A WELL-POSED SEMI-DISCRETIZATION OF ELASTODYNAMIC CONTACT PROBLEMS WITH FRICTION

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Summary

It has been shown numerically in previous works that the well-posedness of the spatial semi-discretization plays a crucial role in obtaining stable numerical schemes for elastodynamic frictionless contact problems. The purpose of this paper is thus to introduce a mass redistribution method adapted to elastodynamic contact with Coulomb friction which guarantees the well-posedness of the semi-discrete problem. It is shown that a differentiated treatment has to be applied to the friction condition. Some numerical tests illustrating the gain in stability for the midpoint time integration scheme are presented. They suggest also that, although the differentiated treatment is necessary for the well-posedness, it is not always mandatory from the numerical viewpoint.

keywords: elasticity, unilateral contact with friction, stability, mass redistribution method.

74M15, 74M10, 65M60, 35L87.

1. Introduction

The aim of this paper is to describe a spatial well-posed semi-discretization for elastodynamic unilateral contact problems with Coulomb friction. This kind of problems is of interest in computational mechanics, where situations with a frictional contact condition are fairly common. Moreover, it is well known that the full discretization of elastodynamic contact problems induces a number of difficulties.

Numerous works have already been dedicated to the construction of numerical schemes that are as much as possible stable, respecting the contact constraint and not leading to spurious oscillations. Among the strategies already proposed in the literature, we refer to (**23**) for a time integration scheme which is adapted to take into account a restitution coefficient coming from an impact law. Although this approach is better fitting to the case of rigid solids, the addition of an impact law makes the semi-discrete problem also well posed. However, the nature of this problem is a measure differential inclusion in time (see (**21**, **22**, **23**, **24**)), which is a very low regular problem. Another proposed strategy is to build energy dissipative schemes. This is the case in (**6**, **2**), where the contact force is implicited. The drawback of this method is that the kinetic energy of the contacting nodes is canceled at each impact. A less radical solution is to build energy conserving schemes. Such schemes are introduced in (**18**, **17**, **9**). However, energy conserving schemes either introduce spurious oscillations on the contact boundary or allow a small interpenetration. It is possible to build energy conserving schemes with a penalized contact condition (**1**, **7**, **9**), but this also leads to important oscillations of the normal stress. In this context, it was early detected that a key point is the satisfaction of the complementarity condition between the sliding velocity and contact pressure, the so-called persistency condition (**11**, **1**, **17**). But a compromise has to be made between the satisfaction of this condition and the nonpenetration condition.

A common point to all these works is that they are focused on finding a good time integration scheme. However, in (**15**) and (**29**), it is shown that this is rather obtaining a well-posed and regular spatial semi-discrete problem which allows for stable schemes (see also (**8**, **5**) for further developments). The spatial semi-discretizations proposed in (**15**) and (**29**) allow the use of any reasonable time integration scheme while almost all time integration schemes are unstable with the standard discretization. However, these works are focused on contact conditions without friction. One might think that the strategies developed there are directly applicable to friction condition. We will see thereafter that this does not provide the well-posedness results and therefore a strategy adapted to the friction condition is needed. This is also reflected in studies presented in (**27**) and (**25**), where it was shown that adding a mass on the contact boundary regularizes the tangential friction problem and prevents the occurrence of multiple solutions in elastodynamics. Note that the semi-discrete problem obtained by finite elements naturally adds a mass on the nodes of the contact boundary (but this is also the main difficulty for the unilateral contact condition!).

The method proposed in this paper is to apply the redistribution mass method introduced in (**15**) only on the unilateral contact condition, not on the friction one. We show that in this case, the semi-discrete problem in space reduces to a differential inclusion with a unique Lipschitz continuous solution (not to a measure differential inclusion as in the standard semi-discretization).

For the sake of simplicity, we limit ourselves to the small deformations framework. However, the same kind of difficulty exists for large deformation problems and similar strategies can be applied.

The outline of the paper is the following. In Section 2, we present a classical semi-discretization of an elastodynamic contact problem with friction. In Section 3, we propose an adaptation of the mass redistribution method, namely to apply it only on the normal component. Then, the well-posedness of the obtained semi-discrete problem is proved in Section 4. The unique solution of the semi-discrete problem is proved to be energy decreasing in Section 5. An elementary example is described in Section 6. It shows that the well-posedness of the fully discrete problem cannot be ensured when the mass redistribution method is applied both to contact and friction conditions. Finally, Section 7 is devoted to a numerical test which confirms the advantage of the mass redistribution method for the stability of the midpoint scheme.

2. A classical finite element approximation

In this section, we introduce a classical spatial semi-discretization based on the finite element method. Since we are mainly interested in the semi-discrete problem, we do not describe the weak formulation of the continuous problem. More details about such a discretization can be found for instance in (**14**, **13**, **16**).

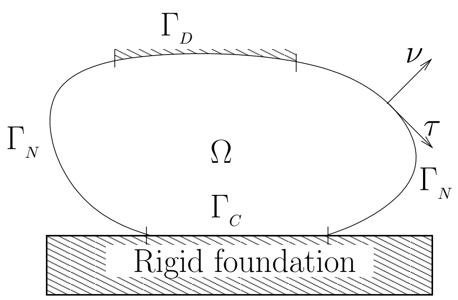

Fig. 1 Elastic body in contact with a rigid foundation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain representing the reference configuration of a linearly elastic body. The Neumann condition is prescribed on Γ_N, the Dirichlet one on Γ_D and a unilateral contact with the Coulomb friction law with respect to a rigid foundation on Γ_C (see Fig. 1). We suppose that Γ_N, Γ_D and Γ_C form a partition of $\partial\Omega$, the boundary of Ω. Let also ρ, $\sigma(u)$, $\varepsilon(u)$ and $\mathcal{A}$ be the mass density, the stress tensor, the linearized strain tensor and the elasticity tensor, respectively. The elastodynamic problem consists in finding the displacement field $u(t,x)$ satisfying

$$\begin{cases} \rho\ddot{u} - \mathrm{div}\ \sigma(u) = f & \text{in } (0,T] \times \Omega, \\ \sigma(u) = \mathcal{A}\ \varepsilon(u) & \text{in } (0,T] \times \Omega, \\ u = 0 & \text{on } (0,T] \times \Gamma_D, \\ \sigma(u)\nu = g & \text{on } (0,T] \times \Gamma_N, \\ u(0) = u^0,\ \dot{u}(0) = v^0 & \text{in } \Omega, \end{cases} \tag{1}$$

where, additionally, $T > 0$ determines the time interval of interest, $\nu \in \mathbb{R}^2$ is the outward unit normal vector to Ω on $\partial\Omega$ and f, g are some given external loads. Assuming the $\mathscr{C}^1$ regularity for Γ_C, we decompose the displacement and the stress vector into normal and tangential components on Γ_C as follows:

$$u_N = u.\nu, \quad u_T = u.\tau,$$
$$\sigma_N(u) = (\sigma(u)\nu).\nu, \quad \sigma_T(u) = (\sigma(u)\nu).\tau,$$

where $\tau \in \mathbb{R}^2$ is a tangent unit vector orthogonal to ν. Without real loss of generality, we also assume that there is no initial gap between the solid and the rigid foundation. Denoting by $\mathscr{F}$ the friction coefficient, the unilateral contact condition with Coulomb friction is expressed as follows:

$$\left.\begin{array}{l} u_N \le 0,\ \sigma_N(u) \le 0,\ u_N\sigma_N(u) = 0, \\ |\sigma_T(u)| \le -\mathscr{F}\sigma_N(u), \\ \sigma_T(u) = \mathscr{F}\sigma_N(u)\dfrac{\dot{u}_T}{|\dot{u}_T|} \quad \text{if } \dot{u}_T \ne 0 \end{array}\right\} \quad \text{on } (0,T] \times \Gamma_C. \tag{2}$$

Now, we consider a vector Lagrange finite element method defined on Ω. Let $a_1, \cdots, a_n$ be the finite element nodes and $\varphi_1, \cdots, \varphi_{n_p}$ the (vector) shape functions of the finite element displacement space. We denote by n_c the number of nodes on Γ_C and by n_p the number of degrees of freedom. Let $\boldsymbol{u}(t)$ be the vector of degrees of freedom of the finite element displacement field $u^h(t,x)$ such that

$$u^h(t,x) = \sum_{1 \le i \le n_p} u_i(t)\varphi_i(x) \quad \text{and} \quad \boldsymbol{u}(t) = (u_i(t)) \in \mathbb{R}^{n_p}.$$

Let a_{α_i}, $i = 1,\ldots,n_c$, be the i-th contact node and $\boldsymbol{\nu}_i, \boldsymbol{\tau}_i \in \mathbb{R}^{n_p}$, $i = 1,\ldots,n_c$, be the vectors linking a displacement vector with its normal and tangential displacements at a_{α_i}, i.e.:

$$\begin{cases} u^h_N(t, a_{\alpha_i}) = \boldsymbol{\nu}_i^T \boldsymbol{u}(t), \\ u^h_T(t, a_{\alpha_i}) = \boldsymbol{\tau}_i^T \boldsymbol{u}(t), \end{cases} \quad i = 1,\ldots,n_c, \tag{3}$$

and satisfying

$$\begin{cases} \|\boldsymbol{\nu}_i\| = 1, \quad \|\boldsymbol{\tau}_i\| = 1, \quad \boldsymbol{\nu}_i^T\boldsymbol{\tau}_j = 0 \quad \forall\, i,j = 1,\ldots n_c, \\ \quad \boldsymbol{\nu}_i^T\boldsymbol{\nu}_j = 0, \quad \boldsymbol{\tau}_i^T\boldsymbol{\tau}_j = 0 \quad \forall\, i,j = 1,\ldots n_c,\ i \ne j. \end{cases} \tag{4}$$

We denote by $\|.\|$ the Euclidean norm of vectors in $\mathbb{R}^n$ as well as the matrix norm in $\mathscr{M}_{n,m}(\mathbb{R})$ generated by the Euclidean vector norm. Using a nodal approximation of the contact condition, the spatial semi-discretization of Problem (1)&(2) can be written as follows:

$$\begin{cases} \text{Find } \boldsymbol{u} : [0,T] \to \mathbb{R}^{n_p},\ \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c} \text{ such that} \\ \boldsymbol{\lambda}_\nu(t) \in \boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau(t) \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \quad \text{a.e. in } (0,T), \\ \boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{A}\boldsymbol{u}(t) = \boldsymbol{f} + \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu(t) + \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau(t) \quad \text{a.e. in } (0,T), \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t))^T\boldsymbol{B}_\nu\boldsymbol{u}(t) \ge 0 \quad \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \text{ a.e. in } (0,T), \\ (\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau(t))^T\boldsymbol{B}_\tau\dot{\boldsymbol{u}}(t) \ge 0 \quad \forall\, \boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)), \text{ a.e. in } (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}^0,\ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0, \end{cases} \tag{5}$$

where

$$A_{ij} = \int_\Omega \mathcal{A}\varepsilon(\varphi_i) : \varepsilon(\varphi_j)\,\mathrm{d}x \quad \text{and} \quad M_{ij} = \int_\Omega \rho\ \varphi_i.\varphi_j\,\mathrm{d}x \qquad (1 \le i,j \le n_p)$$

are the components of the stiffness matrix $\boldsymbol{A} \in \mathscr{M}_{n_p}(\mathbb{R})$ and of the mass matrix $\boldsymbol{M} \in \mathscr{M}_{n_p}(\mathbb{R})$, respectively. We assume that the tensor $\mathcal{A}$ of elasticity coefficients obey the usual symmetry and uniform ellipticity conditions, the density ρ is bounded from below by a positive constant and Γ_D is of nonzero measure on $\partial\Omega$. As a consequence, $\boldsymbol{A}$ and $\boldsymbol{M}$ are both symmetric, positive definite matrices. The components of the load vector $\boldsymbol{f} \in \mathbb{R}^{n_p}$ are given by

$$f_i = \int_\Omega f.\varphi_i \,\mathrm{d}x + \int_{\Gamma_C} g.\varphi_i \,\mathrm{d}\Gamma.$$

We assume for simplicity that the load vector is time independent. Finally, $\boldsymbol{\lambda}_\nu = (\lambda_{\nu,1}, \ldots, \lambda_{\nu,n_c})^T$ and $\boldsymbol{\lambda}_\tau = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n_c})^T$ are the normal and tangential Lagrange multipliers, respectively, $\boldsymbol{B}_\nu = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{n_c})^T$, $\boldsymbol{B}_\tau = (\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n_c})^T \in \mathscr{M}_{n_c,n_p}(\mathbb{R})$ and

$$\begin{aligned} \boldsymbol{\Lambda}_\nu &= \mathbb{R}^{n_c}_-, \\ \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\mu}_\nu) &= \{\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c} \,:\, |\mu_{\tau,i}| \le -\mathscr{F}\mu_{\nu,i} \;\forall\, i = 1, \ldots, n_c\}, \quad \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \end{aligned}$$

stand for the Lagrange multiplier sets, $\mathbb{R}^{n_c}_-$ being the cone of all non-positive vectors in $\mathbb{R}^{n_c}$.

