CSMA 2011
10e Colloque National en Calcul des Structures
9-13 Mai 2011, Presqu'île de Giens (Var)

Numerical convergence and stability of mixed formulation with X-FEM cut-off

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Résumé — The aim of this paper is to study numerical convergence and stability of mixed formulation of X-FEM cut-off for incompressible isotropic linear plane elasticity problem in a cracked domain using a cut-off function to localize the singular enrichment area. The difficulty is caused by both, the so called inf-sup condition which depends on the connection between the approximation spaces for the displacement and pressure, and the discontinuity of the displacement field across the crack. We prove that a quasi-optimal convergence rate holds in spite of the presence of elements cut by the crack.

1 Introduction

The Finite Element Methods (F.E.M.) are widely used in engineering analysis, and has deep mathematical foundations. However, the treatment of incompressible or nearly incompressible problems, with only one unknown field variable, leads to numerical locking of this approximation. For the analysis of such problems, it is necessary to use a mixed formulation. However, mixed finite element methods are not stable in all cases, some of them show spurious pressure oscillations if displacement and pressure spaces are not properly chosen [1]. For a mixed formulation to be stable it must verify consistency, ellipticity and the so called inf-sup (or LBB) condition. The later is a severe condition which depends on the connection between the approximation spaces for the displacement and pressure [2]. The presence of a crack in a structure reveals two types of discontinuities : a strong discontinuity that requires an adapted mesh to the shape of the crack, hence the domain is meshed at each time step ; and a weak discontinuity that requires refinement at the crack tip. These two operations lead to a huge computational cost. In order to overcome these difficulties and to make the finite element method more flexible, many alternative methods had been developed. Apart from mixing meshless methods and finite elements, another alternative consists in using the concept of "partition of unity" introduced by Babuška, to creates an independence between the mesh and the crack. Among the class of partition of unity finite element methods, the Generalised Finite Element Method (GFEM) and the eXtended Finite Element Method (X-FEM) are the most advanced. The X-FEM allows to model cracks, material inclusions and holes on nonconforming meshes. It was introduced by Moës and al [3]. It consists in enriching the basis of the classical finite element method by a step function along the crack line and by some non-smooth functions representing the asymptotic displacement around the crack tip. In spite of the singular enrichment of the finite element basis, the obtained convergence error of X-FEM remains of order if linear finites elements are used. To improve this convergent rate and to obtain an optimal accuracy, Chahine and al introduced a new enrichment strategy [4] : the so called X-FEM cut-off. This enrichment strategy uses a cut-off function to locate the crack tip surface. In their work, Chahine et al have shown that the X-FEM cut-off has an optimal convergence rate of order h and that the conditioning of the stiffness matrix does not deteriorate. In this work, we extend the numerical results given by Chahine et al. [4] to an incompressible isotropic linear plane elasticity problem in fracture mechanics. In particular, this formulation must satisfy the famous inf-sup condition (or LBB).

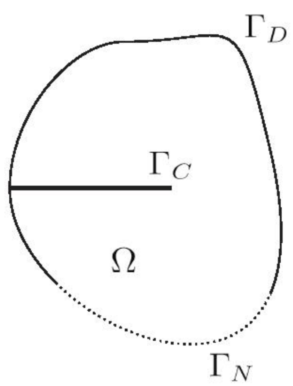


FIGURE 1 – Cracked domain

2 Model problem and descritisation

Let Ω be a two-dimensional cracked domain, Γ_c denotes the crack and Γ the boundary of Ω. We assume that $\Gamma \setminus \Gamma_c$ is partitioned in two parts : Γ_N where a Neumann surface force $\mathbf{t}$ is applied and Γ_D where a Dirichlet condition $\mathbf{u} = \mathbf{0}$ is imposed (see Fig. 1). We assume that we have a traction-free condition on Γ_c. Let $\mathbf{f}$ be the body force applied on Ω. The equilibrium equation and boundary conditions are given by

