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Vibro-Impact of a plate on rigid obstacles: existence theorem, convergence of a scheme and numerical simulations

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The purpose of this paper is to describe a fully discretized approximation and its convergence to the continuum dynamical impact problem for the fourth order Kirchhoff-Love plate model with non-penetration Signorini's condition. We extend to the case of plates the theoretical results of weak convergence due to Y. Dumont and L. Paoli which was stated for Euler-Bernouilli beams. In particular, this provides an existence result for the solution of this problem. Finally we discuss the numerical results we obtain.

Keywords. Variational inequalities; Finite element method; Elastic plate; Dynamics with unilateral constraints; Scheme convergence.

1. Introduction

The impact of linear elastic thin structures, such as beams, membranes or plates, is a domain where there are still fondamental open questions despite a rather important literature. This includes in particular the existence and uniqueness of solutions, the convergence and stability of numerical schemes, the modelization of a restitution coefficient and the construction of energy conserving schemes.

In the particular case of the vibro-impact problem between an Euler-Bernouilli beam and a rigid obstacle, an existence result was shown by Y. Dumont & L. Paoli in (1). They established the convergence of the solution of a fully discretized problem to the continuum model. But there were no result whether energy is conserved in the limit or not. Indeed, it can be easily shown that uniqueness does not hold for this system (see (2) for a counter-example). Moreover, it is generally not possible to prove that each solution to this problem is energy conserving. This is due to the weak regularity involved, since, in particular, velocities may be discontinuous.

The dynamic contact problem for Von Karman plates is studied in (3) and (4). In the first paper, the authors show the existence of a solution, using penalization technics, while other existence results are given in the second by the introduction of a viscosity term. Here, our main goal is to extend Dumont and Paoli results to the case of Kirchhoff-Love plates. We present a convergence result of a fully discrete scheme toward one solution of the continuous problem. This establishes both an existence result for the solution of the continuous problem and ensures that one subsequence weakly converges toward this solution. We do not establish any uniqueness result. Such result would certainly requires the ability to

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express an additional impact law (see (5), (6)). Although the consideration of an impact law is something very natural for modelization of rigid body impacts, this concept seems to be rather difficult to extend to the framework of thin deformable bodies, especially concerning the discretization.

The paper is organized as follows. In the next section, an elastodynamical Kirchhoff-Love plate model is described as well as the vibro-impact model. In section 3, the fully discretized approximation of the problem (finite element model and time scheme) is introduced. Section 4 gives the most important result of this paper, namely a convergence result for fully discretized schemes. Finally, in section 5 we present and discuss some numerical experiments.

2. Notations and statement of the problem

2.1 *Variational formulation of the plate model*

Let us consider a thin elastic plate *i.e.* a plane structure for which one dimension, called the thickness, is very small compared to the others. For this kind of structures, starting from *a priori* hypotheses on the expression of the displacement fields, a two-dimensional problem is usually derived from the three-dimensional elasticity formulation by means of integration along the thickness. Then, the unknown variables are set down on the mid-plane of the plate.

Let Ω be an open, bounded, connected subset of the plane $\mathbb{R}^2$, with Lipschitz boundary. It will define the middle plane of the plate. Then, the plate in its stress free reference configuration coincides with domain :

$$\Omega^{\varepsilon} \;=\; \Omega \times \,]-\varepsilon\;,\;+\varepsilon[\, = \,\left\{\,(x_1,x_2,x_3) \in \mathbb{R}^3 \;/\; (x_1,x_2) \in \Omega \text{ and } x_3 \in \,]-\varepsilon\;;\;\varepsilon[\,\right\}$$

where $2\,\varepsilon > 0$ is called the thickness.

In plate theory, it is usual to consider the following approximation of the three-dimensional displacements for $(x_1,x_2,x_3) \in \Omega^{\varepsilon}$

$$\left\{\begin{array}{lcl} u_1(x_1,x_2,x_3) & = & \overline{u}_1(x_1,x_2) \;+\; x_3\;\psi_1(x_1,x_2) \\ u_2(x_1,x_2,x_3) & = & \overline{u}_2(x_1,x_2) \;+\; x_3\;\psi_2(x_1,x_2) \\ u_3(x_1,x_2,x_3) & = & u_3(x_1,x_2). \end{array}\right. \tag{2.1}$$

In these expressions, $\overline{u}_1$ and $\overline{u}_2$ are the membrane displacements of the mid-plane points, u_3 is the deflection, while ψ_1 and ψ_2 are the section rotations. In the case of an homogeneous isotropic material, the variational plate model splits into two independent problems: the first, called the membrane problem, deals only with membrane displacements, while the second, called the bending problem, concerns deflection and rotations. In this paper, we shall only adress the bending problem, and we shall consider the Kirchhoff-Love model, which can be seen as a particular case of (2.1) obtained by introducing the so-called Kirchhoff-Love assumptions :

$$\psi \;=\; -\nabla\, u_3 \quad \Leftrightarrow \quad \left\{\begin{array}{lcl} \psi_1 & = & -\partial_1\, u_3 \\ \psi_2 & = & -\partial_2\, u_3 \end{array}\right.$$

where ∂_α stands for the partial derivative with respect to x_α , for $\alpha \;=\; 1$ or 2. Consequently, the deflection is the only unknown for the bending Kirchhoff-Love plate problem. For convenience, it will be denoted by u all along the following of this paper. As far as loading is concerned, the plate is subject to a volume force F and two surface forces, say G^+ and G^-, applied on the top and bottom surfaces.

Then, if we assume that the previous forces are purely perpendicular to the mid-plane, the resulting transverse loading reads

$$f_R = G_3^+ + G_3^- + \int_{-\varepsilon}^{\varepsilon} F_3 \, dx_3$$

where G_3^+, G_3^- and F_3 are respectively the third components of G^+, G^- and F. So, the variational formulation of the Kirchhoff-Love elastodynamical model for a thin elastic clamped/free plate reads as

$$\begin{aligned} &\text{Find } u = u(x,t) \text{ with } (x,t) \in \Omega \times [0,T] \text{ such that for any } w \in \mathbb{V} \\ &\int_\Omega \partial^2_{tt} u(x,t) \, w(x) \, dx + \int_\Omega \frac{D}{2\rho\varepsilon} \left[(1-\nu) \, \partial^2_{\alpha\beta} u + \nu \, \Delta u \, \delta_{\alpha\beta} \right] \partial^2_{\alpha\beta} w \, dx = \int_\Omega f \, w \, dx \end{aligned} \tag{2.2}$$

with $f = \dfrac{f_R}{2\rho\varepsilon}$ (ρ and ε are assume to be constant all along the plate), and

$$u(x,0) = u_0(x) \quad , \quad \partial_t u(x,0) = v_0(x) \quad , \quad \forall x \in \Omega \tag{2.3}$$

where $\partial^2_{tt} u = \dfrac{\partial^2 u}{\partial t^2}$, $\partial^2_{\alpha\beta} u = \dfrac{\partial^2 u}{\partial x_\alpha \, \partial x_\beta}$ and the bending modulus is $D = \dfrac{2 \, E \, \varepsilon^3}{3 \, (1-\nu^2)}$, for a plate made of a homogeneous and isotropic material, which mechanical constants are its Young's modulus E, its Poisson's ratio ν and its mass density ρ. As usual, we have: $E > 0$, $0 < \nu < 0.5$ and $\rho > 0$. Moreover, $\delta_{\alpha\beta}$ is the Kronecker's symbol and the summation convention over repeated indices is adopted, Greek indices varying in $\{1,2\}$. The plate is assumed to be clamped on a non-zero Lebesgue measure part of the boundary $\partial\Omega$ denoted Γ_c and free on Γ_f , such as $\partial\Omega = \Gamma_c \cup \Gamma_f$. Then the space of admissible displacements is

$$\mathbb{V} = \{ \, w \in H^2(\Omega) \, / \, w(x) = 0 = \partial_n w(x) \, , \, \forall x \in \Gamma_c \, \} \tag{2.4}$$

where $\partial_n w$ is the normal derivative along Γ_c .

In order to guarantee that (2.2) is well-posed, we use the following result.

LEMMA 2.1 The bilinear form $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ defined by

$$a(u,v) = \int_\Omega \frac{D}{2\rho\varepsilon} \left[(1-\nu) \, \partial^2_{\alpha\beta} u + \nu \, \Delta u \, \delta_{\alpha\beta} \right] \partial^2_{\alpha\beta} v \, dx \tag{2.5}$$

is a scalar product on $\mathbb{V}$ which is equivalent to the canonical scalar product of $H^2(\Omega)$ defined on $\mathbb{V}$.

The bilinear map a is obviously continuous in $\mathbb{V}$. Then there exists a strictly positive constant, say M, such that for any $u \in \mathbb{V}$: $a(u,u) \leqslant M \, \| \, u \, \|^2_{\mathbb{V}}$. The reciprocical inequality is due to the coercivity of $a(.,.)$, which can be established by using Petree-Tartar's lemma we recall here.

LEMMA 2.2 (7) Let $\mathtt{X},\mathtt{Y},\mathtt{Z}$ be three Banach spaces, $A \in \mathscr{L}(\mathtt{X},\mathtt{Y})$ injective, $T \in \mathscr{L}(\mathtt{X},\mathtt{Z})$ compact. If there exists $c > 0$ such that $c \, \|x\|_{\mathtt{X}} \leqslant \|Ax\|_{\mathtt{Y}} + \|Tx\|_{\mathtt{Z}}$, for any $x \in \mathtt{X}$, then there exists $\alpha > 0$ such that

$$\alpha \, \|x\|_{\mathtt{X}} \leqslant \|Ax\|_{\mathtt{Y}} \ , \ \forall \, x \in \mathtt{X}.$$

Proof. (Lemma 2.1) Let us remark that

$$\begin{aligned} a(u,u) &= \int_\Omega \frac{D}{2\rho\varepsilon} \left[(1-\nu)\,\partial^2_{\alpha\beta}u \,+\, \nu\,\Delta u\,\delta_{\alpha\beta}\right]\,\partial^2_{\alpha\beta}u\,dx \\ &= \int_\Omega \frac{D}{2\rho\varepsilon} \left[(1-\nu)\,\partial^2_{\alpha\beta}u\,\partial^2_{\alpha\beta}u \,+\, \nu\,(\Delta u)^2\right]\,dx \\ &\geqslant \frac{(1-\nu)D}{2\rho\varepsilon} \int_\Omega \partial^2_{\alpha\beta}u\,\partial^2_{\alpha\beta}u\,dx \quad \text{as } \nu > 0 \\ &= \frac{(1-\nu)D}{2\rho\varepsilon}\,\|\,Hess(u)\,\|^2_{\mathtt{Y}} \end{aligned}$$

$Hess(u)$ being the Hessian matrix of u and with $\mathtt{Y} = \left(L^2(\Omega)\right)^4$. Now, the Petree-Tartar's lemma is applied with A defined from $\mathtt{X} = \mathbb{V}$ to $\mathtt{Y}$ by $Au = Hess(u)$ which is injective because of the boundary conditions, Γ_c having a non-zero measure in $\partial\Omega$. Setting $\mathtt{Z} = H^1(\Omega)$ and $T = id_{\mathtt{X},\mathtt{Z}}$, which is compact, we obtain the $\mathbb{V}$-coercivity of A, and consequently of a as $\nu < 1$. □

2.2 *Vibro-impact formulation of the plate model*

Let us now introduce the dynamic frictionless Kirchhoff-Love equation with Signorini contact conditions along the plate. We assume that the plate motion is limited by rigid obstacles, located above and below the plate (see Figure 1). So, the displacement is constrained to belong to the convex set $\mathbb{K} \subset \mathbb{V}$ given by

$$\mathbb{K} = \{v \in \mathbb{V} \;/\; g_1(x) \leqslant v(x) \leqslant g_2(x) \;\;,\;\; \forall x \in \Omega\} \tag{2.6}$$

where g_1 and g_2 are two mappings from Ω to $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ such that there exists $g > 0$ with

$$g_1(x) \leqslant -g < 0 < g \leqslant g_2(x) \quad , \quad \forall x \in \Omega. \tag{2.7}$$

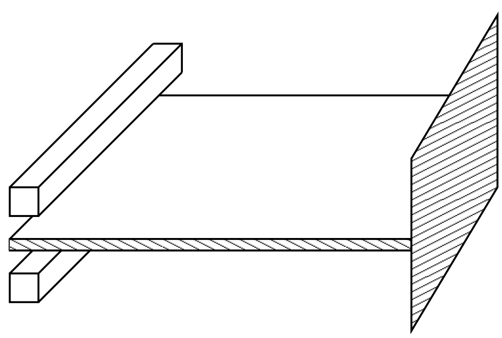

Figure 1. Example of bending clamped plate between rigid obstacles.

Since impact will occur in this system, classical regular solutions cannot be expected. In particular the velocities may be discontinuous. To set the weak formulation, the following functional spaces are introduced

$$
\begin{array}{rcl}
\mathbb{H} & = & L^2(\Omega) \\
\mathbb{V} & = & \{\, w \in H^2(\Omega) \;/\; w(x) \;=\; 0 \;=\; \partial_n w(x) \;,\; \forall x \;\in\; \Gamma_c \,\} \\
\tilde{\mathbb{U}} & = & \{\, w \in L^2(0,T;\mathbb{K}) \;\;,\;\; \dot{w} \in L^2(0,T;L^2(\Omega)) \,\} \\
\mathbb{U} & = & \{\, w \in L^\infty(0,T;\mathbb{K}) \;\;,\;\; \dot{w} \in L^\infty(0,T;L^2(\Omega)) \,\}
\end{array}
$$

where $\dot{w} \;=\; \partial_t w \;=\; \dfrac{\partial w}{\partial t}$, and $T \;>\; 0$ is the length of the plate motion study. The norm in $\mathbb{H}$ will be denoted by $|.|_{\mathbb{H}}$.

The frictionless elastodynamic problem for a plate between two rigid obstacles consists in finding $u \;\in\; \tilde{\mathbb{U}}$ with $u(.,0) \;=\; u_0$ in $\mathbb{K}$ and $\dot{u}(.,0) \;=\; v_0$ such that

$$
\left\{
\begin{array}{l}
-\displaystyle\int_0^T \int_\Omega \partial_t u \; \partial_t(\tilde{w} \;-\; u)\; dxdt \;+\; \int_0^T a(u(.,t),(\tilde{w}-u)(.,t))\; dt \\[2ex]
\geqslant \displaystyle\int_\Omega v_0(x)\; (\tilde{w}(x,0) \;-\; u_0(x))\; dx \;+\; \int_0^T \int_\Omega f\; (\tilde{w}-u)\; dxdt \\[2ex]
\forall \tilde{w} \;\in\; \tilde{\mathbb{U}} \;,\; \tilde{w}(.,T) = u(.,T).
\end{array}
\right.
\tag{2.8}
$$

REMARK 2.1 The discretization of (2.8) does not describe completely the motion (6). In addition, it would require an impact law. For an impact at (x_0,t_0), this law is given by a relation between velocities before and after impact, as

$$
\frac{\partial u}{\partial t}(x_0,t_0^+) \;=\; -e\; \frac{\partial u}{\partial t}(x_0,t_0^-)
\tag{2.9}
$$

where e belongs to $[0,1]$. Since one can only guarantee that the velocity is $L^2(\Omega)$ in space, it is not easy to express (2.9) rigorously. Moreover, in (5), the authors observe that the restitution coefficient for a bar is a rather ill-defined concept. They observed the apparent restitution coefficient depends very strongly on the initial angle of the bar with horizontal. In the particular case of a slender bar dropped on a rigid foundation, the chosen value of the restitution coefficient does not seem to have great influence on the limit displacement when the space step tends to zero, as it has been shown in (5). The idea to explicitly incorporate the restitution coefficient into (2.8) seems a rather problematic task since it would need to separate the post-impact normal velocity at a point due to the impact force from the post-impact normal velocity due to elastic waves. Therefore, knowing whether our schemes will simulate the experimental behavior is an interesting question.