Problem (5) can be viewed as a measure differential inclusion (see (**22**, **23**)). It is ill-posed unless an impact law is added on each contact node. Even in this case, the solutions have a very low regularity.

3. The mass redistribution method

The analysis presented in (**15**) highlights the fact that the main cause of ill-posedness is due to the inertia of the nodes on the contact boundary. It is proposed a method which consists in the redistribution of the mass near the contact boundary. This technique allows to recover the well-posedness of the semi-discrete problem and ensures the solution to be energy conserving. Moreover, it transforms the measure differential inclusion corresponding to (5) into a regular Lipschitz continuous ordinary differential equation, which can be approximated by any reasonable time integration scheme.

The singular dynamic method introduced in (**29**) for unilateral conditions is similar and more general than the mass redistribution method since, for instance, it can be applied to thin structures. However, we use here the mass redistribution method. The reason is that we need a differentiated treatment of unilateral and friction conditions, which would be more difficult to obtain with the singular dynamic method. In Section 6, an elementary example illustrates the fact that an undifferentiated treatment leads to a potential multiplicity of solutions.

Let $\boldsymbol{\mathcal{N}} := \mathrm{span}\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{n_c}\}$ and $\boldsymbol{\mathcal{N}}^\perp$ denote the space spanned by $\boldsymbol{\nu}_i$ and its orthogonal complement, respectively. We shall consider the redistributed mass matrix $\boldsymbol{M}_r \in \mathscr{M}_{n_p}(\mathbb{R})$ satisfying:

$$\begin{cases} (i)\ \boldsymbol{M}_r = \boldsymbol{M}_r^T; \\ (ii)\ \mathrm{Ker}(\boldsymbol{M}_r) = \boldsymbol{\mathcal{N}}; \\ (iii)\ \boldsymbol{w}^T \boldsymbol{M}_r \boldsymbol{w} > 0 \quad \forall\, \boldsymbol{w} \in \boldsymbol{\mathcal{N}}^\perp,\ \boldsymbol{w} \ne \boldsymbol{0}; \end{cases} \tag{6}$$

i.e. being symmetric, positive semi-definite with the kernel equal to $\boldsymbol{\mathcal{N}}$. In (**15**) a simple algorithm is proposed to build the redistributed mass matrix preserving the main characteristics of the mass matrix (total mass, center of gravity and moments of inertia).

Using the decomposition $\boldsymbol{u}(t) = \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t)$, $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) \in \boldsymbol{\mathcal{N}}^\perp$, $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t) \in \boldsymbol{\mathcal{N}}$, of the displacement vector for any time t and replacing $\boldsymbol{M}$ with $\boldsymbol{M}_r$, Problem (5) becomes:

$$\begin{cases} \text{Find } \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp} : [0,T] \to \boldsymbol{\mathcal{N}}^\perp,\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}} : [0,T] \to \boldsymbol{\mathcal{N}},\ \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c} \text{ such that} \\ \boldsymbol{\lambda}_\nu(t) \in \boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau(t) \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \quad \text{a.e. in } (0,T), \\ \boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{A}(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t)) = \boldsymbol{f} + \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu(t) + \boldsymbol{B}_\tau^T \boldsymbol{\lambda}_\tau(t) \\ \hfill \text{a.e. in } (0,T), \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t))^T \boldsymbol{B}_\nu \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t) \geq 0 \quad \forall \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \text{ a.e. in } (0,T), \\ (\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau(t))^T \boldsymbol{B}_\tau \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t) \geq 0 \quad \forall \boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)), \text{ a.e. in } (0,T), \\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp},\ \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp}, \end{cases} \tag{7}$$

where $\boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp}$, $\boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp}$ are the projections of the initial values of the displacement and velocity vectors into $\boldsymbol{\mathcal{N}}^\perp$, respectively. Moreover, it is possible to express the unilateral contact and friction conditions in an equivalent way (see (**13**), for instance) and rewrite the problem as follows:

$$\begin{cases} \text{Find } \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp} : [0,T] \to \boldsymbol{\mathcal{N}}^\perp,\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}} : [0,T] \to \boldsymbol{\mathcal{N}},\ \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c} \text{ such that} \\ \boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{A}(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t)) = \boldsymbol{f} + \sum_{i=1}^{n_c} \lambda_{\nu,i}(t)\boldsymbol{\nu}_i + \sum_{i=1}^{n_c} \lambda_{\tau,i}(t)\boldsymbol{\tau}_i \\ \hfill \text{a.e. in } (0,T), \\ -\lambda_{\nu,i}(t) \in N_{\mathbb{R}_-}(\boldsymbol{\nu}_i^T \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t)) \quad \forall i = 1, \ldots, n_c, \text{ a.e. in } (0,T), \\ \lambda_{\tau,i}(t) \in \mathscr{F}\lambda_{\nu,i}(t)\,\mathrm{Sgn}(\boldsymbol{\tau}_i^T \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t)) \quad \forall i = 1, \ldots, n_c, \text{ a.e. in } (0,T), \\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp}, \quad \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp}, \end{cases} \tag{8}$$

where $N_{\mathbb{R}_-}$ denotes the normal cone of $\mathbb{R}_-$ and the multifunction $\mathrm{Sgn} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the sub-differential of the function $a \mapsto |a|$, i.e.:

$$\mathrm{Sgn}(a) = \begin{cases} \frac{a}{|a|} & \text{if } a \neq 0, \\ [-1, 1] & \text{if } a = 0, \end{cases} \quad a \in \mathbb{R}.$$

Remark 3.1. From (4) it immediately follows that there exists $\beta > 0$ such that

$$\sup_{\mathbf{0} \neq \boldsymbol{w} \in \mathbb{R}^{n_p}} \frac{\boldsymbol{w}^T \boldsymbol{B}_\nu^T \boldsymbol{\mu}_\nu + \boldsymbol{w}^T \boldsymbol{B}_\tau^T \boldsymbol{\mu}_\tau}{\|\boldsymbol{w}\|} \geq \beta(\|\boldsymbol{\mu}_\nu\| + \|\boldsymbol{\mu}_\tau\|) \quad \forall \boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c}. \tag{9}$$

4. Well-posedness result

In this section we shall establish the well-posedness of Problem (7). First, owing to (4) and (6), the first three variables of any $(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solving (7) have to satisfy:

$$\begin{cases} \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) \in \boldsymbol{\mathcal{N}}^\perp,\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t) \in \boldsymbol{\mathcal{N}},\ \boldsymbol{\lambda}_\nu(t) \in \boldsymbol{\Lambda}_\nu, \\ \boldsymbol{w}^T \boldsymbol{A}(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t)) = \boldsymbol{w}^T \boldsymbol{f} + \boldsymbol{w}^T \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu(t) \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{N}}, \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t))^T \boldsymbol{B}_\nu \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t) \geq 0 \quad \forall \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \end{cases} \tag{10}$$

for almost all $t \in (0,T)$. From here, $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}}$ and $\boldsymbol{\lambda}_\nu$ are uniquely determined by $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}$ as states the following assertion.

Lemma 4.1. Let (4) be satisfied and $\boldsymbol{f} \in \mathbb{R}^{n_p}$ be arbitrary. Then there exist unique functions $\boldsymbol{g}_1 : \boldsymbol{\mathcal{N}}^\perp \to \boldsymbol{\mathcal{N}}$ and $\boldsymbol{g}_2 : \boldsymbol{\mathcal{N}}^\perp \to \boldsymbol{\Lambda}_\nu$ such that the triplet $(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t), \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t), \boldsymbol{\lambda}_\nu(t))$ with $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t) := \boldsymbol{g}_1(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t))$, $\boldsymbol{\lambda}_\nu := \boldsymbol{g}_2(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t))$, satisfies (10) for any $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) \in \boldsymbol{\mathcal{N}}^\perp$ and any $t \in [0,T]$. Moreover, the functions $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are Lipschitz continuous:

$$\exists\, L_1, L_2 > 0 : \quad \|\boldsymbol{g}_i(\boldsymbol{w}^1) - \boldsymbol{g}_i(\boldsymbol{w}^2)\| \leq L_i \|\boldsymbol{w}^1 - \boldsymbol{w}^2\| \quad \forall \boldsymbol{w}^1, \boldsymbol{w}^2 \in \boldsymbol{\mathcal{N}}^\perp,\ i = 1, 2. \tag{11}$$

Proof. In fact, it suffices to analyze the static problem:

$$\begin{cases} \text{Find } (\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}}, \tilde{\boldsymbol{\lambda}}_\nu) := (\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}}(\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}), \tilde{\boldsymbol{\lambda}}_\nu(\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp})) \in \boldsymbol{\mathcal{N}} \times \boldsymbol{\Lambda}_\nu \text{ such that} \\ \boldsymbol{w}^T \boldsymbol{A} \tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}} = \boldsymbol{w}^T(\boldsymbol{f} - \boldsymbol{A}\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}) + \boldsymbol{w}^T \boldsymbol{B}_\nu^T \tilde{\boldsymbol{\lambda}}_\nu \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{N}}, \\ (\boldsymbol{\mu}_\nu - \tilde{\boldsymbol{\lambda}}_\nu)^T \boldsymbol{B}_\nu \tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}} \geq 0 \quad \forall \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \end{cases} \tag{12}$$

for $\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp} \in \boldsymbol{\mathcal{N}}^\perp$ given. It is readily seen that this problem is equivalent to finding a saddle-point $(\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}}, \tilde{\boldsymbol{\lambda}}_\nu)$ of the Lagrangian

$$\mathscr{L}(\boldsymbol{w}, \boldsymbol{\mu}_\nu) := \frac{1}{2}\boldsymbol{w}^T \boldsymbol{A}\boldsymbol{w} - \boldsymbol{w}^T(\boldsymbol{f} - \boldsymbol{A}\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}) - \boldsymbol{w}^T \boldsymbol{B}_\nu^T \boldsymbol{\mu}_\nu, \quad (\boldsymbol{w}, \boldsymbol{\mu}_\nu) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c},$$

on $\boldsymbol{\mathcal{N}} \times \boldsymbol{\Lambda}_\nu$. Since $\boldsymbol{A}$ is positive definite and

$$\beta \|\boldsymbol{\mu}_\nu\| \leq \sup_{\boldsymbol{0} \neq \boldsymbol{w} \in \mathbb{R}^{n_p}} \frac{\boldsymbol{w}^T \boldsymbol{B}_\nu^T \boldsymbol{\mu}_\nu}{\|\boldsymbol{w}\|} = \sup_{\boldsymbol{0} \neq \boldsymbol{w} \in \boldsymbol{\mathcal{N}}} \frac{\boldsymbol{w}^T \boldsymbol{B}_\nu^T \boldsymbol{\mu}_\nu}{\|\boldsymbol{w}\|} \quad \forall \boldsymbol{\mu}_\nu \in \mathbb{R}^{n_c} \tag{13}$$

due to (4), where β is the constant from (9), Problem (12) possesses a unique solution for any $\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp} \in \boldsymbol{\mathcal{N}}^\perp$, which depends Lipschitz continuously on the data $\tilde{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}$ (see (**4**) for instance). This yields the existence, the uniqueness and the Lipschitz continuity of the functions $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. ☐

