A local projection stabilization of fictitious domain method for elliptic boundary value problems

Saber AMDOUNI *, Maher MOAKHER †, Yves RENARD‡

Oct. 25, 2013

Abstract

In this paper, a new consistent method based on local projections for the stabilization of a Dirichlet condition is presented in the framework of finite element method with a fictitious domain approach. The presentation is made on the Poisson problem but the theoretical and numerical results can be straightforwardly extended to any elliptic boundary value problem. A numerical comparison is performed with the Barbosa-Hughes stabilization technique. The advantage of the new stabilization technique is to affect only the equation on multipliers and thus to be equation independent.

Keywords: Local projection stabilization method, X-FEM, fictitious domain method, Poisson problem.

Introduction

The fictitious domain method is a technique allowing the use of regular structured meshes on a simple shaped fictitious domain containing the real domain. Generally, this technique is used for solving elliptic problems in domains with unknown or moving boundary without having to build a body fitted mesh. There exist two main approaches of fictitious domain method. The "thin" interface approach where the approached interface has the same dimension as the original interface. This approach was initiated by V.K. Saul'ev in [30]. In this context, there exist different techniques to take account of the boundary condition: the technique where the fictitious domain mesh is modified locally to take account of the boundary condition (see for instance reference [30, 21]); The technique of penalization which allows to conserve the Cartesian mesh of the fictitious domain (see for instance reference [2, 18]) and the technique of Lagrange multiplier introduced by R. Glowinski et al. [13, 16, 18, 17] where a second mesh is considered to conserve the Cartesian mesh of the fictitious domain and to take account of the boundary condition.

The second approach of fictitious domain method is the "Spread" interface approach where the approximate interface is larger than the physical interface. The approximate interface has one dimension more than the original one. It was introduced by Rukhovets [29]. For example, the following methods can be found in this group: Immersed boundary method [26, 27] and Fat boundary method [22, 8].

* *Laboratoire LAMSIN, Ecole Nationale d'Ingénieurs de Tunis, Université Tunis El Manar, B.P. 37, 1002 Tunis-Belvédère, Tunisie & INSA-Lyon, ICJ UMR5208-France.* Saber.Amdouni@insa-lyon.fr.

† *Laboratoire LAMSIN, Ecole Nationale d'Ingénieurs de Tunis, Université Tunis El Manar, B.P. 37, 1002 Tunis-Belvédère, Tunisie.* maher.moakher@enit.rnu.tn.

‡ Université de Lyon, CNRS, INSA-Lyon, ICJ UMR5208, LaMCoS UMR5259, F-69621, Villeurbanne, France. Yves.Renard@insa-lyon.fr.

Recently, fictitious domain methods with a thin interface have been proposed in the context of the extended finite element method (X-FEM) introduced by Moes, Dolbow and Belytscko [24]. Different approaches are proposed in [23, 32, 6] to directly enforce an inf-sup condition on a multiplier to prescribe a Dirichlet boundary condition. Another possibility is the use of the stabilized Nitsche's method [25] which is close to a penalization technique but preserving the consisting and avoiding large penalty terms that would otherwise deteriorate the conditioning of the system matrix [12]. We can cite also the method introduced in [11] which uses a stabilized Lagrange multipliers method using piecewise constant multipliers and an additional stabilization term employing the inter-element jumps of the multipliers. Finally let us mention [19] where an a priori error estimate for non-stabilized Dirichlet problem is given and an optimal method is developed using a Barbosa-Hughes stabilization (see [3, 4]).

In this paper, we perform a study similar to [19] for a local projection stabilization applied to the fictitious domain method inspired by the X-FEM. To our knowledge, this technique was used for the first time by Becker and Braack in [7]. Recently, this new technique was proposed and analyzed by Burman [14, 10] in the context of the Lagrange finite element method and by Barrenechea et al. [5] in the context of a more classical fictitious domain approch (uncut mesh). The principle of the used local projection stabilization is to penalize the difference of the multiplier with its projection on some pre-defined patches. The advantage of this technique is of at least threefold: the method is consistent, there is no use of mesh other than the (possibly Cartesian) one of the fictitious domain and the additional term concerns only the multiplier and is not model dependent such as the Barbosa-Hughes stabilization technique.

The paper is organized as follows. In Section 1 we introduce the Poisson model problem and in Section 2, the non-stabilized fictitious domain method. We present our new stabilization technique in Section 3 together with the theoretical convergence analysis. Finally, Section 4 is devoted to two and three-dimensional numerical experiments and the comparison with the use of Barbosa-Hughes stabilization technique.

1 The model problem

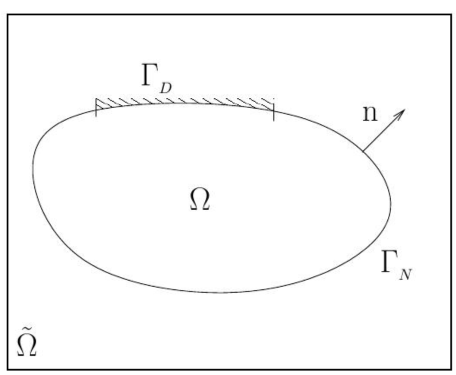


Figure 1: *Fictitious* $\tilde{\Omega}$ *and real* Ω *domains.*

For the sake of simplicity, the presentation and the theoretical analysis is made for a two-dimensional regular domain Ω, although the method extends naturally to higher dimensions. Let $\widetilde{\Omega} \subset \mathbb{R}^2$ be a fictitious domain containing Ω in its interior (and generally assumed to have a simple shape). We consider that the boundary Γ of Ω is split into two parts Γ_N and Γ_D (see Fig. 1). It is assumed that Γ_D has a nonzero one-dimensional Lebesgue measure. Let us consider the following elliptic problem in Ω:

$$\begin{cases} \text{Find } u : \Omega \mapsto \mathbb{R} \text{ such that:} \\ -\Delta u = f \qquad \text{in } \Omega, \\ \quad u = 0 \qquad \text{on } \Gamma_D, \\ \ \partial_n u = g \qquad \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$ are given data. Conciderin a Lagrange multiplier to prescribe the Dirichlet boundary condition, a classical weak formulation of this problem reads as follows:

$$\begin{cases} \text{Find } u \in V \text{ and } \lambda \in W \text{ such that} \\ a(u,v) + \langle \lambda, v \rangle_{W,X} = l(v) \quad \forall v \in V, \\ \langle \mu, u \rangle_{W,X} = 0 \quad \forall \mu \in W, \end{cases} \tag{2}$$

where

$$V = H^1(\Omega), \; X = \left\{ w \in L^2(\Gamma_D) : \exists v \in V, w = v_{|\Gamma_D} \right\}, \quad W = X',$$

$$a(u,v) = \int_\Omega \nabla u . \nabla v d\Omega, \;\; l(v) = \int_\Omega f \, v \, d\Omega + \int_{\Gamma_N} g \, v \, d\Gamma,$$

and $\langle \mu, v \rangle_{W,X}$ denotes the duality pairing between W and X, endowed with the following norms:

$$\|v\|_V = (a(v,v))^{1/2}, \quad \|f\|_X = \inf_{v \in V; f = v_{|\Gamma_D}} \|v\|_V, \quad \|\mu\|_W = \sup_{v \in V} \frac{\langle \mu, v \rangle_{W,X}}{\|v\|_V}.$$

With the following norms we prove easy that

$$\begin{aligned} \|v\|_X &\leq \|v\|_V \; \forall v \in V, \\ c_1 \|\mu\|_W &\leq \|\mu\|_{0,\Gamma_D} \; \forall \mu \in L^2(\Gamma_D), \end{aligned}$$

where c_1 is the inverse of the trace constant. Let $V_0 = \left\{ v \in V : \int_{\Gamma_D} v \, d\Gamma = 0 \right\}$. Then, a direct consequence of Peetre-Tartar lemma (see [15]) is that $a(.,.)$ is coercive on V_0 *i.e.* there exists $\alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V_0. \tag{3}$$

From this, the existence and uniqueness of a solution to Problem (2) follows. Classically, Problem (2) is also equivalent to the problem of finding the saddle point of the Lagrangian

$$\mathcal{L}(v,\mu) = \frac{1}{2} a(v,v) + \langle \mu, v \rangle_{W,X} - l(v), \tag{4}$$

defined on $V \times X$. The existence and uniqueness of a solution to Problem (2) is obtained by standard techniques.

2 The fictitious domain method

The fictitious domain approach requires the introduction of two finite-element spaces on the fictitious domain $\widetilde{\Omega}$. Namely $\widetilde{V}^h \subset H^1(\widetilde{\Omega})$ and $\widetilde{W}^h \subset L^2(\widetilde{\Omega})$. Note that $\widetilde{\Omega}$ may always be chosen as a sufficiently large rectangle $(a,b) \times (c,d)$ such that $\Omega \subset (a,b) \times (c,d)$ which allows $\widetilde{V}^h$ and $\widetilde{W}^h$ to be defined on the same structured mesh $\mathcal{T}^h$ (see Fig. 2). In what follows, we shall assume that

$$\widetilde{V}^h = \{ v^h \in \mathcal{C}(\overline{\widetilde{\Omega}}) : v^h_{|T} \in P(T) \; \forall T \in \mathcal{T}^h \}, \tag{5}$$

where $P(T)$ is a finite-dimensional space of regular functions satisfying $P(T) \supseteq P_k(T)$ for some integer $k \geq 1$.