$$-\operatorname{div} \sigma(\mathbf{u}) = \mathbf{f}, \qquad \text{in} \quad \Omega, \tag{1}$$
$$\sigma(\mathbf{u}) = \mathbb{C} : \varepsilon(\mathbf{u}), \qquad \text{in} \quad \Omega, \tag{2}$$
$$\mathbf{u} = \mathbf{0}, \qquad \text{on} \quad \Gamma_D, \tag{3}$$
$$\sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}, \qquad \text{on} \quad \Gamma_N, \tag{4}$$
$$\sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0}, \qquad \text{on} \quad \Gamma_c. \tag{5}$$

with $\mathbf{n}$ is the outside normal to the domain Ω. Let p be the hydrostatic pressure defined in two dimensions by :

$$p = -\frac{\operatorname{tr}(\sigma)}{2}.$$

Now we decompose the stress tensor σ in two parts : the spherical part and the deviatoric part σ^d given by :

$$\sigma^d(\mathbf{u}) = \sigma(\mathbf{u}) + p\mathbf{I} = 2\mu\varepsilon^d(\mathbf{u}),$$

where

$$\varepsilon^d(\mathbf{u}) = \varepsilon(\mathbf{u}) - \frac{\operatorname{div}(u)}{2}\mathbf{I}.$$

Then, the strong mixed formulation is written as follows

$$-\operatorname{div}[\sigma^d - p\mathbf{I}] = \mathbf{f} \qquad \text{in} \qquad \Omega, \tag{6}$$
$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in} \qquad \Omega. \tag{7}$$

Let $V = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \quad \text{with } \mathbf{u} = 0 \text{ on } \Gamma_D\}$ and $Q = \mathbf{L}^2(\Omega)$. Taking the inner poduct of (6) with a test function $\mathbf{v} \in V$, and multiplying (6) by a test function $q \in Q$. On applying Green's formula for elasticity, we find the weak mixed formulation

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in (V, Q) \text{ such that :} \\ \int_\Omega \sigma^d(\mathbf{u}) : \varepsilon(\mathbf{v})\, d\Omega - \int_\Omega p \operatorname{div}\mathbf{v}\, d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\Omega + \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{v}\, d\Gamma, & \forall \mathbf{v} \in V, \\ \int_\Omega q \operatorname{div}\mathbf{u}\, d\Omega = 0, & \forall q \in Q. \end{cases}$$

Accordingly, the weak mixed formulation of the isotropic incompressible linear elastic problem is written as :

$$\left\{\begin{array}{ll} \text{Find } (\mathbf{u},p)\in(V,Q) \text{ such that :} & \\ a(\mathbf{u},\mathbf{v})-b(\mathbf{v},p)=L(v), & \forall \mathbf{v}\in V, \\ b(\mathbf{u},q)=0, & \forall q\in Q, \end{array}\right. \tag{8}$$

with
$a(\mathbf{u},\mathbf{v})=\int_\Omega \sigma^d(\mathbf{u}):\varepsilon(\mathbf{v})\,d\Omega$,
$b(\mathbf{v},p)=\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\Omega$,
$L(\mathbf{v})=\int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega+\int_{\Gamma_N}\mathbf{t}\cdot\mathbf{v}\,d\Gamma$.

Discretization of the elasticity problem follows the usual steps. We approximate $(\mathbf{u},p)$ by $(\mathbf{u}_h,p_h)\in V_h\times Q_h$. The subspaces V_h and Q_h are finite dimensional spaces that will be defined later. The discretized problem is then :

$$\left\{\begin{array}{ll} \text{Find } (\mathbf{u}_h,p_h)\in(V_h,Q_h) \text{ such that} & \\ a(\mathbf{u}_h,v_h)-b(\mathbf{v}_h,p_h)=L(\mathbf{v}_h), & \forall \mathbf{v}_h\in V_h, \\ b(\mathbf{u}_h,q_h)=0, & \forall q_h\in Q_h. \end{array}\right. \tag{9}$$

