Xfem in [19], we chose a level-set approximation of the boundary. This means that the boundary will be represented by the zero level-set of a function approximated on a convenient finite element space.

The advantage of this approach is to obtain both an optimal approximation of the elasticity problem together with an accurate location of boundary of the real domain. Note that to keep the optimality of the approximation, the level-set functions have to be approximated at the same order than the displacement u.

5.2. Optimization algorithm

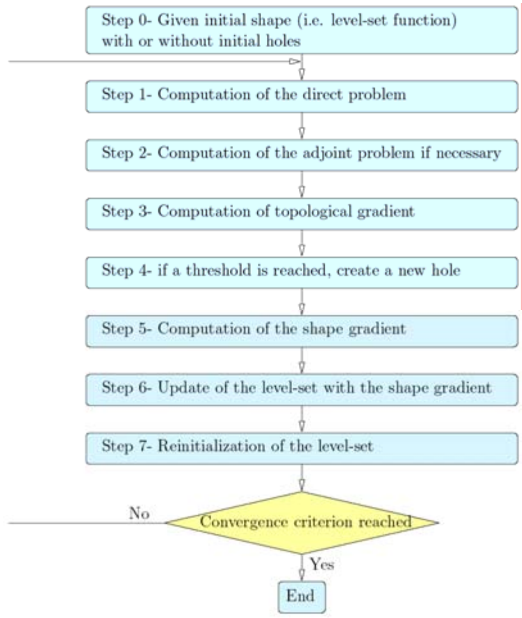

Figure 2. Structural optimization algorithm.

The optimization algorithm is summarized in Figure 2. Since we use the topological gradient to create holes during the optimization process, it is possible to start with a shape containing some initial holes or not. A very small penalization is used when solving the direct problem and the adjoint one to avoid the indeterminacy of the rigid motions of eventual isolated part. Concerning Step 4, a new hole of a given radius is created by the simple operation on the level-set function which can be written on each finite element node x_i,

$$\overline{\psi}(x_i) := \max(\psi(x_i), (r^2 - \| x_i - c \|^2)/(2r)),$$

where $\psi(x)$ is the level-set function, $\overline{\psi}(x_i)$ is its new value, r is the radius of the created hole and c is its center.

At Step 6, the update of the level-set is done directly thanks to the shape derivative applying the following evolution equation for the level-set function:

$$\frac{\partial \psi}{\partial t} = g(x), \quad \text{in} \quad \widetilde{\Omega}, \tag{20}$$

where $g(x)$ corresponds to the function in front of $\theta \cdot \nu$ in the integral of (5). This evolution equation is integrated on a small time interval. In our simulations, the gradient is extended by zero on the complementary of Ω in $\widetilde{\Omega}$. However, a smoother extension could be considered. This method is simpler than the classical way which is to integrate a Hamilton-Jacobi equation (see [1]). It seems also to be numerically more robust.

Note that it is convenient to apply a threshold on the gradient to avoid some incoherent values where the shape gradient may have a singularity (corner, transition form Dirichlet to Neumann condition, ...).

In order to regularize the level-set function, the reinitilization Step 7 is considered. It consists classically in solving

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + \operatorname{sign}(\psi_0)(|\nabla \psi| - 1) = 0 \quad \text{in} \quad \widetilde{\Omega} \times \mathbb{R}_+, \\ \psi(0, x) = \psi_0(x) \quad \text{in} \quad \widetilde{\Omega}, \end{cases} \tag{21}$$

whose stationary solution is a signed distance. This Hamilton-Jacobi equation is known to admit multiple nonsmooth solutions. Classically, a smooth solution is computed thanks to an upwind scheme. Since the fictitious domain $\widetilde{\Omega}$ can be a rectangular/parallelepiped domain, it is possible to use a classical upwind scheme on a cartesian grid (see [21]). However, to keep the possibility of having a non-structured mesh, for instance to proceed to a local refinement, we use a different strategy. Equation (21) is solved on a small time interval $]0, \Delta t]$ integrating the following equation where the non-linearity is made explicit:

$$\begin{cases} \dfrac{\partial \overline{\psi}}{\partial t} + \operatorname{sign}(\psi_0) \dfrac{\nabla \psi^n}{|\nabla \psi^n|} \nabla \overline{\psi} = \operatorname{sign}(\psi_0) \quad \text{in} \quad \widetilde{\Omega} \times]0, \Delta t], \\ \overline{\psi}(0, x) = \psi^n(x). \end{cases} \tag{22}$$