3. Full discretization of the problem

3.1 *Finite element model for the plate problem*

Let us begin by the space discretization of the displacement. The Kirchhoff-Love model corresponds to a fourth order partial differential equation. Consequently, a conformal finite element method needs the use of $\mathscr{C}^1$ (continuously differentiable) finite elements. Here, we consider the classical Argyris triangle, which uses P_5 polynomials, and Fraeijs de Veubeke-Sanders quadrilateral (reduced FVS), see (8). For the FVS element, the quadrangle is divided into four sub-triangles (see Figure 2). The basis functions are P_3 polynomials on each sub-triangle and matched $\mathscr{C}^1$ across each internal edge. In addition, to decrease the number of degrees of freedom, the normal derivative is assumed to vary linearly along the external edges of the elements (this assumption does not hold on the internal edges). Finally, for FVS quadrangles, there are only three degrees of freedom on each node: the value of the function and its first derivatives. Let us assume from now on that $h > 0$ stands for the mesh parameter and that $\mathbb{V}^h$ is a finite dimensional subspace of $\mathbb{V}$ built using the previous finite element methods. Then, following (8) for Argyris triangle and (9) for FVS quadrangles, for all $w \in \mathbb{V}$, there exists a sequence $(w^h)_{h>0}$ of elements of $\mathbb{V}^h$ such that

$$\|w^h - w\|_{\mathbb{V}} \;\to\; 0 \;\;,\;\; \text{when} \;\; h \;\to\; 0.$$

Finally, let us remark that there also exists some non conformal approximations (see (10)), but we do not use them here because we develop our theory within the frame of conformal methods.

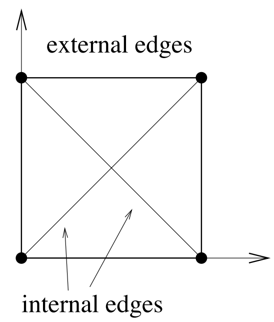

Figure 2. FVS quadrangle. Location of degrees of freedom and sub-triangles.

3.2 *Time discretization*

Now, we consider the time discretization of problem (2.8). For $N \in \mathbb{N}^*$, the time step is denoted by $\Delta t = T/N$. The time scheme is initialized by choosing u_0^h and u_1^h in $\mathbb{K}^h = \mathbb{K} \cap \mathbb{V}^h$ such that

$$\lim_{h\to 0\;,\;\Delta t\to 0} \|u_0^h - u_0\|_{\mathbb{V}} \;+\; \left|\frac{u_1^h - u_0^h}{\Delta t} - v_0\right|_{\mathbb{H}} \;=\; 0. \tag{3.1}$$

As far as the loading is concerned, we assume that f belongs to $L^2(0,T;L^2(\Omega))$. Then, for all $x \in \Omega$ and $n \in \{1,\ldots,N-1\}$, we set

$$f_n(x) \;=\; \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} f(x,s)\; ds. \tag{3.2}$$

For time discretization, we consider the corresponding fully discretized scheme which consists in finding u_{n+1}^h, for all $n \in \{2,\ldots,N-1\}$ solution of the following inequality

$$\left\{\begin{array}{l} \text{Find } u_{n+1}^h \in \mathbb{K}^h \text{ such that} \\[2ex] (w - u_{n+1}^h)^T\, \mathbf{M}\left(\dfrac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{\Delta t^2}\right) \\[2ex] \quad + (w - u_{n+1}^h)^T\, \mathbf{K}\, (\beta u_{n+1}^h + (1-2\beta)u_n^h + \beta u_{n-1}^h) \;\geqslant\; (w - u_{n+1}^h)^T\; f^{n\beta} \end{array}\right. \tag{3.3}$$

which is a classical Newmark scheme of parameters β and $\gamma = 1/2$. If $(\psi_i)_i$ stand for the finite element basis functions, in the previous expression

- $f^{n\beta}$ is the loading which generic term reads

$$f_i^{n\beta} \;=\; \int_\Omega \Big(\beta\, f_{n+1} + (1-2\beta)\, f_n + \beta\, f_{n-1}\Big)\; \psi_i \; dx;$$

- **M** is the mass matrix which generic term reads

$$\mathbf{M}_{ij} \;=\; \int_\Omega \psi_i \,.\, \psi_j \; dx; \tag{3.4}$$

- **K** is the rigidity matrix which generic term reads

$$\mathbf{K}_{ij} \;=\; a(\psi_i \,,\, \psi_j). \tag{3.5}$$

Let us remark that the previous inequality is also equivalent to the inclusion

$$\left\{\begin{array}{l} \text{Find } u_{n+1}^h \;\in\; \mathbb{K}^h \text{ such that} \\[2ex] (\mathbf{M} + \beta\; \Delta t^2\; \mathbf{K}) u_{n+1}^h \;+\; \Delta t^2\; \partial \mathbb{I}_{\mathbb{K}^h}(u_{n+1}^h) \ni f_n^h \\[2ex] \text{where} \\ f_n^h \;=\; \Big(2\mathbf{M} - (1-2\beta)\Delta t^2 \mathbf{K}\Big)\; u_n^h \;-\; \Big(\mathbf{M} + \beta \Delta t^2 \mathbf{K}\Big)\; u_{n-1}^h \;+\; \Delta t^2\; f^{n\beta}. \end{array}\right. \tag{3.6}$$

As $\mathbb{K}^h$ is a non-empty closed convex subset of $\mathbb{W}^h$ and thanks to lemma 2.1, we easily obtain by induction on n that u_{n+1}^h is uniquely defined for all $n \;\in\; \{1,\ldots,N-1\}$. This kind of variational inequality has been intensively studied by Paoli and Schatzman (see (6) and (5)).

4. A convergence result for a Newmark-Dumont-Paoli kind scheme

The discrete problem associated to (3.3) reads

$$\left\{\begin{array}{l} \text{Find } u_{n+1}^h \;\in\; \mathbb{K}^h \text{ such that for all } w^h \;\in\; \mathbb{K}^h \\[2ex] \displaystyle\int_\Omega \frac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{\Delta t^2} \,.\, (w^h - u_{n+1}^h)\; dx \;+\; a(\beta u_{n+1}^h + (1-2\beta) u_n^h + \beta u_{n-1}^h \,,\; w^h - u_{n+1}^h) \\[2ex] \qquad\qquad \geqslant\; \displaystyle\int_\Omega \Big[\beta\, f_{n+1} + (1-2\beta)\, f_n + \beta\, f_{n-1}\Big] \,.\, (w^h - u_{n+1}^h)\; dx. \end{array}\right.$$

In (1), Dumont and Paoli studied the same kind of problem, corresponding to a fully discretized beam problem. They established unconditional stability and gave a convergence result for $\beta \;=\; 1/2$, whereas a conditional stability result is obtained when $\beta \;\in\; [0, 1/2[$. In the following, we shall adapt their proof to the case of a Kirchhoff-Love plate, restricting ourselves to the case $\beta \;=\; 1/2$. So the fully discretized scheme we consider reads

$$\left\{\begin{array}{l} \text{Find } u_{n+1}^h \;\in\; \mathbb{K}^h \text{ such that for all } w^h \;\in\; \mathbb{K}^h \\[2ex] \displaystyle\int_\Omega \frac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{\Delta t^2} \,.\, (w^h - u_{n+1}^h)\; dx \;+\; a\left(\frac{u_{n+1}^h + u_{n-1}^h}{2} \,,\; w^h - u_{n+1}^h\right) \\[2ex] \qquad\qquad\qquad \geqslant\; \displaystyle\int_\Omega \frac{f_{n+1} + f_{n-1}}{2} \,.\, (w^h - u_{n+1}^h)\; dx. \end{array}\right. \tag{4.1}$$

The following result, which states that the discrete solution is uniformly bounded in time, is straighforwardly obtained by adapting the proof of Proposition 3.1 of (1).

LEMMA 4.1 Let $\beta = 1/2$, then there exists a positive constant $C(f,u_0,v_0)$ depending only on the data such that for all $h > 0$ and for all $N \geqslant 1$

$$\Big|\frac{u_{n+1}^h - u_n^h}{\Delta t}\Big|_{\mathbb{H}}^2 + \frac{1}{2}\, a(u_n^h, u_n^h) + \frac{1}{2}\, a(u_{n+1}^h, u_{n+1}^h) \leqslant C(f,u_0,v_0) \tag{4.2}$$

for $n \in \{1,\dots,N-1\}$, where $(u_{n+1}^h)_{1 \leqslant n \leqslant N-1}$ are solutions of problem (4.1).

Now, let us build the sequence of approximate solutions $(u_{h,N})_{h>0,N\geqslant 1}$ of problem (4.1) by linear interpolation

$$\begin{cases} \text{If } t \in [n\Delta t,(n+1)\Delta t] \ , \ 0 \leqslant n \leqslant N-1 \ , \text{ we set} \\ \\ u_{h,N}(x,t) = u_n^h(x)\, \dfrac{(n+1)\Delta t - t}{\Delta t} + u_{n+1}^h(x)\, \dfrac{t - n\Delta t}{\Delta t} \end{cases} \tag{4.3}$$

which is defined on $\Omega \times [0,T]$. Let us observe that these functions are continuous in time (obvious) and space (for all n, u_n^h belongs to $H^2(\Omega)$ which is included in $\mathscr{C}^0(\bar{\Omega})$). Moreover, because of (4.2), for all $h > 0$ and $N \geqslant 1$, functions $u_{h,N}$ belongs to $L^\infty(0,T;\mathbb{V})$ and are uniformly bounded in this space. As $\dot{u}_{h,N}(x,t) = \dfrac{u_{n+1}^h(x) - u_n^h(x)}{\Delta t}$ for $t \in [n\Delta t,(n+1)\Delta t]$, using again (4.2), functions $\dot{u}_{h,N}$ belong to $L^\infty(0,T;L^2(\Omega))$ and are also uniformly bounded in this space. So there exists a subsequence still denoted $(u_{h,N})_{h>0,N\geqslant 1}$ and $u \in \mathbb{U}$ such that we have the following convergences

$$\begin{aligned} u_{h,N} &\rightharpoonup u \text{ weakly* in } L^\infty(0,T;\mathbb{V}), \\ \dot{u}_{h,N} &\rightharpoonup \dot{u} \text{ weakly* in } L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

As the injection $H^2(\Omega)) \hookrightarrow H^{1+\xi}(\Omega))$ is compact (Rellich's lemma, for $\xi < 1$), and with Simon's lemma ((11), Corollary 4, page 85), we deduce that $\{\ w \in L^\infty(0,T;\mathbb{V}) \ \_ \ \dot{w} \in L^\infty(0,T;L^2(\Omega))\ \}$ is compactly embedded in $\mathscr{C}^0(0,T;H^{1+\xi}(\Omega))$, and then in $\mathscr{C}^0([0,T] \times \bar{\Omega})$. Therefore, after another subsequence extraction if necessary, we have

$$u_{h,N} \rightarrow u \ \text{ strongly in } \mathscr{C}^0(0,T;H^{1+\xi}(\Omega)) \text{ and in } \mathscr{C}^0([0,T] \times \bar{\Omega}).$$

Consequently, we obtain the following results.

- As $L^\infty(0,T;L^2(\Omega))$ is included in $L^2(0,T;L^2(\Omega))$, $\dot{u}$ belongs to this space. Moreover, as all functions $u_{h,N}$ belong to $L^2(0,T;\mathbb{K})$, u also belongs to it. So u belongs to $\tilde{\mathbb{U}}$.

- For every h and N, $u_{h,N}(x,0) = u_0^h(x)$ which converges towards u_0 in $\mathbb{V}$ (see (3.1)). As $\mathbb{V} \subset H^{1+\xi}(\Omega)$ with continuous injection, then $u(.,0) = u_0$.

Then, we shall prove the following result.

THEOREM 4.1 Let $\beta = 1/2$. Then, the sequence of approximate solutions $(u_{h,N})_{h>0,N\geqslant 1}$ given by (4.3) converges weakly* to u in $\{\ w \in L^\infty(0,T;\mathbb{V}) \ / \ \dot{w} \in L^\infty(0,T;L^2(\Omega))\ \}$. Moreover, u belongs to $\tilde{\mathbb{U}}$, is such that $u(.,0) = u_0$ and is a solution of Problem (2.8).

The corollary is that the frictionless elastodynamic problem for a Kirchhoff-Love bending plate between two rigid obstacles has at least one solution.

Proof of Theorem 4.1.

Construction of a discrete test-function.

To obtain (2.8) from (4.1), a first point is to associate to any test-function $\tilde{w}$ a discrete one which is close to it. A natural idea would be to define w_n^h as the linear projection, defined by the bilinear form a, on space $\mathbb{V}^h$ of an approximate value of $\tilde{w}$ at time $n\Delta t$. Unfortunately, this projection does not preserve unilateral constraints. Then, this choice would not necessarily give a test-function in $\mathbb{K}^h$.

So, let $\tilde{w}$ be a test-function such that $\tilde{w} \in \tilde{\mathbb{U}}$ and $\tilde{w}(.,T) = u(.,T)$. For $\varepsilon \in]0,T/2[$, we define ϕ as a $\mathscr{C}^1$-function such that

$$\left\{ \begin{array}{lll} 0 \leqslant \phi(t) \leqslant 1 & , & t \in [0,T] \\ \phi(t) = 0 & , & t \in [T-3\varepsilon/2,T] \\ \phi(t) = 1 & , & t \in [0,T-2\varepsilon]. \end{array} \right. \tag{4.4}$$

We set $w = (1-\phi)\, u + \phi\, \tilde{w}$. Then, by construction, $w(.,t) = u(.,t)$ for all $t \in [T-3\varepsilon/2,T]$. And, since $\mathbb{K}$ is convex, we have immediately $w \in \tilde{\mathbb{U}}$.

Now, let $\eta \in]0,\varepsilon/2[$ and $\chi \in]0,1[$. Following (1), we define $w_{\eta,\chi}$ by

$$w_{\eta,\chi}(x,t) = u(x,t) + \frac{1}{\eta}\int_t^{t+\eta} \big((1-\chi)w(x,s) - u(x,s) \big)\, ds \quad , \quad t \in [0,T-\varepsilon/2]. \tag{4.5}$$

Since $u \in \mathbb{U}$ and $w \in \tilde{\mathbb{U}}$, we have clearly

$$\left\{ \begin{array}{l} w_{\eta,\chi} - u \in \mathscr{C}^0(0,T-\varepsilon/2;\mathbb{V}) \\ w_{\eta,\chi} \in L^\infty(0,T-\varepsilon/2;\mathbb{V}) \cap \mathscr{C}^0(0,T-\varepsilon/2;H^{1+\xi}(\Omega)) \\ \dot{w}_{\eta,\chi} \in L^2(0,T-\varepsilon/2;L^2(\Omega)). \end{array} \right.$$

Moreover, we can select η such that $w_{\eta,\chi}$ satisfies strictly the constraint. More precisely, for all $t \in [0,T-\varepsilon/2]$ and for all $x \in \Omega$

$$w_{\eta,\chi}(x,t) = \frac{1}{\eta}\int_t^{t+\eta}(1-\chi)w(x,s)\, ds + u(x,t) - \frac{1}{\eta}\int_t^{t+\eta} u(x,s)\, ds.$$

Let us recall that, in the definition of convex $\mathbb{K}$, it is introduced a scalar g such that

$$g_1(x) \leqslant -g < 0 < g \leqslant g_2(x) \quad , \quad \forall x \in \Omega.$$

First, as $w \in \tilde{\mathbb{U}}$, we have $g_1(x) \leqslant w(x,t) \leqslant g_2(x)$ for all x and t. So,

$$g_1(x)+\chi g \leqslant (1-\chi)g_1(x) \leqslant \frac{1}{\eta}\int_t^{t+\eta}(1-\chi)w(x,s)\, ds \leqslant (1-\chi)g_2(x) \leqslant g_2(x)-\chi g.$$

Second, let us recall that u belongs to $\mathscr{C}^0([0,T] \times \bar{\Omega})$. Thus, by uniform continuity on a compact set, for all $\delta \in]0,\chi g/2[$ (constant $g>0$ is defined by (2.7)), there exists $\eta > 0$ such that for all x,

$|u(x,t) - u(x,s)| < \delta$ whenever $|t - s| < \eta$. Then

$$\left|u(x,t) - \frac{1}{\eta}\int_t^{t+\eta} u(x,s)\,ds\right| \leqslant \frac{1}{\eta}\int_t^{t+\eta} \left|u(x,t) - u(x,s)\right| ds \leqslant \frac{1}{\eta}\,\eta\,\delta = \delta < \chi\,\frac{g}{2}.$$

Finally, we have

$$g_1(x) + \frac{\chi\,g}{2} \leqslant w_{\eta,\chi}(x,t) \leqslant g_2(x) - \frac{\chi\,g}{2} \ ,\ \forall\, x \in \Omega \ ,\ \forall\, t \in [0, T-\varepsilon/2] \qquad (4.6)$$

and it ensures that $w_{\eta,\chi}(x,t) \in [g_1(x)+\chi g/2, g_2(x)-\chi g/2]$.