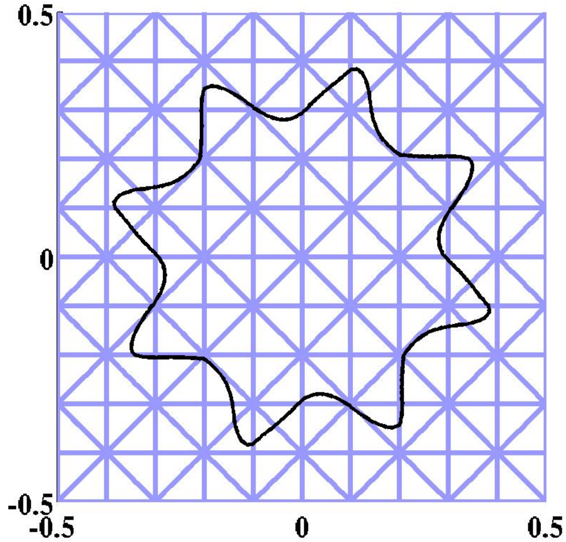


Figure 2: *Example of a real domain and a structured mesh of the fictitious domain.*

For the approximation on the real domain Ω, we consider the following restriction of $\widetilde{V}^h$ and $\widetilde{W}^h$ on Ω and Γ_D, respectively:

$$V^h = \widetilde{V}^h|_\Omega \ , \quad \text{and} \quad W^h = \widetilde{W}^h|_{\Gamma_D},$$

which are natural discretization of V and W. An approximation of Problem (2) is then defined as follows:

$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that} \\ a(u^h, v^h) + \displaystyle\int_{\Gamma_D} \lambda^h v^h d\Gamma = l(v^h) \ \ \forall v^h \in V^h, \\ \displaystyle\int_{\Gamma_D} \mu^h u^h d\Gamma = 0 \ \ \forall \mu^h \in W^h. \end{array} \right. \tag{6}$$

We choose $\widetilde{W}^h$ and $\widetilde{V}^h$ in such a way that the following condition is satisfied:

$$1|_{\Gamma_D} \in W^h. \tag{7}$$

Let us define the following space:

$$V_0^h = \{v^h \in V^h : \int_{\Gamma_D} \mu^h v^h d\Gamma = 0 \ \ \forall \mu^h \in W^h\}. \tag{8}$$

Then $a(.,.)$ is V_0^h-elliptic since $V_0^h \subset V_0$. Without any additional treatment, the following result is proved in [19]:

Proposition 1 *Let $\widetilde{V}^h$ defined by (5), assume (7) is satisfied and, in addition*

$$\inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_W \leq h^\beta, \quad \beta \geq 1/2. \tag{9}$$

$$\overline{\mu}^h \in W^h : \int_{\Gamma_D} \overline{\mu}^h v^h d\Gamma = 0 \ \ \forall v^h \in V^h \Longrightarrow \overline{\mu}^h = 0. \tag{10}$$

Then, one has the following error estimate:

$$\|u^h - u\|_V \leq C\sqrt{h}, \quad h \to 0+.$$

This means that, without any treatment, the guaranteed rate of convergence is limited to $O(\sqrt{h})$ which is confirmed is some numerical situations. This reflects a certain kind of numerical locking phenomenon.

3 A local projection stabilized formulation

In this section, we present a stabilization technique consisting in the addition of a supplementary term involving the local orthogonal projection of the multiplier on a patch decomposition of the mesh.

Let S^h be the one-dimensional mesh resulting in the intersection of $\mathcal{T}^h$ and Γ_D. The idea is to aggregate the possibly very small elements of S^h in order to obtain a set of patches having a minimal and a maximal size (for instance between $3h$ and $6h$). In practice, this operation can be done rather easily (even for three-dimensional problems). A practical way to obtain such a patch decomposition will be described in the next section. An example of patch aggregation is presented in Fig. 3.

Let H be the minimum length of these patches and denote by S^H the corresponding subdivision of Γ_D. Let

$$W^H = \left\{\mu^H \in L^2(\Gamma_D) : \mu^H_{|S} \in P_0(S), \, \forall S \in S_H\right\},$$

be the space of piecewise constants on this mesh. A classical result, presented in [16], states that under a reasonable regularity assumption on Γ_D, an inf-sup condition is satisfied between W^H and V^h for minimal size of $3h$ for the patches. This implies in particular that an optimal convergence can be reached if the multiplier is taken in W^H. However, this assumes a relatively coarse approximation of the multiplier. Our approach is to use this result in order to stabilize the approximation obtained with the multiplier defined on the finer discretization W^h.

Let us first recall the result of Girault and Glowinski in [16]. Under the assumption that Γ_D is of class $\mathscr{C}^{1,1}$ and a condition for the patches $S \in S^H$ to be approximated by a fixed set of line segments having approximatively the same length (see [16], condition (4.17)) with a length between $3h$ and ηh for a constant $\eta > 3$, then the following inf-sup (or LBB) condition holds for a constant $\beta^* > 0$, independent of h and H:

$$\forall \mu^H \in W^H, \qquad \sup_{v^h \in V^h} \frac{\displaystyle\int_{\Gamma_D} v^h \, \mu^H \, d\Gamma}{\|v^h\|_V} \geq \beta^* \|\mu^H\|_W. \tag{11}$$

Remark 1. *The inf-sup condition in [16] is given for the whole boundary $\Gamma = \partial\Omega$. However,* (11) *is easily obtained by restricting to multipliers μ^H being nonzero only on Γ_D and for coarse meshes compatible with Γ_D.*

In the following, we will assume that the conditions to obtain inf-sup condition (11) are satisfied.

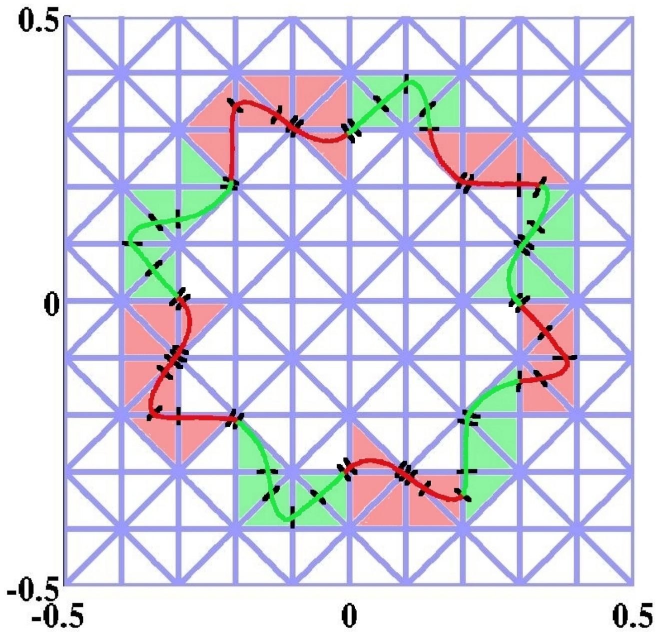


Figure 3: *Example of a patch aggregation (in red and green) of size approximatively* $2h$ *of the intersection of the boundary of the real domain and the mesh. Note the practically inevitable presence of very small intersections.*

Let P_{W^H} be the local orthogonal projection operator from $L^2(\Gamma_D)$ onto W^H which is defined by

$$\forall \mu \in L^2(\Gamma_D),\ \forall S \in S^H \qquad P_{W^H}(\mu)\,|_S \ = \frac{1}{mes(S)} \int_S \mu d\Gamma.$$

The stabilized formulation consists in approximate the Lagrangian (4) by:

$$\mathcal{L}_h(v^h, \mu^h) = \mathcal{L}(v^h, \mu^h) - \frac{\gamma}{2} \int_{\Gamma_D} (\mu_h - P_{W^H}(\mu^h))^2 d\Gamma,$$

where, for the sake of simplicity, γ is a chosen constant. The corresponding optimality system reads as follows:

$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that} \\ a(u^h, v^h) + \displaystyle\int_{\Gamma_D} \lambda^h v^h d\Gamma = l(v^h) \ \ \forall v^h \in V^h, \\ \displaystyle\int_{\Gamma_D} \mu^h u^h d\Gamma - \gamma \int_{\Gamma_D} (\lambda^h - P_{W^H}(\lambda^h))(\mu^h - P_{W^H}(\mu^h)) d\Gamma = 0 \ \ \forall \mu^h \in W^h. \end{array} \right. \tag{12}$$