Here ψ^n is the level-set function at the previous time step and ψ^{n+1} is given by $\overline{\psi}(\Delta t, \cdot)$. The problem (22) is a pure convection one. This problem can be solved for instance with the simple Galerkin-characteristic scheme proposed in [24]. This scheme is unconditionally stable but rather dissipative. The effect is that the level-sets are a little bit smoothed.

5.3. Numerical results

In the two-dimension, the working domain is a rectangle of size 2×1 discretized by a regular triangle mesh and an affine finite element method. For the boundary conditions, we fix Γ_D with homogeneous Dirichlet boundary condition and apply a Neumann condition on the right size with force of intensity g.

Numerical results confirm that the method is efficient and gives better result compared with the classical shape optimization techniques.

5.3.1. The two dimensional case

We give here some numerical results obtained for the two dimensional case at two steps in a 80×160 mesh. We give, left, the shape with initial holes and right, without initial hole.

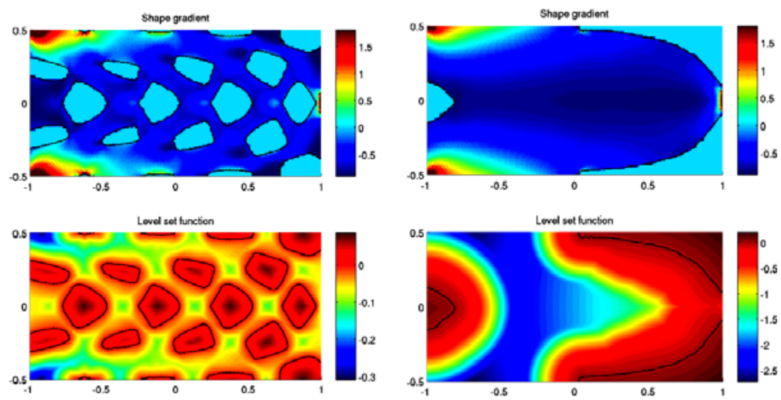

Figure 3. Second step.

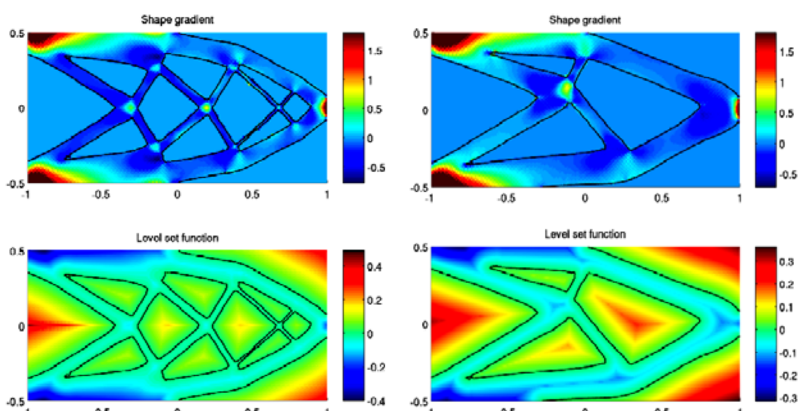

Figure 4. Fourth step (optimal design).

We give here some numerical results obtained for the two dimensional case in different initial meshes. We give, left, the shape with initial holes and right, without initial hole.