REMARK : The existence of a stable finite element approximate solution $(\mathbf{u}_h,p_h)$ depends on choosing a pair of spaces V_h and Q_h such that the following condition holds :

$$\inf_{q\in Q_h}\sup_{\mathbf{v}\in V_h}\frac{b(\mathbf{v}_h,q_h)}{\|q_h\|_{0,\Omega}\|\mathbf{v}_h\|_{1,\Omega}}\geq\beta_0.$$

where $\beta_0>0$ independent of h. This is the “Ladyzhenskaya-Brezzi-Babuška condition” (LBB) also called the “inf-sup condition” [1]. The verification of this condition for a couple (V_h,Q_h) is very difficult to prove in practical situations. Therefore, the numerical evaluation of the inf-sup has been widely studied (see Chapelle and Bathe [2]). The numerical evaluation gives an indication of the verification of the LBB condition for a given finite element discretization. The numerical inf-sup test is based on the following proposition.

Proposition 1 ([2]) *Let $[M]_{uu}$ and $[M]_{pp}$ be the matrices associated with the $\mathbf{H}^1$-inner product in V_h and the $\mathbf{L}^2$-inner product in Q_h, respectively, and let μ_{min} be the smallest nonzero eigenvalue of the eigenvalue problem :*

$$[B]^T\,[M]_{uu}^{-1}\,[B]\,\mathbf{V}=\mu^2\,[M]_{pp}\,\mathbf{V}.$$

Then the value of β_0 in the LBB condition is simply μ_{min}.

The numerical test proposed in [2] is to test the stability of the mixed formulation by calculating β_h with increasingly refined meshes. Indeed, if $\log(\beta_h)$ continuously decreases as h goes to 0, Chapelle and Bathe [2] predicted that the finite element violates the LBB condition. Otherwise, if β_h stabilizes when the number of elements increases, then the numerical inf-sup test is verified.

3 X-FEM cut off

The idea of XFEM is to use a classical finite element space enriched by some additional functions. These functions result from the product of global enrichment functions and some classical finite element functions. This variant of X-FEM uses a cut-off function to find the singular enrichment surface. Indeed, for this method, the enrichment is introduced on all crack tip surface and not only on the nodes. The classical enrichment strategy for this problem is to use the asymptotic expansion of the displacement and pressure fields at the crack tip area. Indeed, the displacement is enriched by the Westergaard functions.

$$F=\{F_j(x),1\leq j\leq 4\}=\left\{\sqrt{r}\sin\frac{\theta}{2},\sqrt{r}\cos\frac{\theta}{2},\sqrt{r}\sin\frac{\theta}{2}\sin\theta,\sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\}.$$

For the pressure, the asymptotic expansion at the crack tip is given by [5]

$$p(r,\theta) = \frac{2K_I}{3\sqrt{2\pi r}}\cos\frac{\theta}{2} + \frac{2K_{II}}{3\sqrt{2\pi r}}\sin\frac{\theta}{2}.$$

This expression is used to obtain the basis of enrichment of the pressure in the area of the crack tip [5] :

$$\begin{cases} F_1^p = \frac{1}{\sqrt{r}}\cos\frac{\theta}{2}, \\ F_2^p = \frac{1}{\sqrt{r}}\sin\frac{\theta}{2}. \end{cases}$$

We note that the displacement and pressure are also enriched with a Heaviside function at the nodes whose shape functions support is totally cut by the crack. Using this enrichment strategy, the discretisation spaces V_h and Q_h take the following forms :

$$V_h = \left\{ \mathbf{v}_h = \sum_{i\in I}\alpha_k\psi_{u,k} + \sum_{i\in I_H}\beta_k H\psi_{u,k} + \sum_{j=1}^{4}\gamma_j F_j^u\chi; \quad \alpha_k,\beta_k,\gamma_j \in \mathbb{R}\ \forall i,j \right\},$$