Lemma 1 *Assume that (7) and (11) hold, then for any* $\gamma > 0$ *there exists a unique solution of the stabilized problem (12).*

Proof. Suppose (u_1^h, λ_1^h) and (u_2^h, λ_2^h) are two solutions to Problem (12). Let us denote $\bar{u}^h = u_1^h - u_2^h$, $\bar{\lambda}^h = \lambda_1^h - \lambda_2^h$ and $\bar{\lambda}^H = P_{W^H}(\lambda_1^h) - P_{W^H}(\lambda_2^h)$. Then, from Problem (12) we obtain

$$\left\{ \begin{array}{l} a(\bar{u}^h, \bar{u}^h) + \displaystyle\int_{\Gamma_D} \bar{\lambda}^h \bar{u}^h d\Gamma = 0, \\ \displaystyle\int_{\Gamma_D} \bar{\lambda}^h \bar{u}^h d\Gamma - \gamma \int_{\Gamma_D} (\bar{\lambda}^h - \bar{\lambda}^H)^2 d\Gamma = 0 \ \ \forall \mu^h \in W^h. \end{array} \right. \tag{13}$$

Consequently,

$$a(\bar{u}^h,\bar{u}^h)+\gamma\int_{\Gamma_D}(\bar{\lambda}^h-\bar{\lambda}^H)^2 d\Gamma=0, \tag{14}$$

which implies that $\bar{u}^h=0$ and $\bar{\lambda}^h=\bar{\lambda}^H$ (i.e. $\bar{\lambda}^h\in W^H$). Moreover, it follows from (11) that there exists $v^h\in V^h$ such that

$$\int_{\Gamma_D}\bar{\lambda}^H v^h \geq \beta^*\|\bar{\lambda}^H\|_W\|v^h\|_V, \tag{15}$$

and thus

$$\beta^*\|\bar{\lambda}^H\|_W\leq\frac{1}{\|v^h\|_V}\int_{\Gamma_D}\bar{\lambda}^H v^h d\Gamma=\frac{1}{\|v^h\|_V}\int_{\Gamma_D}\bar{\lambda}^h v^h d\Gamma=\frac{1}{\|v^h\|_V}a(\bar{u}^h,v^h)=0.$$

This implies the uniqueness of the solution and, since the dimension of the linear system (12) is finite, the existence as well. ■

3.1 Convergence analysis

In this section, we establish an optimal *a priori* error estimate for the following standard finite element spaces:

$$\widetilde{V}^h=\{v^h\in\mathcal{C}(\overline{\widetilde{\Omega}}): v^h_{|T}\in P(T)\ \forall T\in\mathcal{T}^h\}, \tag{16}$$

$$\widetilde{W}^h=\{\mu^h\in L^2(\widetilde{\Omega}): \mu^h_{|T}\in P'(T)\ \forall T\in\mathcal{T}^h\}, \tag{17}$$

where $P(T)$ (resp. $P'(T)$) is a finite-dimensional space of regular functions satisfying $P(T)\supseteq P_k(T)$ (resp. $P(T)\supseteq P_{k'}(T)$) for an integer $k\geq 1$ (resp. $k'\geq 0$).

Theorem 1 *Let $\widetilde{V}^h$ and $\widetilde{W}^h$ be defined by (16) and (17), respectively such that (7) is satisfied. Let (u,λ) be the solution of the continuous problem (2) such that $u\in H^2(\Omega)$ and $\lambda\in H^{1/2}(\Gamma_D)$. Assume that (11) is satisfied and assume also the existence of a constant $\eta>3$ with $3h\leq H\leq\eta h$. Then, the following estimate holds for $C>0$ a constant independent of h:*

$$|||(u-u^h,\lambda-\lambda^h)|||\leq Ch\Big(\|u\|_{2,\Omega}+\|\lambda\|_{1/2,\Gamma_D}\Big), \tag{18}$$

where $|||(u,\lambda)|||^2=\|u\|_V^2+\|\lambda\|_W^2$ and (u^h,λ^h) is the solution to Problem (12).

Proof. Let $\lambda^H=P_{W^H}(\lambda^h)$. As u and u^h are both in V^0 then for all $v^h\in V^h$ and $\mu^H\in W^H$ we have:

$$\begin{aligned}
\alpha\|u^h-u\|_V^2 &\leq a(u^h-u,u^h-u)=a(u^h-u,v^h-u)+a(u^h-u,u^h-v^h),\\
&\leq \|u^h-u\|_V\|v^h-u\|_V-\int_{\Gamma_D}(\lambda^h-\lambda)(u^h-v^h)d\Gamma,\\
&= \|u^h-u\|_V\|v^h-u\|_V-\int_{\Gamma_D}\lambda^h u^h d\Gamma+\int_{\Gamma_D}\lambda u^h d\Gamma+\int_{\Gamma_D}(\lambda^h-\lambda)(v^h-u)d\Gamma,\\
&= \|u^h-u\|_V\|v^h-u\|_V-\gamma\|\lambda^h-\lambda^H\|_{0,\Gamma_D}^2+\int_{\Gamma_D}(\lambda-\mu^H)(u^h-u)d\Gamma\\
&\quad+\int_{\Gamma_D}(\lambda^h-\lambda)(v^h-u)d\Gamma,
\end{aligned}$$

because in particular $\int_{\Gamma_D} (\lambda^h - \lambda) u \, d\Gamma = 0$. Then, still for all $v^h \in V^h$ and $\mu^H \in W^H$, we deduce that

$$\begin{aligned}\alpha\|u^h - u\|_V^2 \quad + \quad & \gamma c_1 \|\lambda^h - \lambda^H\|_W^2 \le M\|u^h - u\|_V \|v^h - u\|_V \\ & + \|\lambda - \mu^H\|_W \|u^h - u\|_V + \|\lambda^h - \lambda\|_W \|u - v^h\|_V.\end{aligned} \tag{19}$$

Besides,

$$\int_{\Gamma_D} (\lambda - \lambda^h) v^h d\Gamma = a(u^h - u, v^h) \qquad \forall v^h \in V^h,$$

and therefore one obtains

$$\int_{\Gamma_D} (\bar{\mu}^h - \lambda^h) v^h d\Gamma = a(u^h - u, v^h) + \int_{\Gamma_D} (\bar{\mu}^h - \lambda) v^h d\Gamma \qquad \forall v^h \in V^h; \ \forall \bar{\mu}^h \in W^h. \tag{20}$$

Now, for $\mu^H = \lambda^H - \bar{\mu}^H \in W^H$ with $\bar{\mu}^H \in W^H$, the inf-sup condition (11) ensures the existence of $v^h \in V^h$ such that together with (20) we get

$$\begin{aligned}\beta^* \|\lambda^H - \bar{\mu}^H\|_W \quad & \le \quad \frac{1}{\|v^h\|_V} \int_{\Gamma_D} (\bar{\mu}^H - \lambda^H) v^h d\Gamma, \\ & \le \quad \frac{1}{\|v^h\|_V} \int_{\Gamma_D} (\bar{\mu}^h - \lambda^h) v^h d\Gamma + \frac{1}{\|v^h\|_V} \int_{\Gamma_D} (\bar{\mu}^H - \lambda^H - (\bar{\mu}^h - \lambda^h)) v^h d\Gamma, \\ & \le \quad \|u^h - u\|_V + \|\bar{\mu}^h - \lambda\|_W + \|\bar{\mu}^H - \lambda^H - (\bar{\mu}^h - \lambda^h)\|_W.\end{aligned}$$

As a consequence, one has

$$\begin{aligned}\beta^* \|\lambda^H - \lambda\|_W \quad \le \quad & \beta^* \|\lambda - \bar{\mu}^H\|_W + \|u^h - u\|_V + \|\bar{\mu}^h - \lambda\|_W \\ & + \|\bar{\mu}^H - \bar{\mu}^h\|_W + \|\lambda^H - \lambda^h\|_W,\end{aligned}$$

and

$$\begin{aligned}\beta^{*2} \|\lambda^H - \lambda\|_W^2 \quad \le \quad & 5\|u - u^h\|_V^2 + 5\beta^{*2} \|\lambda - \bar{\mu}^H\|_W^2 + 5\|\lambda - \bar{\mu}^h\|_W^2 \\ & + 5\|\bar{\mu}^H - \bar{\mu}^h\|_W^2 + 5\|\lambda^H - \lambda^h\|_W^2 \quad \forall \bar{\mu}^h \in W^h.\end{aligned} \tag{21}$$