From the other side, if $(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves (8) then

$$\begin{cases} \boldsymbol{w}^T \boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{w}^T \boldsymbol{A}(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) + \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t)) = \boldsymbol{w}^T \boldsymbol{f} + \boldsymbol{w}^T \Big(\sum_{i=1}^{n_c} \lambda_{\tau,i}(t) \boldsymbol{\tau}_i \Big) \\ \qquad\qquad\qquad\qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{N}}^\perp, \text{ a.e. in } (0,T), \\ \lambda_{\tau,i}(t) \in \mathscr{F} \lambda_{\nu,i}(t) \operatorname{Sgn}(\boldsymbol{\tau}_i^T \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t)) \quad \forall i = 1, \ldots, n_c, \text{ a.e. in } (0,T), \\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp}, \quad \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp}. \end{cases} \tag{14}$$

Substituting the inclusion for $\lambda_{\tau,i}(t)$ into the equality and taking $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}}(t) = \boldsymbol{g}_1(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t))$, $\lambda_{\nu,i}(t) = g_{2,i}(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t))$ according to Lemma 4.1, this becomes:

$$\begin{cases} \boldsymbol{w}^T \boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t) \in \boldsymbol{w}^T(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t) - \boldsymbol{A}\boldsymbol{g}_1(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t))) \\ \qquad + \boldsymbol{w}^T \Big(\sum_{i=1}^{n_c} \mathscr{F} g_{2,i}(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(t)) \operatorname{Sgn}(\boldsymbol{\tau}_i^T \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(t)) \boldsymbol{\tau}_i \Big) \quad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{N}}^\perp, \text{ a.e. in } (0,T), \\ \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp}, \quad \dot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp}(0) = \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp}. \end{cases} \tag{15}$$

LEMMA 4.2. Let (4) and (6) be fulfilled and $\boldsymbol{f} \in \mathbb{R}^{n_p}$, $\boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp} \in \boldsymbol{\mathcal{N}}^\perp$ be arbitrary. Then there exists a unique Lipschitz continuous function $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp} : [0,T] \to \boldsymbol{\mathcal{N}}^\perp$ with $\ddot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp} \in L^1(0,T;\mathbb{R}^{n_p})$ solving (15).

Proof. Introducing the matrix $\boldsymbol{P} \in \mathscr{M}_{n_p,\bar{n}}(\mathbb{R})$, $\bar{n} := \dim \boldsymbol{\mathcal{N}}^\perp$, columns of which form an orthonormal basis of $\boldsymbol{\mathcal{N}}^\perp$, any vector $\boldsymbol{w} \in \boldsymbol{\mathcal{N}}^\perp$ can be represented by $\bar{\boldsymbol{w}} \in \mathbb{R}^{\bar{n}}$ with

$$\bar{\boldsymbol{w}} = \boldsymbol{P}^T \boldsymbol{w}, \quad \boldsymbol{w} = \boldsymbol{P}\boldsymbol{P}^T \boldsymbol{w} = \boldsymbol{P}\bar{\boldsymbol{w}}$$

and (15) is equivalent to

$$\begin{cases} \bar{\boldsymbol{w}}^T \bar{\boldsymbol{M}}_r \ddot{\bar{\boldsymbol{u}}}(t) \in \bar{\boldsymbol{w}}^T(\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t))) + \bar{\boldsymbol{w}}^T \Big(\sum_{i=1}^{n_c} \mathscr{F} \bar{g}_{2,i}(\bar{\boldsymbol{u}}(t)) \operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T \dot{\bar{\boldsymbol{u}}}(t)) \bar{\boldsymbol{\tau}}_i \Big) \\ \qquad\qquad\qquad\qquad \forall \bar{\boldsymbol{w}} \in \mathbb{R}^{\bar{n}}, \text{ a.e. in } (0,T), \\ \bar{\boldsymbol{u}}(0) = \bar{\boldsymbol{u}}^0, \quad \dot{\bar{\boldsymbol{u}}}(0) = \boldsymbol{P}^T \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp}, \end{cases}$$

where

$$\bar{\boldsymbol{M}}_r = \boldsymbol{P}^T \boldsymbol{M}_r \boldsymbol{P}, \quad \bar{\boldsymbol{A}} = \boldsymbol{P}^T \boldsymbol{A} \boldsymbol{P}, \quad \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)) = \boldsymbol{P}^T \boldsymbol{A} \boldsymbol{g}_1(\boldsymbol{P}\bar{\boldsymbol{u}}(t)),$$
$$\bar{\boldsymbol{g}}_2(\bar{\boldsymbol{u}}(t)) = \boldsymbol{g}_2(\boldsymbol{P}\bar{\boldsymbol{u}}(t)), \quad \bar{\boldsymbol{g}}_2(\bar{\boldsymbol{u}}(t)) = (\bar{g}_{2,1}(\bar{\boldsymbol{u}}(t)), \dots, \bar{g}_{2,n_c}(\bar{\boldsymbol{u}}(t)))^T,$$
$$\bar{\boldsymbol{u}} = \boldsymbol{P}^T \boldsymbol{u}_{\mathcal{N}^\perp}, \quad \bar{\boldsymbol{u}}^0 = \boldsymbol{P}^T \boldsymbol{u}^0_{\mathcal{N}^\perp}, \quad \bar{\boldsymbol{f}} = \boldsymbol{P}^T \boldsymbol{f}, \quad \bar{\boldsymbol{\tau}}_i = \boldsymbol{P}^T \boldsymbol{\tau}_i, \ i = 1, \dots, n_c.$$

Having in mind (6), this can be written as

$$\begin{cases} \ddot{\bar{\boldsymbol{u}}}(t) \in \bar{\boldsymbol{M}}_r^{-1}\Big[\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)) + \displaystyle\sum_{i=1}^{n_c} \mathscr{F} \bar{g}_{2,i}(\bar{\boldsymbol{u}}(t)) \operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T \dot{\bar{\boldsymbol{u}}}(t)) \bar{\boldsymbol{\tau}}_i\Big] \\ \hfill \text{a.e. in } (0,T), \\ \bar{\boldsymbol{u}}(0) = \bar{\boldsymbol{u}}^0, \quad \dot{\bar{\boldsymbol{u}}}(0) = \boldsymbol{P}^T \boldsymbol{v}^0_{\mathcal{N}^\perp}, \end{cases}$$

and denoting $\bar{\boldsymbol{v}} = \bar{\boldsymbol{M}}_r^{1/2} \dot{\bar{\boldsymbol{u}}}$, $\bar{\boldsymbol{v}}^0 = \bar{\boldsymbol{M}}_r^{1/2} \boldsymbol{P}^T \boldsymbol{v}^0_{\mathcal{N}^\perp}$, this leads to the following differential inclusion of the first order:

$$\begin{cases} \begin{pmatrix} \dot{\bar{\boldsymbol{u}}}(t) \\ \dot{\bar{\boldsymbol{v}}}(t) \end{pmatrix} \in \begin{pmatrix} \bar{\boldsymbol{M}}_r^{-1/2} \bar{\boldsymbol{v}}(t) \\ \bar{\boldsymbol{M}}_r^{-1/2}\big[\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)) + \sum_{i=1}^{n_c} \mathscr{F} \bar{g}_{2,i}(\bar{\boldsymbol{u}}(t)) \operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2} \bar{\boldsymbol{v}}(t)) \bar{\boldsymbol{\tau}}_i\big] \end{pmatrix} \\ \hfill \text{a.e. in } (0,T), \\ \begin{pmatrix} \bar{\boldsymbol{u}}(0) \\ \bar{\boldsymbol{v}}(0) \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{u}}^0 \\ \bar{\boldsymbol{v}}^0 \end{pmatrix}. \end{cases}$$

Thus we have to solve:

$$\begin{cases} \dot{\boldsymbol{x}}(t) \in \boldsymbol{h}(\boldsymbol{x}(t)) \quad \text{a.e. in } (0,T), \\ \boldsymbol{x}(0) = \boldsymbol{x}^0 \end{cases} \tag{16}$$

with the multifunction $\boldsymbol{h} : \mathbb{R}^{2\bar{n}} \rightrightarrows \mathbb{R}^{2\bar{n}}$ defined by

$$\boldsymbol{h}(\boldsymbol{y}) = \begin{pmatrix} \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{y}_2 \\ \bar{\boldsymbol{M}}_r^{-1/2}\big[\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\boldsymbol{y}_1 - \bar{\boldsymbol{g}}_1(\boldsymbol{y}_1) + \sum_{i=1}^{n_c} \mathscr{F} \bar{g}_{2,i}(\boldsymbol{y}_1) \operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{y}_2) \bar{\boldsymbol{\tau}}_i\big] \end{pmatrix},$$
$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} \in R^{2\bar{n}},$$

and $\boldsymbol{x}^0 = ((\bar{\boldsymbol{u}}^0)^T, (\bar{\boldsymbol{v}}^0)^T)^T$.

Obviously, $\boldsymbol{h}$ is upper semicontinuous, i.e. $\boldsymbol{h}^{-1}(\boldsymbol{C})$ is closed whenever $\boldsymbol{C} \subset \mathbb{R}^{2\bar{n}}$ is closed, and has closed convex values. Furthermore, there exists $c > 0$ such that

$$\|\boldsymbol{h}(\boldsymbol{y})\| \equiv \sup\{\|\boldsymbol{z}\| \mid \boldsymbol{z} \in \boldsymbol{h}(\boldsymbol{y})\} \le c(1 + \|\boldsymbol{y}\|) \quad \forall\, \boldsymbol{y} \in \mathbb{R}^{2\bar{n}}. \tag{17}$$

Indeed,

$$\begin{aligned} &\|\boldsymbol{h}(\boldsymbol{y})\| \\ &\quad \le \|\bar{\boldsymbol{M}}_r^{-1/2}\| \Big[\|\boldsymbol{y}_2\|^2 + \Big\|\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\boldsymbol{y}_1 - \bar{\boldsymbol{g}}_1(\boldsymbol{y}_1) + \sum_{i=1}^{n_c} \mathscr{F} \bar{g}_{2,i}(\boldsymbol{y}_1) \operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{y}_2) \bar{\boldsymbol{\tau}}_i\Big\|^2\Big]^{1/2} \\ &\quad \le \|\bar{\boldsymbol{M}}_r^{-1/2}\| \\ &\qquad \times \Big[\|\boldsymbol{y}_2\|^2 + \Big(\|\bar{\boldsymbol{f}}\| + \|\bar{\boldsymbol{A}}\|\|\boldsymbol{y}_1\| + \|\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1)\| + \Big\|\sum_{i=1}^{n_c} \mathscr{F} \bar{g}_{2,i}(\boldsymbol{y}_1) \operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{y}_2) \bar{\boldsymbol{\tau}}_i\Big\|\Big)^2\Big]^{1/2}. \end{aligned}$$