$$Q_h = \left\{ p_h = \sum_{i\in I} p_i\,\varphi_{p,i} + \sum_{i\in I_H} b_i^p H\varphi_{p,i} + \sum_{j=1}^{2} c_j^p F_j^p\chi; \qquad p_i, b_i^p, c_j^p \in \mathbb{R}\ \forall i,j \right\},$$

with $\psi_{u,k}$ is the vector shape function defined by :

$$\psi_{u,k} = \begin{cases} \begin{pmatrix} \varphi_{u,i} \\ 0 \end{pmatrix} & \text{if } i = \frac{k+1}{2}, \\ \\ \begin{pmatrix} 0 \\ \varphi_{u,i} \end{pmatrix} & \text{if } i = \frac{k}{2}, \end{cases} \tag{10}$$

and $\varphi_{u,i}$ (resp.$\varphi_{p,i}$) is a scalar schape function for displacement (resp. for pressure).
χ is a is a polynom of order 5 between r_0 and R_1 such that :

$$\begin{cases} \chi(r) = 1 & \text{if r} < \text{r}_0, \\ \chi(r) = 0 & \text{if r} > \text{r}_1. \end{cases} \tag{11}$$

4 Numerical study

In this section we numerically study the inf-sup condition and a comparison between the convergence rates of the X-FEM fixed area and X-FEM cut off. The numerical test is made on a non-cracked domain

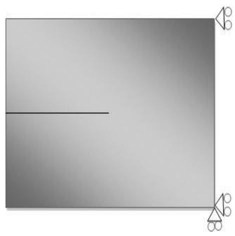


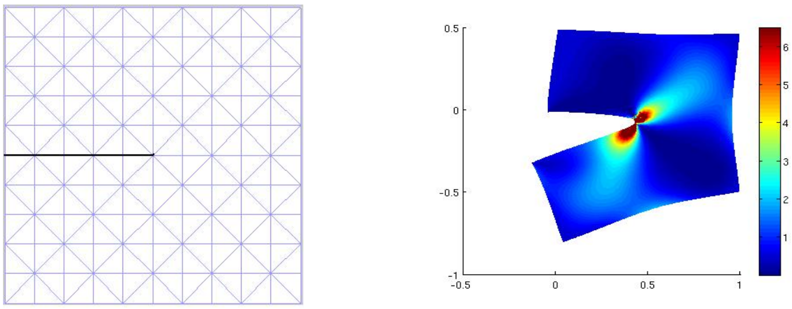
FIGURE 2 – Cracked specimen

defined by

$$\bar{\Omega} =]-0.5, 0.5[\times]-0.5, 0.5[,$$

and the considered crack is the line segment $\Gamma_c =]-0.5; 0[\times\{0\}$ (see Fig. 2). To remove rigid body motions, we must eliminate three degrees of freedom, see Fig. 2. In this numerical test, we impose a boundary condition of Neumann type to avoid this additional error in the transition zone between the boundary conditions of Dirichlet and Neumann. The finite element method is defined on a structured triangulation of $\bar{\Omega}$, (see Fig. 3(a)).

The Von Mises stress for this test is presented in Fig. 3(b). As expected the Von Mises stress is concentrated at the crack tip.



(a) Structured triangulation

(b) Von Mises stress for mixed modes using P_1^+/P_1 elements

FIGURE 3 – Structured triangulation and Von Mises stress

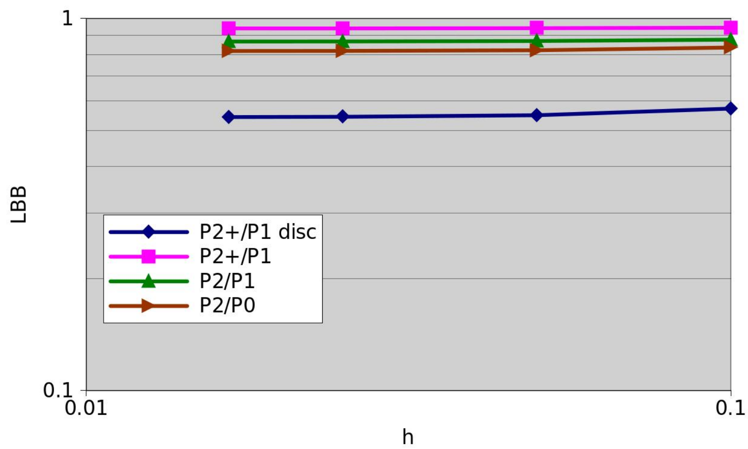


FIGURE 4 – Evaluation of the inf-sup condition for the mixed problem.