By combining inequalities (19) and (21) one obtains for all $\bar{\mu}^h \in W^h$, $\mu^H \in W^H$, $\bar{\mu}^H \in W^H$ and $v^h \in V^h$

$$\begin{aligned}& (\alpha - 5\delta)\|u - u^h\|_V^2 + \delta\beta^{*2} \|\lambda - \lambda^H\|_W^2 + (\gamma c_1 - 5\delta)\|\lambda^h - \lambda^H\|_W^2 \\ \le \quad & \|u^h - u\|_V \|v^h - u\|_V + \|\lambda - \mu^H\|_W \|u^h - u\|_V + \|\lambda - \lambda^h\|_W \|u - v^h\|_V \\ & + 5\delta\beta^{*2} \|\lambda - \bar{\mu}^H\|_W^2 + 5\delta\|\lambda - \bar{\mu}^h\|_W^2 + 5\delta\|\bar{\mu}^h - \bar{\mu}^H\|_W^2, \\ \le \quad & \frac{\delta}{2}\|u - u^h\|_V^2 + \frac{1}{2\delta}\|u - v^h\|_V^2 + \frac{\delta}{2}\|u - u^h\|_V^2 + \frac{1}{2\delta}\|\lambda - \mu^H\|_W^2 + \frac{\xi}{2}\|\lambda - \lambda^h\|_W^2 \\ & + \frac{1}{2\xi}\|u - v^h\|_V^2 + 5\delta\beta^{*2}\|\lambda - \bar{\mu}^H\|_W^2 + 5\delta\|\lambda - \bar{\mu}^h\|_W^2 + 5\delta\|\bar{\mu}^h - \bar{\mu}^H\|_W^2.\end{aligned}$$

Let δ and ξ be such that $\delta < \min\left(\frac{\alpha}{6}; \frac{\gamma c_1}{5}\right)$ and $\xi < \min\left(2\delta\beta^{*2}; 2(\gamma c_1 - 5\delta)\right)$, then, still for all $\bar{\mu}^h \in W^h$, $\mu^H \in W^H$, $\bar{\mu}^H \in W^H$ and $v^h \in V^h$, one deduces that

$$\begin{aligned}& (\alpha - 6\delta)\|u - u^h\|_V^2 + (\gamma c_1 - 5\delta - \frac{\xi}{2})\|\lambda^h - \lambda^H\|_W^2 + (\delta\beta^{*2} - \frac{\xi}{2})\|\lambda - \lambda^H\|_W^2 \\ \le \quad & (\frac{1}{2\delta} + \frac{1}{2\xi})\|u - v^h\|_V^2 + \frac{1}{2\delta}\|\lambda - \mu^H\|_W^2 + 8\delta\beta^{*2}\|\lambda - \bar{\mu}^H\|_W^2 + 8\delta\|\lambda - \bar{\mu}^h\|_W^2 \\ & + 8\delta\|\bar{\mu}^h - \bar{\mu}^H\|_W^2.\end{aligned}$$

Denoting by Π^h (resp. P_{W^h}) the Lagrange interpolation operator (resp. the $L^2(\Gamma_D)$-projection) in V^h (resp. in W^h), we have the following standard finite-element estimates:

$$\begin{aligned} \|u - \Pi^h u\|_V &\leq Ch\|u\|_{2,\Omega}, \\ \|\lambda - P_{W^h}(\lambda)\|_W &\leq Ch\|\lambda\|_{1/2,\Gamma_D}, \\ \|\lambda - P_{W^H}(\lambda)\|_W &\leq CH\|\lambda\|_{1/2,\Gamma_D}. \end{aligned}$$

Finally, the theorem is established by taking $v^h = \Pi^h u$, $\bar{\mu}^h = P_{W^h}(\lambda)$, $\bar{\mu}^H = P_{W^H}(\lambda)$ and $\mu^H = P_{W^H}(\lambda)$. ■

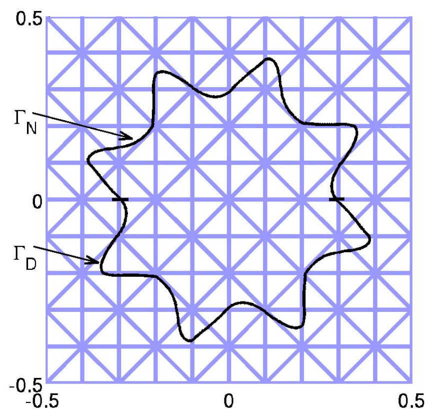
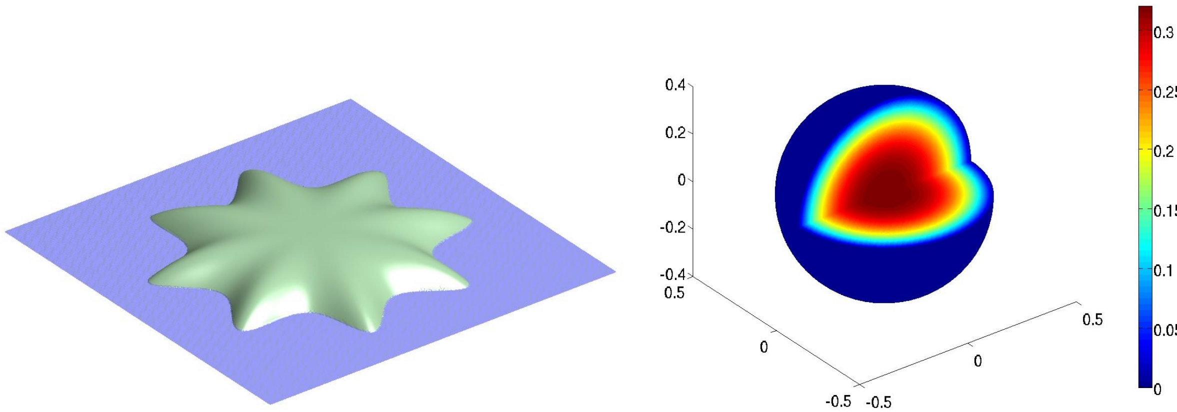


Figure 4: Example of a two-dimensional triangular structured mesh used for the numerical test and partition of the boundary for Neumann and Dirichlet conditions.

4 Numerical tests

In this section, we present 2D and 3D-numerical tests for a fictitious domain being $\widetilde{\Omega} =]-1/2, 1/2[^d$ for $d = 2$ and $d = 3$, respectively. The two-dimensional exact solution is chosen to be $u(x) = -5(R^4 - r^4(2.5 + 1.5\sin(8\theta + \frac{2\pi}{9})))$ where $r = \sqrt{x_1^2 + x_2^2}$, $R = 0.47$ and the three-dimensional one is $u(x) = 5(\rho^3 - R^3)$ with $\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In both cases, the real domain is $\Omega = \{x \in \mathbb{R}^d : u(x) < 0\}$ and the Dirichlet and Neumann boundary conditions are defined on $\Gamma_D = \Gamma \cap \{x \in \mathbb{R}^d : x_d < 0\}$ and $\Gamma_N = \Gamma \cap \{x \in \mathbb{R}^d : x_d > 0\}$, respectively. The two-dimensional domain is represented in Fig. 4 with an example of a triangular structured mesh. The exact solutions are shown in Fig. 5.



(a) Two-dimensional exact solution (b) Three-dimensional exact solution

Figure 5: Exact solutions

The numerical tests are performed with GETFEM++, the C++ finite-element library developed by our team (see [28]).

4.1 Numerical solving

The algebraic formulation of Problem (12) reads

$$\begin{cases} \text{Find } U \in \mathbb{R}^{N_u} \text{ and } \; L \in \mathbb{R}^{N_\lambda} \text{ such that} \\ KU + B^T L = F, \\ BU - M_\gamma L = 0, \end{cases} \tag{22}$$

where U is the vector of degrees of freedom for u^h, L the one for the multiplier λ^h, N_u and N_λ the dimensions of V^h and W^h, respectively, K is the stiffness matrix coming from the term $a(u^h, v^h)$, F is the right-hand side corresponding to the term $\ell(v^h)$, and B and M_γ are the matrices corresponding to the terms $\int_{\Gamma_D} \lambda^h v^h d\Gamma$ and $\gamma \int_{\Gamma_D} (\lambda^h - P_{W^H}(\lambda^h))(\mu^h - P_{W^H}(\mu^h))d\Gamma$, respectively.

Before presenting the numerical experiments, we shall describe in details two important aspects of the implementation of the method. Namely, the extraction of a basis for W^h and the repartition of the elements having an intersection with Γ_D into patches.

The extraction of a basis of W^h could be non-trivial in some cases, except when a piecewise constants (P_0) finite-element method is used to approximate the multiplier or in some other cases when Γ_D is curved. Indeed, if one selects all the shape functions of $\widetilde{W}^h$ whose supports intersect Γ_D, some of them can be linearly dependent, especially when Γ_D is a straight line. In order to eliminate linearly dependent shape functions, the choice here is to consider the mass matrix $\int_{\Gamma_D} \psi_i \psi_j d\Gamma$ where the ψ_i are the finite-element shape functions of $\widetilde{W}^h$. A block-wise Gram-Schmidt algorithm is used to eliminate local dependencies and then the potential remaining kernel of the mass matrix is detected by a Lanczos algorithm. In the presented numerical tests, since curved boundaries are considered the kernel of the mass matrix is either reduced to 0 or is very small. In [1] some numerical experiments are presented for a straight line in 2D using the same technique. The selection of a basis of W^h using this technique took far less computational time than the assembly of the stiffness matrix.