Firstly,

$$\Big\|\sum_{i=1}^{n_c}\mathscr{F}\bar{g}_{2,i}(\boldsymbol{y}_1)\operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2)\bar{\boldsymbol{\tau}}_i\Big\| \le \Big(\sum_{i=1}^{n_c}(\mathscr{F}\bar{g}_{2,i}(\boldsymbol{y}_1))^2\Big)^{1/2} = \mathscr{F}\|\bar{\boldsymbol{g}}_2(\boldsymbol{y}_1)\|$$

in virtue of the orthonormality of $\bar{\boldsymbol{\tau}}_i$ and the definition of the mapping Sgn. Secondly, making use of (11) and of the form of $\boldsymbol{P}$, we have:

$$\|\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1)\| = \|\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{y}_1)\| \le \|\boldsymbol{A}\|\|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{y}_1)\|,$$
$$\|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{y}_1)\| - \|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{0})\| \le L_1\|\boldsymbol{P}(\boldsymbol{y}_1-\boldsymbol{0})\| = L_1\|\boldsymbol{y}_1\|,$$

consequently

$$\|\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1)\| \le \|\boldsymbol{A}\|(\|\boldsymbol{g}_1(\boldsymbol{0})\| + L_1\|\boldsymbol{y}_1\|)$$

and in a similar way one can show that

$$\|\bar{\boldsymbol{g}}_2(\boldsymbol{y}_1)\| \le \|\boldsymbol{g}_2(\boldsymbol{0})\| + L_2\|\boldsymbol{y}_1\|.$$

Hence,

$$\begin{aligned}\|\boldsymbol{h}(\boldsymbol{y})\| &\le \|\bar{\boldsymbol{M}}_r^{-1/2}\| \\ &\quad\times\Big[\|\boldsymbol{y}_2\|^2 + \big(\|\bar{\boldsymbol{f}}\| + \|\bar{\boldsymbol{A}}\|\|\boldsymbol{y}_1\| + \|\boldsymbol{A}\|(\|\boldsymbol{g}_1(\boldsymbol{0})\| + L_1\|\boldsymbol{y}_1\|) + \mathscr{F}(\|\boldsymbol{g}_2(\boldsymbol{0})\| + L_2\|\boldsymbol{y}_1\|)\big)^2\Big]^{1/2},\end{aligned}$$

from which the expression for the constant c in (17) follows. Therefore, Theorem 5.1 in (**3**) guarantees that (16) has an absolutely continuous solution $\boldsymbol{x}$ in $[0,T]$ for any $\boldsymbol{x}^0 \in \mathbb{R}^{2\bar{n}}$, i.e. a function $\boldsymbol{x} : [0,T] \to \mathbb{R}^{2\bar{n}}$ with $\dot{\boldsymbol{x}} \in L^1(0,T;\mathbb{R}^{2\bar{n}})$ satisfying

$$\boldsymbol{x}(t) = \boldsymbol{x}^0 + \int_0^t \dot{\boldsymbol{x}}(s)\,\mathrm{d}s \quad \text{for all } t\in[0,T] \quad \text{and} \quad \dot{\boldsymbol{x}}(t) \in \boldsymbol{h}(\boldsymbol{x}(t)) \ \text{ a.e. in } \ (0,T).$$

This gives the existence part of the assertion. To prove the uniqueness, it suffices to show that $\boldsymbol{h}$ is one-sided Lipschitz (see for instance Theorem 10.4 in (**3**)), i.e.:

$$\exists\, k>0: \quad (\boldsymbol{y}^1-\boldsymbol{y}^2)^T(\boldsymbol{h}(\boldsymbol{y}^1)-\boldsymbol{h}(\boldsymbol{y}^2)) \le k\|\boldsymbol{y}^1-\boldsymbol{y}^2\|^2 \quad \forall\,\boldsymbol{y}^1,\boldsymbol{y}^2\in\mathbb{R}^{2\bar{n}}.$$

From the definition of $\boldsymbol{h}$

$$\begin{aligned}&(\boldsymbol{y}^1-\boldsymbol{y}^2)^T(\boldsymbol{h}(\boldsymbol{y}^1)-\boldsymbol{h}(\boldsymbol{y}^2)) \\ &\quad= (\boldsymbol{y}_1^1-\boldsymbol{y}_1^2)^T\bar{\boldsymbol{M}}_r^{-1/2}(\boldsymbol{y}_2^1-\boldsymbol{y}_2^2) + (\boldsymbol{y}_2^1-\boldsymbol{y}_2^2)^T\bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{A}}(\boldsymbol{y}_1^2-\boldsymbol{y}_1^1) \\ &\qquad+ (\boldsymbol{y}_2^1-\boldsymbol{y}_2^2)^T\bar{\boldsymbol{M}}_r^{-1/2}(\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1^2)-\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1^1)) \\ &\qquad+ (\boldsymbol{y}_2^1-\boldsymbol{y}_2^2)^T\bar{\boldsymbol{M}}_r^{-1/2}\Big(\sum_{i=1}^{n_c}\mathscr{F}\big(\bar{g}_{2,i}(\boldsymbol{y}_1^1)\operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^1) - \bar{g}_{2,i}(\boldsymbol{y}_1^2)\operatorname{Sgn}(\bar{\boldsymbol{\tau}}_i^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^2)\big)\bar{\boldsymbol{\tau}}_i\Big) \\ &\quad=: S_1+S_2+S_3+S_4.\end{aligned}$$

Clearly,

$$S_1 \le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{y}^1-\boldsymbol{y}^2\|^2, \quad S_2 \le \|\bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{A}}\|\|\boldsymbol{y}^1-\boldsymbol{y}^2\|^2$$

and

$$\begin{aligned}S_3 &\le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{y}^1-\boldsymbol{y}^2\|\|\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1^2)-\bar{\boldsymbol{g}}_1(\boldsymbol{y}_1^1)\| \\ &\le \|\boldsymbol{A}\|\|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{y}^1-\boldsymbol{y}^2\|\|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{y}_1^2)-\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{y}_1^1)\| \le L_1\|\boldsymbol{A}\|\|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{y}^1-\boldsymbol{y}^2\|^2\end{aligned}$$

by (11). Furthermore,

$$S_4 = \sum_{i=1}^{n_c} \mathscr{F}\big(\bar{g}_{2,i}(\boldsymbol{y}_1^1)\,\mathrm{Sgn}(\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^1) - \bar{g}_{2,i}(\boldsymbol{y}_1^2)\,\mathrm{Sgn}(\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^2)\big) \\ \times \big({\boldsymbol{y}_2^1}^T \bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{\tau}}_i - {\boldsymbol{y}_2^2}^T \bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{\tau}}_i\big).$$

Hence, fixing i and setting

$$a^1 = \bar{g}_{2,i}(\boldsymbol{y}_1^1), \quad a^2 = \bar{g}_{2,i}(\boldsymbol{y}_1^2), \quad b^1 = \bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^1, \quad b^2 = \bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^2,$$

the i-th summand of S_4 takes the form:

$$\mathscr{F}(a^1\,\mathrm{Sgn}(b^1) - a^2\,\mathrm{Sgn}(b^2))(b^1 - b^2).$$

Note that $a^1, a^2 \le 0$. We claim that in this case

$$(a^1\,\mathrm{Sgn}(b^1) - a^2\,\mathrm{Sgn}(b^2))(b^1 - b^2) \le |a^1 - a^2||b^1 - b^2|. \tag{18}$$

Indeed, for $\zeta \in \mathrm{Sgn}(b^1)$ and $\xi \in \mathrm{Sgn}(b^2)$ we get

$$(a^1\zeta - a^2\xi)(b^1 - b^2) = (a^1\zeta - a^1\xi + a^1\xi - a^2\xi)(b^1 - b^2) \le (a^1 - a^2)\xi(b^1 - b^2)$$

due to the monotonicity of the multifunction Sgn. And of course (18) can be deduced from

$$(a^1 - a^2)\xi(b^1 - b^2) \le |a^1 - a^2||b^1 - b^2|.$$

Applying this together with the Cauchy-Schwarz inequality and (11) we get:

$$S_4 \le \mathscr{F}\sum_{i=1}^{n_c}\big|\bar{g}_{2,i}(\boldsymbol{y}_1^1) - \bar{g}_{2,i}(\boldsymbol{y}_1^2)\big|\big|\bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^1 - \bar{\boldsymbol{\tau}}_i^T \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{y}_2^2\big| \\ \le \mathscr{F}\|\bar{\boldsymbol{g}}_2(\boldsymbol{y}_1^1) - \bar{\boldsymbol{g}}_2(\boldsymbol{y}_1^2)\|\|\boldsymbol{B}_\tau \bar{\boldsymbol{M}}_r^{-1/2}(\boldsymbol{y}_2^1 - \boldsymbol{y}_2^2)\| \le \mathscr{F}L_2\|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{y}^1 - \boldsymbol{y}^2\|^2.$$

All in all, the one-sided Lipschitz property of $\boldsymbol{h}$ is verified. □

On the basis of the previous two lemmas we arrive at the announced well-posedness result.

THEOREM 4.3. Let $\boldsymbol{f} \in \mathbb{R}^{n_p}$, $\boldsymbol{u}^0_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{v}^0_{\boldsymbol{\mathcal{N}}^\perp} \in \boldsymbol{\mathcal{N}}^\perp$ be arbitrary. If (4) and (6) are satisfied then there exist a unique Lipschitz continuous function $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp} : [0,T] \to \boldsymbol{\mathcal{N}}^\perp$ with $\ddot{\boldsymbol{u}}_{\boldsymbol{\mathcal{N}}^\perp} \in L^1(0,T;\mathbb{R}^{n_p})$ and unique functions $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}} : [0,T] \to \boldsymbol{\mathcal{N}}$ and $\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c}$ such that the quadruplet $(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves (7). In addition, $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}}$, $\boldsymbol{\lambda}_\nu$ are Lipschitz continuous in $[0,T]$ and $\boldsymbol{\lambda}_\tau \in L^\infty(0,T;\mathbb{R}^{n_c})$.

Proof. The existence and uniqueness as well as the Lipschitz continuity of $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}$ and $\boldsymbol{u}_{\boldsymbol{\mathcal{N}}}$, $\boldsymbol{\lambda}_\nu$ are ensured by Lemmas 4.2 and 4.1, respectively. Consequently, the existence of $\boldsymbol{\lambda}_\tau$ is readily seen from the relation between (14) and (15). If $(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau^1)$ and $(\boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}, \boldsymbol{u}_{\boldsymbol{\mathcal{N}}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau^2)$ were two solutions to (7) then

$$\boldsymbol{w}^T \boldsymbol{B}_\tau^T(\boldsymbol{\lambda}_\tau^1(t) - \boldsymbol{\lambda}_\tau^2(t)) = 0 \quad \forall\, \boldsymbol{w} \in \mathbb{R}^{n_p}, \text{ a.e. in } (0,T)$$

by the first equation in (7) and

$$\beta\|\boldsymbol{\lambda}_\tau^1(t) - \boldsymbol{\lambda}_\tau^2(t)\| \le \sup_{\boldsymbol{0}\ne\boldsymbol{w}\in\mathbb{R}^{n_p}} \frac{\boldsymbol{w}^T \boldsymbol{B}_\tau^T(\boldsymbol{\lambda}_\tau^1(t) - \boldsymbol{\lambda}_\tau^2(t))}{\|\boldsymbol{w}\|} = 0 \quad \text{a.e. in } (0,T)$$

due to (9). In a similar way one also obtains that $\boldsymbol{\lambda}_\tau \in L^\infty(0,T;\mathbb{R}^{n_c})$ from the Lipschitz continuity of $\boldsymbol{\lambda}_\nu$ and the second inclusion of (8). □

Remark 4.4. The well-posedness result can easily be extended to three-dimensional problems, since the key point of the proof is the monotonicity of the multifunction Sgn. For the three-dimensional problems, the friction condition can be expressed by means of the sub-differential of the function $a \mapsto \|a\|$, which is also monotonic. This allows to obtain an equivalent relation to (18). An extension of the well-posedness result can also be obtained for a load vector which is a Lipschitz continuous function of time.

Remark 4.5. The spatial semi-discrete problem (7) being equivalent to the one-sided Lipschitz regular differential inclusion (16), most of the classical time integration schemes will be convergent (for a fixed mesh) due to, for instance, the result obtained in (**19**). Moreover, the fully discrete problem is also ensured to be well-posed for a sufficiently small time step (because of its monotonicity).