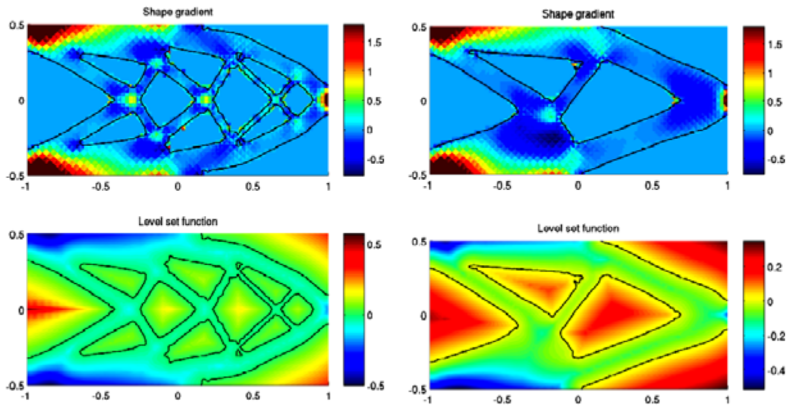

Figure 5. Optimal shape design in a 40×80 mesh.

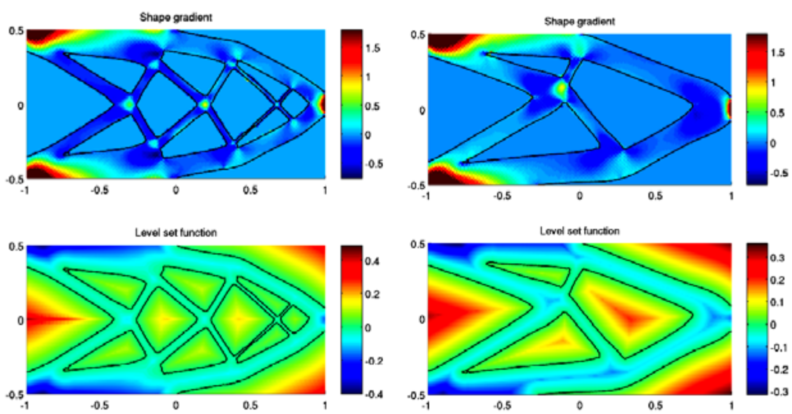

Figure 6. Optimal shape design in a 80×160 mesh.

5.3.2. The three dimensional case

Here are different steps to reach the optimal shape design in a $30 \times 30 \times 60$ mesh. We represent two steps of the evolution of the shape obtained, left, the shape with initial holes and right, without initial hole.

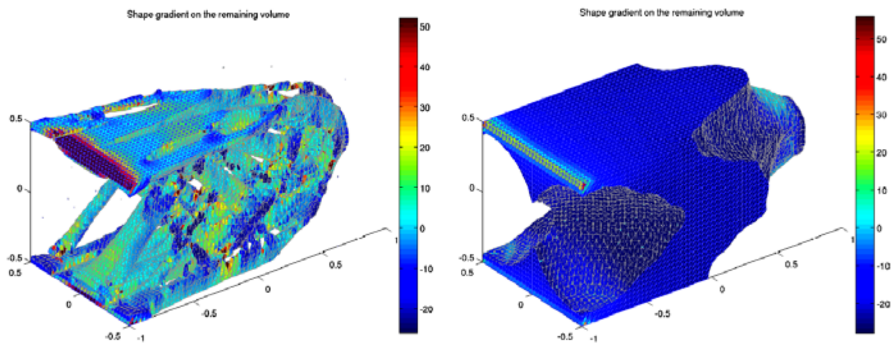

Figure 7. Second step.

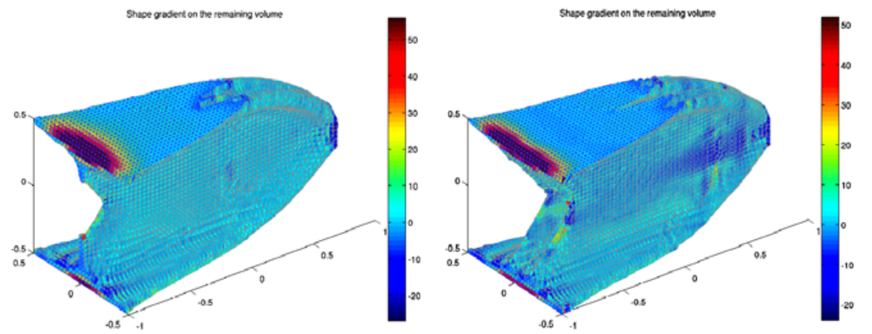

Figure 8. Fourth step (optimal shape).

References

[1] G. Allaire, F. Jouve and A.-M. Toader, Structural optimization using sensitivity analysis and a level-set method, J. Comput. Phys. 194 (2004), 363-393.