The decomposition into patches is made using a graph partitioner algorithm. In the presented numerical tests we use the free software METIS [20]. The nodes of the graph consist in the elements having an intersection with Γ_D and the edges connect adjacent elements. Additionally, a load corresponding to the size of the intersection is considered on each elements. The partition is a very fast operation.

4.2 Comparison with the Barbosa-Hughes stabilization technique

In our numerical test, we compare the new stabilization technique to the one studied in [19] in the same framework which use the technique introduced by Barbosa and Hughes in [3, 4]. For the self consistency of the paper, we briefly recall the principle of the symmetric version of the Barbosa-Hughes stabilization technique applied to Problem (6) as it is presented in [19].

This technique is based on the addition of a supplementary term involving an approximation of the normal derivative on Γ_D. Let us assume that we have at our disposal an operator

$$R^h : V^h \longrightarrow L^2(\Gamma_D),$$

which approximates the normal derivative on Γ_D (*i.e.* for $v^h \in V^h$ converging to a sufficiently smooth function v, $R^h(v^h)$ tends to $\partial_n v$ in an appropriate sense). Several choices of R^h are proposed in [19]. To obtain the stabilized problem, the Lagrangian (4) is approximated by the following one

$$\mathcal{L}_h(v^h, \mu^h) = \mathcal{L}(v^h, \mu^h) - \frac{\gamma}{2} \int_{\Gamma_D} (\mu^h + R^h(v^h))^2 d\Gamma, \quad v^h \in V^h, \mu^h \in W^h,$$

where the stabilization parameter γ depends on the mesh parameter $\gamma := h\gamma_0$, with γ_0 a positive constant over Ω. The corresponding discrete problem reads as follows:

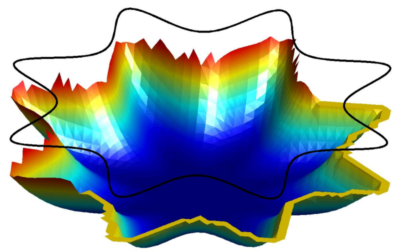
$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h \text{ and } \lambda^h \in W^h \text{ such that} \\ a(u^h, v^h) + \displaystyle\int_{\Gamma_D} \lambda^h v^h d\Gamma - \gamma \int_{\Gamma_D} (\lambda^h + R^h(u^h)) R^h(v^h) d\Gamma = l(v^h) \quad \forall v^h \in V^h, \\ \displaystyle\int_{\Gamma_D} \mu^h u^h d\Gamma - \gamma \int_{\Gamma_D} (\lambda^h + R^h(u^h)) \mu^h d\Gamma = 0 \quad \forall \mu^h \in W^h. \end{array} \right. \tag{23}$$

More details and a convergence analysis can be found in [19]. Note that this is also a consistent modification of the Lagrangian and that a close relationship between Barbosa-Hughes stabilization technique and Nitsche's one [25] has been explained in [31].

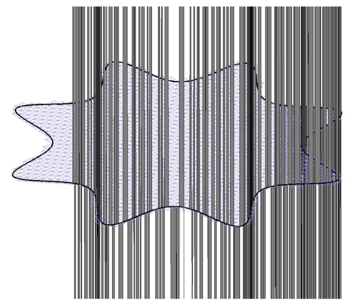
4.3 Two-dimensional numerical tests

A comparison is done between the non-stabilized problem (6), the local projection stabilized problem (12) and the Barbosa-Hughes stabilized one (23) in the two-dimensional case. Additionally, we test different pairs of elements for the main unknown u and the multiplier. Namely, we test the following methods: P_2/P_1, P_1/P_1, P_1/P_0, P_1/P_2, Q_1/Q_0 and Q_1/Q_0. The notation P_i/P_j (resp. Q_i/Q_j) means that solution u is approximated with a P_i finite-element method (resp. a Q_i finite-element method) and the multiplier with a continuous P_j finite-element method (resp. continuous Q_j finite-element method).

Without stabilization. A solution is plotted in Fig. 6 for a P_1/P_2 method. Of course, for this pair of elements, a uniform discrete inf-sup cannot be satisfied since the multiplier is discretized with a reacher element than the main unknown. As a consequence, a local locking phenomenon (Fig. 6(a)) on the Dirichlet boundary (flat part of the solution) holds together with a very noisy multiplier (Fig. 6(b)). This indicates the presence of spurious modes. Some similar results can be observed with the P_1/P_1 and P_1/P_0 methods.

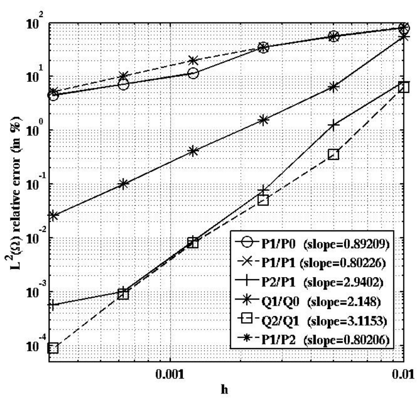


(a) Solution on Ω with no stabilization for the P_1/P_2 method.

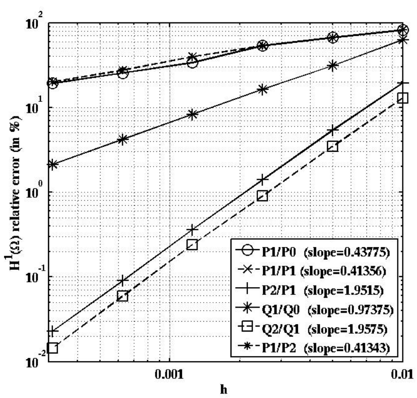


(b) Multiplier on Γ_D with no stabilization for the P_1/P_2 method.

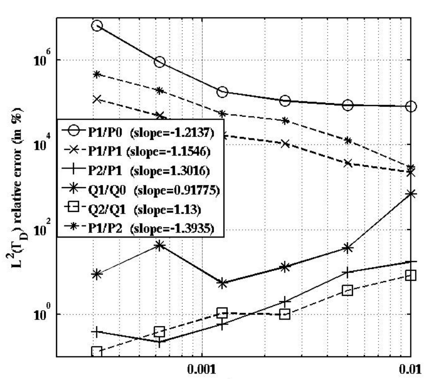
Figure 6: Non-stabilized case with the P_1/P_2 method.



(a) Convergence of $\|u - u^h\|_{0,\Omega}$



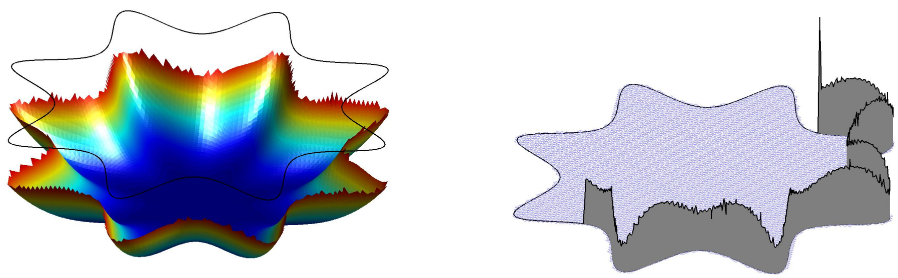
(b) Convergence of $\|u - u^h\|_{1,\Omega}$



(c) Convergence of $\|\lambda^h - \lambda\|_{0,\Gamma_D}$

Figure 7: Convergence curves in the non-stabilized case.

The convergence curves in the non-stabilized case are given in Fig. 7(a) for the error in the $L^2(\Omega)$-norm on u, in Fig. 7(b) for the error in the $H^1(\Omega)$-norm on u and in Fig. 7(c) for the error in the $L^2(\Gamma_D)$-norm on the multiplier. One notes that the convergence rates for the P_1/P_2, P_1/P_1 and P_1/P_0 methods in $H^1(\Omega)$-norm are close to 0.5 which is in good agreement with the general result of Proposition 1. In this cases, there is no convergence of the multiplier (still due to the presence of some spurious modes). Conversely, for the P_2/P_1, Q_2/Q_1 and Q_1/Q_0 methods, one observes a nearly optimal convergence rate. This does not imply that a mesh independent inf-sup condition is systematically satisfied in these cases. In [19], some numerical experiments show that the solution can be deteriorated in the vicinity of very small intersections between the mesh and Γ_D (especially for the multiplier).



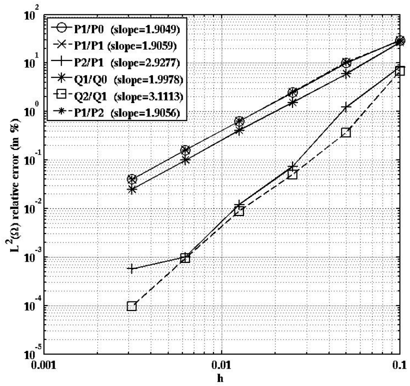
(a) Solution on Ω with Barbosa-Hughes stabilization for the P_1/P_1 method.