5. Energy decreasing result

First, note that the result of Proposition 1 in (**15**) is still valid, which means that the so-called persistency condition holds:

$$\lambda_{\nu,i}(t)(\boldsymbol{\nu}_i^T \dot{\boldsymbol{u}}(t)) = 0 \quad \text{a.e. in } (0,T), \quad i = 1, \cdots, n_c.$$

This allows to prove the following result:

Proposition 5.1. Still assuming the load vector $\boldsymbol{f}$ to be constant in time and denoting $\boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{\mathcal{N}}} + \boldsymbol{u}_{\boldsymbol{\mathcal{N}}^\perp}$ the solution to (7), the energy

$$E(t) = \frac{1}{2}\, \dot{\boldsymbol{u}}^T(t)\boldsymbol{M}_r\dot{\boldsymbol{u}}(t) + \frac{1}{2}\, \boldsymbol{u}^T(t)\boldsymbol{A}\boldsymbol{u}(t) - \boldsymbol{u}^T(t)\boldsymbol{f} \tag{19}$$

is decreasing in time.

Proof. The first equation in (8) implies

$$\dot{\boldsymbol{u}}^T(t)\boldsymbol{M}_r\ddot{\boldsymbol{u}}(t) + \dot{\boldsymbol{u}}^T(t)\boldsymbol{A}\boldsymbol{u}(t) = \dot{\boldsymbol{u}}^T(t)\boldsymbol{f} + \sum_{i=1}^{n_c} \lambda_{\nu,i}(t)\dot{\boldsymbol{u}}^T(t)\boldsymbol{\nu}_i + \sum_{i=1}^{n_c} \lambda_{\tau,i}(t)\dot{\boldsymbol{u}}^T(t)\boldsymbol{\tau}_i.$$

Integrating from t_0 to t_1, it follows:

$$E(t_1) = E(t_0) + \sum_{i=1}^{n_c}\int_{t_0}^{t_1} \lambda_{\nu,i}(t)\dot{\boldsymbol{u}}^T(t)\boldsymbol{\nu}_i \,\mathrm{d}t + \sum_{i=1}^{n_c}\int_{t_0}^{t_1} \lambda_{\tau,i}(t)\dot{\boldsymbol{u}}^T(t)\boldsymbol{\tau}_i \,\mathrm{d}t.$$

Due to the fact that $\lambda_{\tau,i}(t) \in \mathscr{F}\lambda_{\nu,i}(t)\mathrm{Sgn}(\dot{\boldsymbol{u}}^T(t)\boldsymbol{\tau}_i)$ a.e. in $(0,T)$, one has

$$\int_{t_0}^{t_1} \lambda_{\tau,i}(t)\dot{\boldsymbol{u}}^T(t)\boldsymbol{\tau}_i \,\mathrm{d}t \le 0, \quad i = 1, \cdots, n_c.$$

Together with the persistency condition, this gives the result. □

6. An elementary example

This section presents the mass redistribution method for an elementary contact problem involving a single linear triangular finite element depicted in Fig. 2. The aim is to show that an undifferentiated treatment of the contact and friction condition may lead to an ill-posedness of the fully discrete problem whatever is the lenght of the time step. Using the time discretization by the midpoint scheme we shall compare different possibilities of the redistribution of the mass. The studied contact problem is in fact a dynamic case of the elementary example studied e.g. in (**10**) and (**20**).

Denoting the lengths of the sides of the triangle by ℓ, ℓ, $\sqrt{2}\ell$ and employing Hooke's constitutive

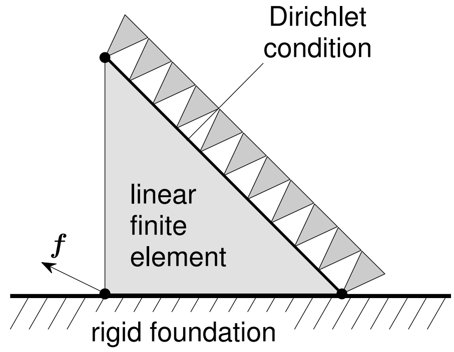

Fig. 2 Geometry of the elementary example.

law for homogeneous, isotropic material, the formulation of the problem in inclusions reads as follows:

$$\begin{cases} \text{Find } \boldsymbol{u} : [0,T] \to \mathbb{R}^2, \lambda_\nu, \lambda_\tau : [0,T] \to \mathbb{R} \text{ such that} \\ \boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{A}\boldsymbol{u}(t) = \boldsymbol{f}(t) + \lambda_\nu(t)\boldsymbol{\nu} + \lambda_\tau(t)\boldsymbol{\tau} \quad \text{a.e. in } (0,T), \\ -\lambda_\nu(t) \in N_{\mathbb{R}_-}(\boldsymbol{\nu}^T\boldsymbol{u}(t)) \quad \text{a.e. in } (0,T), \\ \lambda_\tau(t) \in \mathscr{F}\lambda_\nu(t)\,\mathrm{Sgn}(\boldsymbol{\tau}^T\dot{\boldsymbol{u}}(t)) \quad \text{a.e. in } (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0, \end{cases}$$

where

$$\boldsymbol{M} = \begin{pmatrix} \frac{\rho\ell^2}{12} & 0 \\ 0 & \frac{\rho\ell^2}{12} \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \frac{\lambda+3\mu}{2} & -\frac{\lambda+\mu}{2} \\ -\frac{\lambda+\mu}{2} & \frac{\lambda+3\mu}{2} \end{pmatrix}, \quad \boldsymbol{\nu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here $\rho > 0$ is constant, $\lambda \geq 0$, $\mu > 0$ are the Lamé coefficients and $\boldsymbol{f}$ is assumed to be dependent on t.

Obviously, the mass redistribution method consists in replacing the matrix $\boldsymbol{M}$ by $\boldsymbol{M}_r = \left(\begin{smallmatrix} m_1 & 0 \\ 0 & m_2 \end{smallmatrix}\right)$ with $m_1, m_2 \geq 0$. Further, to do the time discretization by the midpoint method, we divide the interval $[0,T]$ uniformly into n_t subintervals and set $\Delta t = T/n_t$ and $t_k = k\Delta t$ for $k = 0, \dots, n_t$. We seek the approximations $\boldsymbol{u}^{k+1}$ and $\boldsymbol{v}^{k+1}$ of $\boldsymbol{u}(t_{k+1})$ and $\dot{\boldsymbol{u}}(t_{k+1})$, respectively, for $k = 0, \dots, n_t - 1$ so that

$$\begin{cases} \boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \Delta t \boldsymbol{v}^{k+1/2}, & \boldsymbol{v}^{k+1} = \boldsymbol{v}^k + \Delta t \boldsymbol{a}^{k+1/2}, \\ \boldsymbol{u}^{k+1/2} = \dfrac{\boldsymbol{u}^{k+1} + \boldsymbol{u}^k}{2}, & \boldsymbol{v}^{k+1/2} = \dfrac{\boldsymbol{v}^{k+1} + \boldsymbol{v}^k}{2} \end{cases} \tag{20}$$

and

$$\begin{cases} \boldsymbol{M}_r \boldsymbol{a}^{k+1/2} + \boldsymbol{A}\boldsymbol{u}^{k+1/2} = \boldsymbol{f}\big(\dfrac{t_k + t_{k+1}}{2}\big) + \lambda_\nu^{k+1/2}\boldsymbol{\nu} + \lambda_\tau^{k+1/2}\boldsymbol{\tau}, \\ -\lambda_\nu^{k+1/2} \in N_{\mathbb{R}_-}(\boldsymbol{\nu}^T\boldsymbol{u}^{k+1/2}), \\ \lambda_\tau^{k+1/2} \in \mathscr{F}\lambda_\nu^{k+1/2}\,\mathrm{Sgn}(\boldsymbol{\tau}^T\boldsymbol{v}^{k+1/2}). \end{cases} \tag{21}$$

From (20) one can express $\boldsymbol{v}^{k+1/2}$ and $\boldsymbol{a}^{k+1/2}$ as

$$\boldsymbol{v}^{k+1/2} = \frac{2}{\Delta t}\boldsymbol{u}^{k+1/2} - \frac{2}{\Delta t}\boldsymbol{u}^k, \quad \boldsymbol{a}^{k+1/2} = \frac{4}{\Delta t^2}\boldsymbol{u}^{k+1/2} - \frac{4}{\Delta t^2}\boldsymbol{u}^k - \frac{2}{\Delta t}\boldsymbol{v}^k, \tag{22}$$

which inserted into (21) leads to

$$\begin{cases} \big(\frac{4}{\Delta t^2}\boldsymbol{M}_r + \boldsymbol{A}\big)\boldsymbol{u}^{k+1/2} = \hat{\boldsymbol{f}}^{k+1/2} + \lambda_\nu^{k+1/2}\boldsymbol{\nu} + \lambda_\tau^{k+1/2}\boldsymbol{\tau}, \\ -\lambda_\nu^{k+1/2} \in N_{\mathbb{R}_-}(\boldsymbol{\nu}^T\boldsymbol{u}^{k+1/2}), \\ \lambda_\tau^{k+1/2} \in \mathscr{F}\lambda_\nu^{k+1/2}\operatorname{Sgn}\big(\frac{2}{\Delta t}\boldsymbol{\tau}^T(\boldsymbol{u}^{k+1/2} - \boldsymbol{u}^k)\big) \end{cases}$$

with

$$\hat{\boldsymbol{f}}^{k+1/2} = \boldsymbol{f}\big(\frac{t_k + t_{k+1}}{2}\big) + \frac{4}{\Delta t^2}\boldsymbol{M}_r\boldsymbol{u}^k + \frac{2}{\Delta t}\boldsymbol{M}_r\boldsymbol{v}^k.$$

Finally, we consider the decomposition

$$\boldsymbol{u}^i = (u_\nu^i, u_\tau^i), \quad \boldsymbol{v}^i = (v_\nu^i, v_\tau^i), \quad \boldsymbol{a}^i = (a_\nu^i, a_\tau^i), \quad \hat{\boldsymbol{f}}^i = (\hat{f}_\nu^i, \hat{f}_\tau^i)$$

and denote

$$a := \Big(\frac{4}{\Delta t^2}m_1 + \frac{\lambda + 3\mu}{2}\Big), \quad b := \frac{\lambda + \mu}{2}, \quad c := \Big(\frac{4}{\Delta t^2}m_2 + \frac{\lambda + 3\mu}{2}\Big).$$

In each time step we then obtain the following problem:

$$\begin{cases} \text{Find } (u_\nu^{k+1/2}, u_\tau^{k+1/2}, \lambda_\nu^{k+1/2}, \lambda_\tau^{k+1/2}) \in \mathbb{R}^4 \text{ such that} \\ au_\nu^{k+1/2} - bu_\tau^{k+1/2} = \hat{f}_\nu^{k+1/2} + \lambda_\nu^{k+1/2}, \\ -bu_\nu^{k+1/2} + cu_\tau^{k+1/2} = \hat{f}_\tau^{k+1/2} + \lambda_\tau^{k+1/2}, \\ -\lambda_\nu^{k+1/2} \in N_{\mathbb{R}_-}(u_\nu^{k+1/2}), \\ \lambda_\tau^{k+1/2} \in \mathscr{F}\lambda_\nu^{k+1/2}\operatorname{Sgn}(u_\tau^{k+1/2} - u_\tau^k), \end{cases} \tag{23}$$

after resolution of which the values of $(u_\nu^{k+1}, u_\tau^{k+1})$ and $(v_\nu^{k+1}, v_\tau^{k+1})$ are determined by (20) and (22).

We shall derive exact solutions of problem (23) for an arbitrary $k \in \{0, \dots, n_t - 1\}$ by considering all possible situations occurring in the inclusions $(23)_4$ and $(23)_5$.