LEMMA 4.2 **Construction of a discrete test-function**
For $x \in \Omega$, let w_n^h be

$$w_n^h(x) = \begin{cases} u_{n+1}^h(x) + \pi_h(w_{\eta,\chi}(x,n\Delta t) - u(x,n\Delta t)) & \text{if } n\Delta t \leqslant T - \varepsilon \\ u_{n+1}^h(x) & \text{if } n\Delta t > T - \varepsilon \end{cases}$$

where π_h is linear projection, defined by the bilinear form a, on space $\mathbb{V}^h$. Then there exists $h_0 > 0$ and $N_0 \geqslant 1$ such that, for all $h \in]0, h_0[$ and for all $N \geqslant N_0$, w_n^h belongs to $\mathbb{K}^h$, for all $n \in \{1, \ldots, N-1\}$.

Proof of Lemma 4.2.

- It is obvious that w_n^h belongs to $\mathbb{V}^h$ and $\mathbb{K}^h$ when $n\Delta t > T - \varepsilon$.

- Otherwise, when $n\Delta t \leqslant T - \varepsilon$, w_n^h is written as follows

$$\begin{aligned} w_n^h(x) &= u_{h,N}(x,(n+1)\Delta t) - u(x,(n+1)\Delta t) \\ &+ u(x,(n+1)\Delta t) - u(x,n\Delta t) \\ &+ w_{\eta,\chi}(x,n\Delta t) \\ &+ (\pi_h - Id)(w_{\eta,\chi}(x,n\Delta t) - u(x,n\Delta t)). \end{aligned}$$

First, as $(u_{h,N})_{h>0,N\geqslant 1}$ converges strongly to u in $\mathscr{C}^0(0,T;H^{1+\xi}(\Omega))$, and using the continuity of the canonical injection from $H^{1+\xi}(\Omega)$ into $\mathscr{C}^0(\overline{\Omega})$, for h small enough and N large enough, we obtain

$$\sup_{x\in\overline{\Omega}} |u_{h,N}(x,(n+1)\Delta t) - u(x,(n+1)\Delta t)| \leqslant C\,\|u_{h,N} - u\|_{\mathscr{C}^0(0,T;H^{1+\xi}(\Omega))} \leqslant \frac{\chi\,g}{6}.$$

Second, u is continuous on the compact set $[0,T] \times \overline{\Omega}$. So, by uniform continuity, there exists Δt_0 or $N_0 = T/\Delta t_0$, such that if $\Delta t \leqslant \Delta t_0$ or $N \geqslant N_0$, we have

$$\sup_{x\in\overline{\Omega}} |u(x,(n+1)\Delta t) - u(x,n\Delta t)| \leqslant \frac{\chi\,g}{6}.$$

Third, let us introduce the following constant γ_h , which depends on h. Because of the canonical embedding from $\mathbb{V}$ to $H^{1+\xi}(\Omega)$ and the convergence of the finite element scheme, for all $h > 0$, it exists γ_h such that

$$\forall w \in \mathbb{V} \ ,\ \|\pi_h\,w - w\|_{H^{1+\xi}(\Omega)} \leqslant \gamma_h\,\|w\|_{\mathbb{V}} \ \text{ and } \ \lim_{h\to 0}\gamma_h = 0. \qquad (4.7)$$

Then, for h small enough

$$\begin{aligned} \sup_{x\in\bar{\Omega}} |(\pi_h - Id)(w_{\eta,\chi}(x,n\Delta t) - u(x,n\Delta t))| &\leqslant C\,\|(\pi_h - Id)(w_{\eta,\chi}(.,n\Delta t) - u(.,n\Delta t))\|_{H^{1+\xi}(\Omega)} \\ &\leqslant C\,\gamma_h\,\|w_{\eta,\chi} - u\|_{L^\infty(0,T-\varepsilon/2;\mathbb{V})} \leqslant \frac{\chi\,g}{6}. \end{aligned}$$

Finally, using the previous results, for h small enough and N large enough, we have

$$-\frac{\chi\,g}{2} \leqslant w_n^h(x) - w_{\eta,\chi}(x,n\Delta t) \leqslant \frac{\chi\,g}{2}$$

for all $x \in \Omega$ and $n \in \{1,\dots,N-1\}$, which leads to

$$g_1(x) \leqslant w_{\eta,\chi}(x,n\Delta t) - \frac{\chi\,g}{2} \leqslant w_n^h(x) \leqslant w_{\eta,\chi}(x,n\Delta t) + \frac{\chi\,g}{2} \leqslant g_2(x)$$

by using (4.6). And we can conclude that w_n^h belongs to $\mathbb{K}^h$. □

Transformation of inequality (4.1).

Now, our goal is to show that the limit u is solution of the continuous impact problem (2.8). So, to use the previous lemma, in all the following, we will assume that $h \in]0,h_0[$ and $N \geqslant N_0$. Thus, we set $\Delta t = T/N$. In (4.1), we take $w^h = w_n^h$, we multiply by Δt and add on n to obtain

$$\left\{ \begin{aligned} & \sum_{n=1}^{N-1} \left(\int_\Omega \frac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{\Delta t^2} \,.\, (w_n^h - u_{n+1}^h)\,dx \right) \Delta t \\ + \;& \sum_{n=1}^{N-1} \left(\frac{1}{2}\, a\left(u_{n+1}^h + u_{n-1}^h \,,\, w_n^h - u_{n+1}^h\right) \right) \Delta t \\ \geqslant \;& \sum_{n=1}^{N-1} \left(\frac{1}{2} \int_\Omega (f_{n+1} + f_{n-1}) \,.\, (w_n^h - u_{n+1}^h)\,dx \right) \Delta t. \end{aligned} \right. \tag{4.8}$$

From the definition of the discrete test-function (lemma 4.2), we have $w_n^h - u_{n+1}^h = 0$ as far as $n\Delta t > T - \varepsilon$. So the above sums end to integer N' which is the integer part of $\dfrac{T-\varepsilon}{\Delta t}$.

Moreover, up to the coefficient Δt, the first term of (4.8) can be rewritten

$$\sum_{n=1}^{N'} \int_\Omega \frac{u_{n+1}^h - 2u_n^h + u_{n-1}^h}{\Delta t^2} \,.\, (w_n^h - u_{n+1}^h)\, dx$$

$$= \sum_{n=1}^{N'} \int_\Omega \left(\frac{u_{n+1}^h - u_n^h}{\Delta t} - \frac{u_n^h - u_{n-1}^h}{\Delta t} \right) . \frac{w_n^h - u_{n+1}^h}{\Delta t}\, dx$$

$$= \int_\Omega \frac{u_{N'+1}^h - u_{N'}^h}{\Delta t} \,.\, \frac{w_{N'}^h - u_{N'+1}^h}{\Delta t}\, dx \;-\; \sum_{n=1}^{N'} \int_\Omega \frac{u_n^h - u_{n-1}^h}{\Delta t} \,.\, \frac{(w_n^h - u_{n+1}^h) - (w_{n-1}^h - u_n^h)}{\Delta t}\, dx$$

$$- \int_\Omega \frac{u_1^h - u_0^h}{\Delta t} \,.\, \frac{w_0^h - u_1^h}{\Delta t}\, dx$$

$$= -\sum_{n=1}^{N'+1} \int_\Omega \frac{u_n^h - u_{n-1}^h}{\Delta t} \,.\, \frac{(w_n^h - u_{n+1}^h) - (w_{n-1}^h - u_n^h)}{\Delta t}\, dx \;-\; \int_\Omega \frac{u_1^h - u_0^h}{\Delta t} \,.\, \frac{w_0^h - u_1^h}{\Delta t}\, dx$$

as $w_{N'+1}^h - u_{N'+2}^h = 0$. Finally, we have

$$\left\{ \begin{array}{ll} & \displaystyle\int_\Omega \frac{u_1^h - u_0^h}{\Delta t} \,.\, (w_0^h - u_1^h)\, dx \\ + & \displaystyle\sum_{n=1}^{N'} \left(\frac{1}{2} \int_\Omega (f_{n+1} + f_{n-1}) \,.\, (w_n^h - u_{n+1}^h)\, dx \right) \Delta t \\ \leqslant & \displaystyle\sum_{n=1}^{N'} \left(\frac{1}{2}\, a\left(u_{n+1}^h + u_{n-1}^h \,,\, w_n^h - u_{n+1}^h \right) \right) \Delta t \\ - & \displaystyle\sum_{n=1}^{N'+1} \left(\int_\Omega \frac{u_n^h - u_{n-1}^h}{\Delta t} \,.\, \frac{(w_n^h - u_{n+1}^h) - (w_{n-1}^h - u_n^h)}{\Delta t}\, dx \right) \Delta t. \end{array} \right. \tag{4.9}$$

The goal of the remainder of this proof is to make h and Δt tend to zero. So each term of the previous expression will be examined separately in the four following steps.

Step 1. By definition, $w_0^h(x) - u_1^h(x) = \pi_h(w_{\eta,\chi}(x,0) - u(x,0))$. Then

$$\int_\Omega \frac{u_1^h - u_0^h}{\Delta t}\, (w_0^h - u_1^h)\, dx \;=\; \int_\Omega \frac{u_1^h - u_0^h}{\Delta t}\, (\pi_h - Id)(w_{\eta,\chi}(x,0) - u(x,0))\, dx$$

$$+ \int_\Omega \frac{u_1^h - u_0^h}{\Delta t}\, (w_{\eta,\chi}(x,0) - u(x,0))\, dx.$$

So, (4.7) leads to

$$\begin{aligned} |(\pi_h - Id)(w_{\eta,\chi}(.,0) - u(.,0))|_{\mathbb{H}} &\leqslant \|(\pi_h - Id)(w_{\eta,\chi}(.,0) - u(.,0))\|_{H^{1+\xi}(\Omega)} \\ &\leqslant \gamma_h \,\|(w_{\eta,\chi}(.,0) - u(.,0))\|_{\mathbb{V}} \end{aligned}$$

with $\lim_{h\to 0}\gamma_h = 0$. Finally, from (3.1), it is known that $\lim_{h\to 0\ ,\ \Delta t\to 0}\left|\dfrac{u_1^h-u_0^h}{\Delta t}-v_0\right|_{\mathbb{H}} = 0$, and we obtain

$$\int_\Omega \frac{u_1^h-u_0^h}{\Delta t}\,(w_0^h-u_1^h)\,dx \quad \overset{h\ ,\ \Delta t\ \to\ 0}{\longrightarrow} \quad \int_\Omega v_0(x)\ .\ (w_{\eta,\chi}(x,0)\ -\ u(x,0))\,dx. \tag{4.10}$$

Step 2. The second term of (4.9) can be split in two parts of the same following form

$$\begin{aligned}
&\sum_{n=1}^{N'}\int_\Omega f_{n'}\,(w_n^h-u_{n+1}^h)\,dx\,\Delta t \;=\; \sum_{n=1}^{N'}\int_\Omega f_{n'}(x)\ \pi_h(w_{\eta,\chi}(x,n\Delta t)\ -\ u(x,n\Delta t))\,dx\,\Delta t\\
&\quad = \sum_{n=1}^{N'}\int_\Omega f_{n'}(x)\ (\pi_h-Id)(w_{\eta,\chi}(x,n\Delta t)\ -\ u(x,n\Delta t))\,dx\,\Delta t\\
&\quad + \sum_{n=1}^{N'}\int_{n'\Delta t}^{(n'+1)\Delta t}\int_\Omega f(x,s)\ \big[(w_{\eta,\chi}(x,n\Delta t)\ -\ u(x,n\Delta t))\ -\ (w_{\eta,\chi}(x,s)\ -\ u(x,s))\big]\ dx\,ds\\
&\quad + \sum_{n=1}^{N'}\int_{n'\Delta t}^{(n'+1)\Delta t}\int_\Omega f(x,s)\ (w_{\eta,\chi}(x,s)\ -\ u(x,s))\ dx\,ds\\
&\quad \equiv\ S_1\ +\ S_2\ +\ S_3
\end{aligned}$$

from the definition of $f_{n'}$ (see (3.2)), and those of the discrete test-function w_n^h. Here, we have $n' = n+1$ or $n' = n-1$. Let us examine successively each of these terms.

(1) As in step 1, (4.7) leads to

$$\begin{aligned}
|(\pi_h-Id)(w_{\eta,\chi}(.,n\Delta t)\ -\ u(.,n\Delta t))|_{\mathbb{H}} \quad &\leqslant\quad \gamma_h\ \|(w_{\eta,\chi}(.,n\Delta t)\ -\ u(.,n\Delta t))\|_{\mathbb{V}}\\
&\leqslant\quad \gamma_h\ \|w_{\eta,\chi}\ -\ u\|_{L^\infty(0,T-\varepsilon/2;\mathbb{V})}
\end{aligned}$$

for all $n\ \in\ \{1,\ldots,N'\}$. Then we deduce :

$$\begin{aligned}
|S_1| \quad &= \quad \left|\sum_{n=1}^{N'}\int_\Omega f_{n'}(x)\ (\pi_h-Id)(w_{\eta,\chi}(x,n\Delta t)\ -\ u(x,n\Delta t))\,dx\,\Delta t\right|\\
&\leqslant \quad \left(\sum_{n=1}^{N'}|f_{n'}|_{\mathbb{H}}\ \Delta t\right)\ \gamma_h\ \|w_{\eta,\chi}\ -\ u\|_{L^\infty(0,T-\varepsilon/2;\mathbb{V})}\\
&\leqslant \quad \left(\sqrt{T}\ \|f\|_{L^2(0,T;\mathbb{H})}\right)\ \gamma_h\ \|w_{\eta,\chi}\ -\ u\|_{L^\infty(0,T-\varepsilon/2;\mathbb{V})} \overset{h\ \to\ 0}{\longrightarrow}\ \ 0.
\end{aligned}$$

(2) The definition of $w_{\eta,\chi}$ (4.5) leads to

$$\begin{aligned}
&(w_{\eta,\chi}(x,n\Delta t) - u(x,n\Delta t)) - (w_{\eta,\chi}(x,s) - u(x,s)) \\
&\quad = \frac{1}{\eta}\int_{n\Delta t}^{n\Delta t+\eta} \big((1-\chi)w(x,t) - u(x,t) \big)\, dt - \frac{1}{\eta}\int_{s}^{s+\eta} \big((1-\chi)w(x,t) - u(x,t) \big)\, dt \\
&\quad = \frac{1}{\eta}\int_{n\Delta t}^{s} \big((1-\chi)w(x,t) - u(x,t) \big)\, dt - \frac{1}{\eta}\int_{n\Delta t+\eta}^{s+\eta} \big((1-\chi)w(x,t) - u(x,t) \big)\, dt.
\end{aligned}$$

Moreover, if φ belongs to $L^2(0,T;\mathbb{H})$, a and b being such that $0 \leqslant a < b \leqslant T$, one has

$$\Big| \int_a^b \varphi(.,t)\, dt \Big|_{\mathbb{H}}^2 = \int_\Omega \left(\int_a^b \varphi(x,t)\, dt \right)^2 dx \leqslant (b-a) \int_\Omega \int_a^b \varphi^2(x,t)\, dt dx \leqslant (b-a)\, \|\varphi\|^2_{L^2(0,T;\mathbb{H})}$$

or else

$$\Big| \int_a^b \varphi(.,t)\, dt \Big|_{\mathbb{H}} \leqslant \sqrt{b-a}\, \|\varphi\|_{L^2(0,T;\mathbb{H})}.$$