(b) Multiplier on Γ_D with Barbosa-Hughes stabilization for the P_1/P_1 method.

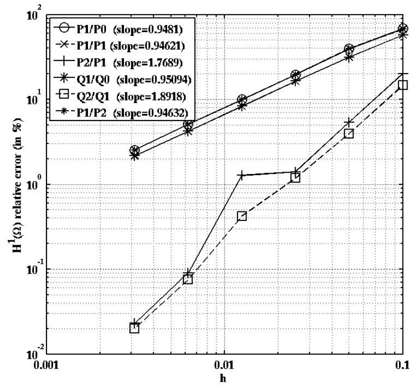
Figure 8: Barbosa-Hughes stabilized case with the P_1/P_1 method.

Barbosa-Hughes stabilization. Fig. 8 shows that the Barbosa-Hughes stabilization technique eliminates the locking phenomenon (Fig. 8(a)) and the spurious modes on the multiplier (Fig. 8(b)). The convergence curves in the Barbosa-Hughes stabilized case are given in Fig. 9(a) for the error in the $L^2(\Omega)$-norm on u, in Fig. 9(b) for the error in the $H^1(\Omega)$-norm on u and in Fig. 9(c) for the error in the $L^2(\Gamma_D)$-norm on the multiplier. The rates of convergence for the error in $L^2(\Omega)$-norm (resp. $H^1(\Omega)$-norm) on u with Barbosa-Hughes stabilization are optimal: of order 3 (resp. of order close to 2) for both P_2/P_1 and Q_2/Q_1 and of order 2 (resp. order 1) for the remaining pairs of elements. Fig. 9(c) shows that the approximation of the multiplier is considerably improved. Concerning the error in $L^2(\Gamma_D)$-norm for the multiplier the rate of convergence is also close to optimality for all pairs of elements.
We refer to [1] for the study of the influence of the stabilization parameter. A rather small influence is noted on the error in $L^2(\Omega)$ and $H^1(\Omega)$-norms on u. Concerning the error in $L^2(\Gamma_D)$-norm of the multiplier, the value of the stabilization parameter can be divided into two zones. A coercive zone where the error decreases when the stabilization parameter γ_0 increases and a non-coercive zone for large values of the stabilization parameter where the error evolves randomly according to the stabilization parameter.

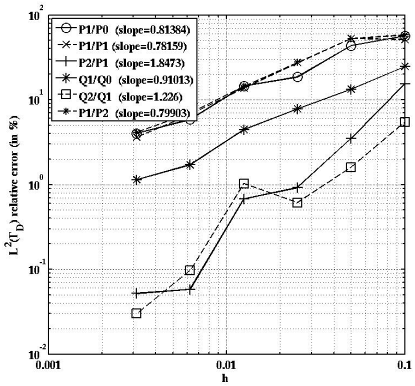
Local projection stabilization. Similarly to the Barbosa-Hughes stabilization, the local projection stabilization gives some optimal rates of convergence for all pairs of elements and eliminates the locking phenomena (Fig. 10(a)) and the spurious modes on the multiplier (Fig. 8(b)). The convergence curves are shown in Fig. 11(a) for the error in the $L^2(\Omega)$-norm on u, in Fig. 11(b) for the error in the $H^1(\Omega)$-norm on u and in Fig. 11(c) for the error in the $L^2(\Gamma_D)$-norm on the multiplier. The rates of convergence for the P_1/P_2, P_1/P_1, P_1/P_0 and Q_1/Q_0 methods are in good agreement with the theoretical result of Theorem 1. For the P_2/P_1 and Q_2/Q_1



(a) Convergence of $\|u - u^h\|_{0,\Omega}$

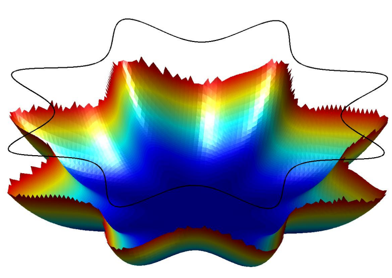


(b) Convergence of $\|u - u^h\|_{1,\Omega}$

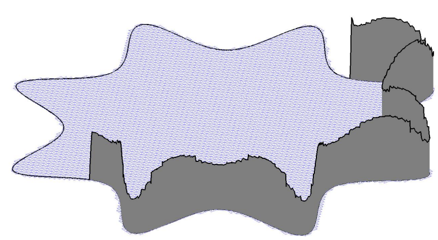


(c) Convergence of $\|\lambda^h - \lambda\|_{0,\Gamma_D}$

Figure 9: Convergence curves in the Barbosa-Hughes stabilized case ($\gamma = 0.0001$ for P_2/P_1 and Q_2/Q_1 methods and $\gamma = 0.1$ for the remaining methods).

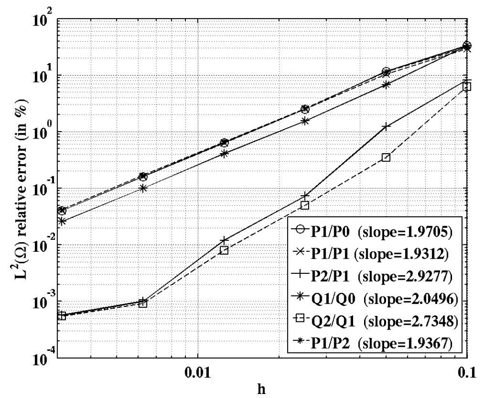


(a) Solution on Ω with local projection stabilization for the P_1/P_1 method.

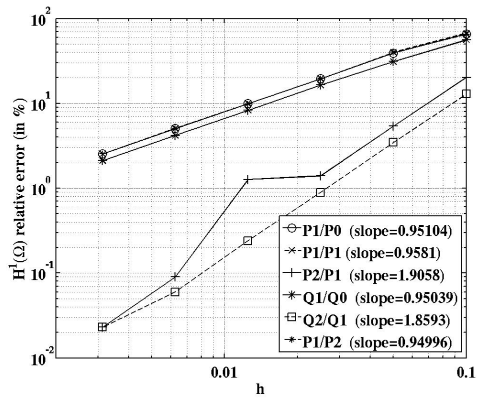


(b) Multiplier on Γ_D with local projection stabilization for the P_1/P_1 method.

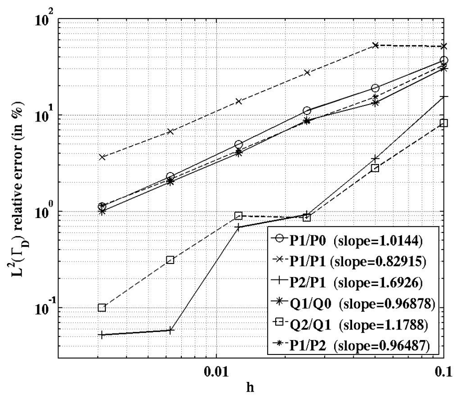
Figure 10: Local projection stabilized case with the P_1/P_1 method.



(a) Convergence of $\|u - u^h\|_{0,\Omega}$



(b) Convergence of $\|u - u^h\|_{1,\Omega}$



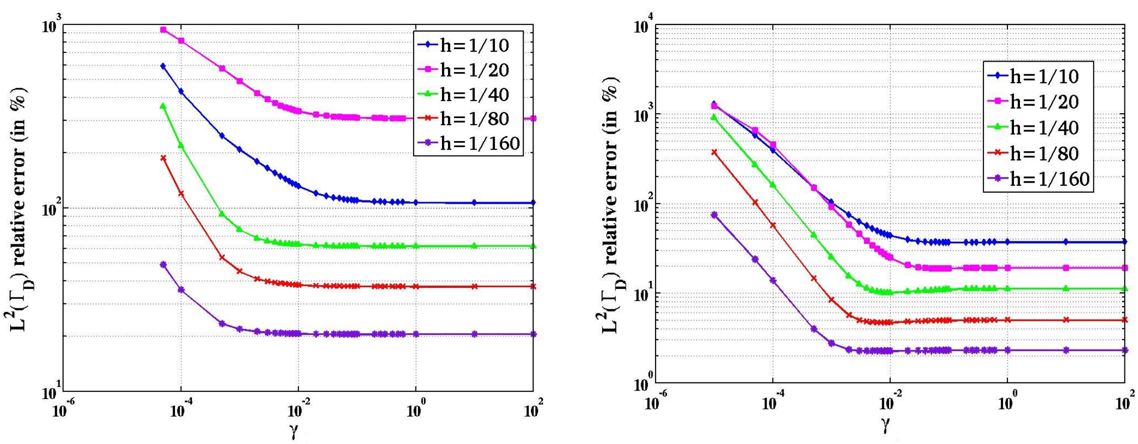
(c) Convergence of $\|\lambda^h - \lambda\|_{0,\Gamma_D}$

Figure 11: Convergence curves in the local projection stabilized case ($\gamma = 0.00001$ for P_2/P_1 and Q_2/Q_1 methods and $\gamma = 0.01$ for the remaining methods).

methods, the rates are close to optimality. For these methods, if one tries to extend the result of Theorem 1 to a $H^3(\Omega)$ regular exact solution, one finds that the rate of convergence of the error estimate depends on the interpolation error of the local orthogonal projection which limits the rate of convergence to 3/2 for the $H^1(\Omega)$-norm and 1 for the $L^2(\Gamma_D)$-norm on the multiplier (The same observation was shown in the case of Stokes and Darcy's equations by Burman [9]). This limitation is observed on Fig. 11(c) on the multiplier of the Q_2/Q_1 method, but not for the P_2/P_1 method (for an unknown reason).