(*i*) Let $\lambda_\nu^{k+1/2} = 0$. From $(23)_5$ it follows that $\lambda_\tau^{k+1/2} = 0$ and solving the equations $(23)_2$ and $(23)_3$ we get:

$$u_\nu^{k+1/2} = \frac{c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2}}{ac - b^2}, \quad u_\tau^{k+1/2} = \frac{a\hat{f}_\tau^{k+1/2} + b\hat{f}_\nu^{k+1/2}}{ac - b^2}.$$

Since $u_\nu^{k+1/2} \le 0$ by $(23)_4$, this solution is valid under the following constraint:

$$c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2} \le 0.$$

(*ii*) If $\lambda_\nu^{k+1/2} < 0$ and $u_\tau^{k+1/2} = u_\tau^k$ then $u_\nu^{k+1/2} = 0$ according to $(23)_4$ and $(23)_{2,3}$ yield:

$$\lambda_\nu^{k+1/2} = -bu_\tau^k - \hat{f}_\nu^{k+1/2}, \quad \lambda_\tau^{k+1/2} = cu_\tau^k - \hat{f}_\tau^{k+1/2}.$$

Our assumption $\lambda_\nu^{k+1/2} < 0$ and the condition $\mathscr{F}\lambda_\nu^{k+1/2} \le \lambda_\tau^{k+1/2} \le -\mathscr{F}\lambda_\nu^{k+1/2}$ implied by $(23)_5$ give the restrictions:

$$\hat{f}_\nu^{k+1/2} > -bu_\tau^k, \quad (c - b\mathscr{F})u_\tau^k - \mathscr{F}\hat{f}_\nu^{k+1/2} \le \hat{f}_\tau^{k+1/2} \le (c + b\mathscr{F})u_\tau^k + \mathscr{F}\hat{f}_\nu^{k+1/2}.$$

(*iii*) Let us pose $\lambda_\nu^{k+1/2} < 0$, $u_\tau^{k+1/2} > u_\tau^k$. Making use of $(23)_{4,5}$ we have $u_\nu^{k+1/2} = 0$, $\lambda_\tau^{k+1/2} = \mathscr{F}\lambda_\nu^{k+1/2}$ and $(23)_{2,3}$ lead to

$$u_\tau^{k+1/2} = \frac{\hat{f}_\tau^{k+1/2} - \mathscr{F}\hat{f}_\nu^{k+1/2}}{c + b\mathscr{F}}, \quad \lambda_\nu^{k+1/2} = -\frac{c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2}}{c + b\mathscr{F}}.$$

From $\lambda_\nu^{k+1/2} < 0$ and $u_\tau^{k+1/2} > u_\tau^k$ one can see that

$$c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2} > 0, \quad \hat{f}_\tau^{k+1/2} > (c + b\mathscr{F})u_\tau^k + \mathscr{F}\hat{f}_\nu^{k+1/2}.$$

(*iv*) Suppose that $\lambda_\nu^{k+1/2} < 0$ and $u_\tau^{k+1/2} < u_\tau^k$. Consequently, $u_\nu^{k+1/2} = 0$ and $\lambda_\tau^{k+1/2} = -\mathscr{F}\lambda_\nu^{k+1/2}$. If $\mathscr{F} \neq c/b$ then we obtain

$$u_\tau^{k+1/2} = \frac{\hat{f}_\tau^{k+1/2} + \mathscr{F}\hat{f}_\nu^{k+1/2}}{c - b\mathscr{F}}, \quad \lambda_\nu^{k+1/2} = -\frac{c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2}}{c - b\mathscr{F}}$$

under the condition

$$\mathscr{F} < \frac{c}{b}, \quad c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2} > 0, \quad \hat{f}_\tau^{k+1/2} < (c - b\mathscr{F})u_\tau^k - \mathscr{F}\hat{f}_\nu^{k+1/2}$$

or

$$\mathscr{F} > \frac{c}{b}, \quad c\hat{f}_\nu^{k+1/2} + b\hat{f}_\tau^{k+1/2} < 0, \quad \hat{f}_\tau^{k+1/2} > (c - b\mathscr{F})u_\tau^k - \mathscr{F}\hat{f}_\nu^{k+1/2}.$$

In the case of $\mathscr{F} = c/b$ there exists the whole solution set

$$\{(u_\tau^{k+1/2}, \lambda_\nu^{k+1/2}) \in \mathbb{R}^2 \mid \lambda_\nu^{k+1/2} = -bu_\tau^{k+1/2} - \hat{f}_\nu^{k+1/2}\}$$

with the following restrictions:

$$\hat{f}_\tau^{k+1/2} + \mathscr{F}\hat{f}_\nu^{k+1/2} = 0, \quad -\frac{\hat{f}_\nu^{k+1/2}}{b} < u_\tau^{k+1/2} < u_\tau^k, \quad -bu_\tau^k < \hat{f}_\nu^{k+1/2}.$$

To summarize the results, introduce the linear functions $\boldsymbol{S}_i^{k+1/2} : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}^4$, $1 \le i \le 4$, and the multi-valued function $\boldsymbol{S}_5^{k+1/2} : \mathbb{R}^2 \times (0, +\infty) \rightrightarrows \mathbb{R}^4$ by

$$\begin{aligned}
\boldsymbol{S}_1^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) &= \Big(\frac{c\hat{f}_\nu + b\hat{f}_\tau}{ac - b^2}, \frac{a\hat{f}_\tau + b\hat{f}_\nu}{ac - b^2}, 0, 0\Big)^T, && \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} > 0,\\
\boldsymbol{S}_2^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) &= (0, u_\tau^k, -(\hat{f}_\nu + bu_\tau^k), cu_\tau^k - \hat{f}_\tau)^T, && \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} > 0,\\
\boldsymbol{S}_3^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) &= \Big(0, \frac{\hat{f}_\tau - \mathscr{F}\hat{f}_\nu}{c + b\mathscr{F}}, -\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c + b\mathscr{F}}, -\mathscr{F}\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c + b\mathscr{F}}\Big)^T, && \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} > 0,\\
\boldsymbol{S}_4^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) &= \Big(0, \frac{\hat{f}_\tau + \mathscr{F}\hat{f}_\nu}{c - b\mathscr{F}}, -\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c - b\mathscr{F}}, \mathscr{F}\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c - b\mathscr{F}}\Big)^T, && \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} \in (0, +\infty) \setminus \Big\{\frac{c}{b}\Big\},\\
\boldsymbol{S}_5^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) &= \Big\{(u_\nu, u_\tau, \lambda_\nu, \lambda_\tau)^T \in \mathbb{R}^4 \mid u_\nu = 0,\ -\frac{\hat{f}_\nu}{b} \le u_\tau \le u_\tau^k,\\
&\qquad \lambda_\nu = -(\hat{f}_\nu + bu_\tau), \lambda_\tau = \mathscr{F}(\hat{f}_\nu + bu_\tau)\Big\}, && \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} = \frac{c}{b},
\end{aligned}$$

and for $\mathscr{F} > 0$ define the sets:

$$\begin{aligned}
&\boldsymbol{\sigma}_1^{k+1/2}(\mathscr{F}) = \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, c\hat{f}_\nu + b\hat{f}_\tau \le 0\},\\
&\boldsymbol{\sigma}_2^{k+1/2}(\mathscr{F}) = \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu_\tau^k,\, (c - b\mathscr{F})u_\tau^k - \mathscr{F}\hat{f}_\nu \le \hat{f}_\tau \le (c + b\mathscr{F})u_\tau^k + \mathscr{F}\hat{f}_\nu\},\\
&\boldsymbol{\sigma}_3^{k+1/2}(\mathscr{F}) = \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, c\hat{f}_\nu + b\hat{f}_\tau \ge 0,\, \hat{f}_\tau \ge (c + b\mathscr{F})u_\tau^k + \mathscr{F}\hat{f}_\nu\},\\
&\boldsymbol{\sigma}_4^{k+1/2}(\mathscr{F}) = \begin{cases} \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu_\tau^k,\, c\hat{f}_\nu + b\hat{f}_\tau \ge 0,\, \hat{f}_\tau \le (c - b\mathscr{F})u_\tau^k - \mathscr{F}\hat{f}_\nu\} \\ \hfill \text{if } \mathscr{F} \in \left(0, \dfrac{c}{b}\right), \\ \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu_\tau^k,\, c\hat{f}_\nu + b\hat{f}_\tau \le 0,\, \hat{f}_\tau \ge (c - b\mathscr{F})u_\tau^k - \mathscr{F}\hat{f}_\nu\} \\ \hfill \text{if } \mathscr{F} \in \left(\dfrac{c}{b}, +\infty\right), \end{cases}\\
&\boldsymbol{\sigma}_5^{k+1/2}(\mathscr{F}) = \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu_\tau^k,\, c\hat{f}_\nu + b\hat{f}_\tau = 0\}, \quad \mathscr{F} = \frac{c}{b}.
\end{aligned}$$

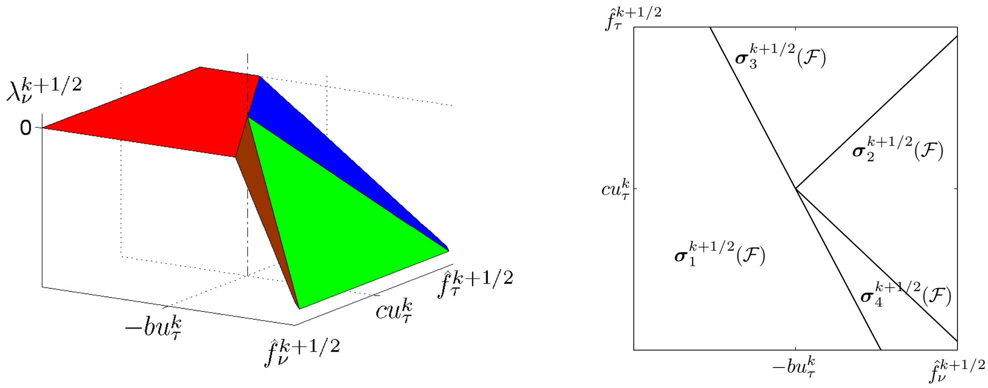

Fig. 3 *Structure of the solution for* $\mathscr{F} \in (0, c/b)$.

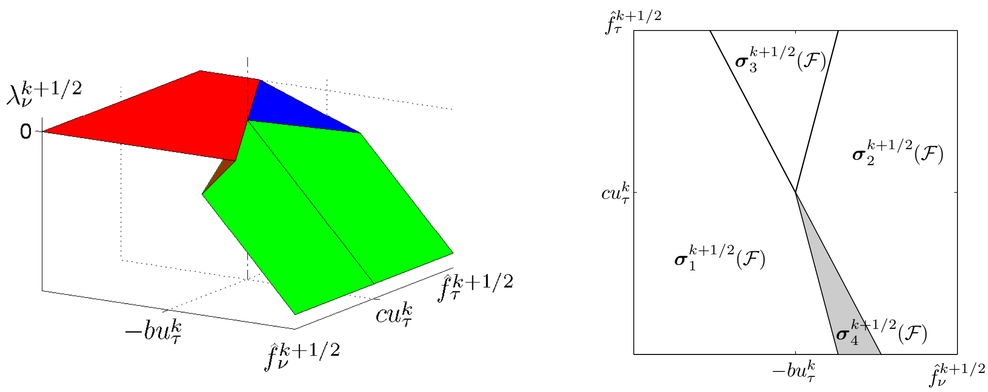

Fig. 4 *Structure of the solution for* $\mathscr{F} \in (c/b, +\infty)$.