This result implies that

$$\begin{aligned}
&|(w_{\eta,\chi}(.,n\Delta t) - u(.,n\Delta t)) - (w_{\eta,\chi}(.,s) - u(.,s))|_{\mathbb{H}} \\
&\qquad \leqslant \frac{2\sqrt{|s-n\Delta t|}}{\eta}\, \|(1-\chi)w-u\|_{L^2(0,T;\mathbb{H})} \leqslant \frac{2\sqrt{|s-n\Delta t|}}{\eta}\, \|(1-\chi)w-u\|_{L^2(0,T;\mathbb{V})}.
\end{aligned}$$

As s belongs to $[(n-1)\Delta t, n\Delta t]$ or $[(n+1)\Delta t, (n+2)\Delta t]$, in all cases, we obtain

$$\begin{aligned}
|S_2| &= \sum_{n=1}^{N'} \int_{n'\Delta t}^{(n'+1)\Delta t} \int_\Omega f(x,s)\, \big[(w_{\eta,\chi}(x,n\Delta t) - u(x,n\Delta t)) - (w_{\eta,\chi}(x,s) - u(x,s)) \big]\, dxds \\
&\leqslant \left(\sum_{n=1}^{N'} \int_{n'\Delta t}^{(n'+1)\Delta t} |f(.,s)|_{\mathbb{H}}\, ds \right) \frac{2\sqrt{2\,\Delta t}}{\eta}\, \|(1-\chi)w-u\|_{L^2(0,T;\mathbb{V})} \\
&\leqslant \sqrt{T}\, \|f\|_{L^2(0,T;\mathbb{H})}\, \frac{2\sqrt{2\,\Delta t}}{\eta}\, \|(1-\chi)w-u\|_{L^2(0,T;\mathbb{V})} \xrightarrow{\Delta t \to 0} 0.
\end{aligned}$$

(3) Finally, as $w_{\eta,\chi} - u$ belongs to $\mathscr{C}^0(0,T-\varepsilon/2;\mathbb{V})$ which is contained in $L^2(0,T;\mathbb{H})$, as f is in

$L^2(0,T;\mathbb{H})$ and N' is the integer part of $\dfrac{T\ -\ \varepsilon}{\Delta t}$, then we can make Δt go to zero and obtain

$$\begin{aligned}
S_3 \quad &= \quad \sum_{n=1}^{N'} \int_{n'\Delta t}^{(n'+1)\Delta t} \int_\Omega f(x,s)\ (w_{\eta,\chi}(x,s)\ -\ u(x,s))\ dx\ ds \\
&= \quad \begin{cases} \displaystyle\int_{2\Delta t}^{(N'+2)\Delta t} \int_\Omega f(x,s)\ (w_{\eta,\chi}(x,s)\ -\ u(x,s))\ dx\ ds & \text{if } n' = n+1 \\ \displaystyle\int_0^{N'\Delta t} \int_\Omega f(x,s)\ (w_{\eta,\chi}(x,s)\ -\ u(x,s))\ dx\ ds & \text{if } n' = n-1 \end{cases} \\
&\overset{\Delta t\ \to\ 0}{\longrightarrow} \quad \int_0^{T-\varepsilon} \int_\Omega f(x,s)\ (w_{\eta,\chi}(x,s)\ -\ u(x,s))\ dx\ ds.
\end{aligned}$$

So that we can conclude this step and have

$$\sum_{n=1}^{N'} \left(\frac{1}{2} \int_\Omega (f_{n+1} + f_{n-1})\ .\ (w_n^h - u_{n+1}^h)\ dx \right) \Delta t \quad \overset{h\ ,\ \Delta t\ \to\ 0}{\longrightarrow} \quad \int_0^{T-\varepsilon} \int_\Omega f\ (w_{\eta,\chi}\ -\ u)\ dx\ ds. \tag{4.11}$$

Step 3. We carry on the convergence of the third term of (4.9). Here, we shall use some results we recall hereafter.
- The bilinear form a defines a scalar product on $\mathbb{V}$ which is equivalent to the canonical scalar product (see lemma 2.1). So there exists $C\ >\ 0$ such that $|a(w,w)|\ \leqslant\ C\ \|w\|_{\mathbb{V}}$, for all $w\ \in\ \mathbb{V}$.
- π_h is the linear projection on space $\mathbb{V}^h$ defined by the bilinear form a. In particular, for all $w^h\ \in\ \mathbb{V}^h$ and $v\ \in\ \mathbb{V}$, $a(w^h,\pi_h v)\ =\ a(w^h,v)$.

Now, let us observe that

$$\begin{aligned}
\frac{1}{2} \sum_{n=1}^{N'} a\left(u_{n+1}^h + u_{n-1}^h, w_n^h - u_{n+1}^h\right)\ \Delta t \quad &= \quad \frac{1}{2}\ a(u_0^h, w_0^h - u_1^h)\ \Delta t \\
&+ \quad \frac{1}{2} \sum_{n=1}^{N'+1} a\left(u_{n-1}^h, (w_n^h - u_{n+1}^h)\ -\ (w_{n-1}^h - u_n^h)\right)\ \Delta t \\
&+ \quad \frac{1}{2} \sum_{n=1}^{N'} a\left(u_{n+1}^h + u_n^h, w_n^h - u_{n+1}^h\right)\ \Delta t \\
&\equiv \quad \frac{1}{2}\ S_1\ +\ \frac{1}{2}\ S_2\ +\ S_3
\end{aligned}$$

as $w_{N'+1}^h\ -\ u_{N'+2}^h\ =\ 0$. Now, each of these terms will be studied.

(1) By definition, $w_0^h(x)\ - u_1^h(x)\ =\ \pi_h(w_{\eta,\chi}(x,0)\ -\ u(x,0))$. So

$$\begin{aligned}
|S_1| \quad &= \quad |a(u_0^h, w_0^h - u_1^h)|\ \Delta t\ =\ |a(u_0^h, w_{\eta,\chi}(.,0)\ -\ u(.,0)|\ \Delta t \\
&\leqslant \quad C^2\ \|u_0^h\|_{\mathbb{V}}\ \|w_{\eta,\chi}(.,0)\ -\ u(.,0)\|_{\mathbb{V}}\ \Delta t \overset{h\ ,\ \Delta t\ \to\ 0}{\longrightarrow} \quad 0
\end{aligned}$$

$(u_0^h)_h$ being bounded as the time scheme is initialized by choosing u_0^h such that $\lim\limits_{h\to 0} \|u_0^h - u_0\|_{\mathbb{V}}\ =\ 0$ (see (3.1)).

(2) Here again, from the definitions of the test functions w_p^h and the projection π_h, we have

$$
\begin{aligned}
S_2 &= \sum_{n=1}^{N'+1} a\left(u_{n-1}^h, (w_n^h - u_{n+1}^h) - (w_{n-1}^h - u_n^h)\right) \Delta t \\
&= \sum_{n=1}^{N'+1} a\left(u_{n-1}^h, (w_{\eta,\chi} - u)(.,n\Delta t) - (w_{\eta,\chi} - u)(.,(n-1)\Delta t)\right) \Delta t.
\end{aligned}
$$

Following Step 2-(2), with $s = (n-1)\Delta t$, we obtain

$$
|(w_{\eta,\chi} - u)(.,n\Delta t) - (w_{\eta,\chi} - u)(.,(n-1)\Delta t)|_{\mathbb{H}} \leqslant \frac{2\sqrt{\Delta t}}{\eta} \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{H})}.
$$

This property can be extended to the space derivatives (in the distribution sense) of $(1-\chi)w - u$ exactly in the same way and leads to

$$
\|(w_{\eta,\chi} - u)(.,n\Delta t) - (w_{\eta,\chi} - u)(.,(n-1)\Delta t)\|_{\mathbb{V}} \leqslant \frac{2\sqrt{\Delta t}}{\eta} \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})}. \tag{4.12}
$$

Then, using this inequality and (4.2), we have

$$
\begin{aligned}
|S_2| &\leqslant \sum_{n=1}^{N'+1} \sqrt{a(u_{n-1}^h, u_{n-1}^h)}\, \frac{2\,C\,\sqrt{\Delta t}}{\eta} \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})} \Delta t \\
&\leqslant \sum_{n=1}^{N'+1} \sqrt{2\,C(f,u_0,v_0)}\, \frac{2\,C\,\sqrt{\Delta t}}{\eta} \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})} \Delta t \\
&\leqslant T\,\sqrt{2\,C(f,u_0,v_0)}\, \frac{2\,C\,\sqrt{\Delta t}}{\eta} \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})} \xrightarrow{\Delta t \to 0} 0.
\end{aligned}
$$

(3) As function $u_{h,N}$ is linear in time on each interval $[n\Delta t, (n+1)\Delta t]$ (see (4.3)), we have

$$
\int_{n\Delta t}^{(n+1)\Delta t} u_{h,N}(.,s)\, ds = \frac{1}{2}\,(u_{n+1}^h + u_n^h)\,\Delta t
$$

which allows to rewrite the third term as

$$
\begin{aligned}
S_3 &= \frac{1}{2}\sum_{n=1}^{N'} a\left(u_{n+1}^h + u_n^h \,,\, w_n^h - u_{n+1}^h\right)\,\Delta t \;=\; \sum_{n=1}^{N'} \int_{n\Delta t}^{(n+1)\Delta t} a\left(u_{h,N}(.,s)\,,\, w_n^h - u_{n+1}^h\right)\,ds \\
&= \sum_{n=1}^{N'} \int_{n\Delta t}^{(n+1)\Delta t} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,n\Delta t)\right)\,ds \\
&= \sum_{n=1}^{N'} \int_{n\Delta t}^{(n+1)\Delta t} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,n\Delta t) \,-\, (w_{\eta,\chi} - u)(.,s)\right)\,ds \\
&+ \int_0^{T-\varepsilon} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds \\
&- \int_0^{\Delta t} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds \,-\, \int_{(N'+1)\Delta t}^{T-\varepsilon} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds
\end{aligned}
$$

With (4.12) in which $(n-1)\Delta t$ is replaced by s, that belongs to $[n\Delta t,(n+1)\Delta t]$, we obtain

$$
\begin{aligned}
&\sum_{n=1}^{N'} \int_{n\Delta t}^{(n+1)\Delta t} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,n\Delta t) \,-\, (w_{\eta,\chi} - u)(.,s)\right)\,ds \\
&\qquad \leqslant C^2\, T\, \|u_{h,N}\|_{L^\infty(0,T;\mathbb{V})}\, \frac{2\sqrt{\Delta t}}{\eta}\, \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})} \overset{\Delta t\,\to\,0}{\longrightarrow} 0
\end{aligned}
$$

as functions $(u_{h,N})_{h>0,N\geqslant 1}$ are uniformly bounded because of (4.2). The same reason leads to

$$
\begin{aligned}
&\left|\int_0^{\Delta t} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds\right| \\
&\qquad \leqslant \int_0^{\Delta t} C^2\, \|u_{h,N}(.,s)\|_{\mathbb{V}} \|(w_{\eta,\chi} - u)(.,s)\|_{\mathbb{V}}\, ds \\
&\qquad \leqslant \Delta t\, C^2\, \|u_{h,N}\|_{L^\infty(0,T;\mathbb{V})} \|(w_{\eta,\chi} - u)\|_{L^\infty(0,T-\varepsilon/2;\mathbb{V})} \overset{\Delta t\,\to\,0}{\longrightarrow} 0
\end{aligned}
$$

and, in a similar way

$$
\int_{(N'+1)\Delta t}^{T-\varepsilon} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds \overset{\Delta t\,\to\,0}{\longrightarrow} 0.
$$

Finally, as the inclusion of $L^\infty(0,T;\mathbb{V})$ into $L^2(0,T;\mathbb{V})$ is continuous, functions $(u_{h,N})_{h>0,N\geqslant 1}$, being uniformly bounded in $L^\infty(0,T;\mathbb{V})$, are also uniformly bounded in $L^2(0,T;\mathbb{V})$. So, up to a possible subsequence extraction, $(u_{h,N})_{h>0,N\geqslant 1}$ converges weakly in this space towards u (uniqueness of the limit). So that we obtain

$$
\int_0^{T-\varepsilon} a\left(u_{h,N}(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds \overset{h\,,\,\Delta t\,\to\,0}{\longrightarrow} \int_0^{T-\varepsilon} a\left(u(.,s)\,,\, (w_{\eta,\chi} - u)(.,s)\right)\,ds
$$

and then

$$\frac{1}{2}\sum_{n=1}^{N'} a\left(u_{n+1}^h + u_{n-1}^h, w_n^h - u_{n+1}^h\right)\,\Delta t \overset{h\,,\,\Delta t\,\to\,0}{\longrightarrow} \int_0^{T-\varepsilon} a\left(u(.,s)\,,\,(w_{\eta,\chi}-u)(.,s)\right)\,ds. \tag{4.13}$$

Step 4. At last, let us study the convergence of the fourth term of (4.9). To simplify the presentation, we introduce the notation

$$\psi_{\Delta t}(x,t) \;=\; \frac{(w_{\eta,\chi}-u)(x,t+\Delta t)\;-\;(w_{\eta,\chi}-u)(x,t)}{\Delta t}\;\;,\;\;\forall t\in[0,T-\varepsilon/2]\;\;,\;\;\forall x\in\Omega$$

and we recall that, by definition of N', $w_{N'+1}^h \;-\; u_{N'+2}^h \;=\; 0$ and that, by definition of the discrete test-functions (see lemma 4.2), $w_p^h(x)-u_{p+1}^h(x) \;=\; \pi_h(w_{\eta,\chi}(x,p\Delta t)\;-\;u(x,p\Delta t))$. Then, we deduce the following decomposition

$$\begin{aligned}
&\sum_{n=1}^{N'+1}\left(\int_\Omega \frac{u_n^h-u_{n-1}^h}{\Delta t}\,.\,\frac{(w_n^h-u_{n+1}^h)-(w_{n-1}^h-u_n^h)}{\Delta t}\,dx\right)\Delta t\\
&\qquad = -\int_\Omega \frac{u_{N'+1}^h-u_{N'}^h}{\Delta t}\,.\,(w_{N'}^h-u_{N'+1}^h)\,dx\\
&\qquad + \sum_{n=1}^{N'}\left(\int_\Omega \frac{u_n^h-u_{n-1}^h}{\Delta t}\,.\,(\pi_h-Id)\psi_{\Delta t}(.,(n-1)\Delta t)\,dx\right)\Delta t\\
&\qquad + \sum_{n=1}^{N'}\int_{(n-1)\Delta t}^{n\Delta t}\int_\Omega \frac{u_n^h-u_{n-1}^h}{\Delta t}\,.\,(\psi_{\Delta t}(.,(n-1)\Delta t)-\psi_{\Delta t}(.,t))\,dxdt\\
&\qquad + \sum_{n=1}^{N'}\int_{(n-1)\Delta t}^{n\Delta t}\int_\Omega \frac{u_n^h-u_{n-1}^h}{\Delta t}\,.\,\psi_{\Delta t}(.,t)\,dxdt\\
&\qquad \equiv\; S_1\;+\;S_2\;+\;S_3\;+\;S_4.
\end{aligned}$$

(1) First, using (4.2) and the definition of $w^h_{N'}$, we have

$$
\begin{aligned}
|S_1| &= \Big| \int_\Omega \frac{u^h_{N'+1} - u^h_{N'}}{\Delta t} \,.\, (w^h_{N'} - u^h_{N'+1})\, dx \Big| \\
&= \Big| \int_\Omega \frac{u^h_{N'+1} - u^h_{N'}}{\Delta t} \,.\, \pi_h(w_{\eta,\chi}(x, N'\Delta t) \;-\; u(x, N'\Delta t))\, dx \Big| \\
&\leqslant \Big| \frac{u^h_{N'+1} - u^h_{N'}}{\Delta t} \Big|_{\mathbb{H}} \Big| \pi_h(w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t))\, dx \Big|_{\mathbb{H}} \\
&\leqslant \sqrt{C(f, u_0, v_0)}\, \Big| \pi_h(w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t))\, dx \Big|_{\mathbb{H}} \\
&\leqslant \sqrt{C(f, u_0, v_0)}\, \Big| (\pi_h - Id)\, (w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t))\, dx \Big|_{\mathbb{H}} \\
&+ \sqrt{C(f, u_0, v_0)}\, \Big| w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t)\, dx \Big|_{\mathbb{H}}.
\end{aligned}
$$