Concerning the error in $L^2(\Gamma_D)$-norm the value of the stabilization parameter can also be divided into two zones (see Figs. 12, 13 and 14). The first zone where the error decreases when the stabilization parameter γ increases. The second zone, for large values of the parameter, where the error increases (Figs. 13, 14) or remains almost constant (Fig. 12). Figure 12 for the P_1/P_0 elements indicates that a large value of the stabilization parameter does not affect too much the quality of the solution. This behavior has been noted whenever a piecewise constant multiplier is considered. Conversely, for all remaining couples of elements, an excessive value of the stabilization parameter leads to a bad quality solution (see Figs. 13, 14).

Now, concerning the minimal patch size, the inf-sup condition is proven to be satisfied in [16] for a size equal or greater to $3h$. Numerically, the inf-sup condition seems to be satisfied for smaller values of the minimal patch size. In our numerical experiments we found an optimal value between h and $2h$. For the P_1/P_0 method, a minimal patch size equal to h seems to be inadequate (Fig. 12(a)). A value of $2h$ is found to be more optimal (Fig. 12(b)). Conversely, a value of h is slightly more optimal for the $P1/P1$ pair of elements (Fig. 13).



(a) With a minimal patch size equal to h

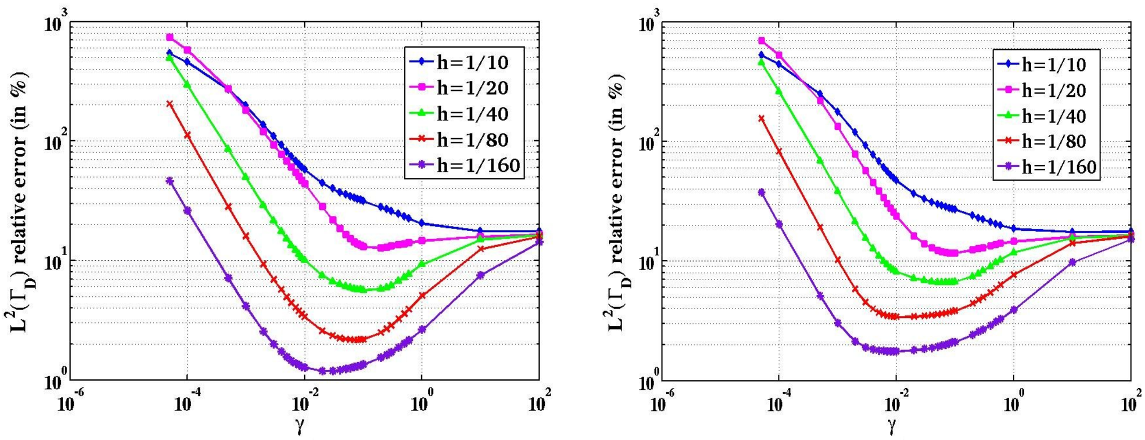
(b) With a minimal patch size equal to 2h

Figure 12: Influence of the stabilization parameter for the error in the $L^2(\Gamma_D)$-norm of the multiplier for the P_1/P_0-element.

4.4 Three-dimensional numerical tests

In this section, we compare the non-stabilized three-dimensional case to the local projection stabilized three-dimensional case with the following pairs of finite-element methods: P_2/P_1, P_1/P_1, P_1/P_0, P_1/P_2, Q_2/Q_1 and Q_1/Q_0.

Without stabilization. Convergence curves in the non-stabilized case are shown in Fig. 15. Perhaps due to the simple chosen geometry and exact solution, no locking phenomenon is



(a) With a minimal patch size equal to h (b) With a minimal patch size equal to 2h

Figure 13: Influence of the stabilization parameter for the error in the $L^2(\Gamma_D)$-norm of the multiplier for the P_1/P_1-element.

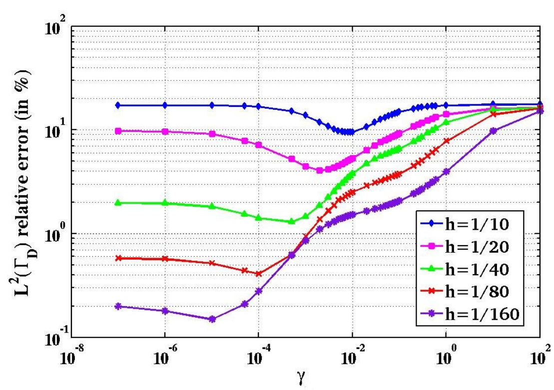
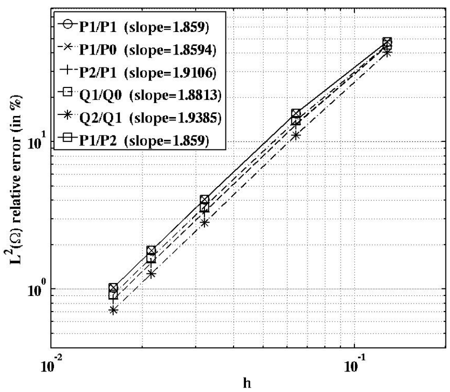
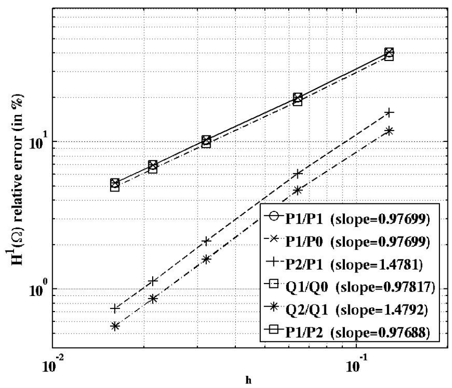


Figure 14: Influence of the stabilization parameter for the error in the $L^2(\Gamma_D)$-norm of the multiplier for the P_2/P_1-element (with a minimal patch size equal to h).

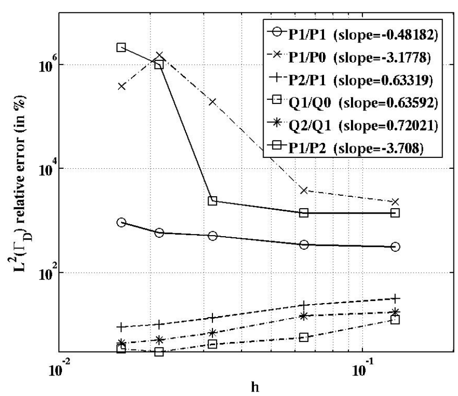
observed for the P_1/P_2, P_1/P_1 and P_1/P_0 methods. However, in these cases, the multiplier does not converge probably due to the presence of spurious modes. The rate of convergence in the $H^1(\Omega)-$ norm on u is optimal for the P_1/P_1, P_1/P_0, P_1/P_2 and Q_1/Q_0 methods (see Fig. 15(b)). For the remaining elements (Q_2/Q_1 and P_2/P_1) the rate of convergence is limited to 3/2.



(a) Convergence of $\|u - u^h\|_{0,\Omega}$



(b) Convergence of $\|u - u^h\|_{1,\Omega}$

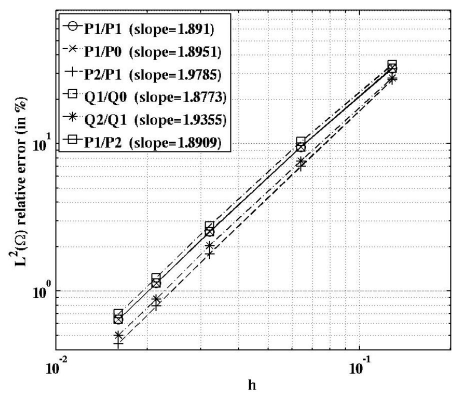


(c) Convergence of $\|\lambda^h - \lambda\|_{0,\Gamma_D}$

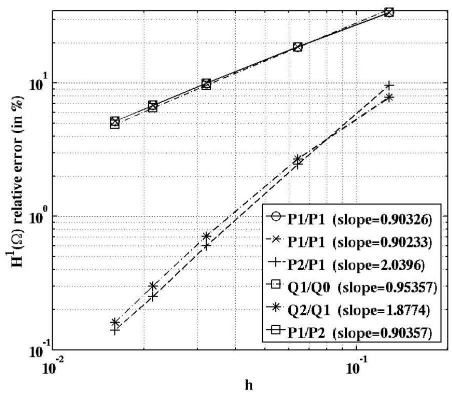
Figure 15: Convergence curves in the three-dimensional non-stabilized case.