4.1 Numerical inf-sup test

In this section we numerically study the inf-sup condition and its dependence on the position of the crack. First, the inf-sup condition is approximated using gradually refined structured triangulation meshes. The evolution of the numerical inf-sup value is plotted in Fig. 4 with respect to the element size. From this figure we can conclude that the numerical inf-sup value is stable for all studied formulations ($P_2^+/P_1 disc$, P_2^+/P_1, P_2/P_1, P_2/P_0).

Let δ be the crack position as shown in Fig. 5. To test the influence of the position of the crack on the inf-sup condition, we calculate the LBB condition by decreasing δ. The tests are made, on a P_1^+/P_1 formulation, with $h = 1/60$ (see Fig. 6(a)) and $h = 1/40$ (see Fig. 6(b)).
The results presented on Figs. 6(a) and 6(b) show that the inf-sup condition remains bounded regardless of the position of the crack. Hence, one can conclude that our formulation is stable independently of the position of the crack.

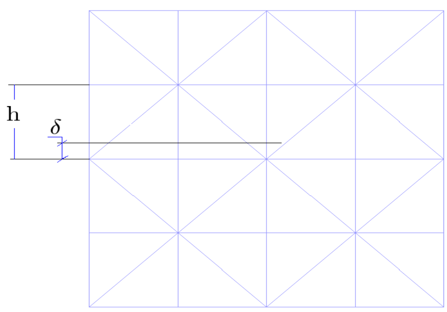
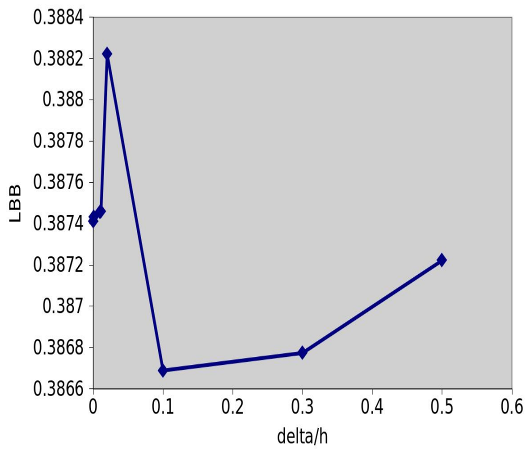
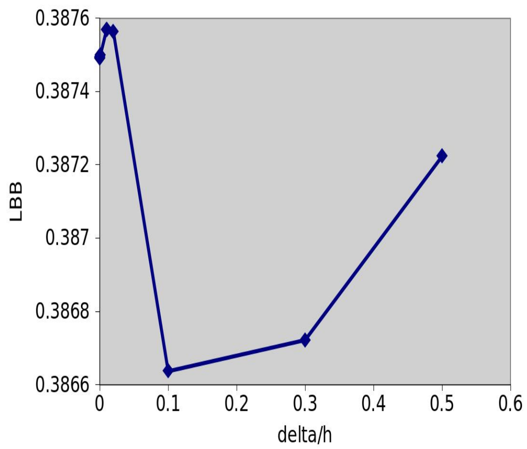


FIGURE 5 – Position of the crack.