Let us recall that, by construction, $w(.,t) \;=\; u(.,t)$ for all $t \;\in\; [T - 3\varepsilon/2, T]$ and that N' is the integer part of $\frac{T-\varepsilon}{\Delta t}$. So, for Δt small enough, it is possible to have $N'\Delta t \;\geqslant\; T - 3\varepsilon/2$. Consequently, the definition of $w_{\eta,\chi}$ (4.5) leads to

$$
w_{\eta,\chi}(., N'\Delta t) - u(., N'\Delta t) \;=\; \frac{1}{\eta} \int_{N'\Delta t}^{N'\Delta t + \eta} \big((1-\chi) w(.,t) \;-\; u(.,t) \big)\, dt \;=\; \frac{\chi}{\eta} \int_{N'\Delta t}^{N'\Delta t + \eta} u(.,t)\, dt
$$

Moreover, following Step 2-(2), if φ belongs to $L^\infty(0,T;\mathbb{H})$, a and b being such that $0 \;\leqslant\; a \;<\; b \;\leqslant\; T$, one has

$$
\Big| \int_a^b \varphi(.,t)\, dt \Big|^2_{\mathbb{H}} \;\leqslant\; (b-a) \int_\Omega \int_a^b \varphi^2(x,t)\, dt dx \;\leqslant\; (b-a)^2 \sup_t |\varphi(.,t)|^2_{\mathbb{H}} \;=\; (b-a)^2\, \|\varphi\|^2_{L^\infty(0,T;\mathbb{H})}
$$

or else

$$
\Big| \int_a^b \varphi(.,t)\, dt \Big|_{\mathbb{H}} \;\leqslant\; (b-a)\, \|\varphi\|_{L^\infty(0,T;\mathbb{H})}.
$$

As u belongs to $L^\infty(0,T;\mathbb{V})$, this result implies that

$$
\Big| w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t) \Big|_{\mathbb{H}} \;=\; \frac{\chi}{\eta} \Big| \int_{N'\Delta t}^{N'\Delta t + \eta} u(.,t)\, dt \Big|_{\mathbb{H}} \;\leqslant\; \chi\, \|u\|_{L^\infty(0,T;\mathbb{H})} \;\leqslant\; \chi\, \|u\|_{L^\infty(0,T;\mathbb{V})}.
$$

Finally, using (4.7), as γ_h goes to zero when h goes to zero, we have

$$
\begin{aligned}
\Big| (\pi_h - Id)\, (w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t))\, dx \Big|_{\mathbb{H}} &\leqslant \|(\pi_h - Id)(w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t))\|_{H^{1+\xi}(\Omega)} \\
&\leqslant \|(\pi_h - Id)(w_{\eta,\chi}(., N'\Delta t) \;-\; u(., N'\Delta t))\|_{\mathbb{V}} \\
&\leqslant \gamma_h\, \|w_{\eta,\chi} \;-\; u\|_{L^\infty(0,T-\varepsilon/2;\mathbb{V})} \;\leqslant\; \chi\, \|u\|_{L^\infty(0,T;\mathbb{V})}
\end{aligned}
$$

if h is chosen small enough. Hence, it leads to

$$
|S_1| \;\leqslant\; 2\,\chi\, \sqrt{C(f, u_0, v_0)}\, \|u\|_{L^\infty(0,T;\mathbb{V})} \;\equiv\; \chi\, C\, \|u\|_{L^\infty(0,T;\mathbb{V})}. \tag{4.14}
$$

(2) Let us now derive an estimate for S_2 .

$$\begin{aligned}
|S_2| &= \Big|\sum_{n=1}^{N'} \Big(\int_\Omega \frac{u_n^h - u_{n-1}^h}{\Delta t} \,.\, (\pi_h - Id)\psi_{\Delta t}(.,(n-1)\Delta t)\, dx \Big)\, \Delta t\Big| \\
&\leqslant \sum_{n=1}^{N'} \Big|\frac{u_n^h - u_{n-1}^h}{\Delta t}\Big|_{\mathbb{H}} \, |(\pi_h - Id)\psi_{\Delta t}(.,(n-1)\Delta t)|_{\mathbb{H}}\, \Delta t \\
&\leqslant \sqrt{C(f,u_0,v_0)} \sum_{n=1}^{N'} |(\pi_h - Id)\psi_{\Delta t}(.,(n-1)\Delta t)|_{\mathbb{H}}\, \Delta t \\
&\leqslant \gamma_h \sqrt{C(f,u_0,v_0)} \sum_{n=1}^{N'} \|\psi_{\Delta t}(.,(n-1)\Delta t)\|_{\mathbb{V}}\, \Delta t \quad , \\
&\leqslant \gamma_h \sqrt{C(f,u_0,v_0)}\, \sqrt{N} \left(\sum_{n=1}^{N'} \|\Delta t\; \psi_{\Delta t}(.,(n-1)\Delta t)\|_{\mathbb{V}}^2 \right)^{1/2}
\end{aligned}$$

thanks to (4.2) and (4.7). Moreover, the definitions of $\psi_{\Delta t}$ and $w_{\eta,\chi}$ (4.5) lead to

$$\begin{aligned}
&\|\Delta t\; \psi_{\Delta t}(.,(n-1)\Delta t)\|_{\mathbb{V}}^2 = \|(w_{\eta,\chi} - u)(x, n\Delta t) - (w_{\eta,\chi} - u)(x,(n-1)\Delta t)\|_{\mathbb{V}}^2 \\
&= \Big\| \int_{n\Delta t}^{n\Delta t+\eta} \frac{(1-\chi)w(x,t) - u(x,t)}{\eta}\, dt - \int_{(n-1)\Delta t}^{(n-1)\Delta t+\eta} \frac{(1-\chi)w(x,t) - u(x,t)}{\eta}\, dt \Big\|_{\mathbb{V}}^2 \\
&= \Big\| \int_{(n-1)\Delta t+\eta}^{n\Delta t+\eta} \frac{(1-\chi)w(x,t) - u(x,t)}{\eta}\, dt - \int_{(n-1)\Delta t}^{n\Delta t} \frac{(1-\chi)w(x,t) - u(x,t)}{\eta}\, dt \Big\|_{\mathbb{V}}^2 \\
&\leqslant 2 \Big\| \int_{(n-1)\Delta t+\eta}^{n\Delta t+\eta} \frac{(1-\chi)w(x,t) - u(x,t)}{\eta}\, dt \Big\|_{\mathbb{V}}^2 \\
&\qquad + 2 \Big\| \int_{(n-1)\Delta t}^{n\Delta t} \frac{(1-\chi)w(x,t) - u(x,t)}{\eta}\, dt \Big\|_{\mathbb{V}}^2 .
\end{aligned} \tag{4.15}$$

Now, if φ belongs to $L^2(0,T;\mathbb{H})$, one has

$$\begin{aligned}
\sum_{n=1}^{N'} \Big| \int_{(n-1)\Delta t}^{n\Delta t} \varphi(.,t)\, dt \Big|_{\mathbb{H}}^2 &= \sum_{n=1}^{N'} \int_\Omega \left(\int_{(n-1)\Delta t}^{n\Delta t} \varphi(x,t)\, dt \right)^2 dx \\
&\leqslant \sum_{n=1}^{N'} \Delta t \int_\Omega \int_{(n-1)\Delta t}^{n\Delta t} \varphi^2(x,t)\, dt dx \\
&\leqslant \Delta t \int_\Omega \int_0^T \varphi^2(x,t)\, dt dx = \Delta t\, \|\varphi\|_{L^2(0,T;\mathbb{H})}^2 .
\end{aligned}$$

In a similar way, as $\eta < \varepsilon/2$ and $N'\Delta t \leqslant T-\varepsilon$ (definition of N'), we have $N'\Delta t+\eta \leqslant T$ and then

$$\sum_{n=1}^{N'} \Big| \int_{(n-1)\Delta t+\eta}^{n\Delta t+\eta} \varphi(.,t)\, dt \Big|_{\mathbb{H}}^2 \leqslant \Delta t\, \|\varphi\|^2_{L^2(0,T;\mathbb{H})}.$$

If φ belongs to $L^2(0,T;\mathbb{V})$, the previous properties can be extended to its space derivatives (in the distribution sense) exactly in the same way and lead to

$$\left(\sum_{n=1}^{N'} \Big\| \int_{(n-1)\Delta t}^{n\Delta t} \varphi(.,t)\, dt \Big\|_{\mathbb{V}}^2 + \sum_{n=1}^{N'} \Big\| \int_{(n-1)\Delta t+\eta}^{n\Delta t+\eta} \varphi(.,t)\, dt \Big\|_{\mathbb{V}}^2 \right)^{1/2} \leqslant \sqrt{2\,\Delta t}\, \|\varphi\|_{L^2(0,T;\mathbb{V})}.$$

Setting $\varphi = \dfrac{1}{\eta}\,((1-\chi)w - u)$ in the above inequality, this result and (4.15) implie that

$$\begin{aligned}
|S_2| &\leqslant \gamma_h \sqrt{C(f,u_0,v_0)}\, \sqrt{N} \left(\sum_{n=1}^{N'} \|\Delta t\ \psi_{\Delta t}(.,(n-1)\Delta t)\|_{\mathbb{V}}^2 \right)^{1/2} \\
&\leqslant \gamma_h \sqrt{C(f,u_0,v_0)}\, \sqrt{N}\, \frac{2\sqrt{\Delta t}}{\eta}\, \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})} \\
&\leqslant \gamma_h \sqrt{C(f,u_0,v_0)}\, \frac{2\sqrt{T}}{\eta}\, \|(1-\chi)w - u\|_{L^2(0,T;\mathbb{V})} \overset{h \to 0}{\longrightarrow} 0.
\end{aligned}$$

(3) To treat the third term, we begin by the following transformation. First, let us recall that the definitions of $\psi_{\Delta t}$ and $w_{\eta,\chi}$ lead, for all $\tau \in [0, T-\varepsilon/2]$, to

$$\begin{aligned}
\psi_{\Delta t}(x,\tau) &= \frac{(w_{\eta,\chi}-u)(x,\tau+\Delta t) - (w_{\eta,\chi}-u)(x,\tau)}{\Delta t} \\
&= \frac{1}{\eta\Delta t} \int_{\tau+\Delta t}^{\tau+\Delta t+\eta} ((1-\chi)w-u)(x,s)\, ds - \frac{1}{\eta\Delta t}\int_{\tau}^{\tau+\eta} ((1-\chi)w-u)(x,s)\, ds \\
&= \frac{1}{\eta\Delta t} \int_{\tau}^{\tau+\eta} \big(((1-\chi)w-u)(x,s+\Delta t) - ((1-\chi)w-u)(x,s)\big)\, ds \\
&= \frac{1}{\eta\Delta t} \int_{\tau}^{\tau+\eta} \left(\int_s^{s+\Delta t} ((1-\chi)\dot{w}-\dot{u})(x,r)\, dr \right) ds.
\end{aligned} \tag{4.16}$$

Hence, we obtain

$$\begin{aligned}
&\psi_{\Delta t}(x,(n-1)\Delta t) - \psi_{\Delta t}(x,t) \\
&= \frac{1}{\eta\Delta t}\left(\int_{(n-1)\Delta t}^{(n-1)\Delta t+\eta} \int_s^{s+\Delta t} ((1-\chi)\dot{w}-\dot{u})(x,r)\, drds - \int_t^{t+\eta} \int_s^{s+\Delta t} ((1-\chi)\dot{w}-\dot{u})(x,r)\, drds \right) \\
&= \frac{1}{\eta\Delta t}\left(\int_{(n-1)\Delta t}^{t} \int_s^{s+\Delta t} ((1-\chi)\dot{w}-\dot{u})(x,r)\, drds - \int_{(n-1)\Delta t+\eta}^{t+\eta} \int_s^{s+\Delta t} ((1-\chi)\dot{w}-\dot{u})(x,r)\, drds \right)
\end{aligned}$$

Now, a and b being such that $0 \leqslant a < b \leqslant T$, and setting $\varphi = (1-\chi)\dot{w}-\dot{u}$ which belongs to

$L^2(0,T;\mathbb{H})$, one has

$$\begin{aligned}
\Big|\int_a^b\int_s^{s+\Delta t}\varphi(.,r)\,drds\Big|_{\mathbb{H}}^2 &= \int_\Omega\left(\int_a^b\int_s^{s+\Delta t}\varphi(x,r)\,drds\right)^2 dx\\
&\leqslant \int_\Omega\left(\int_a^b\sqrt{\Delta t}\left[\int_s^{s+\Delta t}\varphi^2(x,r)\,dr\right]^{1/2} ds\right)^2 dx\\
&\leqslant \int_\Omega(b-a)\,\Delta t\,\int_a^b\int_s^{s+\Delta t}\varphi^2(x,r)\,dr\,ds\,dx\\
&= (b-a)\,\Delta t\,\int_a^b\left(\int_\Omega\int_s^{s+\Delta t}\varphi^2(x,r)\,drdx\right)ds\\
&\leqslant (b-a)^2\,\Delta t\,\|\varphi\|_{L^2(0,T;\mathbb{H})}^2 \qquad (4.17)
\end{aligned}$$

and then

$$|\psi_{\Delta t}(.,(n-1)\Delta t)-\psi_{\Delta t}(.,t)|_{\mathbb{H}}\leqslant 2\,\frac{|t-(n-1)\Delta t|}{\eta\,\sqrt{\Delta t}}\,\|((1-\chi)\dot{w}-\dot{u})\|_{L^2(0,T;\mathbb{H})}.$$

Finally, using again (4.2), we obtain from these results

$$\begin{aligned}
|S_3| &\leqslant \sum_{n=1}^{N'}\int_{(n-1)\Delta t}^{n\Delta t}\Big|\frac{u_n^h-u_{n-1}^h}{\Delta t}\Big|_{\mathbb{H}}\,|\psi_{\Delta t}(.,(n-1)\Delta t)-\psi_{\Delta t}(.,t)|_{\mathbb{H}}\,dt\\
&\leqslant \sqrt{C(f,u_0,v_0)}\,\sum_{n=1}^{N'}\int_{(n-1)\Delta t}^{n\Delta t}2\,\frac{|t-(n-1)\Delta t|}{\eta\,\sqrt{\Delta t}}\,\|(1-\chi)\dot{w}-\dot{u}\|_{L^2(0,T;\mathbb{H})}\,dt\\
&\leqslant \sqrt{C(f,u_0,v_0)}\,\sum_{n=1}^{N'}\frac{\Delta t^2}{\eta\,\sqrt{\Delta t}}\,\|(1-\chi)\dot{w}-\dot{u}\|_{L^2(0,T;\mathbb{H})}\\
&\leqslant \sqrt{C(f,u_0,v_0)}\,\frac{T\,\sqrt{\Delta t}}{\eta}\,\|(1-\chi)\dot{w}-\dot{u}\|_{L^2(0,T;\mathbb{H})}\overset{\Delta t\to 0}{\longrightarrow}0.
\end{aligned}$$

(4) Finally, from the definition of $u_{h,N}$ (4.3), we have $\dot{u}_{h,N}(x,t)=\dfrac{u_n^h(x)-u_{n-1}^h(x)}{\Delta t}$ when t belongs to $[(n-1)\Delta t,n\Delta t]$. Hence, S_4 can be rewritten

$$\begin{aligned}
S_4 &= \sum_{n=1}^{N'}\int_{(n-1)\Delta t}^{n\Delta t}\int_\Omega\frac{u_n^h-u_{n-1}^h}{\Delta t}\,.\,\psi_{\Delta t}(.,t)\,dxdt\\
&= \int_0^{T-\varepsilon}\int_\Omega\dot{u}_{h,N}\,.\,\psi_{\Delta t}\,dxdt-\int_{N'\Delta t}^{T-\varepsilon}\int_\Omega\frac{u_{N'}^h-u_{N'-1}^h}{\Delta t}\,.\,\psi_{\Delta t}(.,t)\,dxdt.
\end{aligned}$$