[2] H. J. C. Barbosa and T. J. R. Hughes, The finite element method with Lagrange multipliers on the boundary: circumventing the Babuška-Brezzi condition, Comput. Meth. Appl. Mech. Engrg. 85 (1991), 109-128.

[3] H. J. C. Barbosa and T. J. R. Hughes, Boundary Lagrange multipliers in finite element methods: error analysis in natural norms, Numer. Math. 62 (1992), 1-15.

[4] D. Bucur and G. Buttazo, Variational Methods in Shape Optimization Problem, Progress in Nonlinear Differential Equations and its Applications, Birkhäuser, 2005.

[5] J. Cea, Conception optimale on identification de forme, Calcul rapide de la dérivée directionnelle de la fonction coût, RAIRO. Math. Anal. Numèr. 20 (1986), 371-402.

[6] Ph. Ciarlet, The Finite Element Methode for Elliptic Problems, North-Hollande, Amsrerdam, 1978.

[7] E. Chahine, P. Labore and Y. Renard, Crack-tip enrichment in the Xfem method using a cut-off function, Submitted.

[8] P. Fulkmański, A. Laurain, J. F. Schied and J. Sokolowski, A level-set in shape and topology optimization for variational inequalities, Int. J. Math. Comput. Sci. 17(3) (2007), 413-430.

[9] S. Garreau, Ph. Guillaume and M. Masmoudi, The topological asymptotic for PDE systems: the elastic case, SIAM Control Optim. 39(6) (2001), 1756-1778.

[10] J. Haslinger and Y. Renard. A new fictitious domain approach inspired by the extended finite element method, SIAM J. Numer. Anal. 47(2) (2009), 1474-1499.

[11] An. Henrot and M. Pierre, Variation et Optimisation de formes: Une Analyse Géométrique, Springer, Berlin, Heidelberg, 2005.

[12] M. Masmoudi, The topological asymptotic, Computational Methods for Control Applications, H. Kawarada and J. Periaux, eds., GAKUTO Internat. Ser. Math. Sci. Appli. Gakkotōsho, Tokyo, 2002.

[13] N. Moës, J. Dolbow and T. Belytschko, A finite element method for crack growth without remeshing, Int. J. Numer. Meth. Engng. 46 (1999), 131-150.

[14] N. Moës, A. Gravouil and T. Belytschko, Non-planar 3D crack growth by the extended finite element method and level sets, Part I: Mechanical model, Int. J. Numer. Meth. Engng. 53 (2002), 2549-2568.

[15] F. Murat and J. Simon, Sur le Contrôle par un Domaine Géométrique, Habilitation de l'Université de Paris, 1976.

[16] J. Nitsche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen Unterworfen Sind, Abh. Math. Univ. Hamburg 36 (1971), 9-15.

[17] Y. Renard and J. Pommier, Getfem. An open source generic library for finite element methods, http://home.gna.org/getfem.

[18] F. L. Stazi, E. Budyn, J. Chessa and T. Belytschko, An extended finite element method with higher-order for curved cracks, Comput. Mech. 31 (2003), 38-48.

[19] N. Sukumar, D. L. Chopp, N. Moës and T. Belytschko, Modeling holes and inclusions by level sets in the extended finite-element method, Comput. Methods Appl. Mech. Engrg. 190 (46-47) (2001), 6183-6200.

[20] J. Serrin, A symmetric problem in potential theory, Arch. Rational Mech. Anal. 43 (1971), 304-318.

[21] J. A. Sethian, Level-set Methods and Fast Marching Methods: Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision and Material Science, Cambridge University Press, Cambridge, 1999.

[22] N. Sukumar, N. Moës and T. Belytschko, Extended finite element method for three dimensional crack modeling, Int. J. Numer. Engrg. 48 (2000), 1549-1570.

[23] R. Stenberg, On some techniques for approximating boundary conditions in the finite element method, J. Comput. Appl. Math. 63 (1995), 139-148.

[24] O. C. Zienkiewicz and R. L. Taylor, The Finite Element Method, Fluids Dynamics, Sixth Edition, Vol. 3, Elsevier, 2005.