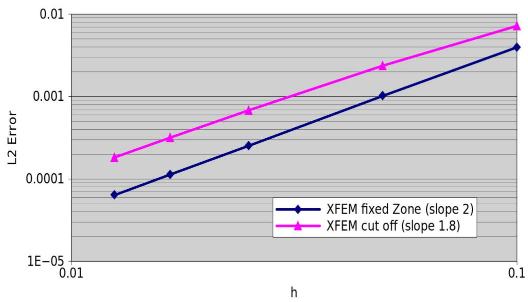
(a) $h = 1/60$



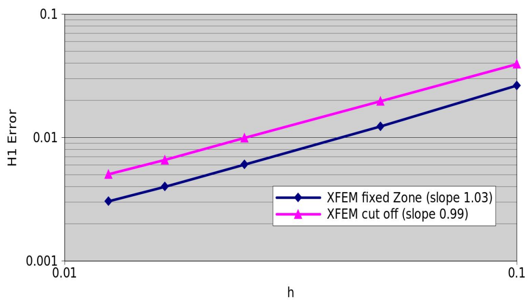
(b) $h = 1/40$

FIGURE 6 – Evolution of the inf-sup condition as, a function of the position of the crack

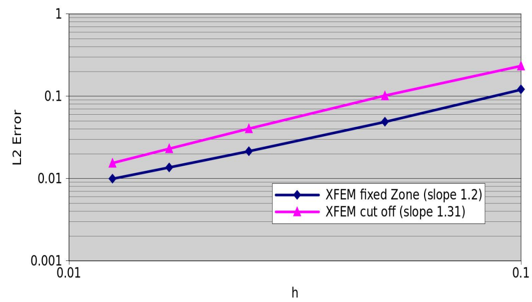
4.2 convergence rate



(a) $\mathbf{L}^2$-displacement



(b) $\mathbf{H}^1$-displacement



(c) L^2-pressure error

FIGURE 7 – Error for the mixed problem with enriched P_1^+/P_1 elements (logarithmic scales)

Figures 7(a), 7(b) and 7(c) show a comparison between the convergence rates of the X-FEM fixed area and X-FEM cut off for the $\mathbf{L}^2$-norm and $\mathbf{H}^1$-norm ($P1^+/P_1$ element are used). These errors are obtained by running the test problem for some values of the parameter n_s, where n_s is the number of subdivision (number of cells) in each direction $h = \dfrac{1}{n_s}$.

Figure 7(b) confirms that the convergence rate for the energy norm is of order h for both variants, of the X-FEM : with fixed area and cut-off. Figure 7(a) shows that the convergence rates for the $\mathbf{L}^2$-norm in displacement is of order h^2 for both variants. Figure 7(c) shows that the convergence rates for the $\mathbf{L}^2$-norm in pressure is h for both variants. Furthermore, the rate of convergence for the $\mathbf{L}^2$-norm is better than the one for the $\mathbf{H}^1$-norm, which is quite usual.

Compared to the X-FEM method with a fixed enrichment area, the convergence rate is very close but the

Number of cells in each direction	Number of degrees of freedom	
	X-FEM fixed enrichment area	X-FEM Cut Off
40	13456	11516
60	30046	25666
80	53376	45416

TABLE 1 – Number of degrees of freedom

error values are large.

In order to test the computational cost of X-FEM cut-off, Table (1) shows a comparison between the number of degrees of freedom for different refinements of the classical method X-FEM with fixed enrichment area and the cut off method. This latter enrichment leads to a significant decrease in the number of degrees of freedom. Figure 8 shows that the conditioning of the linear system associated to the cut-off enrichment is much better than the one associated with the X-FEM with a fixed enrichment area. We can conclude that, similarly to the X-FEM with fixed enrichment area, the X-FEM cut off leads

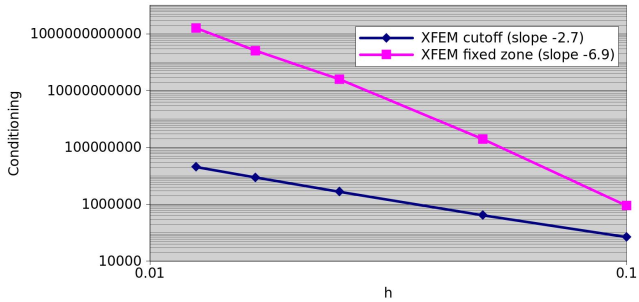


FIGURE 8 – Conditioning number of the stiffness matrix for the mixed problem.

to an optimal convergence rate but it reduces the approximation errors without significant additional costs.