Exactly as in the previous point, using (4.2) and (4.16)-(4.17), we obtain

$$\begin{aligned}\Big|\int_{N'\Delta t}^{T-\varepsilon}\int_\Omega \frac{u^h_{N'}-u^h_{N'-1}}{\Delta t}\cdot\psi_{\Delta t}(.,t)\,dxdt\Big| &\leqslant \int_{N'\Delta t}^{T-\varepsilon}\Big|\frac{u^h_{N'}-u^h_{N'-1}}{\Delta t}\Big|_{\mathbb{H}}\,|\psi_{\Delta t}(.,t)|_{\mathbb{H}}\,dt\\ &\leqslant \int_{N'\Delta t}^{T-\varepsilon}\sqrt{C(f,u_0,v_0)}\,\frac{1}{\sqrt{\Delta t}}\,\|(1-\chi)\dot w-\dot u\|_{L^2(0,T;\mathbb{H})}\,dt\\ &\leqslant \sqrt{C(f,u_0,v_0)}\,\sqrt{\Delta t}\,\|(1-\chi)\dot w-\dot u\|_{L^2(0,T;\mathbb{H})}\ \xrightarrow{\Delta t\,\to\,0}\ 0.\end{aligned}$$

Moreover, following (4.16) with $\tau = t$ belonging to $[0,T-\varepsilon]$, we have

$$\begin{aligned}\psi_{\Delta t}(x,t) &= \frac{1}{\eta\Delta t}\int_{t+\Delta t}^{t+\Delta t+\eta}((1-\chi)w-u)(x,s)\,ds - \frac{1}{\eta\Delta t}\int_t^{t+\eta}((1-\chi)w-u)(x,s)\,ds\\ &= \frac{1}{\eta\Delta t}\int_{t+\eta}^{t+\eta+\Delta t}((1-\chi)w-u)(x,s)\,ds - \frac{1}{\eta\Delta t}\int_t^{t+\Delta t}((1-\chi)w-u)(x,s)\,ds\\ &\xrightarrow{\Delta t\,\to\,0}\ \frac{1}{\eta}\,(((1-\chi)w-u)(x,t+\eta) - ((1-\chi)w-u)(x,t))\end{aligned}$$

and this convergence is strong in $L^2(0,T-\varepsilon;\mathbb{V})$ as $(1-\chi)w-u \in L^2(0,T;\mathbb{V})$. Furthermore, as the inclusion of $L^\infty(0,T;\mathbb{H})$ into $L^2(0,T;\mathbb{H})$ is continuous, functions $(\dot u_{h,N})_{h>0,N\geqslant 1}$, being uniformly bounded in $L^\infty(0,T;\mathbb{H})$, are also uniformly bounded in $L^2(0,T;\mathbb{H})$. So, up to a possible subsequence extraction, $(\dot u_{h,N})_{h>0,N\geqslant 1}$ converges weakly in this space towards $\dot u$ (uniqueness of the limit). So that we obtain

$$\int_0^{T-\varepsilon}\int_\Omega \dot u_{h,N}\,\psi_{\Delta t}\,dxdt \ \xrightarrow{h,\Delta t\to 0}\ \int_0^{T-\varepsilon}\int_\Omega \dot u(x,t)\,\frac{((1-\chi)w-u)(x,t+\eta)-((1-\chi)w-u)(x,t)}{\eta}\,dxdt$$

and then

$$S_2+S_3+S_4 \ \xrightarrow{h,\Delta t\,\to\,0}\ \int_0^{T-\varepsilon}\int_\Omega \dot u(x,t)\,\frac{((1-\chi)w-u)(x,t+\eta)-((1-\chi)w-u)(x,t)}{\eta}\,dxdt. \tag{4.18}$$

Conclusion. Thanks to the previous convergence results (4.10)-(4.11)-(4.13)-(4.14) and (4.18), when h and Δt tend to zero in inequality (4.9), we obtain for all $\varepsilon\in]0,T/2[$ and $\eta\in]0,\varepsilon/2[$

$$\left\{\begin{aligned} &\int_\Omega v_0(x)\,.\,(w_{\eta,\chi}(x,0) - u(x,0))\,dx + \int_0^{T-\varepsilon}\int_\Omega f\,(w_{\eta,\chi} - u)\,dxdt\\ \leqslant\ &\int_0^{T-\varepsilon} a\big(u(.,t)\,,\,(w_{\eta,\chi}-u)(.,t)\big)\,dt + \chi\,C\,\|u\|_{L^\infty(0,T;\mathbb{V})}\\ -\ &\int_0^{T-\varepsilon}\int_\Omega \dot u(x,t)\,\frac{((1-\chi)w-u)(x,t+\eta)-((1-\chi)w-u)(x,t)}{\eta}\,dxdt.\end{aligned}\right. \tag{4.19}$$

First, we shall make η going to zero. As $(1-\chi)\dot w-\dot u \in L^2(0,T;\mathbb{H})$, then

$$\begin{aligned}\frac{((1-\chi)w-u)(x,t+\eta)-((1-\chi)w-u)(x,t)}{\eta} &= \int_t^{t+\eta}\frac{(1-\chi)\dot w(x,s)-\dot u(x,s)}{\eta}\,ds\\ &\xrightarrow{\eta\,\to\,0}\ (1-\chi)\dot w(x,t)-\dot u(x,t)\quad \text{strongly in } L^2(0,T-\varepsilon;\mathbb{H}).\end{aligned}$$

With the same arguments, as $(1-\chi)w-u \in L^2(0,T;\mathbb{V}) \cap \mathscr{C}^0(0,T;\mathbb{H})$, we have first

$$w_{\eta,\chi}(x,t)-u(x,t) \;=\; \frac{1}{\eta}\int_t^{t+\eta}((1-\chi)w(x,s)-u(x,s))\;ds$$

$$\stackrel{\eta\,\to\,0}{\longrightarrow}\;\; (1-\chi)w(x,t)-u(x,t)\quad \text{strongly in } L^2(0,T-\varepsilon;\mathbb{V})$$

and second, for $t \;=\; 0$,

$$w_{\eta,\chi}(x,0)\;-\;u(x,0)\;\;\stackrel{\eta\,\to\,0}{\longrightarrow}\;\;(1-\chi)w(x,0)-u(x,0)\quad \text{strongly in } \mathbb{H}.$$

So, when η goes to zero, inequality (4.19) becomes

$$\left\{\begin{array}{ll} & \displaystyle\int_\Omega v_0(x)\;.\;((1-\chi)w(x,0)-u(x,0))\;dx\;+\;\int_0^{T-\varepsilon}\int_\Omega f\;((1-\chi)w-u)\;dxdt \\ \leqslant & \displaystyle\int_0^{T-\varepsilon} a\,(u(.,t)\;,\;((1-\chi)w-u)(.,t)))\;dt\;+\;\chi\,C\;\|u\|_{L^\infty(0,T;\mathbb{V})} \\ - & \displaystyle\int_0^{T-\varepsilon}\int_\Omega \dot{u}\;((1-\chi)\dot{w}-\dot{u})\;dxdt. \end{array}\right.$$

The proof is achieved by making χ and ε tend to zero, observing that $w-u=\phi(\tilde{w}-u)$, where ϕ is defined by (4.4). □

REMARK 4.1 Let us recall that, in their paper (1), Dumont and Paoli gave a more general result, including in particular a conditional convergence when parameter β belongs to $[0,1/2[$. Actually, we could have follow the same way. As a matter of fact, the coefficient

$$\kappa_h \;=\; \sup_{u^h\,\in\,\mathbb{V}^h\setminus\{0\}} \frac{a(u^h\;,\;u^h)}{|u^h|^2_{\mathbb{H}}}\quad ,$$

they introduced in (1) to lead to a conditional stability, can be used in a same way for plates. It means that the above Lemma 4.1, which states that the discrete solution is uniformly bounded in time, can also be straighforwardly obtained from Proposition 3.1 of (1) under the same hypotheses. Then, up to some technical details, if we follow more closely Dumont and Paoli's proof, Theorem 4.1 remains valid. The only point to discuss is the evaluation of κ_h . In (1), the authors show that $\kappa_h^{beam} \;\sim\; \dfrac{EI}{\rho S}\;\dfrac{1}{\Delta x^4}$, for a homogeneous and isotropic beam, Δx being the mesh size, which is uniform here. In the case of a Kirchhoff-Love plate, if we assume it is made of a homogeneous and isotropic material too, then, definition of bilinear form $a(\cdot,\cdot)$ shows that κ_h is the highest eigenvalue of the bilaplacian operator on the plate mesh. So, first, it proportional to $\dfrac{D}{2\,\rho\,\varepsilon}\;=\;\dfrac{E\;\varepsilon^2}{3\,(1-\nu^2)\,\rho}$. Second, if the mesh is uniform of size h, following for example (12), it is easy to see that the bilaplacian highest eigenvalue is of order $1/h^4$. Consequently, in our case, and under the previous assumptions, $\kappa_h^{plate}\;\sim\;\dfrac{E\;\varepsilon^2}{3\,(1-\nu^2)\,\rho}\;\dfrac{1}{h^4}$, which is quite close to the case of beams. Finally, from a practical point of view, for a similar computational cost, it is better to use an inconditionaly stable scheme. Consequently, we only tested the scheme with $\beta\;=\;1/2$.

5. Numerical results and conclusions

We will consider a steel rectangular panel of length 120 *cm*, width 40 *cm* and thickness equal to $\varepsilon = 0.5\ cm$. The flexural rigidity is $D = 1.923\ 10^4$, corresponding to $E = 210\ GPa$, $\nu = 0.3$ and $\rho = 7.77\ 10^3 kg/m^3$. This plate is clamped along one edge and free along the three others. The numerical tests are performed with GETFEM++ (13) and Matlab, using structured meshes (see Figures 3 and 4).

Let us recall the problem to be solved at each iteration

$$\begin{cases} \text{Find } u^h_{n+1} \in \mathbb{K}^h \text{ such that} \\ (\mathbf{M} + \beta\ \Delta t^2\ \mathbf{K})u^h_{n+1} + \Delta t^2\ \partial \mathbb{I}_{\mathbb{K}^h}(u^h_{n+1}) \ni f^h_n \\ \text{where } f^h_n = \left(2\mathbf{M} - (1-2\beta)\Delta t^2 \mathbf{K}\right) u^h_n - \left(\mathbf{M} + \beta \Delta t^2 \mathbf{K}\right) u^h_{n-1} + \Delta t^2\ f^{n\beta}. \end{cases}$$

In practice, we have chosen $\beta = 1/2$ in all the following computations. Since matrix $\mathbf{A} \equiv \mathbf{M} + \beta\ \Delta t^2\ \mathbf{K}$ is symmetric and positive definite like $\mathbf{M}$ and $\mathbf{K}$, this problem is equivalent to the following minimization problem

$$u^h_{n+1} = \underset{w \in \mathbb{K}^h}{\text{Argmin}} \left(\frac{1}{2}\ w^T\ \mathbf{A}\ w - w^T\ f^h_n \right).$$

As the convex constraints $w \in \mathbb{K}^h$ correspond to linear inequality constraints, such a problem can be solved by using the Lagrange multipliers method, or interior-point methods, for instance. Here, as in (14), we use the Matlab function "quadprog", which lies on the Lagrange multipliers method.

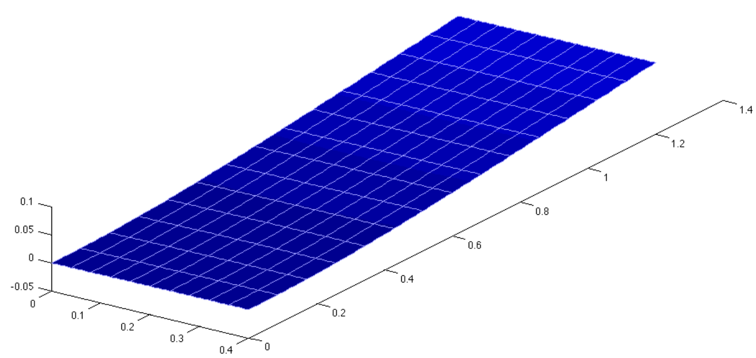

Figure 3. Bending clamped plate under a rigid obstacle: FVS quadrangular mesh.

5.1 *Forced oscillations*

In this section, we consider two flat symmetric obstacles along the plate length

$$g_1(x) = -0.1 = -g_2(x) \quad , \quad \forall x \in \Omega$$

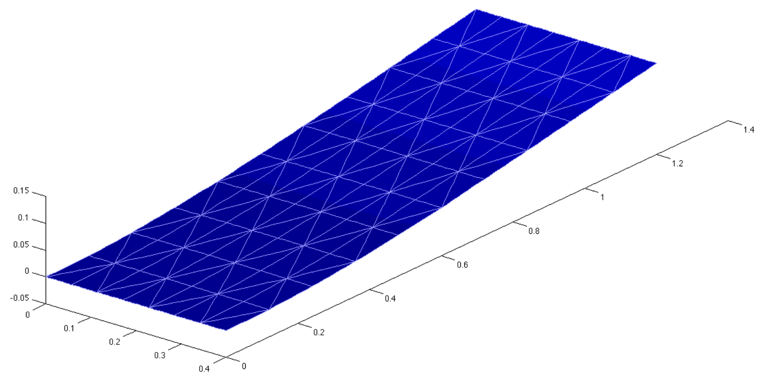

Figure 4. Bending clamped plate under a rigid obstacle: Argyris triangular mesh.

and we prescribe a sine-sweep base forced vibration, by mean of the following boundary conditions on Γ_c

$$u(x,t) = c\ \sin(\omega t)\quad ,\quad \frac{\partial u}{\partial x}(x,t) = 0\quad ,\quad \forall x \in \Gamma_c$$

with $c = 0.09\ m$ and $\omega = 10\ Hz$. The displacements of the two free corners, for different time steps, and for quadrangular and triangular meshes, are plotted on Figures 5, 6 and 7. Not surprisingly, due to symmetry of the problem, the curves, corresponding to the displacements of the two corners, are overlaid. Moreover, there is no significative qualitative difference between the FVS and the Argyris approaches. As far as CPU times are concerned, they are given in Table 1 for the numerical simulations related to the previous test case. They are of same magnitudes for triangles and quadrangles, considering the fact the degrees of freedom and the matrices sizes are different. Finally, analogous results as in (15), for a beam impacting obstacles, are observed.

Time step	10^{-3}	10^{-4}	10^{-5}
140 Argyris triangles	80	870	8880
140 FVS quadrilaterals	120	1220	12220

Table 1. CPU times in seconds (MacBook Pro computer with a 2.2 GHz processor)

To complete this numerical study, some other results are given. First, the case of two flat symmetric obstacles along the plate where $g_1(x) = -0.01 = -g_2(x)$, for all $x \in \Omega$, is considered on Figure 8. Second, the case of various frequencies is investigated (see Figures 9, 10 and 11). All these results confirm the previous conclusions.