One can easily verify that $\boldsymbol{S}_i^{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ solves (23) for $\hat{\boldsymbol{f}}^{k+1/2} \in \boldsymbol{\sigma}_i^{k+1/2}(\mathscr{F})$, $\mathscr{F} > 0$, $1 \le i \le 4$, and $\boldsymbol{S}_5^{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ is the set of solutions to (23) for $\hat{\boldsymbol{f}}^{k+1/2} \in \boldsymbol{\sigma}_5^{k+1/2}(\mathscr{F})$,

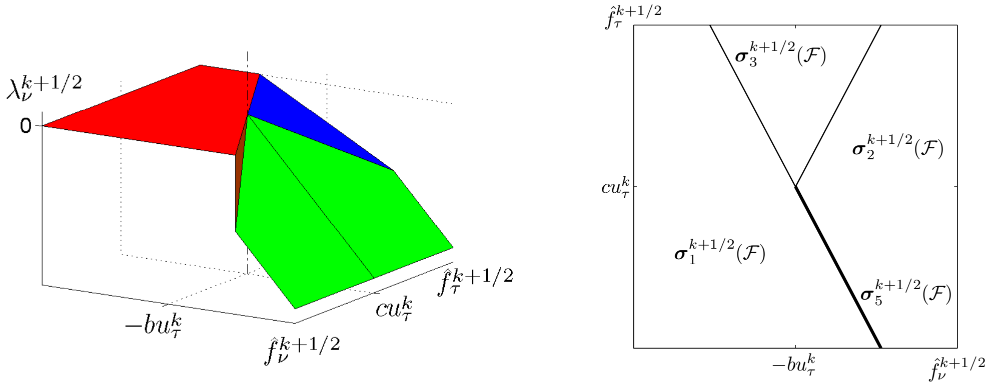

Fig. 5 *Structure of the solution for $\mathscr{F} = c/b$.*

$\mathscr{F} = c/b$. Hence, the structure of the solution set to (23) depends on the mutual position of $\boldsymbol{\sigma}_i^{k+1/2}(\mathscr{F})$, which depends on the magnitude of $\mathscr{F}$.

If $\mathscr{F} \in (0, c/b)$ then the interiors of $\boldsymbol{\sigma}_i^{k+1/2}(\mathscr{F})$ are mutually disjoint for $1 \leq i \leq 4$ and

$$\boldsymbol{S}_i^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) = \boldsymbol{S}_j^{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) \quad \forall\, \hat{\boldsymbol{f}} \in \partial\boldsymbol{\sigma}_i^{k+1/2}(\mathscr{F}) \cap \partial\boldsymbol{\sigma}_j^{k+1/2}(\mathscr{F})\ \forall\, i, j \in \{1, \ldots, 4\}.$$

Consequently, (23) has a unique solution for any $\hat{\boldsymbol{f}}^{k+1/2} \in \mathbb{R}^2$ (see Fig. 3; note that $u_\nu^{k+1/2}, u_\tau^{k+1/2}$ and $\lambda_\tau^{k+1/2}$ are uniquely determined by the values of $\lambda_\nu^{k+1/2}$). If $\mathscr{F} > c/b$ then $\boldsymbol{\sigma}_4^{k+1/2}(\mathscr{F}) = \boldsymbol{\sigma}_1^{k+1/2}(\mathscr{F}) \cap \boldsymbol{\sigma}_2^{k+1/2}(\mathscr{F})$ and its interior is non-empty. In this case there exists a unique solution to (23) if $\hat{\boldsymbol{f}}^{k+1/2} \in (\mathbb{R}^2 \setminus \boldsymbol{\sigma}_4^{k+1/2}(\mathscr{F})) \cup \{(-bu_\tau^k, cu_\tau^k)\}$, there are two solutions on $\partial\boldsymbol{\sigma}_4^{k+1/2}(\mathscr{F}) \setminus \{(-bu_\tau^k, cu_\tau^k)\}$ and three solutions in $\operatorname{Int}\boldsymbol{\sigma}_4^{k+1/2}(\mathscr{F})$ (see Fig. 4). Finally, if $\mathscr{F} = c/b$, $\boldsymbol{\sigma}_1^{k+1/2}(\mathscr{F}) \cap \boldsymbol{\sigma}_2^{k+1/2}(\mathscr{F}) = \boldsymbol{\sigma}_5^{k+1/2}(\mathscr{F})$ is a half-line and there exists a unique solution to (23) for $\hat{\boldsymbol{f}}^{k+1/2} \in (\mathbb{R}^2 \setminus \boldsymbol{\sigma}_5^{k+1/2}(\mathscr{F})) \cup \{(-bu_\tau^k, cu_\tau^k)\}$ whereas the continuous branch $\boldsymbol{S}_5^{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ of solutions connects $\boldsymbol{S}_1^{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ and $\boldsymbol{S}_2^{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ for $\hat{\boldsymbol{f}}^{k+1/2} \in \boldsymbol{\sigma}_5^{k+1/2}(\mathscr{F}) \setminus \{(-bu_\tau^k, cu_\tau^k)\}$ (as depicted in Fig. 5).

Now take the redistributed mass matrix $\boldsymbol{M}_r$ such that $m_1 = 0$ and $m_2 > 0$, i.e. (6) is fulfilled. Then, for any $\mathscr{F} > 0$ given, one can find $\Delta t_0 > 0$ satisfying

$$\frac{c}{b} = \frac{\frac{4}{\Delta t^2} m_2 + \frac{\lambda + 3\mu}{2}}{\frac{\lambda + \mu}{2}} > \mathscr{F} \quad \forall\, \Delta t \in (0, \Delta t_0)$$

and the analysis above ensures the unique solvability of (23) for any $\hat{\boldsymbol{f}}^{k+1/2} \in \mathbb{R}^2$ and any $\Delta t \in (0, \Delta t_0)$. Observe that this is in good accordance with the well-posedness result established in Section 4.

On the contrary, consider $\boldsymbol{M}_r$ with $m_1 > 0$, $m_2 = 0$ or $m_1 = m_2 = 0$, which corresponds to the elimination of the mass in the tangential direction and the total elimination of the mass on the contact zone, respectively. If the coefficient $\mathscr{F}$ is larger than $(\lambda + 3\mu)/(\lambda + \mu) = c/b$, one can always find $\hat{\boldsymbol{f}}^{k+1/2}$ such that (23) possesses multiple solutions whatever small Δt is. This suggests that the well-posedness is not reached in such cases.

7. Numerical tests

The numerical simulations presented in (**15**) show the effectiveness of the mass redistribution method to remove the spurious oscillations caused by the contact condition in discretized dynamical problems. Here we chose a test case where the sliding is much more present with

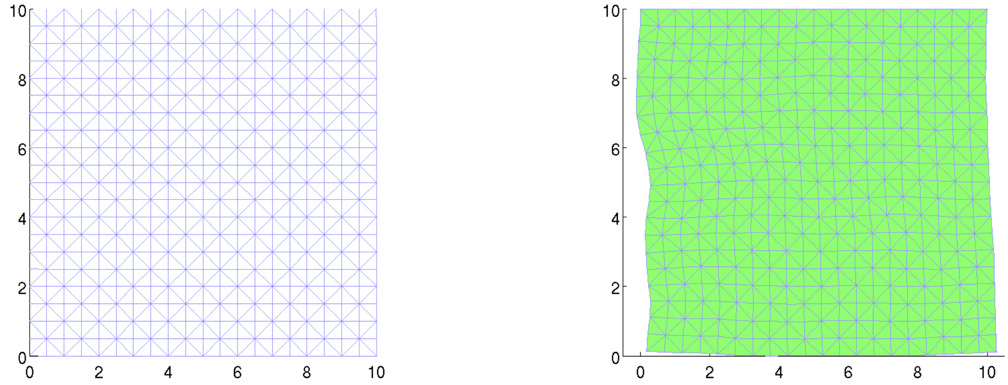

Fig. 6 *Structured mesh of the square* $\{(x_1, x_2) \in\ ]0, 10\,\mathrm{cm}[\ \times\ ]0, 10\,\mathrm{cm}[\}$ *and an example of deformation (at* $t = 0.01\,\mathrm{s}$*). The rigid foundation lies at* $x_2 = 0$ *and has a constant horizontal velocity of* $20\,\mathrm{m/s}$*. The structure is clambed on its top (*$x_2 = 10\,\mathrm{cm}$*).*

Density ρ	$10^3\,\mathrm{kg/m^3}$
Domain	$]0, 10\,\mathrm{cm}[\times]0, 10\,\mathrm{cm}[$
Lamé coefficients	$\lambda = 300\,\mathrm{MPa}$, $\mu = 150\,\mathrm{MPa}$
Simulation time	$0.02\,\mathrm{s}$
Friction coefficient $\mathscr{F}$	1.2
Horizontal velocity of the rigid foundation	$20\,\mathrm{m/s}$

Table 1 *Main characteristics of the simulation.*

still a partial loss of contact (see Fig. 6). An elastic body whose reference configuration is the square $\{(x_1, x_2) \in\ ]0, 10\,\mathrm{cm}[\ \times\ ]0, 10\,\mathrm{cm}[\}$ is in contact with a rigid foundation at $x_2 = 0$. This rigid foundation is moving horizontally at a constant speed of $20\,\mathrm{m/s}$. At the top of the structure ($x_2 = 10\,\mathrm{cm}$) the Dirichlet condition $u_1 = 0, u_2 = -2.5 \times 10^{-3}\,\mathrm{cm}$ ensures an initial compression of the body. The friction coefficient is larger than one ($\mathscr{F} = 1.2$). The other characteristics of the simulation are summarized in Table 1. Additionally, Problem (7) is approximated in time with the midpoint scheme (see Section 6 or (**15**) for more details). The C++ program that performs the tests is available along with Getfem++ (**26**).

Due to friction and the fact that the rigid foundation has a constant horizontal velocity, there is a source of energy in the system. The evolution of the total energy (given by (19)) for an element size $h = 0.5\,\mathrm{cm}$ and various time steps is shown in Fig. 7 for four situations.

The first graph of Fig. 7 corresponds to the standard semi-discretization. It clearly shows the instability of the midpoint scheme applied to the standard semi-discretization. A rather unique feature, reserved to the discretization of dynamic contact problems, is that the smaller the time step is, the most the scheme is unstable. One can also remark that the scheme is reasonably stable in the first half of the simulation period. It is probably due to the fact that, at the beginning of the simulation, the body is pressed down to the foundation and is submitted to a relatively monotone loading due to the friction force. Consequently, the number of transitions between contact and noncontact is low.

The second and third graphs correspond to the mass redistribution applied only on the normal component of the displacement and on both components, respectively. The stability of the midpoint scheme is recovered.

An interesting situation is described on the fourth graph of this figure, which corresponds to the standard semi-discretization with an adaptation of the midpoint scheme where the contact forces have been implicited (see (**6**, **2**)). Even though this scheme is proven to be stable (energy

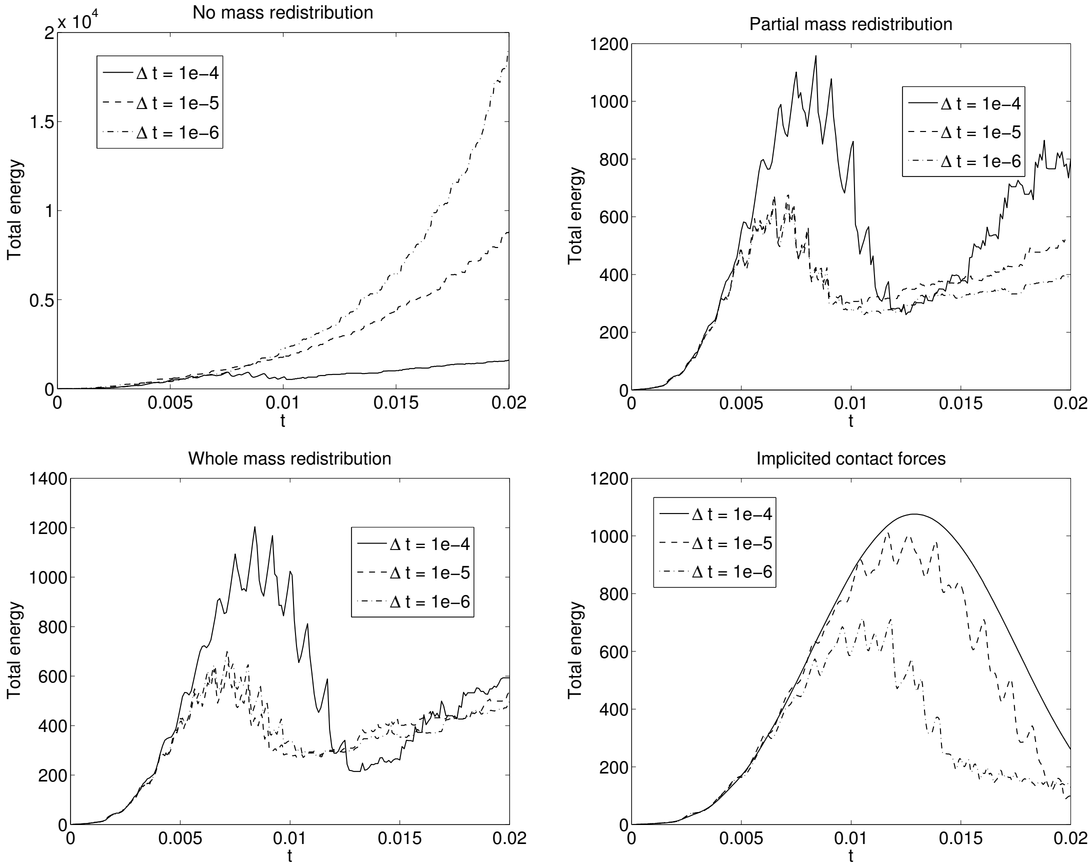

Fig. 7 Evolution of the energy for the following four situations: standard discretization, partial mass redistribution, whole mass redistribution and standard discretization with implicited contact forces.

dissipative in fact) and this is also the case here, one can see that the result is rather different than the two previous graphs.