5 Convergence of the High-order X-FEM

The numerical tests of the higher order X-FEM method (P_2^+/P_1 disc, P_2^+/P_1, P_2/P_1 and P_2/P_0) do not give an optimal order of convergence (see Figs. 9(a), 9(b), 9(c) and 9(d)). This means that the enrichment function does not capture the behavior of the solution at the crack tip.

This result was expected as the main singularity belongs to $\mathbf{H}^{3/2}(\Omega)$. Then, for the X-FEM cut-off, the convergence rate remains limited to $h^{3/2}$ with high order polynomials.

To have an optimal convergence rate, we must make an asymptotic expansion of order 2 to find the correct expression of the enrichment basis for the displacement and pressure.

6 conclusion

By this study we can conclude that the X-FEM cut-off mixed formulation is stable, regardless of the position of the crack. Similar to the X-FEM with fixed enrichment area, the X-FEM cut-off gives an optimal convergence rate but without significant additional costs. For shape functions of higher order, the convergence rate is limited to $h^{3/2}$. This result was expected as the main singularity belongs to $\mathbf{H}^{3/2}(\Omega)$.

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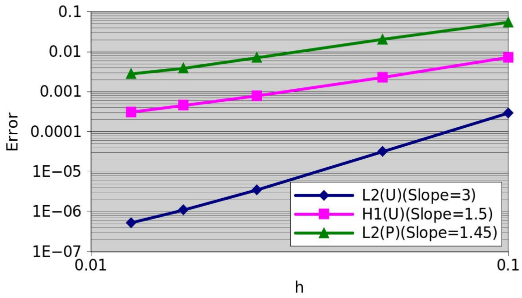
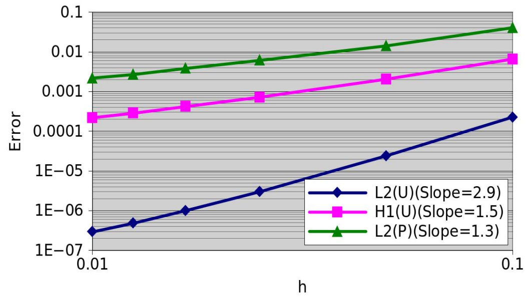
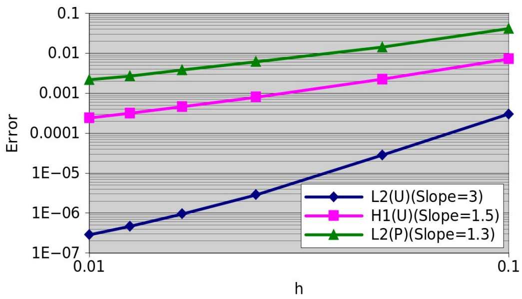


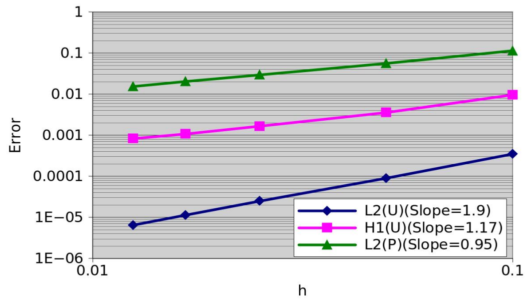
(a) Enriched P_2^+/P_1 disc element



(b) Enriched P_2^+/P_1 element



(c) Enriched P_2/P_1 element



(d) Enriched P_2/P_0 element

FIGURE 9 – Error for the mixed problem with enriched P_1^+/P_1 elements (logarithmic scales)

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