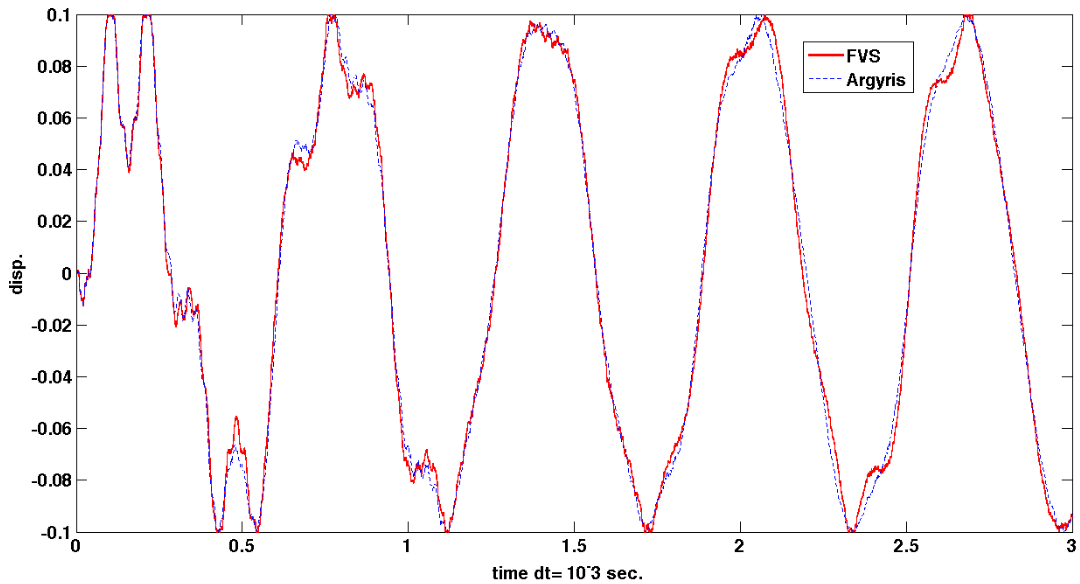

Figure 5. Displacement of a plate impacting flat obstacles - 140 FVS quadrilaterals and 140 Argyris triangles - $\Delta t = 10^{-3}$.

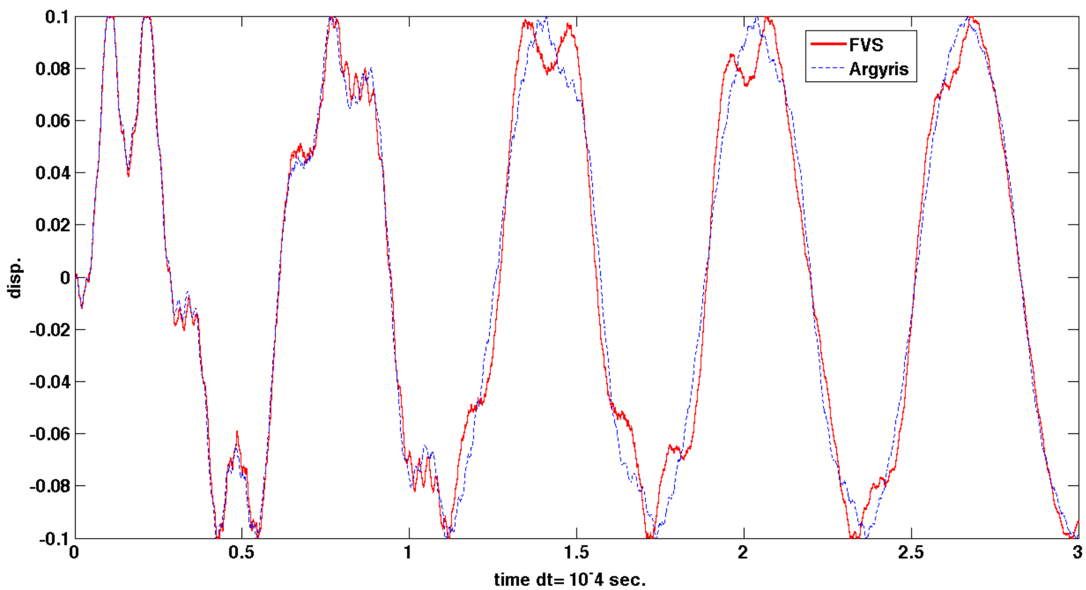

Figure 6. Displacement of a plate impacting flat obstacles - 140 FVS quadrilaterals and 140 Argyris triangles - $\Delta t = 10^{-4}$.

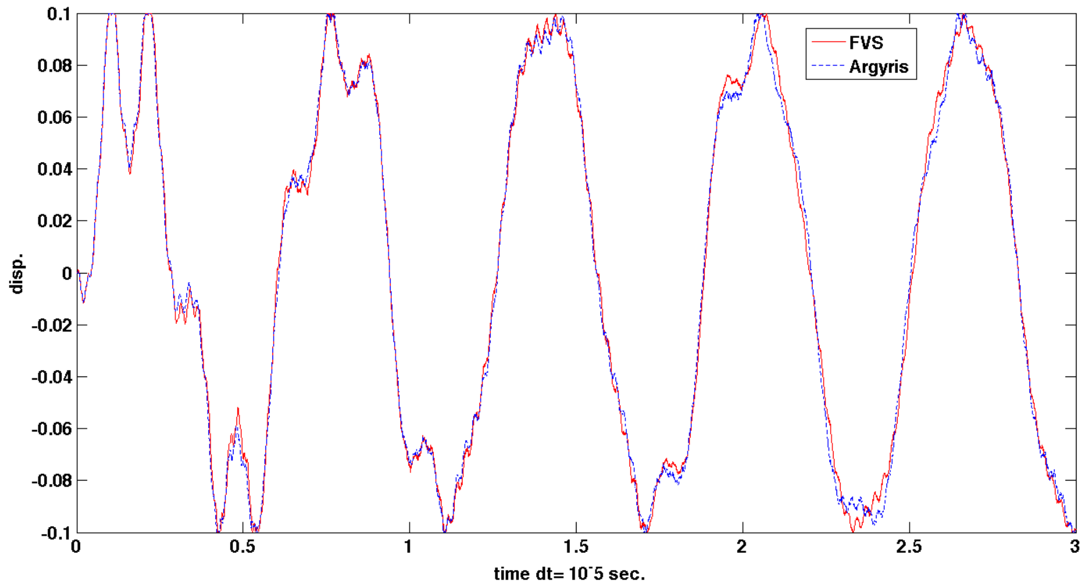

Figure 7. Displacement of a plate impacting flat obstacles - 140 FVS quadrilaterals and 140 Argyris triangles - $\Delta t = 10^{-5}$.

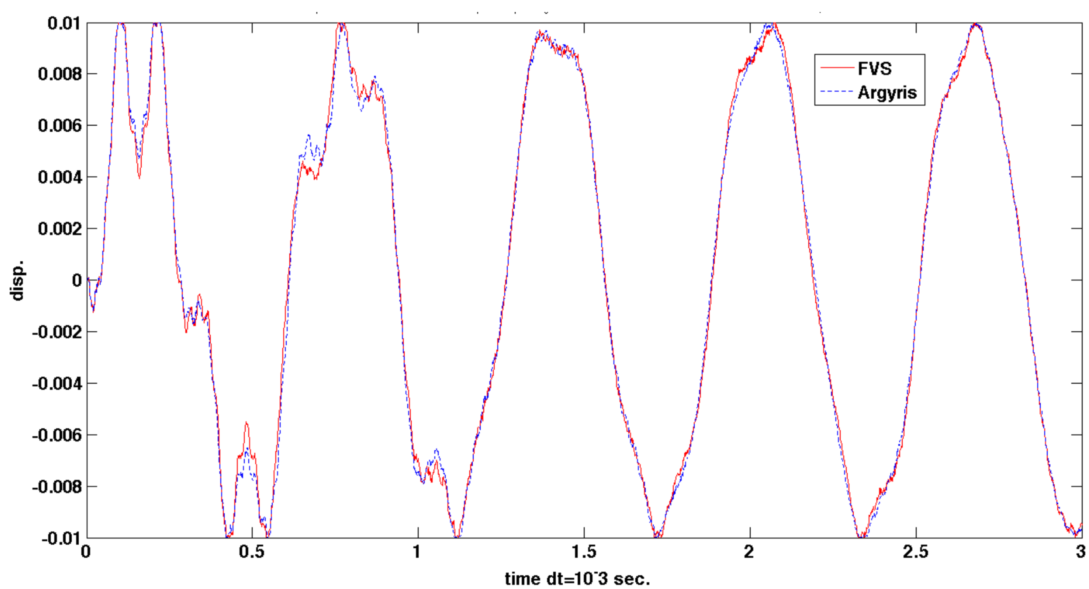

Figure 8. Displacement of a plate impacting flat obstacles - 140 FVS and 140 Argyris elements - $\Delta t = 10^{-3}$ - Obstacle ± 0.01.

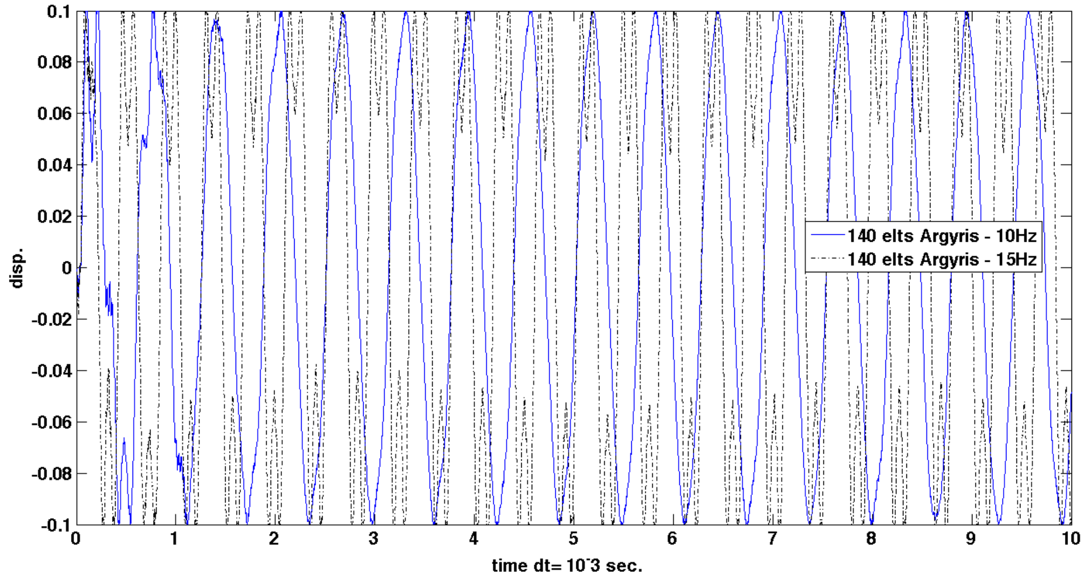

Figure 9. Displacement of a plate impacting flat obstacles - 140 Argyris triangles - $\Delta t = 10^{-3}$ - $\omega = 10\,Hz$ and $\omega = 15\,Hz$.

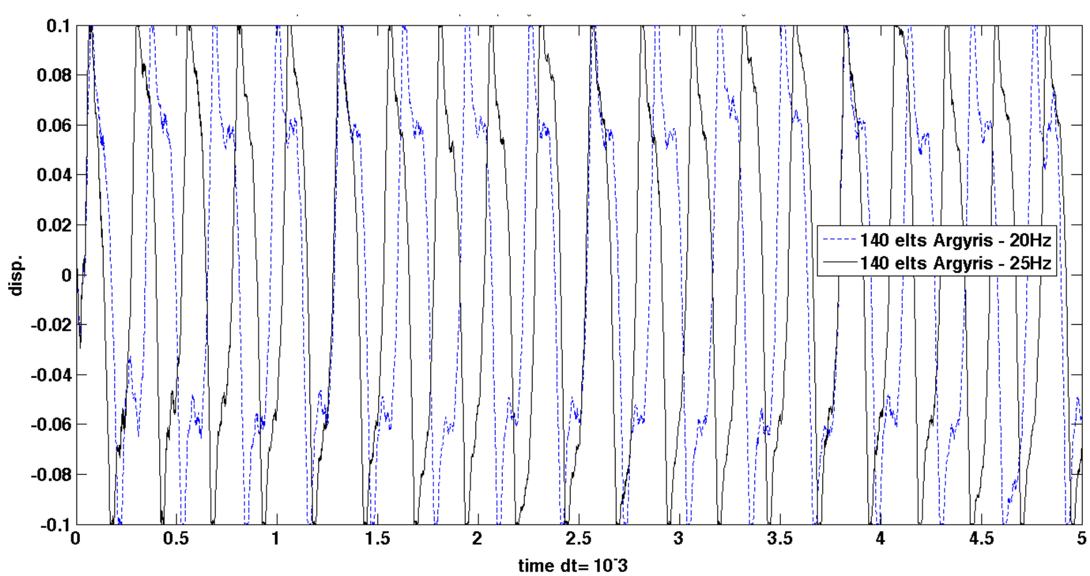

Figure 10. Displacement of a plate impacting flat obstacles - 140 Argyris triangles - $\Delta t = 10^{-3}$ - $\omega = 20\,Hz$ and $\omega = 25\,Hz$.

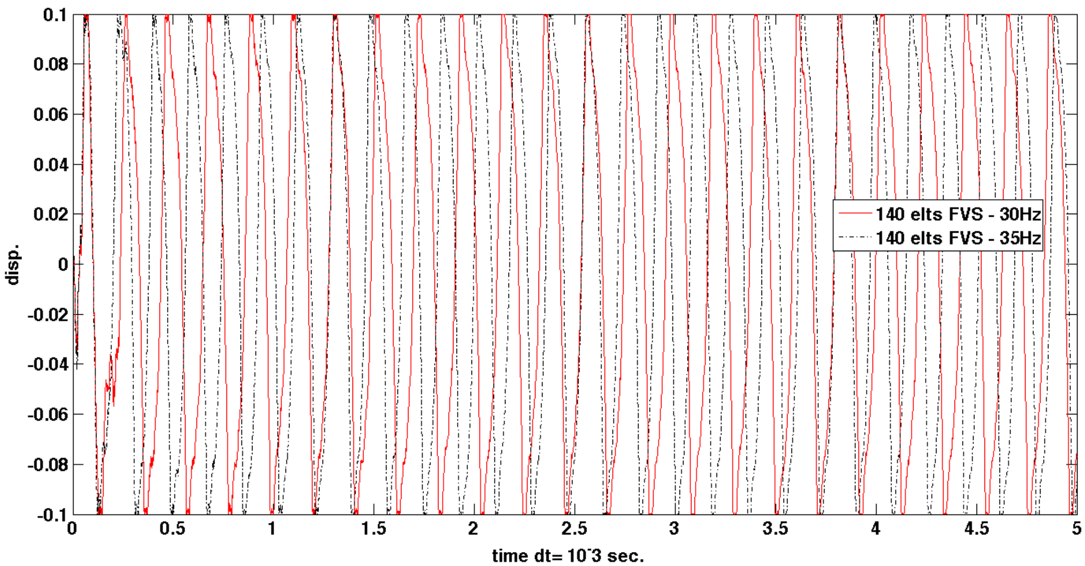

Figure 11. Displacement of a plate impacting flat obstacles - 140 FVS quadrilaterals - $\Delta t = 10^{-3}$ - $\omega = 30\ Hz$ and $\omega = 35\ Hz$.

5.2 *Energy evolution*

This section is devoted to the study of energy variations during the motion. So, here, a forced vibration is not prescribed. The motion is due to an initial displacement u_0, obtained as the static equilibrium of the plate under a constant load $f_0 = 8600\ N$ and an initial velocity $v_0 = 0$. Moreover, the upper obstacle is removed, which corresponds to set $g_2 = +\infty$. The lower obstacle is flat and remains to $g_1 = -0.1\ m$.

First, as in the previous section, the displacements of the two free corners, and also the midpoint between them, are given for rectangular (Figure 12) and triangular (Figure 13) meshes. The results are very close. Here again, the three curves are overlaid. For the two corners, it was expected, but not for the midpoint. To investigate this, a zoom was made on this curves (Figures 14, 15 and 16). They show a so small difference in the motion of this three points that explains that this is not visible on the first figures. Moreover, Figures 15 and 16 illustrate again there is no meaningful difference between triangular and rectangular meshes. Finally, Figures 12 and 13 show the maximum displacements decrease as time passes, which means that impacts create damping during the motion.