This is more clear in Fig. 8, where the graphs correspond to the difference between the total energy and the energy transferred (provided and dissipated) by friction. The latter is given at the time step $l\Delta t$ by

$$\xi_l = \sum_{k=1}^{l} \sum_{i=1}^{n_c} \lambda_{\tau,i}^k (\dot{\boldsymbol{u}}^k)^T \boldsymbol{\tau}_i \Delta t,$$

where $\lambda_{\tau,i}^k$ and $\dot{\boldsymbol{u}}^k$ are the corresponding quantities to $\lambda_{\tau,i}(k\Delta t)$ and $\dot{\boldsymbol{u}}(k\Delta t)$ at the k-th time step of the midpoint scheme. These graphs reflect better the stability of each scheme. Of course, this does not change the conclusion for the midpoint scheme with the standard semi-discretization, which still appears to be unstable. However, from the two graphs corresponding to the mass-redistribution method we can see that for both cases the energy conservation is obtained asymptotically for a time step going to zero. This is not the case for the fourth graph, corresponding to the standard semi-discretization with implicited contact forces. The scheme is stable, but it does not converge toward an energy conserving solution. This is due to the fact that a certain amount of energy is lost at each impact of each node, independently of the length of the time step. Since the number of impacts grows in this simulation for decreasing time steps, the smallest the time step is, the most dissipative the scheme is. In conclusion, we can say that this leads to a non-physical solution.

A convergence test on the time interval $[0, 0.005\,\mathrm{s}]$ has been performed. The results are shown

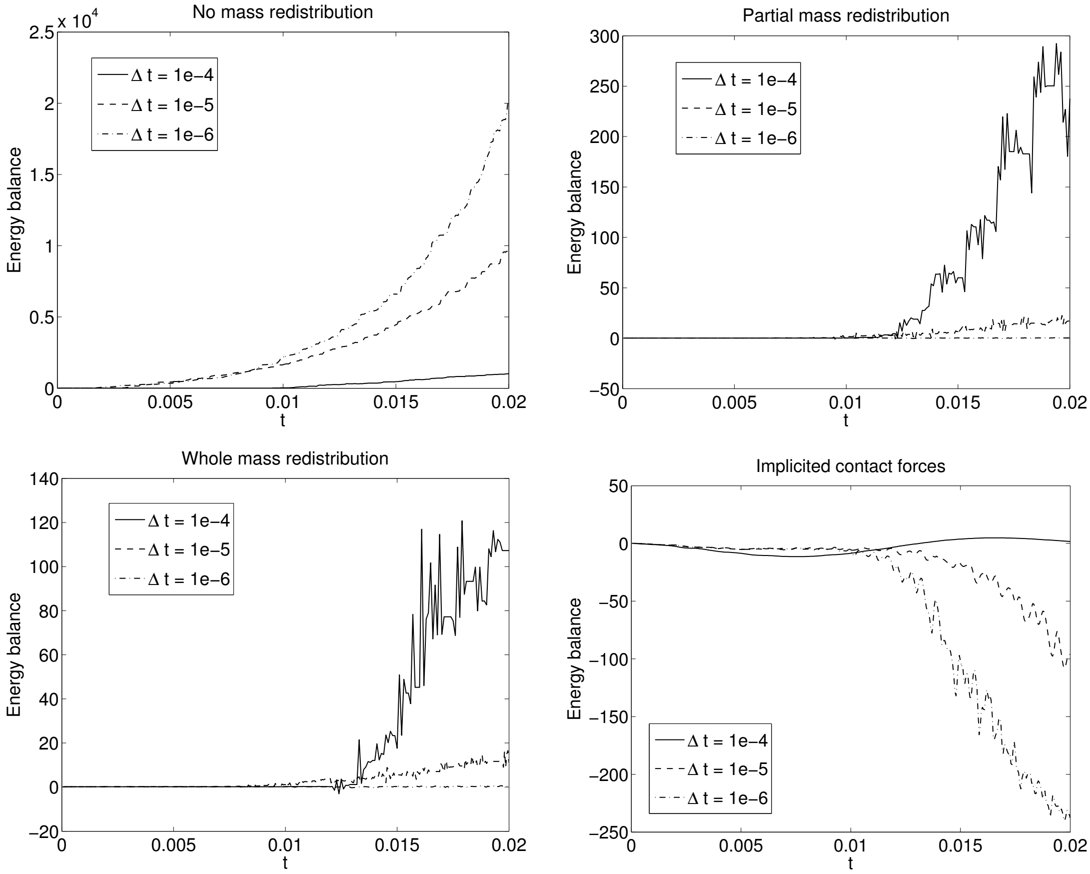

Fig. 8 Evolution of the difference between the total energy and the energy transferred by friction.

	experiment 1	experiment 2	experiment 3	experiment 4	reference solution
Element size h	$2\,\mathrm{cm}$	$1\,\mathrm{cm}$	$0.5\,\mathrm{cm}$	$0.25\,\mathrm{cm}$	$0.0625\,\mathrm{cm}$
Time step Δt	$1.6 \times 10^{-5}\,\mathrm{s}$	$8 \times 10^{-6}\,\mathrm{s}$	$4 \times 10^{-6}\,\mathrm{s}$	$2 \times 10^{-6}\,\mathrm{s}$	$5 \times 10^{-7}\,\mathrm{s}$

Table 2 *Element sizes and time steps for the convergence test.*

in Fig. 9 and Fig. 10. A reference solution has been computed on a refined mesh ($h = 0.0625\,\mathrm{cm}$) for a small time step ($\Delta t = 5 \times 10^{-7}\,\mathrm{s}$) using the partial mass redistribution. Then the differences between this reference solution and four experiments whose characteristics are presented in Table 2 are computed. The curves in Fig. 9 present the maxima of the $L^2(\Omega)$-norm and $H^1(\Omega)$-semi-norm in $[0, 0.005\,\mathrm{s}]$ for the four experiments. Numerical convergence with an order less than one is found. Due to the weak regularity of the solution, a faster rate of convergence cannot reasonably be expected. Note that a mathematical result of convergence of numerical solutions toward a solution of the continuous problem is an open problem. Untill now, neither existence nor uniqueness results have been established on this model (unless a certain number of regularizations).

The convergence is also illustrated in Fig. 10, where the evolution of the density of friction force for different experiments is shown. Both the cases with a redistribution of mass only on the vertical component and on both components of the displacement are shown. In the two cases the numerical solution seems to converge toward the same solution. This means that at least for the presented test case, the differentiated treatment of the two conditions is not strictly mandatory to

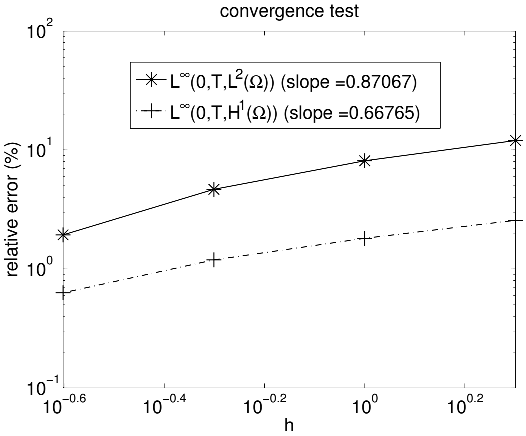

Fig. 9 Convergence test for the partial mass redistribution. Maxima of the $L^2(\Omega)$-norm and $H^1(\Omega)$-semi-norm in $[0, 0.005\,\mathrm{s}]$.

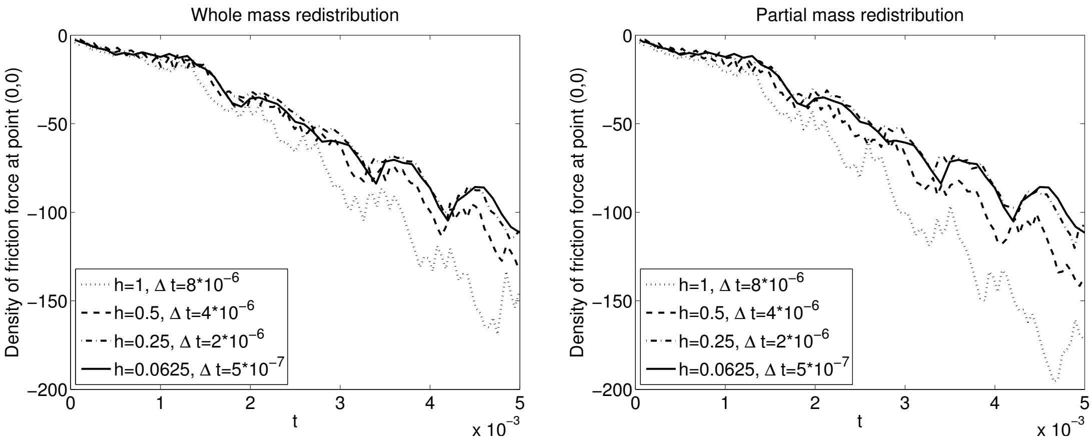

Fig. 10 Comparison of the density of friction force at the point $(0, 0)$ for the convergence test.

obtain reliable numerical results. Of course, with the redistribution of mass on both components, the well-posedness of the semi-discretization is not guaranteed (see the discussion in Section 6). A real difference may occur if there is a dynamical bifurcation. But the exhibition of such a dynamical bifurcation is also still an open problem.

Concluding remarks

We adapted the mass redistribution method for the elastodynamic contact problem with friction. The proposed strategy, which is to apply the mass redistribution only on the normal component corresponding to the contact condition, allows to transform the semi-discrete problem into a regular one-sided Lipschitz differential inclusion. The advantage is that any reasonable time integration scheme is then convergent (see (**19**)) at least for a fixed mesh. Moreover, the fully discrete problem is also well-posed for a sufficiently small time step. The simple example described in Section 6 shows that this is not the case when the mass redistribution is applied on both the contact and friction conditions.

The test case presented in Section 7 confirms that the stability is gained by the mass redistribution method. However, for this simple test case, there is no significant difference between the two strategies how to apply the mass redistribution. One may think that it makes no difference numerically.

For the moment, we do not have any dynamical bifurcation example for the Coulomb friction law with a constant friction coefficient that would test the difference. A perspective of this work would be to consider a coefficient of friction depending on the sliding velocity. In (**27**, **28**) such a friction coefficient has been considered and multi-solutions are given in a one-dimensional case. It is proven that an additional mass on the contact boundary allows to recover the uniqueness of the solution. Moreover, it selects a particular solution which is related to the perfect delay criterion introduced in (**12**) for contact problems with friction. In this context, obtaining a well-posed semi-discrete problem would be more crucial and it would be interesting to see if the same solution is selected.

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