Local projection stabilization. The local projection stabilization gives an optimal rate of convergence for all pairs of elements and eliminates the spurious modes for the P_1/P_1, P_1/P_0 and P_1/P_2 methods. Especially, the rate of convergence in the $H^1(\Omega)$-norm for the Q_2/Q_1 and P_2/P_1 are improved compared to the non-stabilized case.

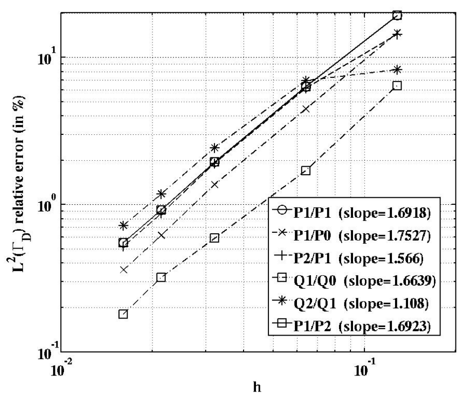
Except for the Q_2/Q_1 pair of elements, the convergence rate for the $L^2(\Gamma_D)$-norm for the multiplier are optimal (more than 1.5). For the Q_2/Q_1 pair of elements, the convergence rate for the $L^2(\Gamma_D)$-norm is optimal but limited to 1.1 (we did not find any interpretation for that). The rate of convergence in the $L^2(\Omega)$-norm is limited to 2 for all methods. For quadratic methods, the fact that we used level set function of order 1 to approximate the curved domain limits theoretically the rate of convergence to 3/2.



(a) Convergence of $\|u - u^h\|_{0,\Omega}$



(b) Convergence of $\|u - u^h\|_{1,\Omega}$



(c) Convergence of $\|\lambda^h - \lambda\|_{0,\Gamma_D}$

Figure 16: Convergence curves in the three-dimensional local projection stabilized case ($\gamma = 0.00001$ for P_2/P_1 and Q_2/Q_1 methods and $\gamma = 0.01$ for the remaining methods).

5 Concluding remarks

In this paper, we presented a stabilization technique based on local projections for the fictitious domain method inspired by the X-FEM introduced in [12, 19].

A main advantage compared to some other stabilization techniques like the Barbosa-Hughes one, is that it only affects the multiplier equation in a manner that is independent of the problem to be solved. This makes the extension to other linear or nonlinear problems very easy.

The two-dimensional theoretical result does not ensure an optimal rate of convergence when a quadratic finite element is used for the main unknown due to the fact that the local projection is made on piecewise constants. The method could be generalized to the projection on (discontinuous) piecewise affine or piecewise quadratic functions for high-order approximations.

The extension to the three-dimensional case of the theoretical result is of course subject to obtaining an inf-sup condition of the same kind of the one obtained in [16].

References

[1] S. Amdouni, P. Hild, V. Lleras, M. Moakher, Y. Renard, A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies, ESAIM Math. Model. Numer. Anal. 46 (4) (2012) 813–839.

[2] P. Angot, C.-H. Bruneau, P. Fabrie, A penalization method to take into account obstacles in incompressible viscous flows, Numerische Mathematik 81 (1999) 497–520.

[3] H. J. Barbosa, T. Hughes, The finite element method with Lagrange multipliers on the boundary: circumventing the Babuška-Brezzi condition, Comput. Methods Appl. Mech. Engrg. 85 (1991) 109–128.

[4] H. J. Barbosa, T. Hughes, Boundary Lagrange multipliers in finite element methods: error analysis in natural norms, Numer. Math. 62 (1992) 1–15.

[5] G. R. Barrenechea, F. Chouly, A local projection stabilized method for fictitious domains, Appl. Math. Lett. 25 (12) (2012) 2071 – 2076.

[6] É. Béchet, N. Moës, B. Wohlmuth, A stable Lagrange multiplier space for stiff interface conditions within the extended finite element method, Internat. J. Numer. Methods Engrg. 78 (8) (2009) 931–954.

[7] R. Becker, M. Braack, A finite element pressure gradient stabilization for the Stokes equations based on local projections, Calcolo 38 (2001) 173–199.

[8] S. Bertoluzza, M. Ismail, B. Maury, The fat boundary method: Semi-discrete scheme and some numerical experiments, in: Domain Decomposition Methods in Science and Engineering, vol. 40 of Lecture Notes in Computational Science and Engineering, Springer Berlin Heidelberg, 2005, pp. 513–520.

[9] E. Burman, Pressure projection stabilizations for galerkin approximations of Stokes' and Darcy's problem, Numer. Methods Partial Differential Equations 24 (1) (2008) 127–143.

[10] E. Burman, Projection stabilisation of lagrange multipliers for the imposition of constraints on interfaces and boundaries, ArXiv e-prints, http://arxiv.org/abs/1203.4116.

[11] E. Burman, P. Hansbo, Fictitious domain finite element methods using cut elements: I. A stabilized Lagrange multiplier method, Comput. Methods Appl. Mech. Engrg. 199 (41-44) (2010) 2680–2686.

[12] E. Burman, P. Hansbo, Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method, Appl. Numer. Math. 62 (4) (2012) 328–341.

[13] E. J. Dean, Q. V. Dinh, R. Glowinski, J. He, T.-W. Pan, J. Périaux, Least squares/domain imbedding methods for Neumann problems: applications to fluid dynamics, in: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991), SIAM, Philadelphia, PA, 1992, pp. 451–475.

[14] C. R. Dohrmann, P. B. Bochev, A stabilized finite element method for the stokes problem based on polynomial pressure projections, International Journal for Numerical Methods in Fluids 46 (2) (2004) 183–201.

[15] A. Ern, J.-L. Guermond, Theory and Practice of Finite Elements, vol. 159, Springer-Verlag, Applied Mathematical Sciences, New York, 2004.

[16] V. Girault, R. Glowinski, Error analysis of a fictitious domain method applied to a Dirichlet problem, Japan J. Indust. Appl. Math. 12 (3) (1995) 487–514.

[17] R. Glowinski, Y. Kuznetsov, On the solution of the Dirichlet problem for linear elliptic operators by a distributed Lagrange multiplier method, C. R. Acad. Sci. Paris Sér. I Math. 327 (7) (1998) 693–698.

[18] R. Glowinski, T. W. Pan, R. O. Wells, Jr., X. Zhou, Wavelet and finite element solutions for the Neumann problem using fictitious domains, J. Comput. Phys. 126 (1) (1996) 40–51.

[19] J. Haslinger, Y. Renard, A new fictitious domain approach inspired by the extended finite element method, SIAM J. Numer. Anal. 47 (2) (2009) 1474–1499.

[20] G. Karypis, V. Kumar, Metis: Unstructured graph partitioning and sparse matrix ordering system., `http://www.cs.umn.edu/~metis`.

[21] G. I. Marchuk, Methods of numerical mathematics, vol. 2 of Applications of Mathematics, 2nd ed., Springer-Verlag, New York, 1982, translated from the Russian by Arthur A. Brown.

[22] B. Maury, A fat boundary method for the Poisson problem in a domain with holes, J. Sci. Comput. 16 (3) (2001) 319–339.

[23] N. Moës, É. Béchet, M. Tourbier, Imposing Dirichlet boundary conditions in the extended finite element method, Internat. J. Numer. Methods Engrg. 67 (12) (2006) 1641–1669.

[24] N. Moës, J. Dolbow, T. Belytschko, A finite element method for cracked growth without remeshing, Inter. J. Numer. Methods Engrg. 46 (1999) 131–150.

[25] J. Nitsche, über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, Abh. Math. Univ. Hamburg 36 (1971) 9–15.

[26] C. Peskin, Flow patterns around heart valves: A numerical method, J. Comput. Phys. 10(2) (1972) 252–271.

[27] C. S. Peskin, The immersed boundary method, Acta Numer. 11 (2002) 479–517.

[28] J. Pommier, Y. Renard, Getfem++, an open source generic C++ library for finite element methods, `http://www-gmm.insa-toulouse.fr/getfem`.

[29] L. Rukhovets, A remark on the method of fictive domains, Diff. Equ. 3,4 (in russian) (1967) 698–701.

[30] V. Saul'ev, On the solution of some boundary value problems on high performance computers by fictitious domain method, Siberian Math. Journal 4:4 (in russian) (1963) 912–925.

[31] R. Stenberg, On some techniques for approximating boundary conditions in the finite element method, J. Comput. Appl. Math. 63 (1-3) (1995) 139–148, international Symposium on Mathematical Modelling and Computational Methods Modelling 94 (Prague, 1994).

[32] N. Sukumar, D. L. Chopp, N. Moës, T. Belytschko, Modeling holes and inclusions by level sets in the extended finite-element method, Comput. Methods Appl. Mech. Engrg. 190 (46-47) (2001) 6183–6200.