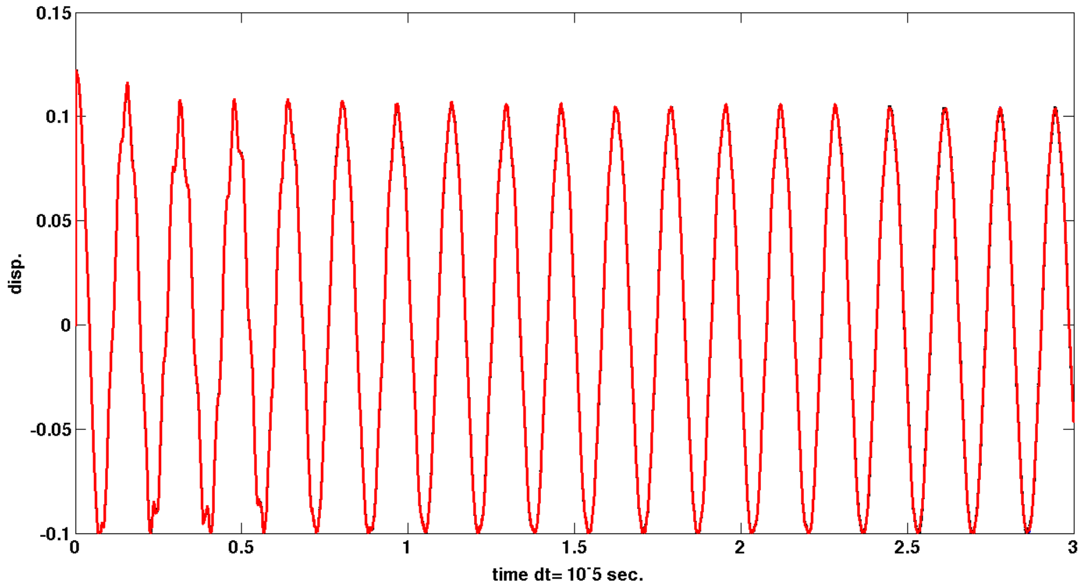

Figure 12. Displacements in free vibrations - 140 FVS quadrilaterals - $\Delta t = 10^{-5}$.

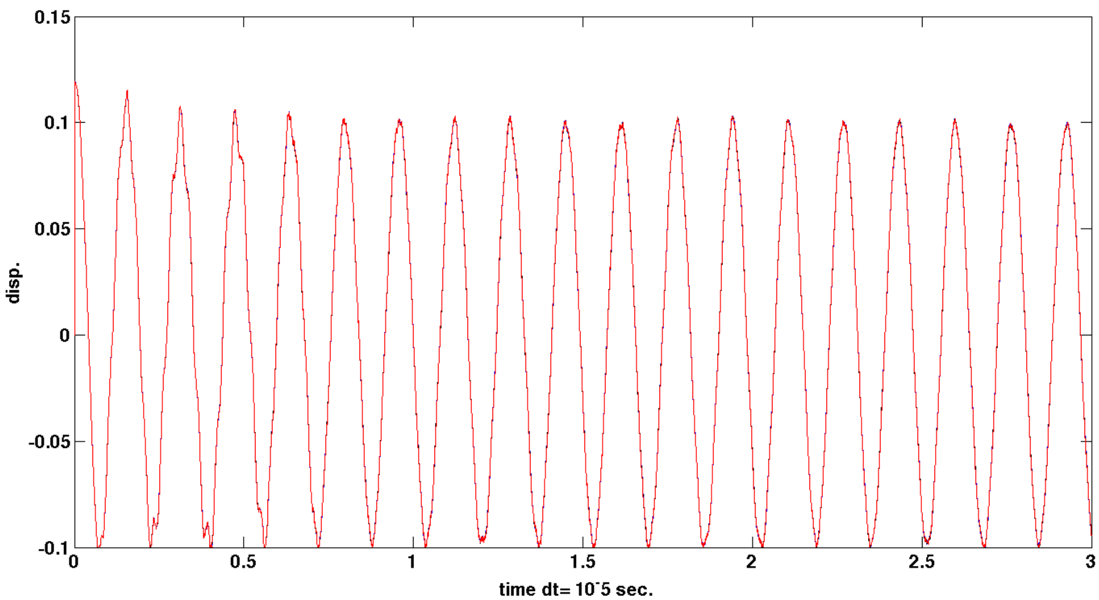

Figure 13. Displacements in free vibrations - 160 Argyris triangles - $\Delta t = 10^{-5}$.

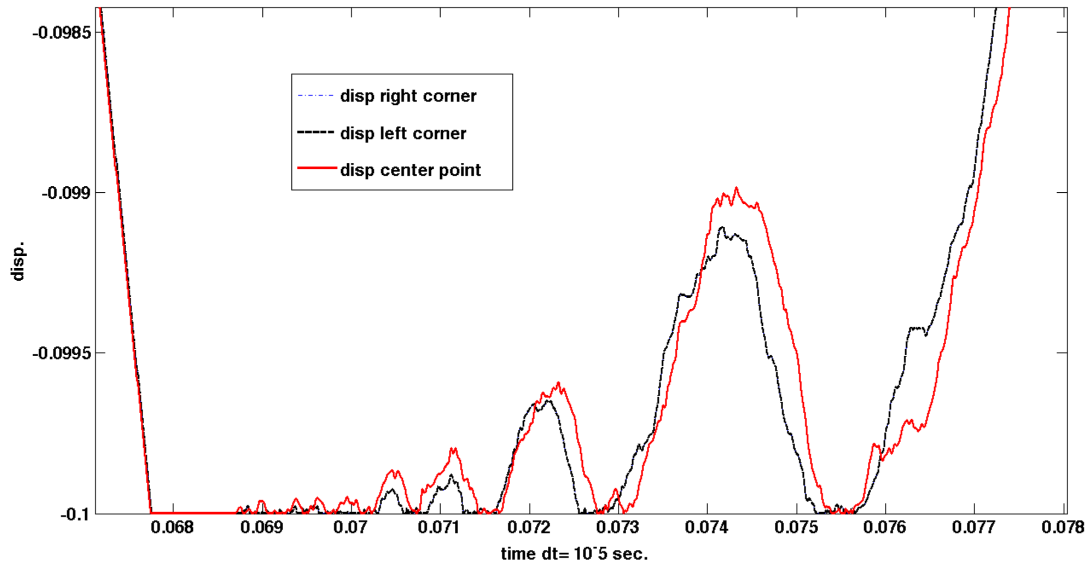

Figure 14. Zoom on displacements in free vibrations - 140 FVS quadrilaterals - $\Delta t = 10^{-5}$.

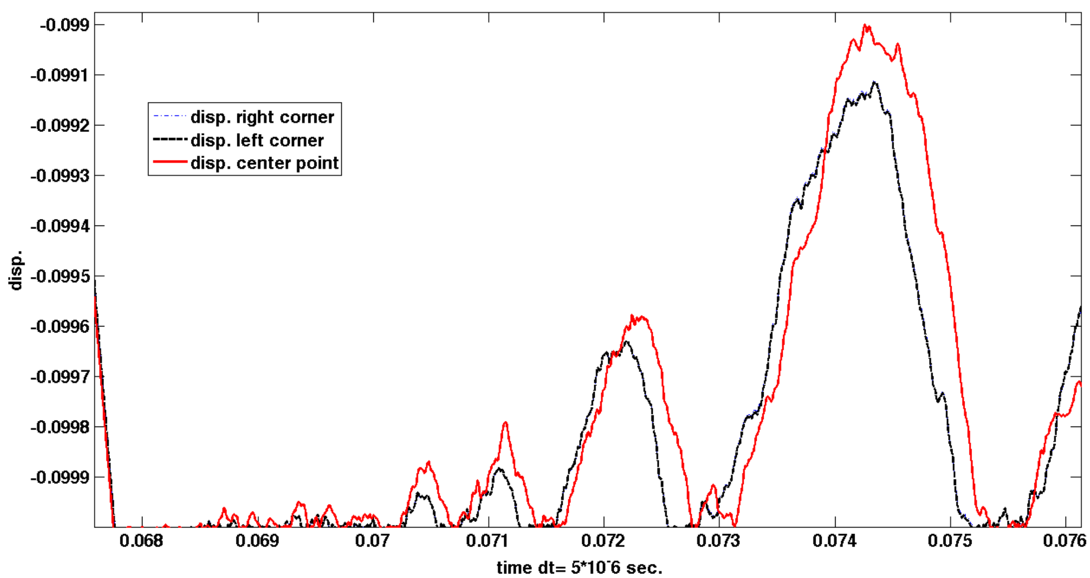

Figure 15. Zoom on displacements in free vibrations - 140 FVS quadrilaterals - $\Delta t = 5\ 10^{-6}$.

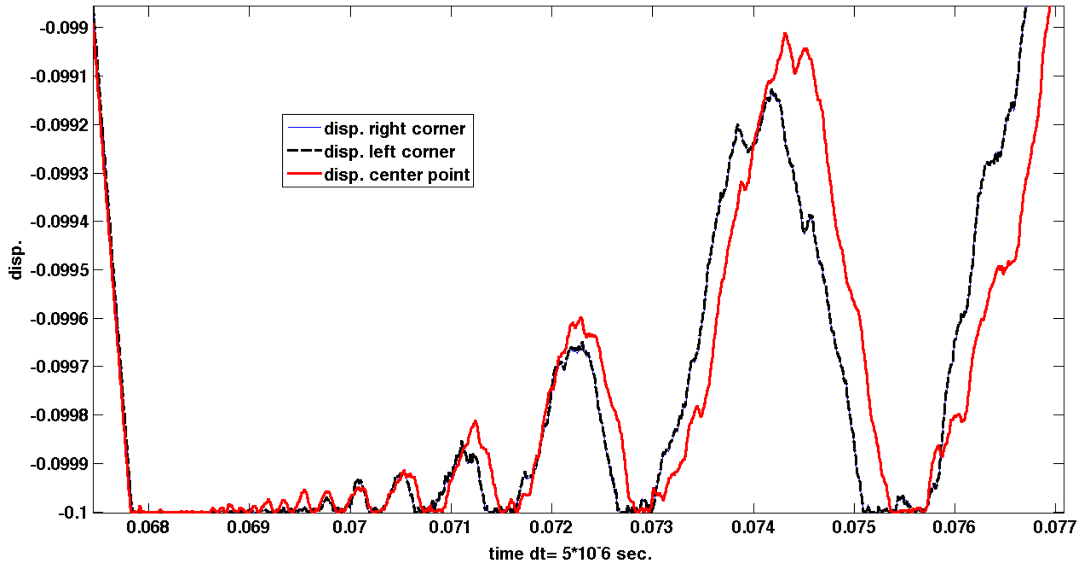

Figure 16. Zoom on displacements in free vibrations - 160 Argyris triangles - $\Delta t = 5\ 10^{-6}$.

Finally, we compare the variations of total energy obtained for different time steps and meshes. This total energy is defined by

$$E(w,t) = \frac{1}{2}\int_\Omega (\dot{w})^2(x,t)\,dx + \frac{1}{2}\,a(w(.,t),w(.,t)) - \int_\Omega f(x,t)\,.\,w(x,t)\,dx.$$

In the case of free vibrations, the loading f is zero. The associated discrete energy reads

$$E(u^h_{n+1},u^h_n) = \frac{1}{2}\int_\Omega \left|\frac{u^h_{n+1}-u^h_n}{\Delta t}\right|^2 dx + \frac{1}{2}\,a(u^h_n,u^h_n).$$

Figures 17 and 18 show the discrete energy decreases. First of all, let us remark these curves exhibit a small difference in the initial energy, which is due to difference of discretizations on the two meshes. But it is a detail. The main point is that, in the two cases, energy is dissipated when the plate reaches the obstacles. The same qualitative results were obtained in (2) and (1). By the way, our numerical model is a fully implicit scheme. It seems that it corresponds to choose a restitution coefficient, defined by (2.9), close to zero. The continuous problem energy will conserve if and only if $e = 1$, which is a totally elastic shock. The results we obtain are then mechanically consistent. To conclude, when the time step decreases, the loss of energy decreases too, which tends to show the scheme creates a too big numerical damping. Looking for energy conserving schemes for plates, as we did for beams in (15), and also studying their convergence properties, is then of particular importance and will be the subject of forthcoming papers.

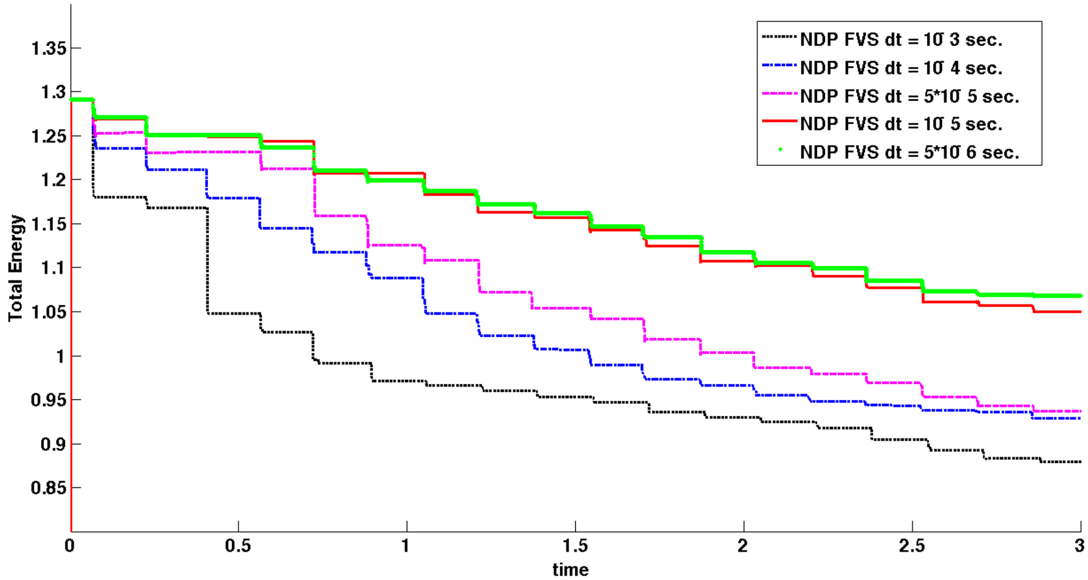

Figure 17. Total energy variations for different values of Δt - 140 FVS quadrilaterals.

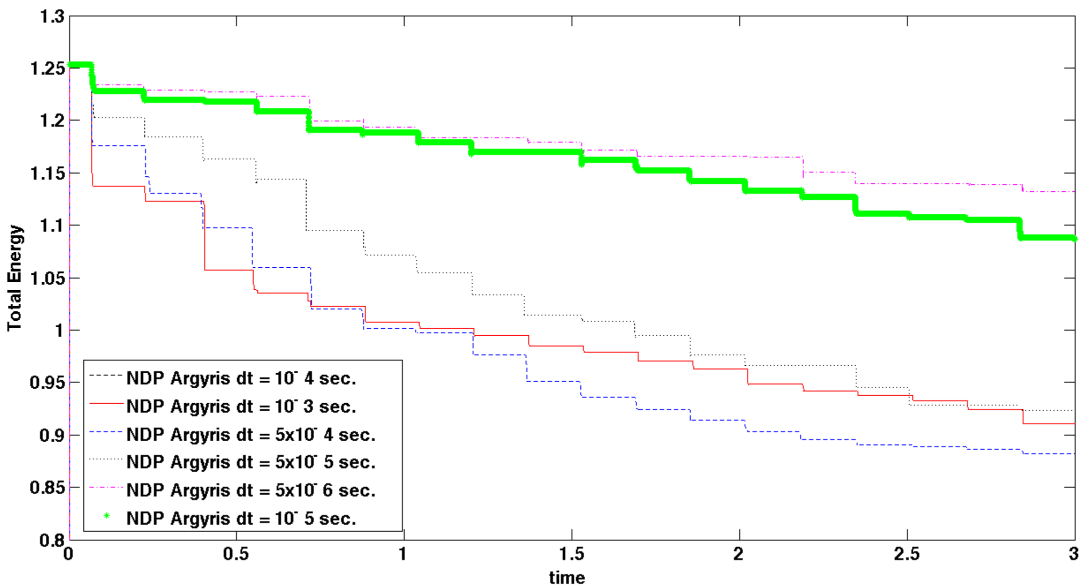

Figure 18. Total energy variations for different values of Δt - 160 Argyris triangles.

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